Optimal design of transport networks

Benedikt Wirth (joint work with Alessio Brancolini)

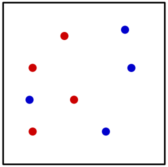
Lyon

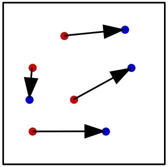
July 4th, 2016

Optimal transport and optimal transport networks

Task: Transport material from sources to sinkes at low cost

Special case: Transport via network with cost-minimizing geometry



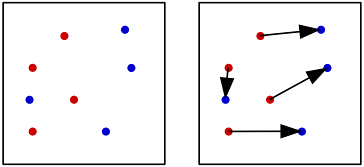


"Monge's problem"

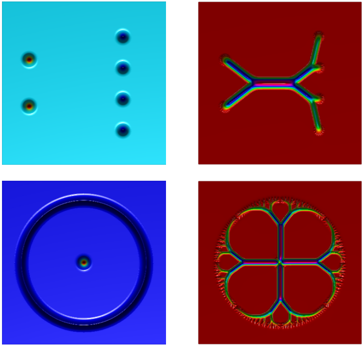
Optimal transport and optimal transport networks

Task: Transport material from sources to sinkes at low cost

Special case: Transport via network with cost-minimizing geometry



"Monge's problem"

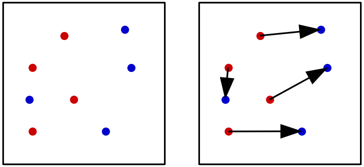


simulation by Edouard Oudet

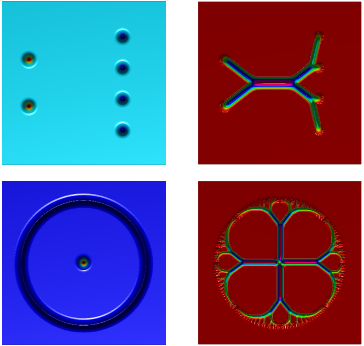
Optimal transport and optimal transport networks

Task: Transport material from sources to sinkes at low cost

Special case: Transport via network with cost-minimizing geometry



“Monge’s problem”



“Urban planning”

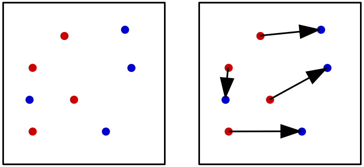
“Branched transport”

simulation by Edouard Oudet

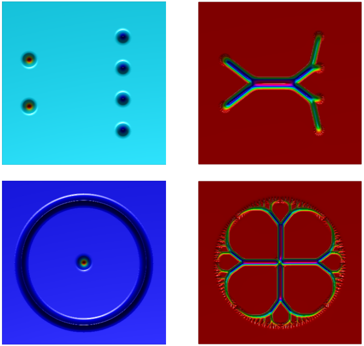
Optimal transport and optimal transport networks

Task: Transport material from sources to sinkes at low cost

Special case: Transport via network with cost-minimizing geometry



"Monge's problem"



simulation by Edouard Oudet

"Urban planning"
"Branched transport"

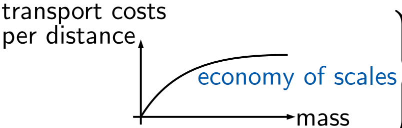
Models for transport networks: Intuition

cost functional = transport costs + network costs

network costs

≀

network length



⇒ branching structures

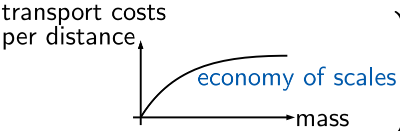
Models for transport networks: Intuition

cost functional = transport costs + network costs

network costs

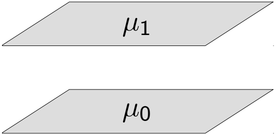
≀

network length



} ⇒ branching structures

sought: 1D pipe network $\Sigma \subset \mathbb{R}^n$ for transport from μ_0 to μ_1



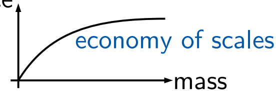
Models for transport networks: Intuition

cost functional = transport costs + network costs

network costs

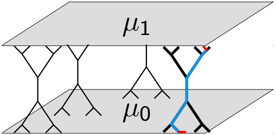
$\wr$

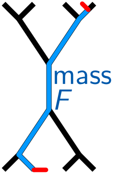
network length



$\Rightarrow$ branching structures

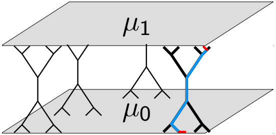
sought: 1D pipe network $\Sigma \subset \mathbb{R}^n$ for transport from μ_0 to μ_1

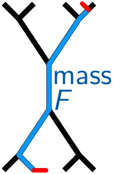




cost per flux & length	urban planning	branched transport
	1	$F^{-\varepsilon}$
	$a > 1$	∞
	ε	0

Are models of qualitatively different type?





cost per flux & length	urban planning	branched transport
	1	$F^{-\varepsilon}$
	$a > 1$	∞
	ε	0

Classical formulation **urban planning**

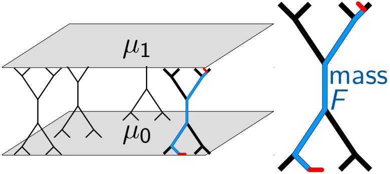
$$d_\Sigma(x,y) = \inf\{a\mathcal{H}^1(\gamma \setminus \Sigma) + \mathcal{H}^1(\gamma \cap \Sigma) \ : \ \gamma \text{ path from } x \text{ to } y\}$$

$$\mathcal{E}^{\varepsilon,a}[\Sigma] = W_{d_\Sigma}(\mu_0,\mu_1) + \varepsilon\mathcal{H}^1(\Sigma)$$

$$= \inf_{\substack{\mu\in\mathrm{fbm}(\mathbb{R}^n\times\mathbb{R}^n) \\ \pi_{1\#}\mu=\mu_0,\quad \pi_{2\#}\mu=\mu_1}} \int_{\mathbb{R}^n\times\mathbb{R}^n} d_\Sigma(x,y)\,\mathrm{d}\mu(x,y) + \varepsilon\mathcal{H}^1(\Sigma)$$

- requires computation of d_Σ or dual formulation
- variation with respect to Σ nontrivial

Are models of qualitatively different type?

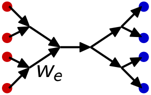


	urban planning	branched transport
cost per flux & length	1	$F^{-\varepsilon}$
	$a > 1$	∞
	ε	0

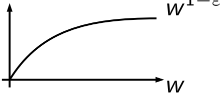
Classical formulation **branched transport**

$G = (V, E) =$ directed weighted graph

$w_e =$ flow through $e \in E$, $\quad l_e =$ length of e

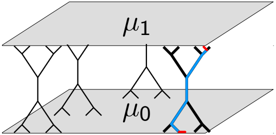


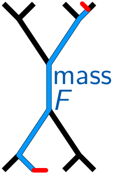
$$\mathcal{M}^\varepsilon[G] = \begin{cases} \sum_{e\in E} w_e^{1-\varepsilon} l_e & \text{if Kirchhoff laws satisfied} \\ \infty & \text{else} \end{cases}$$



$$\mathcal{M}^\varepsilon[\mathcal{F}] = \inf\{\liminf_{n\to\infty} \mathcal{M}^\varepsilon[G_n] \mid G_n \stackrel{*}{\rightharpoonup} \mathcal{F},\ \mathrm{div}\mathcal{F} = \mu_0 - \mu_1\}$$

Are models of qualitatively different type?



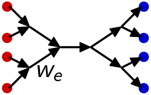


	urban planning	branched transport
cost per flux & length	1	$F^{-\varepsilon}$
	$a > 1$	∞
	ε	0

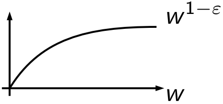
Classical formulation **branched transport**

$G = (V, E) =$ directed weighted graph

$w_e =$ flow through $e \in E$, $\quad l_e =$ length of e



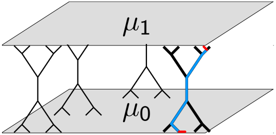
$$\mathcal{M}^\varepsilon[G] = \begin{cases} \sum_{e\in E} w_e^{1-\varepsilon} l_e & \text{if Kirchhoff laws satisfied} \\ \infty & \text{else} \end{cases}$$



$$\mathcal{M}^\varepsilon[\mathcal{F}] = \inf\{\liminf_{n\to\infty} \mathcal{M}^\varepsilon[G_n] \,|\, G_n \overset{*}{\rightharpoonup} \mathcal{F},\ \mathrm{div}\mathcal{F} = \mu_0 - \mu_1\}$$

allows phase field description!

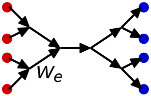
Are models of qualitatively different type?



cost per flux & length	urban planning	branched transport
	1	$F^{-\varepsilon}$
	$a > 1$	∞
	ε	0

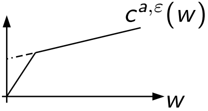
New formulation urban planning [Brancolini, Wirth '15]

$G = (V, E) =$ directed weighted graph



$w_e =$ flow through $e \in E$, $\quad l_e =$ length of e

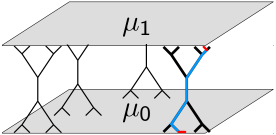
$$\tilde{\mathcal{E}}^{a,\varepsilon}[G] = \begin{cases} \sum_{e\in E} \mathbf{c}^{a,\varepsilon}(\mathbf{w}) l_e & \text{if Kirchhoff laws satisfied} \\ \infty & \text{sonst} \end{cases}$$

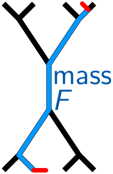


$$\tilde{\mathcal{E}}^{a,\varepsilon}[\mathcal{F}] = \inf\{\liminf_{n\to\infty} \mathcal{M}^{\varepsilon}[G_n] \mid G_n \overset{*}{\rightharpoonup} \mathcal{F},\ \mathrm{div}\mathcal{F} = \mu_0 - \mu_1\}$$

Thm. $\min_\Sigma \mathcal{E}^{a,\varepsilon}[\Sigma] = \min_{\mathcal{F}} \tilde{\mathcal{E}}^{a,\varepsilon}[\mathcal{F}]$ & $\Sigma_{\mathrm{opt}} \subset \mathrm{spt}\mathcal{F}_{\mathrm{opt}}$

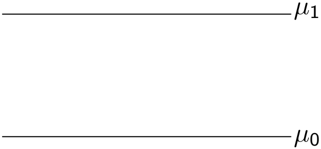
Analysis of optimal geometries



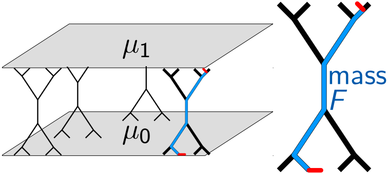


cost per flux & length	urban planning	branched transport
	1	$F^{-\varepsilon}$
	$a > 1$	∞
	ε	0

Near-optimal networks

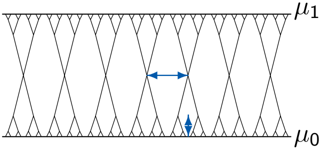


Analysis of optimal geometries



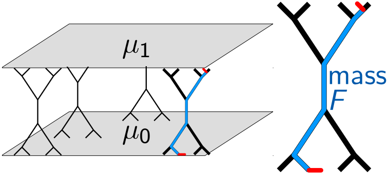
cost per flux & length	urban planning	branched transport
	1	$F^{-\varepsilon}$
	$a > 1$	∞
	ε	0

Near-optimal networks



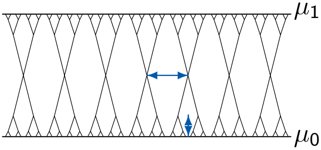
length scales in terms of powers of ε

Analysis of optimal geometries

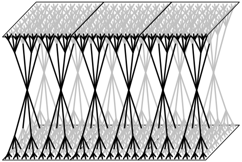


cost per flux & length	urban planning	branched transport
	1	$F^{-\varepsilon}$
	$a > 1$	∞
	ε	0

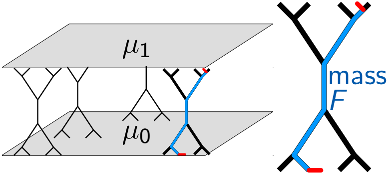
Near-optimal networks



length scales in terms of powers of ε



Analysis of optimal geometries



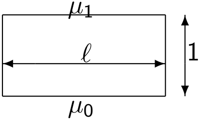
cost per flux & length	urban planning	branched transport
	1	$F^{-\varepsilon}$
	$a > 1$	∞
	ε	0

Thm. $cf(\varepsilon, a) \leq \min_{\Sigma} \mathcal{J}^{\varepsilon,a}[\Sigma] - \mathcal{J}^* \leq Cf(\varepsilon, a)$

$$f(\varepsilon, a) = \begin{cases} \varepsilon^{\frac{2}{3}} & \text{urban planning 2D } (\mathcal{J}^{\varepsilon,a} = \mathcal{E}^{\varepsilon,a}) \\ \sqrt{\varepsilon}(\sqrt{a} + |\log \frac{a-1}{\sqrt{\varepsilon}}|) & \text{urban planning 3D } (\mathcal{J}^{\varepsilon,a} = \mathcal{E}^{\varepsilon,a}) \\ \varepsilon^{\frac{1}{n-1}} \sqrt{a} \sqrt{a-1}^{\frac{n-3}{n-1}} & \text{urban planning } n\text{D } (\mathcal{J}^{\varepsilon,a} = \mathcal{E}^{\varepsilon,a}) \\ \varepsilon |\log \varepsilon| & \text{branched transport } (\mathcal{J}^{\varepsilon,a} = \mathcal{M}^{\varepsilon}) \end{cases}$$

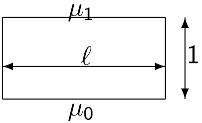
Relaxed energy and upper bound

$\mathcal{J}^\varepsilon[\mathcal{F}] = \int_\Sigma F(x) + \varepsilon\,\mathrm{d}\mathcal{H}^1(x) \qquad \mathcal{F} = \vec{F}\mathcal{H}^1 \llcorner \Sigma$



Thm. $c\ell\varepsilon^{2/3} \le \min_{\mathcal{F}} \mathcal{J}^\varepsilon[\mathcal{F}] - \mathcal{J}^*_{\mu_0,\mu_1} \le C\ell\varepsilon^{2/3}$

Relaxed energy and upper bound



$\mathcal{J}^\varepsilon[\mathcal{F}] = \int_\Sigma F(x) + \varepsilon \, \mathrm{d}\mathcal{H}^1(x) \qquad \mathcal{F} = \vec{F}\mathcal{H}^1 \llcorner \Sigma$

Thm. $$c\ell\varepsilon^{2/3} \leq \min_{\mathcal{F}} \mathcal{J}^\varepsilon[\mathcal{F}] - \mathcal{J}^*_{\mu_0,\mu_1} \leq C\ell\varepsilon^{2/3}$$

Step 1: Relaxation for $\varepsilon = 0$

given source μ_a & sink μ_b,

$\mathcal{J}^*_{\mu_a,\mu_b} := \inf\{\mathcal{J}^0[\mathcal{F}] \,|\, \mathcal{F}$ transports μ_a to $\mu_b\}$

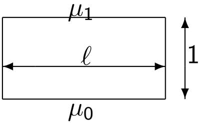
$= \text{Wasserstein-distance}(\mu_a, \mu_b)$

$\mathcal{J}^*_{\mu_a,\mu_b}$ can be computed/accurately estimated!

(e. g. via convex duality) $\quad \Rightarrow \quad \mathcal{J}^*_{\mu_0,\mu_1} = \ell$

Relaxed energy and upper bound

$\mathcal{J}^\varepsilon[\mathcal{F}] = \int_\Sigma F(x) + \varepsilon \,\mathrm{d}\mathcal{H}^1(x)$ $\quad$ $\mathcal{F} = \vec{F}\mathcal{H}^1 \llcorner \Sigma$



Thm. $\quad c\ell\varepsilon^{2/3} \leq \min_{\mathcal{F}} \mathcal{J}^\varepsilon[\mathcal{F}] - \mathcal{J}^*_{\mu_0,\mu_1} \leq C\ell\varepsilon^{2/3}$

Step 1: Relaxation for $\varepsilon = 0$

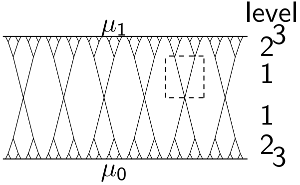
given source μ_a & sink μ_b,

$\mathcal{J}^*_{\mu_a,\mu_b} := \inf\{\mathcal{J}^0[\mathcal{F}] \,|\, \mathcal{F}$ transports μ_a to $\mu_b\}$

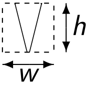
$\quad = \text{Wasserstein-distance}(\mu_a, \mu_b)$

$\mathcal{J}^*_{\mu_a,\mu_b}$ can be computed/accurately estimated!
(e. g. via convex duality) $\quad \Rightarrow \quad \mathcal{J}^*_{\mu_0,\mu_1} = \ell$

Step 2: Upper bound by construction



elementary cell



$w_1 \sim \sqrt[3]{\varepsilon}$

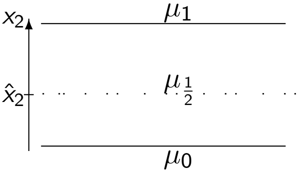
$h_i = \sqrt{w_i^3/\varepsilon}$

Lower bound

$\mathcal{J}^\varepsilon[\Sigma] = \int_\Sigma F(x)\,\mathrm{d}\mathcal{H}^1(x) + \varepsilon \mathrm{length}(\Sigma)$

abbr.: $\hat{\mathcal{J}} \equiv \min_\Sigma \mathcal{J}^\varepsilon[\Sigma]\,,$ $\Delta\mathcal{J} \equiv \hat{\mathcal{J}} - \mathcal{J}^*_{\mu_0,\mu_1}$

Step 3: Lower bound based on relaxation



on $x_2 = \hat{x}_2$

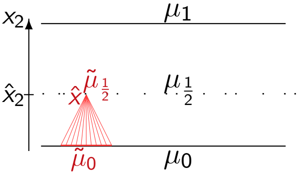
$N \lesssim \Delta\mathcal{J}/\varepsilon$ intersecting pipes

Lower bound

$\mathcal{J}^\varepsilon[\Sigma] = \int_\Sigma F(x)\,\mathrm{d}\mathcal{H}^1(x) + \varepsilon\,\mathrm{length}(\Sigma)$

abbr.: $\hat{\mathcal{J}} \equiv \min_\Sigma \mathcal{J}^\varepsilon[\Sigma]\,,\qquad \Delta\mathcal{J} \equiv \hat{\mathcal{J}} - \mathcal{J}^*_{\mu_0,\mu_1}$

Step 3: Lower bound based on relaxation



on $x_2 = \hat{x}_2$

$N \lesssim \Delta\mathcal{J}/\varepsilon$ intersecting pipes

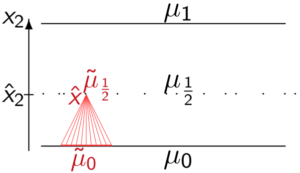
$$\mathcal{J}^*_{\tilde{\mu}_0,\tilde{\mu}_{\frac{1}{2}}} = F(\hat{x})\begin{pmatrix}\text{average}\\ \text{transport}\\ \text{distance}\end{pmatrix} = F(\hat{x})\sqrt{\hat{x}_2^2 + c_1 F(\hat{x})^2} \geq F(\hat{x})\left(\hat{x}_2 + c_2\frac{F(\hat{x})^2}{\hat{x}_2}\right)$$

Lower bound

$\mathcal{J}^\varepsilon[\Sigma] = \int_\Sigma F(x)\,\mathrm{d}\mathcal{H}^1(x) + \varepsilon\,\text{length}(\Sigma)$

abbr.: $\hat{\mathcal{J}} \equiv \min_\Sigma \mathcal{J}^\varepsilon[\Sigma]\,, \qquad \Delta\mathcal{J} \equiv \hat{\mathcal{J}} - \mathcal{J}^*_{\mu_0,\mu_1}$

Step 3: Lower bound based on relaxation



on $x_2 = \hat{x}_2$

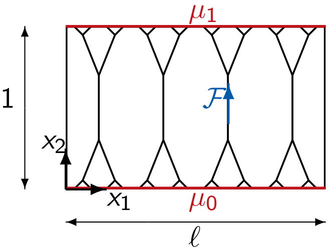
$N \lesssim \Delta\mathcal{J}/\varepsilon$ intersecting pipes

$$\mathcal{J}^*_{\tilde{\mu}_0,\tilde{\mu}_{\frac12}} = F(\hat{x})\begin{pmatrix}\text{average}\\ \text{transport}\\ \text{distance}\end{pmatrix} = F(\hat{x})\sqrt{\hat{x}_2^2 + c_1 F(\hat{x})^2} \geq F(\hat{x})\left(\hat{x}_2 + c_2\frac{F(\hat{x})^2}{\hat{x}_2}\right)$$

$$\hat{\mathcal{J}} \geq \mathcal{J}^*_{\mu_0,\mu_{\frac12}} + \mathcal{J}^*_{\mu_{\frac12},\mu_1} \geq \sum_{\hat{x}\in\text{supp}\mu_0}\left[\hat{x}_2 F(\hat{x}) + c\frac{F(\hat{x})^3}{\hat{x}_2}\right] + \left[(1-\hat{x}_2)F(\hat{x}) + c\frac{F(\hat{x})^3}{1-\hat{x}_2}\right]$$

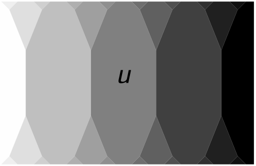
$$\geq \ell + \sum_{\hat{x}\in\text{supp}\mu_0} c\frac{F(\hat{x})^3}{\frac12} \geq \mathcal{J}^*_{\mu_0,\mu_1} + 2c\ell\left(\frac{\ell}{N}\right)^2 \geq \mathcal{J}^*_{\mu_0,\mu_1} + 2c\ell\left(\frac{\varepsilon\ell}{\Delta\mathcal{J}}\right)^2$$

Analysis & numerics in 2D via images



$$\mathcal{F} \in \mathrm{fbm}(\Omega; \mathbb{R}^2)$$

$$\mathrm{div}\mathcal{F} = \mu_0 - \mu_1$$

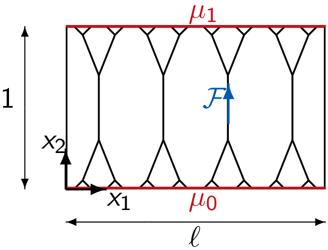


$$u \in \mathrm{BV}(\Omega; \mathbb{R})$$

$$u(\hat{x}_1, 0) = \int_{\{x_2=0, x_1 \leq \hat{x}_1\}} \mathrm{d}\mu_0(x)$$

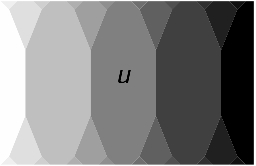
$$u(\hat{x}_1, 1) = \int_{\{x_2=0, x_1 \leq \hat{x}_1\}} \mathrm{d}\mu_1(x)$$

Analysis & numerics in 2D via images



$$\mathcal{F} \in \mathrm{fbm}(\Omega; \mathbb{R}^2)$$

$$\mathrm{div}\mathcal{F} = \mu_0 - \mu_1$$



$$u \in \mathrm{BV}(\Omega; \mathbb{R})$$

$$u(\hat{x}_1, 0) = \int_{\{x_2=0, x_1 \leq \hat{x}_1\}} \mathrm{d}\mu_0(x)$$

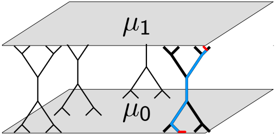
$$u(\hat{x}_1, 1) = \int_{\{x_2=0, x_1 \leq \hat{x}_1\}} \mathrm{d}\mu_1(x)$$

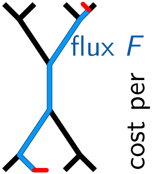
One-to-one relation fluxes $\leftrightarrow$ images: $\mathcal{F}_u = \nabla u^\perp$, $\Sigma = S_u$

$$\int_\Omega \phi \,\mathrm{d}(\mathrm{div}\mathcal{F}_u) = -\int_\Omega \nabla\phi \cdot \mathrm{d}\mathcal{F}_u = \int_\Omega \nabla\phi^\perp \cdot \mathrm{d}\nabla u = -\int_\Omega \mathrm{div}(\nabla\phi^\perp) u \,\mathrm{d}x = 0 \quad \forall \phi \in \mathcal{C}_c^\infty(\Omega)$$

... with boundary terms: $\mathrm{div}\mathcal{F}_u = \mu_0 - \mu_1$

Network functionals in terms of images



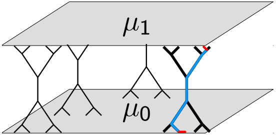


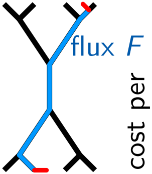
	urban planning	branched transport
cost per flux & length	1	$F^{-\varepsilon}$
	$a > 1$	∞
	ε	0

Versions of Mumford–Shah segmentation ...

M.-S.: $$\mathcal{J}^{\varepsilon,a}[u] = \int_{\overline{\Omega}\setminus S_u} a(u-\hat{u})^2 + |\nabla u|^2 \mathrm{d}x + \int_{S_u} \qquad \varepsilon \qquad \mathrm{d}\mathcal{H}^1$$

Network functionals in terms of images





	urban planning	branched transport
cost per flux & length	1	$F^{-\varepsilon}$
	$a > 1$	∞
	ε	0

Versions of Mumford–Shah segmentation ...

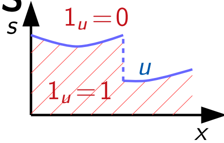
M.-S.: $\mathcal{J}^{\varepsilon,a}[u] = \int_{\overline{\Omega}\setminus S_u} g(x,u,\nabla u)\,\mathrm{d}x + \int_{S_u} \psi(x,u^+,u^-,\nu)\,\mathrm{d}\mathcal{H}^1$

urb. pl.: $\tilde{\mathcal{E}}^{\varepsilon,a}[u] = \begin{cases} \int_{\overline{\Omega}\setminus S_u} a|\nabla u|\,\mathrm{d}x + \int_{S_u} |[u]| + \varepsilon\,\mathrm{d}\mathcal{H}^1 & \text{if } u \text{ satisfies b. c.} \\ \infty & \text{else} \end{cases}$

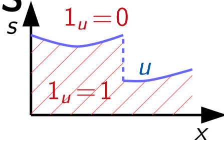
br. tpt.: $\tilde{\mathcal{M}}^{\varepsilon}[u] = \begin{cases} \int_{S_u} |[u]|^{1-\varepsilon}\,\mathrm{d}\mathcal{H}^1 & \text{if } u \text{ satisfies b. c. and } \nabla u \equiv 0 \\ \infty & \text{else} \end{cases}$

By now a classic: Functional lifting for MS

$$1_u : \Omega \times \mathbb{R} \to \{0,1\},\ (x,s) \mapsto \begin{cases} 1 & \text{if } u(x) > s \\ 0 & \text{else} \end{cases}$$



By now a classic: Functional lifting for MS



$$1_u : \Omega \times \mathbb{R} \to \{0,1\},\ (x,s) \mapsto \begin{cases} 1 & \text{if } u(x) > s \\ 0 & \text{else} \end{cases}$$

$$\mathcal{J}^{\varepsilon,a}[u] = \int_{\overline{\Omega}\setminus S_u} g(x,u,\nabla u)\,\mathrm{d}x + \int_{S_u} \psi(x,u^+,u^-,\nu)\,\mathrm{d}\mathcal{H}^1$$

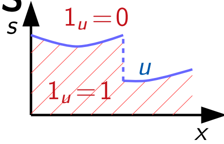
$$= \sup_{\phi\in\mathcal{K}} \int_{\Omega\times\mathbb{R}} \phi \cdot \mathrm{d}D1_u$$

$$\text{with } \mathcal{K} = \Big\{ \phi = (\phi^x,\phi^s) \in \mathcal{C}_0(\Omega\times\mathbb{R};\mathbb{R}^2\times\mathbb{R}) \ : $$

$$\phi^s(x,s) \geq g^*(x,s,\phi^x(x,s)) \quad \forall (x,s) \in \Omega\times\mathbb{R}\,,$$

$$\left| \int_{s_1}^{s_2} \phi^x(x,s)\mathrm{d}s \right| \leq \psi(x,s_1,s_2,\nu)\ \forall x \in \Omega, s_1 < s_2, \nu \in S^1 \Big\}$$

By now a classic: Functional lifting for MS



$$1_u : \Omega \times \mathbb{R} \to \{0,1\},\ (x,s) \mapsto \begin{cases} 1 & \text{if } u(x) > s \\ 0 & \text{else} \end{cases}$$

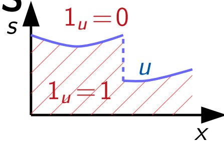
$$\mathcal{J}^{\varepsilon,a}[u] = \int_{\overline{\Omega}\setminus S_u} g(x,u,\nabla u)\,\mathrm{d}x + \int_{S_u} \psi(x,u^+,u^-,\nu)\,\mathrm{d}\mathcal{H}^1$$

$$= \sup_{\phi\in\mathcal{K}} \int_{\Omega\times\mathbb{R}} \phi\cdot \mathrm{d}D1_u$$

urb. pl.	br. tpt.
$a\lvert\nabla u\rvert$	$I_{\nabla u=0}$
$\lvert u^+ - u^-\rvert + \varepsilon$	$\lvert u^+ - u^-\rvert^{1-\varepsilon}$

$$\text{with } \mathcal{K} = \Big\{ \phi = (\phi^x,\phi^s) \in \mathcal{C}_0(\Omega\times\mathbb{R};\mathbb{R}^2\times\mathbb{R}) \ : \ \phi^s(x,s) \ge g^*(x,s,\phi^x(x,s)) \quad \forall (x,s)\in\Omega\times\mathbb{R}, \ \left|\int_{s_1}^{s_2}\phi^x(x,s)\mathrm{d}s\right| \le \psi(x,s_1,s_2,\nu)\ \forall x\in\Omega, s_1<s_2, \nu\in S^1 \Big\}$$

By now a classic: Functional lifting for MS



$$1_u : \Omega \times \mathbb{R} \to \{0,1\},\ (x,s) \mapsto \begin{cases} 1 & \text{if } u(x) > s \\ 0 & \text{else} \end{cases}$$

$$\mathcal{J}^{\varepsilon,a}[u] = \int_{\overline{\Omega}\setminus S_u} g(x,u,\nabla u)\,\mathrm{d}x + \int_{S_u} \psi(x,u^+,u^-,\nu)\,\mathrm{d}\mathcal{H}^1$$

$$= \sup_{\phi\in\mathcal{K}} \int_{\Omega\times\mathbb{R}} \phi \cdot \mathrm{d}D1_u$$

urb. pl.	br. tpt.
$a\lvert\nabla u\rvert$	$I_{\nabla u=0}$
$\lvert u^+ - u^-\rvert + \varepsilon$	$\lvert u^+ - u^-\rvert^{1-\varepsilon}$

$$\text{with } \mathcal{K} = \Big\{ \phi = (\phi^x,\phi^s) \in \mathcal{C}_0(\Omega\times\mathbb{R};\mathbb{R}^2\times\mathbb{R}) \ :$$

$$\phi^s(x,s) \geq g^*(x,s,\phi^x(x,s)) \quad \forall (x,s)\in\Omega\times\mathbb{R}\,,$$

$$\left| \int_{s_1}^{s_2} \phi^x(x,s)\mathrm{d}s \right| \leq \psi(x,s_1,s_2,\nu)\ \forall x\in\Omega, s_1 < s_2, \nu\in S^1 \Big\}$$

$$\inf_{u\in\mathrm{BV}(\Omega)} \mathcal{J}^{\varepsilon,a}[u] \geq \inf_{v\in\mathcal{C}} \sup_{\phi\in\mathcal{K}} \int_{\Omega\times\mathbb{R}} \phi\cdot\mathrm{d}Dv$$

$$\text{with } \mathcal{C} = \Big\{ v\in\mathrm{BV}(\Omega\times\mathbb{R};[0,1]) \ : \lim_{s\to-\infty} v(x,s) = 1, \lim_{s\to\infty} v(x,s) = 0 \Big\}$$

Lower bound on network costs

Urban planning: $g(x,u,\nabla u) = a|\nabla u|$, $\psi(x,s_1,s_2,\nu) = |s_1 - s_2| + \varepsilon$

$$\mathcal{K} = \left\{ \phi = (\phi^x, \phi^s) \,:\, \phi^s \geq 0, |\phi^x| \leq a, \left| \int_{s_1}^{s_2} \phi^x(x,s)\mathrm{d}s \right| \leq |s_2 - s_1| + \varepsilon \right\}$$

Branched transport: $g(x,u,\nabla u) = I_{\nabla u = 0}$, $\psi(x,s_1,s_2,\nu) = |s_1 - s_2|^{1-\varepsilon}$

$$\mathcal{K} = \left\{ \phi = (\phi^x, \phi^s) \,:\, \phi^s \geq 0, \qquad \left| \int_{s_1}^{s_2} \phi^x(x,s)\mathrm{d}s \right| \leq |s_2 - s_1|^{1-\varepsilon} \right\}$$

Lower bound: $\mathcal{J}^{\varepsilon,a} = \mathcal{E}^{\varepsilon,a}$, $\mathcal{J}^{\varepsilon,a} = \mathcal{M}^{\varepsilon}$

$$\min_{\mathrm{div}\mathcal{F} = \mu_0 - \mu_1} \mathcal{J}^{\varepsilon,a}[\mathcal{F}] = \min_{u|_{\partial\Omega}(x) = x_1} \tilde{\mathcal{J}}^{\varepsilon,a}[u] \geq \inf_{v|_{\partial(\Omega\times\mathbb{R})} = 1_{x\mapsto x_1}} \sup_{\phi\in\mathcal{K}} \int_{\overline{\Omega}\times\mathbb{R}} \phi \cdot \mathrm{d}Dv$$

Lower bound on network costs

Urban planning: $g(x,u,\nabla u) = a|\nabla u|$, $\psi(x,s_1,s_2,\nu) = |s_1 - s_2| + \varepsilon$

$$\mathcal{K} = \left\{ \phi = (\phi^x, \phi^s) \,:\, \phi^s \geq 0, |\phi^x| \leq a, \left| \int_{s_1}^{s_2} \phi^x(x,s)\mathrm{d}s \right| \leq |s_2 - s_1| + \varepsilon \right\}$$

Branched transport: $g(x,u,\nabla u) = I_{\nabla u = 0}$, $\psi(x,s_1,s_2,\nu) = |s_1 - s_2|^{1-\varepsilon}$

$$\mathcal{K} = \left\{ \phi = (\phi^x, \phi^s) \,:\, \phi^s \geq 0, \qquad \left| \int_{s_1}^{s_2} \phi^x(x,s)\mathrm{d}s \right| \leq |s_2 - s_1|^{1-\varepsilon} \right\}$$

Lower bound: $\mathcal{J}^{\varepsilon,a} = \mathcal{E}^{\varepsilon,a}$, $\mathcal{J}^{\varepsilon,a} = \mathcal{M}^{\varepsilon}$

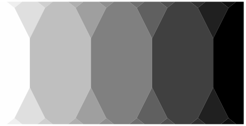
$$\min_{\mathrm{div}\mathcal{F} = \mu_0 - \mu_1} \mathcal{J}^{\varepsilon,a}[\mathcal{F}] = \min_{u|_{\partial\Omega}(x) = x_1} \tilde{\mathcal{J}}^{\varepsilon,a}[u] \geq \sup_{\phi \in \mathcal{K}} \inf_{v|_{\partial(\Omega\times\mathbb{R})} = 1_{x \mapsto x_1}} \int_{\overline{\Omega}\times\mathbb{R}} \phi \cdot \mathrm{d}Dv$$

$$= \sup_{\phi \in \mathcal{K}} \inf_{v|_{\partial(\Omega\times\mathbb{R})} = 1_{x \mapsto x_1}} \int_{\partial\Omega} \int_{-\infty}^{x_1} \phi \cdot \nu \,\mathrm{d}s\,\mathrm{d}x - \int_{\Omega\times\mathbb{R}} v \,\mathrm{div}\phi \,\mathrm{d}(x,s)$$

$$\geq \sup_{\phi \in \mathcal{K}, \mathrm{div}\phi = 0, \phi^s = 0} \int_{\partial\Omega} \int_{-\infty}^{x_1} \phi \cdot \nu \,\mathrm{d}s\,\mathrm{d}x$$

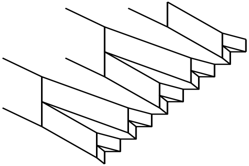
Construction for ϕ

expected optimal image



$$\min_{\mathrm{div}\mathcal{F}=\mu_0-\mu_1} \mathcal{J}^{\varepsilon,a}[\mathcal{F}] = \min_{u|_{\partial\Omega}(x)=x_1} \tilde{\mathcal{J}}^{\varepsilon,a}[u]$$

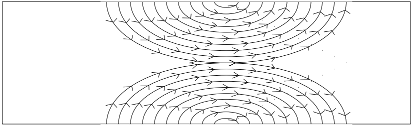
corresponding lifting



$$\min_{\mathrm{div}\mathcal{F}=\mu_0-\mu_1} \mathcal{J}^{\varepsilon,a}[\mathcal{F}] \geq \sup_{\substack{\mathrm{div}^x\phi^x=0,\ |\phi^x|\leq a\\ \left|\int_{s_1}^{s_2}\phi^x(x,s)\mathrm{d}s\right|\leq h(|s_2-s_1|)}} \int_{\partial\Omega}\int_{-\infty}^{x_1} \phi^x\cdot\nu^x \mathrm{d}x\mathrm{d}s$$

$$h(s) = \begin{cases} s+\varepsilon & \text{urb. pl.} \\ s^{1-\varepsilon} & \text{br. tpt.} \end{cases}$$

test field ϕ

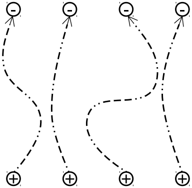


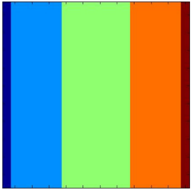
Numerical solution

$$\begin{Bmatrix}\text{2D network}\\\text{optimization}\end{Bmatrix} \Leftrightarrow \begin{Bmatrix}\text{“Mumford–Shah”}\\\text{image segmentation}\end{Bmatrix} \overset{\substack{\text{functional lifting}\\\text{relaxation}}}{\Longrightarrow} \begin{Bmatrix}\text{convex}\\\text{opt. problem}\end{Bmatrix}$$

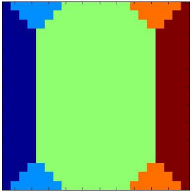
Numerical solution

$$\begin{Bmatrix}\text{2D network}\\ \text{optimization}\end{Bmatrix} \Leftrightarrow \begin{Bmatrix}\text{“Mumford–Shah”}\\ \text{image segmentation}\end{Bmatrix} \overset{\text{functional lifting}}{\underset{\text{relaxation}}{\Longrightarrow}} \begin{Bmatrix}\text{convex}\\ \text{opt. problem}\end{Bmatrix}$$

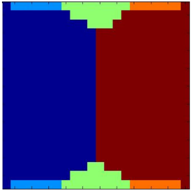




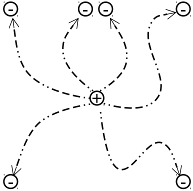
$\varepsilon = 0.25$

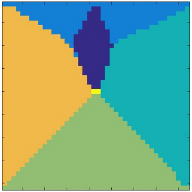


$\varepsilon = 0.35$

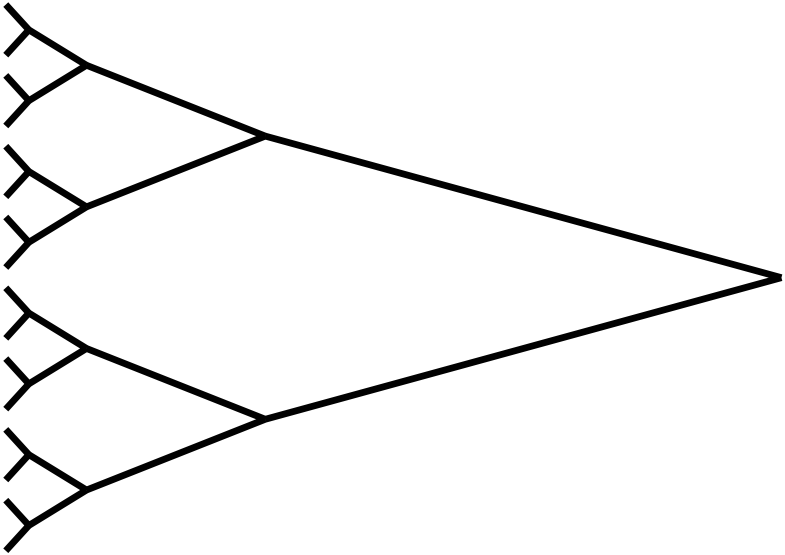


$\varepsilon = 0.45$

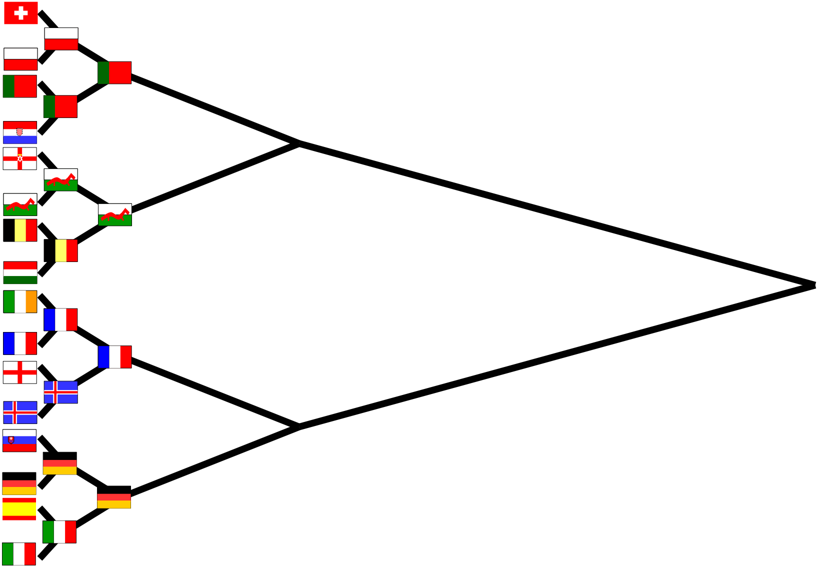




Networks in real life



Networks in real life



Networks in real life