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Connected components of sets of finite perimeter and applications to image processing

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Abstract. This paper contains a systematic analysis of a natural measure theoretic notion of connectedness for sets of finite perimeter in $\mathbb{R}^N$, introduced by H. Federer in the more general framework of the theory of currents. We provide a new and simpler proof of the existence and uniqueness of the decomposition into the so-called M-connected components. Moreover, we study carefully the structure of the essential boundary of these components and give in particular a reconstruction formula of a set of finite perimeter from the family of the boundaries of its components. In the two dimensional case we show that this notion of connectedness is comparable with the topological one, modulo the choice of a suitable representative in the equivalence class. Our strong motivation for this study is a mathematical justification of all those operations in image processing that involve connectedness and boundaries. As an application, we use this weak notion of connectedness to provide a rigorous mathematical basis to a large class of denoising filters acting on connected components of level sets. We introduce a natural domain for these filters, the space $\mathrm{WBV}(\Omega)$ of functions of weakly bounded variation in Ω, and show that these filters are also well behaved in the classical Sobolev and BV spaces.

1. Introduction

Recently, and from different points of view, there has been a renewed interest in measure theoretic notions of connectedness [21, 71] (see also [36]). For the case of BV functions and sets of finite perimeter, we shall present here a theory as much complete as possible, giving at the same time new and simpler proofs of some classical results. We are strongly motivated by the use of such objects as “connected components of level sets”, “Jordan curves”, etc. in digital image technology. One of our aims will be to give a well founded mathematical model

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for the well-spread use, in image processing and image analysis, of connectedness properties to create regions or "shapes" in an image. Also, the description of the regions boundaries in terms of "curves" and the existence of "level lines" in an image will be justified.

The extraction of shapes from images

Image analysis theory admits the existence of "shapes" in an image. There are many theories and algorithms for the extraction of such objects from a digital image. Some theories propose a segmentation of the image into connected regions by a variational principle [52, 53]. Other theories assume that the discontinuity set of the image provides curves which, in some way or another, can be closed by an algorithm (see [8, 50] and the discussion in [7]). Canny's filter [9], for instance, computes a set of discontinuity points in the image which must be thereafter connected by some variational principle. The obtained curves are supposed to be the boundaries of the "shapes" of the image. Many pattern recognition theories directly assume the existence of Jordan curves in the image (without explaining how such shapes should be extracted) and focus on subsequent recognition algorithms [33, 40, 41].

To summarize, most shape analysis methods deal with connected regions and their surrounding curves, and the curves surrounding their holes as well. Now, the ways such regions and curves are extracted are rather diverse and uncertain. Indeed, this extraction is often based on "edge detection theory", a wide galaxy of heuristic algorithms finding boundaries in an image. See [14] for a survey of these techniques and also the book [51] for an attempt of mathematical classification. We shall see, however, that in most practical cases shapes can and should be extracted as connected components of level sets of the image, and Jordan curves as their boundaries.

Why scalar images and not vector (colour) images ?

Let us first define the digital image as raw object. We shall then discuss what the alternatives for the extraction of shapes are. An image can be realistically modelled as a real function $u(x)$ where x represents an arbitrary point of $\mathbb{R}^{\mathrm{N}}$ ($N = 2$ for usual snapshots, 3 for medical images or movies, 4 for moving medical images) and $u(x)$ denotes the grey level, or colour, at x. In general, the image domain is finite (a hyperrectangle) but there will be no loss of generality in assuming that it is defined on the whole euclidean space. An image may be panchromatic; in that case $u(x)$ represents the photonic flux over a wide band of wavelengths and we have a proper grey level image. Now, $u(x)$ may also represent a colour intensity, when the photonic flux is subjected to a colour selective filter. In the following, we always consider scalar images, that is, images with a single channel, be it colour or grey level. When several channels have been captured simultaneously, we obtain naturally vector images, with e.g. three channels (Red, Green, Blue). It may appear at first as a restriction not to consider vector images, but only scalar ones. Indeed, the use of colour images is well-spread in human communication and most image processing and analysis operators must therefore be defined on vector images. Now, the redundancy of the colour images (from the perceptual viewpoint) is high. It is well admitted that the essential geometric features of any natural image are contained in its panchromatic (grey level) representation. Given a colour image, this panchromatic version is simply given as a linear positive combination of the three colour channels. As a consequence of this empiric observation, most image processing operators are defined separately for each channel and most image analysis operators are expected to give essentially the same result no matter whether applied to each one of the colour channel or to the panchromatic (grey level) version of the image. This fact, that geometric information essentially be contained in the grey level representation, can be checked by numerical experimental procedures [11]. These procedures involve discrete implementations of operators computing connected components of level sets, so that they are part of our motivations for investigating connectedness.

Image formation

From now on, and for the reasons just developed, we shall limit ourselves to the problem of connectedness in scalar images. We sketch in the following some aspects of image formation which will be relevant to our discussion. The process of image formation is, in a first approximation, given by the following formula [70]:

$$u = Q\Big\{g(k * O)\Pi + n\Big\} \cdot d, \tag{1}$$

where O represents the photonic flux (in a given wavelength band), k is the point spread function of the optical-captor joint apparatus, $*$ denotes the convolution operator, Π is a sampling operator, i.e. a Dirac comb supported by the centers of the matrix of digital sensors, g is a nonlinear contrast change characterizing the nonlinear response of the sensors, n represents a random perturbation due to photonic or electronic noise, Q is a uniform quantization operator mapping $\mathbb{R}$ to a discrete interval of values, typically $[0, 255]$, and d represents an impulse noise due to transmission. Each one of the operations involved in (1) is at the basis of one of the main theories of signal processing. For instance, Shannon theory fixes the conditions under which we can recover $k * O$ from the sampled signal $(k * O)\Pi$, assuming that $k * O$ is a bandlimited function, i.e., its frequency range has compact support.

Nonlinear contrast changes and level sets

Let us focus on the consequences of the nonlinear contrast change g for image processing. In human communication, none of the camera parameters is known to the observer; in most cases this information is lost when the image u is used. This loss is rather the rule for the contrast change g. The informations about g are inasmuch neglected as they are generally irrelevant: indeed, the contrast of an image widely depends on the sensor's properties but also on the lighting conditions and finally on the objects' temporary reflection properties: these conditions are anyway unknown! This led the physicist and gestaltist M. Wertheimer [68] to state as a principle that the grey level is not an observable. Images are observed up to an arbitrary and unknown contrast change.

An image analysis doctrine, the so called Mathematical Morphology, has recognized contrast invariance as a basic invariance requirement and proposed that image analysis operations should take into account this invariance principle [60]. With this principle, an image u is a representative of an equivalence class of images v obtained from u via a contrast change, i.e., $v = g(u)$ where g, for simplicity, will be a continuous strictly increasing function. Under this assumption, an image is characterized by its upper (or lower) level sets $X_\lambda = \{x : u(x) \geq \lambda\}$ (resp. $X'_\lambda = \{x : u(x) \geq \lambda\}$). Moreover, the image can be recovered from its level sets by the reconstruction formula

$$u(x) = \sup\{\lambda : x \in X_\lambda\}.$$

As it is easily seen, the family of the level sets (upper or lower) of u is invariant under continuous strictly increasing contrast changes. An image operator T is contrast invariant if

$$T(g(u)) = g(T(u)),$$

for any continuous strictly increasing contrast change g and any image u. In particular, many efficient denoising operators respect this principle. See a classification of contrast invariant image multiscale smoothing operators in [2].

Connected components of level sets

Level sets are therefore basic objects for image processing and analysis. They have been acknowledged as such in several shape analysis theories, where thresholding is the basic image analysis operator [34]. Very early in image processing, authors noticed that to find a single and the right threshold in an image was enough to deliver a binary image with most of the relevant shape information. Theories of the "optimal threshold" were even developed [69]. In order to have a more local description of the basic objects of an image, several authors ([12, 60]) proposed to consider the connected components of (upper or lower) level sets as the basic objects of the image. They argue that contrast changes are local and depend upon the reflectance properties of objects. Thus, not only global contrast, but also local contrast is irrelevant. In [12], a notion of local contrast change is defined and it is proved that only connected components of level sets are invariant under such contrast changes. This approach was generalized in [6] where the authors compare different satellite images of the same landscape, taken at different times or in different channels. They show that these images have many connected components of bilevel sets in common (we call bilevel set any set $\{x, a \leq u(x) \leq b\}$). This same technique has been recently extended in [48] to image registration, one of the most basic tools in multiimage processing. Image registration based on connected components of level sets is shown to work efficiently where classical correlation techniques fail: when both registered images do not correspond to almost simultaneous snapshots. If u belongs to a function space such that each connected component of a level set is bounded by a countable or finite number of oriented Jordan curves, we call *topographic map* the family of these Jordan curves [12]. In [44], a disocclusion method is developed, which restores images with spots or missing parts. This method computes Jordan curves in the image as boundaries of level sets and interpolates them in the missing parts.

A nested Jordan curves representation

Following [12], P. Monasse and F. Guichard [49] proposed, in a discrete framework, a fast and consistent discrete algorithm to compute a topographic map: they consider connected components of level sets, then they define a tree, ordered by inclusion, in the following way: they construct (in a discrete framework) a uniquely defined Jordan curve surrounding each connected component of each upper level set. In the same way, they consider all external Jordan curves of all connected components of lower level sets of the same image. Provided connectedness is adequately defined in the discrete grid (this definition is different for the upper level sets and the lower level sets!) they show that both systems of Jordan curves fuse into one, such that no pair of Jordan curves crosses. In this way, they obtain a topographic map, i.e. a system of Jordan lines organized by inclusion as a tree. They call this digital representation "fast level set transform" and it provides a fast numerical access to any connected region of the image and any "shape", understood as a Jordan curve surrounding a region. They let notice by some examples, however, that the inclusion trees of u and $-u$ are not necessarily identical.

WBV: *Functions whose level sets have finite perimeter*

One of the main purposes of this paper is to justify the assumptions underlying the above mentioned methods. We shall define a functional model for u where it is possible to define a notion of connected components for the level sets of u. Boundary of these connected components must consist of a countable or finite number of oriented Jordan curves from which we can recover the set by the obvious filling algorithm. This functional model, called WBV, is a variant of the space of functions of bounded variation. Indeed, WBV functions are BV functions modulo a change of contrast, i.e. for any $u \in \mathrm{WBV}$ there exists a bounded, continuous and strictly increasing contrast change g such that $g(u)$ is a function of bounded variation. The space of functions of bounded variation is a sound model for images which have discontinuities and it has been frequently used as a functional model for the purposes of image denoising, edge detection, etc. [56]. L. Rudin [55] proposed that images should be handled as functions with bounded variation. He used the classical result of geometric measure theory [29] that the essential discontinuity set of a BV function is rectifiable and argued that the "edge set" sought in edge detection theory [42,43], was nothing but this discontinuity set. An indirect confirmation of this thesis is given by the variational image segmentation theory. Indeed, a paradigmatic variational model proposed by Mumford-Shah [52] finds naturally its minima in a class of functions with bounded variation, SBV [3, 4, 17, 18]. A full account can be found in the book [5].

As a consequence of the results discussed in this paper, we shall show that all of the mentioned approaches, Mathematical Morphology, BV model, shapes described by Jordan curves or by connected regions, fast level set transform are compatible with a single underlying functional model, WBV. We shall introduce the "M-connectedness" as the right notion of connectedness for sets of finite perimeter. We shall develop this formalism in full generality for sets of finite perimeter in $\mathbb{R}^N$. For sets of finite perimeter in $\mathbb{R}^2$ a more precise description is possible, since in this case, the essential boundary of each M-connected component can be described as a countable or finite union of rectifiable Jordan curves. Since almost all level sets of functions in WBV are sets of finite perimeter, then level sets of WBV functions can be described in terms of rectifiable Jordan curves and we get a description of the shapes in an image which is both complete and well-founded.

Image denoising or segmentation operators based on connected components

The use of connected components of level sets has become recently very relevant in a series of image filters introduced in Mathematical Morphology. Motivated by the study of a family of filters by reconstruction [37, 38, 57, 64, 65], J. Serra and Ph. Salembier [58, 62] introduced the notion of connected operators. To be precise, Serra and Salembier call connected an operator ψ on sets if, for each family of sets A, the partition of the image domain associated to $\psi(A)$ (i.e., the partition of the image domain made of the connected components of $\psi(A)$ and the connected components of its complement) is less fine than the partition associated to A (i.e., the partition of the image domain made of the connected components of A and the connected components of its complement). Such operators simplify the topographic map of the image. These filters have become very popular because, on an experimental basis, they have been claimed to simplify the image while preserving contours. This property has made them very attractive for a large number of applications such as noise cancellation [64, 65] or segmentation [47, 66]. More recently, they have become the basis of a morphological approach to image and video compression (see [59] and references therein, and more recently [27]).

Application to connected operators

As an application of the theory of M-connected components for sets of finite perimeter developed here, we study the L. Vincent filters (filters which, when defined on sets, remove the connected components of small measure). We show that these filters can be defined on functions of bounded variation and, more generally, in WBV. We prove that they define contrast invariant filtering operators which are well behaved also in the classical Sobolev and BV spaces and simplify the connected components of the upper and/or lower level sets of the image (see also [44]).

An objection to the BV *model*

Before closing with this introduction, it may be useful to answer to an obvious objection: according to the classical model given by (1), the raw image O may be BV, but the digital image $g(k * O)$ is more regular, at least, say, C^1 if g is and if the image formation follows Shannon conditions. Thus, we might as well have worked in a space of continuous functions. In this framework, connected components can be defined in the classical way and Jordan curves obtained in the image by Sard Lemma and the Implicit Functions theorem. To take this assumption would save all of the effort spent here. The answer to this objection comes from technology. There is no evidence in all of the works dedicated to image processing in favour of any advantage taken of a regularity assumption for the images. Because of the three noises present in image caption (transmission impulse noise, gaussian quantum noise of sensors, quantization noise), the image cannot be considered as a continuous function. In many cases, Shannon conditions are imperfectly satisfied. In addition, the BV model makes sense for the subjacent "real" image O, which presents rectifiable discontinuity lines along all apparent contours of objects. Thus, O is at least as discontinuous as a BV function, and probably more. In fact, an experimental procedure can be defined [1] to check whether the subjacent image is in BV or not: the results seem to indicate that most images are too oscillating to belong to BV. We mentioned that both restoration and segmentation models try with success to project back in some more or less nonlinear way the image onto BV [56]. This is also true for the recent "wavelet shrinkage" method for image denoising [19] or image deconvolution [20]. Last but not least, the discrete representations used in Mathematical Morphology [60] are not more regular than BV and the recent image compression standards aim at the delivery of a BV compressed image. To summarize, the BV model is probably too smooth for the "real" subjacent image (i.e. the photonic flux), but seems to be on the way to be acknowledged as the right model to describe the digital images handled in technology. We may add the results of the present work as one more argument in favour of the BV model (and the variant WBV we propose) as a common denominator to image analysis and restoration.

Plan of this paper

This paper is organized as follows. Sections 2 and 3 introduce some basic facts about Caccioppoli sets and BV functions. In Sect. 4 we study in detail a definition of M-connectedness for sets with finite perimeter, first proposed by H. Federer in the more general framework of the theory of currents. We compare this concept with the conventional topological one and give a new proof, based on a simple variational argument, of the existence and uniqueness of the decomposition into M-connected components. Section 5 explains how to "fill the holes", or to "saturate", an indecomposable set. Section 6 defines Jordan boundaries (which correspond in dimension 2 to Jordan curves) and gives a unique decomposition theorem (Theorem 4) of the essential boundary into Jordan boundaries, with their structure. Theorem 5 gives a converse statement and a reconstruction formula of a Caccioppoli set from its set of Jordan boundaries. In Sect. 7 we construct for any Caccioppoli set E a "topographic function", an integer valued BV function whose boundaries of upper level sets yield all Jordan boundaries of E. In this way, the Jordan boundaries of E benefit of the obvious inclusion structure of the upper level sets of u and are numbered in odd and even levels of u, following their level of inclusion and their classification into set, versus hole, boundaries. In Sect. 8, we give the two dimensional interpretation of these results and show that in this case the link with conventional topology is much stronger: indeed, we show that the essential boundary of any simple set E (i.e. such that both E and $\mathbb{R}^2 \setminus E$ are indecomposable) is equivalent, modulo $\mathcal{H}^1$-negligible sets, to a Jordan curve (this result was first proved by W.H. Fleming in [25]) and also that for any indecomposable set E there exists a canonical set F equivalent to E which is connected by rectifiable arcs. Section 9 is devoted, as an illustration, to a case study in image denoising. We show the good definition and properties of the above mentioned Vincent-Serra "connected operators" in WBV and in the classical Sobolev and BV spaces. In particular, we prove that these operators, notwithstanding their nonlocal nature, map $\mathrm{W}^{1,p}$ in $\mathrm{W}^{1,p}$ for any $p \in [1, \infty]$ and do not increase a.e. the modulus of the gradient. In this respect, quite surprisingly, they behave as the usual local truncation operators.

2. Notation and main facts about sets of finite perimeter

We consider a N-dimensional euclidean space $\mathbb{R}^{\mathrm{N}}$, with $N \geq 2$. The Lebesgue measure of a Lebesgue measurable set $E \subseteq \mathbb{R}^{\mathrm{N}}$ will be denoted by $|E|$. For a Lebesgue measurable subset $E \subseteq \mathbb{R}^{\mathrm{N}}$ and a point $x \in \mathbb{R}^{\mathrm{N}}$, the upper and lower densities of E at x are respectively defined by

$$\overline{D}(E,x) := \limsup_{r\to 0^+} \frac{|E \cap B(x,r)|}{|B(x,r)|}, \qquad \underline{D}(E,x) := \liminf_{r\to 0^+} \frac{|E \cap B(x,r)|}{|B(x,r)|}.$$

If the upper and lower densities are equal, their common value will be called the density of E at x and it will be denoted by $D(E,x)$. We shall use the word *measurable* to mean *Lebesgue measurable*.

Using densities we can define the essential interior $\mathring{E}^{\mathrm{M}}$, the essential closure $\overline{E}^{\mathrm{M}}$ and the essential boundary $\partial^{\mathrm{M}} E$ of a measurable set E as follows:

$$\mathring{E}^{\mathrm{M}} := \{x : D(E,x) = 1\}, \qquad \overline{E}^{\mathrm{M}} := \left\{x : \overline{D}(E,x) > 0\right\} \tag{2}$$

$$\partial^{\mathrm{M}} E := \overline{E}^{\mathrm{M}} \cap \overline{\mathbb{R}^{\mathrm{N}} \setminus E}^{\mathrm{M}} = \left\{x : \overline{D}(E,x) > 0,\ \overline{D}(\mathbb{R}^{\mathrm{N}} \setminus E, x) > 0\right\}. \tag{3}$$

Notice also that by the Lebesgue differentiation theorem the symmetric difference $\mathring{E}^{\mathrm{M}} \Delta E$ is Lebesgue negligible, hence the measure theoretic interior of $\mathring{E}^{\mathrm{M}}$ is $\mathring{E}^{\mathrm{M}}$ (in this sense $\mathring{E}^{\mathrm{M}}$ is essentially open), and also that

$$\partial^{\mathrm{M}} E = \mathbb{R}^{\mathrm{N}} \setminus \Big(\mathring{E}^{\mathrm{M}} \cup \overset{\circ}{\widehat{\mathbb{R}^{\mathrm{N}} \setminus E}}{}^{\mathrm{M}}\Big).$$

We also use the notation $E^{1/2}$ to indicate the set of points where the density of E is $1/2$.

Here and in what follows we shall denote by $\mathcal{H}^\alpha$ the Hausdorff measure of dimension α in $\mathbb{R}^N$. In particular, $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure and $\mathcal{H}^N$, the N-dimensional Hausdorff measure, coincides with the (outer) Lebesgue measure in $\mathbb{R}^N$. Given any Borel set $B \subseteq \mathbb{R}^N$ with $\mathcal{H}^\alpha(B) < \infty$, we denote by $\mathcal{H}^\alpha \llcorner B$ the finite Borel measure $\chi_B \mathcal{H}^\alpha$, i.e. $\mathcal{H}^\alpha \llcorner B(C) = \mathcal{H}^\alpha(B \cap C)$ for any Borel set $C \subseteq \mathbb{R}^N$. We recall that

$$\lim_{r \to 0^+} \frac{\mathcal{H}^k\,(B \cap B(x,r))}{r^k} = 0 \qquad \text{for } \mathcal{H}^k\text{-a.e. } x \in \mathbb{R}^N \setminus B \tag{4}$$

holds whenever $B \subseteq \mathbb{R}^N$ is a Borel set with finite k-dimensional Hausdorff measure (see for instance §2.3 of [22]).

Given $A,\ B \subseteq \mathbb{R}^N$, we shall write $E_1 = E_2 \pmod{\mathcal{H}^\alpha}$ if $H^\alpha(E_1 \Delta E_2) = 0$, where $E_1 \Delta E_2 = (E_1 \setminus E_2) \cup (E_2 \setminus E_1)$ is the symmetric difference of E_1 and E_2. We will use an analogous notation for the inclusion and in some cases, in order to simplify the notation, the equivalence or inclusion $(\mathrm{mod}\,\mathcal{H}^N)$ will be tacitly understood.

We say that a measurable set $E \subseteq \mathbb{R}^N$ has *finite perimeter* in $\mathbb{R}^N$ if there exist a positive finite measure μ in $\mathbb{R}^N$ and a Borel function $\nu_E : \mathbb{R}^N \to \mathbf{S}^{N-1}$ (called generalized inner normal to E) such that the following generalized Gauss–Green formula holds

$$\int_E \mathrm{div}\phi\, dx = -\int_{\mathbb{R}^N} \langle \nu_E, \phi \rangle\, d\mu \qquad \forall \phi \in C^1_c(\mathbb{R}^N, \mathbb{R}^N).$$

Hence the measure $\nu_E \mu$ is the distributional derivative of χ_E, which will be denoted by $D\chi_E$, while $\mu = |D\chi_E|$ is its total variation; the *perimeter* $P(E, B)$ of E in a Borel set $B \subseteq \mathbb{R}^N$ is defined by $|D\chi_E|(B)$, and we use the notation $P(E)$ in the case $B = \mathbb{R}^N$.

The main facts concerning sets of finite perimeter that we will use in the following are listed below, for the reader's convenience (see for instance [5, 22, 24, 29, 72]).

- *Criteria for the finiteness of perimeter*

By Riesz theorem, a measurable set $E \subseteq \mathbb{R}^N$ has finite perimeter if and only if

$$\sup \left\{ \int_E \mathrm{div}\,\phi\, dx :\ \phi \in C^1_c(\mathbb{R}^N, \mathbb{R}^N),\ \ |\phi| \le 1 \right\} < \infty \tag{5}$$

and in this case the supremum equals the perimeter. A much deeper criterion is due to Federer: E has finite perimeter in $\mathbb{R}^N$ if and only if $\mathcal{H}^{N-1}(\partial^{\mathrm{M}} E) < \infty$ (if $\mathcal{H}^{N-1}(\partial E) < \infty$ the proof is much simpler, see for instance Proposition 3.62 of [5]).

• *Structure of $D\chi_E$*

According to the De Giorgi and Federer theorems, for any set with finite perimeter the sets $E^{1/2}$ and $\partial^{\mathrm{M}} E$ have the same $\mathcal{H}^{N-1}$ measure, so that $\mathcal{H}^{N-1}(\partial^{\mathrm{M}} E \setminus E^{1/2}) = 0$ and

$$\mathcal{H}^{N-1}\big(\mathbb{R}^{\mathrm{N}} \setminus (\mathring{E}^{\mathrm{M}} \cup E^{1/2} \cup \overbrace{\mathbb{R}^{\mathrm{N}} \setminus E}^{\circ\ \mathrm{M}})\big) = 0. \tag{6}$$

So, at $\mathcal{H}^{N-1}$-a.e. point of $\mathbb{R}^{\mathrm{N}}$ the density exists and belongs to $\{0, 1/2, 1\}$. Moreover

$$|D\chi_E| = \mathcal{H}^{N-1} \llcorner \partial^{\mathrm{M}} E = \mathcal{H}^{N-1} \llcorner E^{1/2}.$$

• *Lower semicontinuity, approximation and compactness*

The functional $E \mapsto P(E)$ (defined by (5), so that $P(E) = \infty$ if E has not finite perimeter) is lower semicontinuous with respect to local convergence in measure in $\mathbb{R}^{\mathrm{N}}$ (i.e. L^1_{loc} convergence of the characteristic functions); moreover, for any set E with $P(E) < \infty$ there exists a sequence of sets E_h with smooth boundary locally converging in measure to E and such that $P(E) = \lim_h P(E_h)$. Any sequence of sets with equibounded perimeters admits subsequences locally converging in measure.

• *Isoperimetric inequalities*

If $E \subseteq \mathbb{R}^{\mathrm{N}}$ has finite perimeter, then either E or $\mathbb{R}^{\mathrm{N}} \setminus E$ have finite measure and the isoperimetric inequality holds:

$$\min \left\{ |E|^{\frac{N-1}{N}}, |\mathbb{R}^{\mathrm{N}} \setminus E|^{\frac{N-1}{N}} \right\} \le \gamma_N P(E).$$

Denoting by ω_N the measure of the unit ball $B(0, 1)$, the optimal isoperimetric constant is $\omega_N^{-1/N}/N$ (see [16]). A local counterpart of this inequality is the relative isoperimetric inequality:

$$\min \{|B(x,r) \cap E|, |B(x,r) \setminus E|\} \le \eta_N r \mathcal{H}^{N-1}\left(\partial^{\mathrm{M}} E \cap B(x,r)\right). \tag{7}$$

3. BV **functions and related spaces**

In this section we recall some definitions and properties related to the space of functions with bounded variation in Ω, denoted by $\mathrm{BV}(\Omega)$.

Given a Borel function $u : \Omega \to [-\infty, +\infty]$, the approximate lower and upper limits $u^-, u^+ : \Omega \to [-\infty, +\infty]$ are Borel functions defined at every point $x \in \Omega$ as follows: $u^-(x)$ is the supremum of all those $t \in [-\infty, +\infty]$ such that $x \in \overbrace{\{u \ge t\}}^{\circ\ \mathrm{M}}$ whereas $u^+(x)$ is the infimum of all those $t \in [-\infty, +\infty]$ such that $x \in \overbrace{\{u \le t\}}^{\circ\ \mathrm{M}}$. The set

$$S_u := \left\{x \in \Omega : \ u^-(x) < u^+(x)\right\}$$

is called the *approximate discontinuity set* of u and is negligible with respect to the Lebesgue measure. The function u is said to be approximatively continuous at any point $x \in \Omega \setminus S_u$ and we shall denote

$$\operatorname{ap}\lim_{y \to x} u(y) = u^-(x) = u^+(x) \qquad \forall x \in \Omega \setminus S_u.$$

Let $x \in \Omega \setminus S_u$ such that $\operatorname{ap}\lim u(x) \in \mathbb{R}$. We say that u is approximatively differentiable at x if there exists a vector $\nabla u(x)$ such that the sets

$$\left\{ y \in \Omega \setminus \{x\} : \frac{|u(y) - \operatorname{ap}\lim u(x) - \langle \nabla u(x), y - x \rangle|}{|y - x|} > \epsilon \right\}$$

have 0 density in x for every $\epsilon > 0$.

We define $\mathrm{BV}(\Omega)$ as the space of all those functions $u \in \mathrm{L}^1(\Omega)$ whose distributional derivative is representable as a $\mathbb{R}^{\mathrm{N}}$-valued measure $Du = (D_1 u, \ldots, D_N u)$ with finite total variation in Ω, i.e.

$$\int_\Omega u \operatorname{div} \phi \, dx = -\sum_{i=1}^{N} \int_\Omega \phi_i \, dD_i u \quad \forall \phi \in \left[\mathrm{C}_c^1(\Omega) \right]^N.$$

The total variation $|Du|$ of a BV function u is defined as the total variation of the vector measure Du. The space $\mathrm{BV}(\Omega)$ is endowed with the norm $\|u\|_{\mathrm{BV}} = \|u\|_{\mathrm{L}^1} + |Du|(\Omega)$. We shall denote by $\mathrm{BV}_{\mathrm{loc}}(\Omega)$ the space of all those functions that belong to $\mathrm{BV}(\tilde{\Omega})$ for every open set $\tilde{\Omega} \subset\subset \Omega$. In view of Sect. 2, it is easily seen that a subset $E \subset \mathbb{R}^{\mathrm{N}}$ has finite perimeter in Ω if and only if $u = \chi_E \in \mathrm{BV}_{\mathrm{loc}}(\Omega)$ and $|Du|(\Omega) < \infty$. Main properties of BV functions are the following (see for instance [5,22,24,29,72]):

- *Lower semicontinuity of the variation measure*

Suppose $\{u_n\}_{n \in \mathbb{N}} \subset \mathrm{BV}(\Omega)$ and $u_n \to u$ in $\mathrm{L}^1_{\mathrm{loc}}(\Omega)$ then

$$|Du|(\Omega) \leq \liminf_{n \to \infty} |Du_n|(\Omega).$$

- *Approximation by smooth functions*

Assume $u \in \mathrm{BV}(\Omega)$. There exist functions $u_n \in \mathrm{BV}(\Omega) \cap \mathrm{C}^\infty(\Omega)$ such that

$$u_n \to u \text{ in } \mathrm{L}^1(\Omega) \quad \text{and} \quad |Du_n|(\Omega) \to |Du|(\Omega) \text{ as } n \to \infty.$$

- *Compactness*

If $\{u_n\}$ is a sequence in $\mathrm{BV}(\Omega)$ satisfying $\sup_n \|u_n\|_{\mathrm{BV}} < \infty$, then there exist a subsequence $\{u_{n_k}\}$ and a function $u \in \mathrm{BV}(\Omega)$ such that

$$u_{n_k} \to u \text{ in } \mathrm{L}^1_{\mathrm{loc}}(\Omega).$$

- *Poincaré inequality*

If Ω is bounded, connected and with Lipschitz boundary, then there exists a constant C such that

$$\int_{\Omega\cap B(x,r)} |u-\overline{u}| \leq C|Du|(B(x,r)\cap\Omega) \text{ for all balls } B(x,r) \subset \mathbb{R}^N \text{ and } u \in \mathrm{BV}(\Omega)$$

where $\overline{u}(x) = \fint_{\Omega\cap B(x,r)} u(y)\, dy$.

- *Coarea formula*

Let $u \in \mathrm{BV}(\Omega)$. Then $\{u > t\}$ has finite perimeter in Ω for L^1-a.e. $t \in \mathbb{R}$ and

$$|Du|(\Omega) = \int_{-\infty}^{+\infty} P(\{u > t\}, \Omega)\, dt.$$

Conversely, if $u \in \mathrm{L}^1(\Omega)$ and $\int_{-\infty}^{+\infty} P(\{u > t\}, \Omega)\, dt < \infty$ then $u \in \mathrm{BV}(\Omega)$. In addition, notice that $P(\{u > t\}, \Omega) = P(\{u < t\}, \Omega)$ since the fact that u is measurable is enough to ensure that $|\{u = t\}| > 0$ for at most countably many $t \in \mathbb{R}$.

- *Rectifiability of S_u and approximate jump set J_u*

Let $u \in \mathrm{BV}(\Omega)$. Then S_u is countably $(N-1)$-rectifiable and $-\infty < u^-(x) \leq u^+(x) < +\infty$ for $\mathcal{H}^{N-1}$-almost every $x \in \Omega$. In addition, for $\mathcal{H}^{N-1}$-a.e. $x \in S_u$ there exists a unique unit vector $\nu_u \in S^{N-1}$ such that, setting $B_r^+(x, \nu_u) := \{y \in B_r(x) : \langle y - x, \nu_u\rangle > 0\}$ and $B_r^-(x, \nu) := \{y \in B_r(x) : \langle y - x, \nu_u\rangle < 0\}$,

$$\lim_{r\downarrow 0} \left[\fint_{B_r^+(x,\nu_u)} |u(y) - u^+(x)|\, dy + \fint_{B_r^-(x,\nu)} |u(y) - u^-(x)|\, dy \right] = 0.$$

The set of points where this equality occurs is called the approximate jump set and denoted as J_u. Hence, $\mathcal{H}^{N-1}(S_u \setminus J_u) = 0$ and Du vanishes on $S_u \setminus J_u$.

- *Decomposition of the derivative*

Let $u \in \mathrm{BV}(\Omega)$. Then Du can be decomposed into three parts:

$$Du = D^a u + D^j u + D^c u$$

where $D^a u$ is the absolutely continuous part of Du with respect to $\mathcal{L}^N$ and, denoting by $D^s u$ the singular part of Du with respect to $\mathcal{L}^N$, $D^j u := D^s \llcorner J_u$ and $D^c u := D^s u \llcorner (\Omega \setminus S_u)$. $D^j u$ is called the jump part of the derivative and $D^c u$ the Cantor part of the derivative. Then $D^a u = \nabla u\, \mathcal{L}^n$, $D^j u = Du \llcorner J_u = (u^+ - u^-)\nu_u \mathcal{H}^{N-1} \llcorner J_u$ and $D^c u$ vanishes on sets which are σ-finite with respect to $\mathcal{H}^{N-1}$.

Several functional spaces were introduced in [3] (see also [54]) to offer a reliable framework for some minimization problems issuing from image processing and the mathematical theory of liquid crystals. We shall concentrate on the space of generalized functions of bounded variation $\mathrm{GBV}(\Omega)$, which can be defined as follows: we say that $u : \Omega \to [-\infty, +\infty]$ is a generalized function of bounded variation if

$$u^N := (-N) \vee u \wedge N \in \mathrm{BV}(\Omega) \qquad \forall N \in \mathbb{N},$$

which means that all truncations of u have bounded variation. For the sake of simplicity, we have chosen to work with $\mathrm{BV}(\Omega)$ rather than $\mathrm{BV}_{\mathrm{loc}}(\Omega)$, which is the definition adopted in [3]. Remark that GBV functions are not summable in general. Let us now define the function $m_u : \mathbb{R} \to [0, \infty]$ as

$$m_u(t) := P(\{u > t\}, \Omega).$$

Lemma 1. *Let $u : \Omega \to [-\infty, +\infty]$ be a Borel function such that $u \not\equiv +\infty$ and $u \not\equiv -\infty$ up to Lebesgue negligible sets. Then the following propositions hold:*

(i) if Ω is bounded then $m_u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ if and only if $u \in \mathrm{GBV}(\Omega)$.
(ii) if Ω is bounded, connected and with Lipschitz boundary then $m_u \in \mathrm{L}^1(\mathbb{R})$ if and only if $u \in \mathrm{BV}(\Omega)$.

Proof. (i) ($\Leftarrow$) By definition, $u^N \in \mathrm{BV}(\Omega)$ for every $N \in \mathbb{N}$. Since, for any $N \in \mathbb{N}$, $\{u > t\} = \{u^N > t\}$ for every $t \in (-N, N)$ we get by the coarea formula applied to the truncated function

$$\int_{-N}^{N} P(\{u > t\}, \Omega)\, dt = \int_{-N}^{N} P(\{u^N > t\}, \Omega)\, dt \le |Du^N|(\Omega) < +\infty \qquad \text{for every } N \in \mathbb{N}.$$

Therefore, $m_u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$.
($\Rightarrow$) First recall the well-known equality for Borel functions

$$u(x) = \int_0^{+\infty} \chi_{\{u>t\}}(x)\, dt - \int_{-\infty}^{0} (1 - \chi_{\{u>t\}})(x)\, dt \qquad \forall x \in \Omega.$$

Given $\phi \in \mathrm{C}^1_{\mathrm{c}}(\Omega; \mathbb{R}^{\mathrm{N}})$ with $\|\phi\|_\infty \le 1$, we use Riesz Theorem applied to the upper level sets, Fubini's Theorem and the fact that the integral of $\operatorname{div}\phi$ is zero to get for every $N \in \mathbb{N}$

$$\begin{aligned}\int_\Omega u^N \operatorname{div}\phi\, dx &= \int_\Omega \int_{-N}^{+N} \chi_{\{u^N>t\}} \operatorname{div}\phi\, dx\, dt\\ &= \int_{-N}^{+N} \int_\Omega \chi_{\{u>t\}} \operatorname{div}\phi\, dx\, dt\\ &\le \int_{-N}^{+N} P(\chi_{\{u>t\}}, \Omega)\, dt < +\infty.\end{aligned}$$

By Riesz Theorem, this implies that $u^N \in \mathrm{BV}(\Omega)$ for every $N \in \mathbb{N}$.

(ii) ($\Leftarrow$) is a straightforward consequence of the coarea formula.
($\Rightarrow$) It follows from (i) that $u \in \mathrm{GBV}(\Omega)$. Using Poincaré inequality we get that for every $N \in \mathbb{N}$

$$\int_\Omega |u^N - \overline{u^N}| dx \le C|Du^N|(\Omega) \le C \int_{-\infty}^{+\infty} P(\{u > t\}, \Omega)\, dt = C_1$$

with $\overline{u^N} = \fint_\Omega u^N(y)\,dy$. Now, let us prove that the sequence $|\overline{u^N}|$ is bounded. Assume that for some sequence $N_i \in \mathbb{N}$, $\overline{u^{N_i}} \to +\infty$ (the argument is analogous if $\overline{u^{N_i}} \to -\infty$) and let $\Omega_M = \{u \in [-\infty, M]\}$. Then, for i large enough

$$(\overline{u^{N_i}} - M)|\Omega_M| \le \int_\Omega |u^N - \overline{u^N}|\,dx \le C_1$$

thus $|\Omega_M| = 0$. It follows that $u \equiv +\infty$ which is contradictory to our assumptions. Thus $|\overline{u^N}|$ is bounded and, possibly by extracting a subsequence, $\overline{u^N} \to z$. Finally, letting $N \to \infty$, we get that

$$\int_\Omega |u - z|\,dx \le C_1$$

which implies that $u \in \mathrm{L}^1(\Omega)$ and u is real-valued. Then $u \in \mathrm{BV}(\Omega)$ by a simple application of the coarea formula. $\square$

4. Decomposability of a set with finite perimeter

Let $E \subseteq \mathbb{R}^\mathrm{N}$ be a set with finite perimeter. We say that E is *decomposable* if there exists a partition (A, B) of E such that $P(E) = P(A) + P(B)$ and both $|A|$ and $|B|$ are strictly positive. We say that E is *indecomposable* if it is not decomposable; notice that the properties of being decomposable or indecomposable are invariant (mod $\mathcal{H}^N$) and that, according to our definition, any Lebesgue negligible set is indecomposable.

It is natural to compare this definition with the topological one of connectedness: no implication is trivial in general, since on one hand in the definition of indecomposability the sets A, B are not required to be relatively open, but on the other hand they are required to be sets of finite perimeter. We will see that in some cases a comparison is possible, especially in the case of subsets of the plane, and that in any case all formal properties satisfied by connected sets are fulfilled in this slightly different setting.

We start our investigation by analyzing the situations in which the equality $P(A \cup B) = P(A) + P(B)$ occurs.

Proposition 1. *Let A, B be sets of finite perimeter. Then*

$$P(A \cup B) + P(A \cap B) \le P(A) + P(B)$$

and

$$P(A) + P(B) = P(A \cup B) + 2\mathcal{H}^{N-1}(\partial^\mathrm{M} A \cap \partial^\mathrm{M} B) \quad \textit{whenever } |A \cap B| = 0.$$

Proof. The following inclusions are a straightforward consequence of the definition of ∂^M:

$$\partial^\mathrm{M}(A \cup B) \cup \partial^\mathrm{M}(A \cap B) \subset \partial^\mathrm{M} A \cup \partial^\mathrm{M} B, \qquad \partial^\mathrm{M}(A \cup B) \cap \partial^\mathrm{M}(A \cap B) \subset \partial^\mathrm{M} A \cap \partial^\mathrm{M} B.$$

Taking into account that $P(E) = \mathcal{H}^{N-1}(\partial^M E)$ for any set of finite perimeter E, the first inequality follows. If $|A \cap B| = 0$ we denote by L the $\mathcal{H}^{N-1}$-negligible set $(\partial^M A \setminus A^{1/2}) \cup (\partial^M B \setminus B^{1/2})$ and notice that

$$\partial^M (A \cup B) \setminus L \subset \partial^M A \Delta \partial^M B, \qquad \partial^M A \Delta \partial^M B \subset \partial^M (A \cup B)$$

hence $P(A \cup B) = \mathcal{H}^{N-1}(\partial^M A \Delta \partial^M B)$. From this fact the second identity easily follows. $\square$

As an application of Proposition 1 we can prove that any open connected set with finite perimeter is indecomposable. We will obtain a converse property in Theorem 2 (see also Theorem 8, for domains in the plane).

Proposition 2 (Connectedness and indecomposability). *Any connected open set $\Omega \subseteq \mathbb{R}^N$ satisfying $\mathcal{H}^{N-1}(\partial^M \Omega) < \infty$ is indecomposable.*

Proof. By Federer's theorem, we know that Ω has finite perimeter. Let (A, B) be a partition of Ω such that $P(\Omega) = P(A) + P(B)$. Then, since

$$\partial^M A \subset \partial^M B \cup \partial^M \Omega$$

and, by Proposition 1, $\partial^M A \cap \partial^M B = \emptyset \pmod{\mathcal{H}^{N-1}}$, we have

$$\mathcal{H}^{N-1}(\Omega \cap \partial^M A) \leq \mathcal{H}^{N-1}(\Omega \cap \partial^M \Omega) = 0$$

hence $D\chi_A = 0$ in Ω. This proves that χ_A is locally equivalent to a constant in Ω, and, being Ω connected, this is true globally. $\square$

Another simple consequence of Proposition 1 is the *subadditivity* of perimeter

$$P\left(\bigcup_{i \in I} A_i\right) \leq \sum_{i \in I} P(A_i)$$

for finite or countable families. For finite families the proof is achieved by induction and for countable ones one can use the lower semicontinuity of the perimeter with respect to the local convergence in measure.

Now we extend our analysis to finite or countable families of sets; this extension is necessary in view of the treatment of the family of indecomposable components of a set. A more comprehensive treatment of the properties of partitions in finitely or countably many sets of finite perimeter (the so-called *Caccioppoli partitions*) is given in the paper [13] by G. Congedo and I. Tamanini (see also Chapter 4 of [5] and [39]); here we only prove the properties that will be needed in the following.

Proposition 3. *Let I be a finite or countable set, let $\{A_i\}_{i \in I}$ be a family of sets of finite perimeter and let A be their union. Then, assuming that $A_i \neq \mathbb{R}^N$ for any $i \in I$ and $\sum_i P(A_i) < \infty$, the following conditions are equivalent:*

(i) $P(A) \geq \sum_i P(A_i)$;
(ii) $P(A) = \sum_i P(A_i)$;
(iii) *for any* $i \neq j$ *we have* $|A_i \cap A_j| = 0$ *and* $\mathcal{H}^{N-1}(\partial^M A_i \cap \partial^M A_j) = 0$;
(iv) *for any* $i \neq j$ *we have* $|A_i \cap A_j| = 0$ *and* $\cup_i \partial^M A_i \subset \partial^M A \pmod{\mathcal{H}^{N-1}}$.

If these conditions are fulfilled we have also $\partial^{\mathrm{M}} A = \cup_i \partial^{\mathrm{M}} A_i \pmod{\mathcal{H}^{N-1}}$ *and*

$$\mathcal{H}^{N-1}\left(\mathring{A}^{\mathrm{M}} \setminus \bigcup_{i\in I} \mathring{A}_i^{\mathrm{M}}\right) = 0. \tag{8}$$

Proof. The equivalence between (i) and (ii) follows by the subadditivity of perimeter.

(ii)$\Longrightarrow$(iii) For any pair of indexes $i,\ j \in I, i \neq j$, we have

$$\begin{aligned} P(A) &\le P(A_i \cup A_j) + P\left(\bigcup_{k\in I\setminus\{i,\,j\}} A_k\right) \le P(A_i) + P(A_j) + P\left(\bigcup_{k\in I\setminus\{i,\,j\}} A_k\right) \\ &\le \sum_{k\in I} P(A_k) = P(A). \end{aligned}$$

Thus $P(A_i \cup A_j) = P(A_i) + P(A_j)$. From Proposition 1 we get $|A_i \cap A_j| = 0$ and $\partial^{\mathrm{M}} A_i \cap \partial^{\mathrm{M}} A_j = \emptyset \pmod{\mathcal{H}^{N-1}}$.

(iii)$\Longrightarrow$(iv) We know that $\mathcal{H}^{N-1}$-a.e. $x \in \partial^{\mathrm{M}} A_i$ belongs to $A_i^{1/2}$ and to $\cap_{j\neq i} \mathbb{R}^{\mathrm{N}} \setminus \partial^{\mathrm{M}} A_j$, hence is a point of density 0 for all sets A_j with $j \neq i$. Let us fix a point x with these properties and assume, in addition, that

$$\lim_{r\to 0^+} \frac{\mathcal{H}^{N-1}\left(\cup_{j\neq i} \partial^{\mathrm{M}} A_j \cap B(x,r)\right)}{r^{N-1}} = 0.$$

By (4) with $B = \cup_{j\neq i} \partial^{\mathrm{M}} A_j$ we know that also this additional condition is fulfilled $\mathcal{H}^{N-1}$-a.e. in $\partial^{\mathrm{M}} A_i$. The relative isoperimetric inequality (7) easily implies the existence of a constant c such that

$$|E \cap B(x,r)| \le cr\mathcal{H}^{N-1}(\partial^{\mathrm{M}} E \cap B(x,r)) \quad \text{whenever} \quad |B(x,r) \setminus E| \ge \frac{|B(x,r)|}{4}.$$

Hence

$$|A_j \cap B(x,r)| \le cr\mathcal{H}^{N-1}\left(\partial^{\mathrm{M}} A_j \cap B(x,r)\right) \qquad \forall j \neq i$$

for any $r > 0$ sufficiently small, such that $|A_i \cap B(x,r)| \ge |B(x,r)|/4$. Adding with respect to j we obtain

$$\lim_{r\to 0^+} \frac{|(A \setminus A_i) \cap B(x,r)|}{r^N} \le c \lim_{r\to 0^+} \frac{\mathcal{H}^{N-1}\left(\cup_{j\neq i} \partial^{\mathrm{M}} A_j \cap B(x,r)\right)}{r^{N-1}} = 0.$$

Hence $x \in A^{1/2} \subset \partial^{\mathrm{M}} A$.

(iv)$\Longrightarrow$(i) Since $(A_i)^{1/2} \cap (A_j)^{1/2} \subset \mathring{A}^{\mathrm{M}}$ whenever $i \neq j$ (because the sets A_i are pairwise disjoint), we obtain that

$$\mathcal{H}^{N-1}\left(\partial^{\mathrm{M}} A_i \cap \partial^{\mathrm{M}} A_j\right) = \mathcal{H}^{N-1}\left(\partial^{\mathrm{M}} A \cap (A_i)^{1/2} \cap (A_j)^{1/2}\right) = 0$$

hence $\sum_i P(A_i) = \sum_i \mathcal{H}^{N-1}(\partial^{\mathrm{M}} A_i) \le \mathcal{H}^{N-1}(\partial^{\mathrm{M}} A) = P(A)$.

The identity $\partial^{\mathrm{M}} A = \cup_i \partial^{\mathrm{M}} A_i$ (mod $\mathcal{H}^{N-1}$) follows by (ii). Since $\mathring{A}^{\mathrm{M}} \cap \partial^{\mathrm{M}} A = \emptyset$, (4) again with $B = \partial^{\mathrm{M}} A$ gives that $\mathcal{H}^{N-1}\left(B(x,r) \cap \partial^{\mathrm{M}} A\right)/r^{N-1}$ tends to 0 as $r \to 0^+$ for $\mathcal{H}^{N-1}$-a.e. $x \in \mathring{A}^{\mathrm{M}}$, Thus, in order to prove (8) we prove the inclusion

$$\mathring{A}^{\mathrm{M}} \setminus \bigcup_{i\in I} \mathring{A}_i^{\mathrm{M}} \subset \left\{ x \in \mathbb{R}^{\mathrm{N}} : \limsup_{r\to 0^+} \frac{\mathcal{H}^{N-1}(B(x,r)\cap \partial^{\mathrm{M}} A)}{r^{N-1}} > 0 \right\}. \tag{9}$$

Let $x \in \mathring{A}^{\mathrm{M}}$ be such that $\mathcal{H}^{N-1}\left(B(x,r) \cap \partial^{\mathrm{M}} A\right)/r^{N-1}$ tends to 0 as $r \to 0^+$. Let $r_0 > 0$ and $\sigma \in (0, 1/2)$ such that $\mathcal{H}^{N-1}(B(x,r) \cap \partial^{\mathrm{M}} A) \le \sigma\omega_N r^{N-1}/\eta_N$ for any $r \in (0, r_0]$. By the relative isoperimetric inequality (7) we infer

$$\min\{|B(x,r) \cap A_i|, |B(x,r) \setminus A_i|\} \le \sigma |B(x,r)| \qquad \forall i \in I,\ \ r \in (0, r_0].$$

Since the sets A_i are pairwise disjoint, the family

$$\begin{aligned} R_i &:= \{r \in (0, r_0] :\ |B(x,r) \cap A_i| \ge (1-\sigma)|B(x,r)|\}, \\ R_\infty &:= \{r \in (0, r_0] :\ |B(x,r) \cap A_i| \le \sigma|B(x,r)|\ \forall i \in I\} \end{aligned}$$

is a partition of $(0, r_0]$ in relatively closed sets. Being $(0, r_0]$ connected, one of these sets coincides with $(0, r_0]$. If $(0, r_0] = R_\infty$ the relative isoperimetric inequality (7) gives

$$|B(x,r) \cap A| = \sum_{i\in I} |B(x,r) \cap A_i| \le r\eta_N \sum_{i\in I} \mathcal{H}^{N-1}\big(B(x,r) \cap \partial^{\mathrm{M}} A_i\big) \le \sigma|B(x,r)|$$

for any $r \in (0, r_0]$, which is a contradiction. If $(0, r_0] = R_i$ for some $i \in I$, then we have that $\underline{D}(A_i, x) \ge 1 - \sigma$. Choose a sequence $\sigma_n \to 0+$ and $i_n \in \mathbb{N}$ such that $\underline{D}(A_{i_n}, x) \ge 1 - \sigma_n$. Then, i_n is constant for n large enough, say $i_n = i$ for n large enough. Thus we conclude that $D(A_i, x) = 1$, i.e, $x \in \mathring{A}_i^{\mathrm{M}}$. □

Remark 1 (Additional properties of partitions). Under the assumptions of the previous proposition, we remark that if $|A| = \infty$, due to the fact that the series of perimeters is convergent, there is exactly one set A_i with infinite measure; indeed, if all of them have finite measure, from the isoperimetric inequality we get

$$\sum_{i:|A_i|\le 1} |A_i|^{\frac{N-1}{N}} + \sum_{i:|A_i|\ge 1} |A_i|^{\frac{N-1}{N}} \le \gamma_N \sum_{i\in I} P(A_i) < \infty$$

and we obtain that $|A_i| \ge 1$ only for finitely many i. Thus

$$\infty = \sum_{i:|A_i|\le 1} |A_i| \le \sum_{i:|A_i|\le 1} |A_i|^{\frac{N-1}{N}} \le \gamma_N \sum_{i\in I} P(A_i) < \infty.$$

This contradiction proves that at least one set has infinite measure. Suppose that at least two of them, say $A_{i_0},\ A_{i_1}$, have infinite measure. Again by the isoperimetric inequality we would get

$$\min\left\{ |A_{i_0}|^{\frac{N-1}{N}}, \left|\mathbb{R}^{\mathrm{N}} \setminus A_{i_0}\right|^{\frac{N-1}{N}} \right\} \le P(A_{i_0}) \le \sum_{j\ne i_0} P(A_j) < \infty.$$

However, the quantity on the left hand side is infinite since $A_{i_1} \subseteq \mathbb{R}^{\mathrm{N}} \setminus A_{i_0}$.

We notice also that the argument used in the proof of (ii)$\Longrightarrow$(iii) gives

$$P\left(\bigcup_{i\in I_1\cup I_2} A_i\right) = P\left(\bigcup_{i\in I_1} A_i\right) + P\left(\bigcup_{i\in I_2} A_i\right) \tag{10}$$

whenever $I_1,\ I_2 \subseteq I$ are disjoint.

As a consequence of Proposition 3 with $A = E$, $A_1 = F$ and $A_2 = E \setminus F$, we obtain that characteristic functions of sets of finite perimeter F are constant inside an indecomposable set E, provided χ_F has no "derivative" in E. This is expressed by saying that $\partial^{\mathrm{M}}(E \cap F) \subset \partial^{\mathrm{M}} E$, or equivalently that $\partial^{\mathrm{M}}(E \cap F) \cap \mathring{E}^{\mathrm{M}} = \emptyset$ (mod $\mathcal{H}^{N-1}$). A more general statement is presented in Remark 2.

Proposition 4. *Let E be an indecomposable set and let $F \subseteq E$ be a set with finite perimeter, such that $\partial^{\mathrm{M}} F \subseteq \partial^{\mathrm{M}} E$ (mod $\mathcal{H}^{N-1}$). Then either $|F| = 0$ or $|E \setminus F| = 0$.*

Remark 2 (Constancy theorem). Since $F \subseteq E$, the assumption $\partial^{\mathrm{M}} F \subseteq \partial^{\mathrm{M}} E$ (mod $\mathcal{H}^{N-1}$) in Proposition 4 is equivalent to $\mathcal{H}^{N-1}(\partial^{\mathrm{M}} F \cap \mathring{E}^{\mathrm{M}}) = 0$. Proposition 4 is a particular case of the following result, proved by G. Dolzmann and S. Müller in [21]: if $u \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^{\mathrm{N}})$ satisfies $|Du|(\mathbb{R}^{\mathrm{N}}) < \infty$ and E is indecomposable, then

$$|Du|(\mathring{E}^{\mathrm{M}}) = 0 \qquad \Longrightarrow \qquad \exists c \in \mathbb{R} : \ u(x) = c \ \text{ for a.e. } x \in E.$$

The proof follows by the coarea formula

$$|Du|(\mathring{E}^{\mathrm{M}}) = \int_{-\infty}^{\infty} \mathcal{H}^{N-1}\big(\partial^{\mathrm{M}}\{u > \lambda\} \cap \mathring{E}^{\mathrm{M}}\big)\, d\lambda$$

noticing that Proposition 4 applies to a.e. level set $F_\lambda = \{u > \lambda\}$.

The main result of this section is the following decomposition theorem; a similar (see Remark 4) decomposition result for integer currents is stated in 4.2.25 of [24]. This result has also been used in G. Dolzmann and S. Müller [21] and B. Kirchheim [36] to prove Liouville type theorems for a class of partial differential inclusions; the second paper contains also an explicit proof of the decomposition theorem, based on Lyapunov convexity theorem (see also Theorem 1 in §3.4 of Chap. 4 of [28]). The proof that we present here is new and based on a simple variational argument.

Theorem 1 (Decomposition theorem). *Let E be a set with finite perimeter in $\mathbb{R}^{\mathrm{N}}$. Then there exists a unique finite or countable family of pairwise disjoint indecomposable sets $\{E_i\}_{i\in I}$ such that $|E_i| > 0$ and $P(E) = \sum_i P(E_i)$. Moreover*

$$\mathcal{H}^{N-1}\left(\mathring{E}^{\mathrm{M}} \setminus \bigcup_{i\in I} \mathring{E}_i^{\mathrm{M}}\right) = 0 \tag{11}$$

and the E_i's are maximal indecomposable sets, i.e. any indecomposable set $F \subseteq E$ is contained (mod $\mathcal{H}^N$) in some set E_i.

Proof. (Existence) Let $\alpha \in (1, N/(N-1))$, let us define

$$\mu(E) := \left(\int_E \exp(-|x|^2)\, dx\right)^{1/\alpha}$$

for any measurable set $E \subseteq \mathbb{R}^N$ and let $\mathcal{P}$ be the collection of all partitions $\{E_i\}_{i\in\mathbb{N}}$ of E such that $|E_i| \geq |E_j|$ for $i \leq j$ and $\sum_i P(E_i) \leq P(E)$. Recall that the condition $\sum_i P(E_i) < \infty$ implies that at most one set E_i (namely E_0) has infinite measure (see Remark 1). The class $\mathcal{P}$ is not empty, since it contains $\{E, \emptyset, \emptyset, \ldots\}$.

We will prove that the problem

$$\max\left\{\sum_{i\in\mathbb{N}} \mu(E_i) :\ \{E_i\}_{i\in\mathbb{N}} \in \mathcal{P}\right\}$$

has a (essentially unique) solution. Indeed, let $\{E_i^n\}_{i\in\mathbb{N}}$ be a maximizing sequence indexed by n; since $P(E_i^n) \leq P(E)$ by the compactness properties of sets of finite perimeter (see Sect. 2) we can assume, possibly extracting a subsequence, that E_i^n locally converge in measure in $\mathbb{R}^N$ to suitable sets E_i as $n \to \infty$. The sets E_i are pairwise disjoint (mod $\mathcal{H}^N$), and the lower semicontinuity of perimeter with respect to local convergence in measure gives $\sum_i P(E_i) \leq P(E)$. In order to show that $\{E_i\}_{i\in\mathbb{N}} \in \mathcal{P}$ we have to prove that $|E \setminus \cup_i E_i| = 0$. To this aim, we first prove that

$$\lim_{p\to\infty} \limsup_{n\to\infty} \sum_{i=p}^{\infty} \mu\big(E_i^n\big) = 0. \tag{12}$$

First, we notice that the isoperimetric inequality and the subadditivity of perimeter give

$$p^{\frac{N-1}{N}} \left|E_p^n\right|^{\frac{N-1}{N}} \leq \left|\bigcup_{i=1}^p E_i^n\right|^{\frac{N-1}{N}} \leq \gamma_N \sum_{i=1}^p P(E_i^n) \leq \gamma_N P(E)$$

for any $p \geq 1$ because $i \mapsto |E_i^n|$ is decreasing. Therefore

$$\sum_{i=p}^{\infty} \mu\big(E_i^n\big) \leq \sum_{i=p}^{\infty} \left|E_i^n\right|^{1/\alpha} \leq \frac{[\gamma_N P(E)]^{\frac{N}{\alpha(N-1)}-1}}{p^{\frac{1}{\alpha}-\frac{(N-1)}{N}}} \sum_{i=p}^{\infty} \left|E_i^n\right|^{\frac{N-1}{N}}$$
$$\leq \frac{[\gamma_N P(E)]^{\frac{N}{\alpha(N-1)}}}{p^{\frac{1}{\alpha}-\frac{(N-1)}{N}}}$$

proving (12).

Since $\alpha > 1$, (12) also holds with $[\mu(E_i)]^\alpha$ in place of $\mu(E_i)$, and since $\mu(E_i^n) \to \mu(E_i)$ as $n \to \infty$ for any $i \in \mathbb{N}$, this implies

$$\sum_{i\in\mathbb{N}} [\mu(E_i)]^\alpha = \lim_{n\to\infty} \sum_{i\in\mathbb{N}} \left[\mu\big(E_i^n\big)\right]^\alpha = [\mu(E)]^\alpha.$$

By the definition of μ, this proves that

$$\int_{E\setminus\cup_i E_i} \exp(-|x|^2)\,dx = 0$$

and hence that $|E \setminus \cup_i E_i| = 0$. Moreover, using (12) again we obtain

$$\lim_{n\to\infty} \sum_{i\in\mathbb{N}} \mu\big(E_i^n\big) = \sum_{i\in\mathbb{N}} \mu(E_i)$$

hence $\{E_i\}_{i\in\mathbb{N}}$ is maximizing. If $\{E_i\}_{i\in\mathbb{N}}$ is any maximizing partition, then any E_i is clearly indecomposable, because $\mu(A)+\mu(B) > \mu(A\cup B)$ whenever $\mu(A)$ and $\mu(B)$ are strictly positive.

(Uniqueness) Let (E_i) be a maximizing partition and let F be an indecomposable set with $|F| > 0$. Since $F \subseteq E$ there exists $i \in I$ such that $|F \cap E_i| > 0$. We will prove that $F \subseteq E_i \pmod{\mathcal{H}^N}$. Since F is indecomposable, to this aim it suffices to prove that $P(F \setminus E_i) + P(F \cap E_i) = P(F)$, or equivalently that

$$(F \cap E_i)^{1/2} \cap (F \setminus E_i)^{1/2} = \emptyset \pmod{\mathcal{H}^{N-1}}. \tag{13}$$

Using Proposition 3 we obtain that $\partial^{\mathrm{M}} E_i \subset \partial^{\mathrm{M}} E \pmod{\mathcal{H}^{N-1}}$ and $P(E) = P(E \setminus E_i) + P(E_i)$. In turn, by Proposition 1, this gives $\partial^{\mathrm{M}} E_i \cap \partial^{\mathrm{M}}(E \setminus E_i) = \emptyset \pmod{\mathcal{H}^{N-1}}$. Hence, (13) would be proved by the inclusion

$$(F \cap E_i)^{1/2} \cap (F \setminus E_i)^{1/2} \subset E_i^{1/2} \cap (E \setminus E_i)^{1/2}. \tag{14}$$

Any point x in the set on the left side clearly belongs to $\mathring{F}^{\mathrm{M}}$ and hence to $\mathring{E}^{\mathrm{M}}$; taking this fact into account, it suffices to prove that $x \in E_i^{1/2}$, and since $x \in (F \cap E_i)^{1/2}$ this easily follows by the fact that $E_i \setminus (F \cap E_i)$ is contained in the complement of F. This proves the maximal character of E_i.

Finally, if $\{E_i\}_{i\in I}$ and $\{F_j\}_{j\in J}$ are two maximizing partitions, we know that any E_i is contained in one (and only one) F_j and any F_j is contained in one (and only one) E_i. Equation (11) follows by (8). □

Definition 1 (M-connected components). *In view of the previous theorem, we call the sets E_i the M-*connected components *of E and denote this family by $\mathcal{CC}^{\mathrm{M}}(E)$; we always choose the index set I as an interval of* $\mathbb{N}$, *with* $0 \in I$.

Notice that $\mathcal{CC}^{\mathrm{M}}(E) = \emptyset$ whenever E is Lebesgue negligible and that Proposition 3 gives

$$\partial^{\mathrm{M}} F \subset \partial^{\mathrm{M}} E \pmod{\mathcal{H}^{N-1}} \qquad \text{for any } F \in \mathcal{CC}^{\mathrm{M}}(E). \tag{15}$$

By (8), for $\mathcal{H}^{N-1}$-a.e. $x \in \mathring{E}^{\mathrm{M}}$ it also makes sense to talk about the *M-connected component of E containing x*, namely the unique set $F \in \mathcal{CC}^{\mathrm{M}}(E)$ such that $x \in \mathring{F}^{\mathrm{M}}$. The necessity to exclude an exceptional $\mathcal{H}^{N-1}$-negligible set is shown by the following example.

Example 1. Let $K \subseteq \{x_2 = 0\} \subseteq \mathbb{R}^2$ be a compact and $\mathcal{H}^1$-negligible set and let $\phi(x_1) = \mathrm{dist}^2(x_1, K)$. Then, the set

$$E := \left\{x = (x_1, x_2) \in \mathbb{R}^2 : \ x_2 < 0 \ \text{ or } \ x_2 > \phi(x_1)\right\}$$

has two M-connected components $E_1, \ E_2$ and it is easy to check that $K \subset \mathring{E}^{\mathrm{M}} \setminus (\mathring{E}_1^{\mathrm{M}} \cup \mathring{E}_2^{\mathrm{M}})$.

In the following theorem we prove that $\mathcal{CC}^{\mathrm{M}}(A)$ coincides with the family of connected components of A for any sufficiently regular open set A; we prove in Remark 3 that for any Lipschitz function $u : \mathbb{R}^{\mathrm{N}} \to \mathbb{R}$ almost every upper level set $\{u > \lambda\}$ has this (weak) regularity property. In general an open indecomposable set needs not be connected: for instance a disk without a diameter is disconnected but indecomposable. Example 2 shows in addition that an indecomposable set need not be equivalent (mod $\mathcal{H}^N$) to an open connected set.

Theorem 2. *Let $A \subseteq \mathbb{R}^{\mathrm{N}}$ be an open set such that $\mathcal{H}^{N-1}(\partial A) = \mathcal{H}^{N-1}(\partial^{\mathrm{M}} A)$. Then $\mathcal{CC}^{\mathrm{M}}(A)$ coincides with the family of connected components of A.*

Proof. The connected components $\{A_i\}_{i\in I}$ of A are pairwise disjoint, indecomposable by Proposition 2 and satisfy

$$\partial^{\mathrm{M}} A_i \subseteq \partial A \subseteq \partial^{\mathrm{M}} A \quad (\mathrm{mod}\ \mathcal{H}^{N-1}) \qquad\qquad \forall i \in I.$$

By Proposition 3 we obtain that $\sum_i P(A_i) \leq P(A)$. Hence, Theorem 1 implies that A_i are the M-connected components of A. □

Example 2. Let $K \subset (0, 1)$ be a compact set with empty interior and strictly positive measure and let $I_i = (a_i, b_i)$ be the connected components of $(0, 1) \setminus K$, indexed by $i \in I$, and let c_i be the central point of I_i. We define

$$A = (0, 1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \setminus \overline{\bigcup_{i\in I} B_i} \subseteq \mathbb{R}^2,$$

where B_i are closed balls centered at $(c_i, 0)$ with radius $b_i - c_i$ (see Fig. 1). Then, since K has empty interior it is easy to check that

$$\overline{\bigcup_{i\in I} B_i} = \bigcup_{i\in I} B_i \cup [0, 1] \times \{0\},$$

hence A is disconnected by the two open sets $A_1 = A \cap \{x_2 > 0\}$ and $A_2 = A \cap \{x_2 < 0\}$. On the other hand, we claim that A is indecomposable: indeed, since A_i are connected open sets, they are also indecomposable and hence are contained in M-connected components of A. Thus, if A were decomposable we would get $\mathcal{CC}^{\mathrm{M}}(A) = \{A_1, A_2\}$, and this contradicts the fact that $\partial^{\mathrm{M}} A_1$ and $\partial^{\mathrm{M}} A_2$ intersect on $K \times \{0\}$, a set with strictly positive $\mathcal{H}^1$ measure.

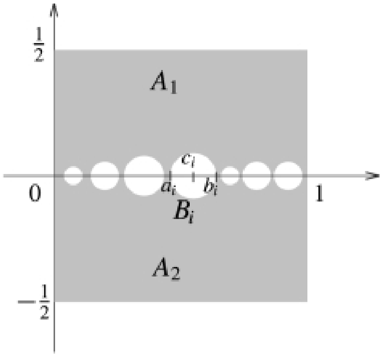

Fig. 1. An example illustrating the fact that the *M*-connected components of an open set do not coincide in general with the classical connected components

Remark 3. For any Lipschitz function $u : \mathbb{R}^{\mathrm{N}} \to \mathbb{R}$ the set $\{u > \lambda\}$ satisfies the assumption of Theorem 2 for a.e. $\lambda \in \mathbb{R}$. Indeed, let $\Omega \subseteq \mathbb{R}^{\mathrm{N}}$ be a bounded open set; by applying both the coarea formula for BV functions (see Sect. 3) and the coarea formula for Lipschitz functions (see for instance [22]) we get

$$\int_{-\infty}^{+\infty} \mathcal{H}^{N-1}\left(\Omega \cap \{u = \lambda\}\right) d\lambda = \int_{\Omega} |\nabla u|\, dx$$
$$= \int_{-\infty}^{+\infty} \mathcal{H}^{N-1}\left(\Omega \cap \partial^{\mathrm{M}}\{u > \lambda\}\right) d\lambda < \infty.$$

Since $\partial\{u > \lambda\} \subset \{u = \lambda\}$ for any $\lambda \in \mathbb{R}$, this proves that

$$\mathcal{H}^{N-1}\left(\Omega \cap \partial^{\mathrm{M}}\{u > \lambda\}\right) = \mathcal{H}^{N-1}\left(\Omega \cap \partial\{u > \lambda\}\right) \qquad \text{for a.e. } \lambda \in \mathbb{R}.$$

Taking a countable family of open sets Ω_h whose union is $\mathbb{R}^{\mathrm{N}}$ our statement follows.

Using the decomposition theorem we can easily prove that indecomposable sets have the same stability properties of connected sets.

Proposition 5 (Stability of indecomposable sets).

(i) If $E_1,\ E_2$ are indecomposable and either $|E_1 \cap E_2| > 0$ or $\mathcal{H}^{N-1}(\partial^{\mathrm{M}} E_1 \cap \partial^{\mathrm{M}} E_2) > 0$, then $E_1 \cup E_2$ is indecomposable.
(ii) If (E_h) is an increasing sequence of indecomposable sets with equibounded perimeters, then $\cup_h E_h$ is indecomposable.

Proof. (i) Let $\{G_i\}_{i\in I}$ be the components of $E_1 \cup E_2$ and let $j_1,\ j_2 \in I$ such that $E_i \subseteq G_{j_i}$. If $|E_1 \cap E_2| > 0$, since the G_i's are pairwise disjoint, we conclude that $j_1 = j_2$, hence $\mathcal{CC}^{\mathrm{M}}(E_1 \cup E_2) = \{G_{j_1}\}$. Otherwise we conclude that $\mathcal{CC}^{\mathrm{M}}(E_1 \cup E_2) = \{E_1, E_2\}$, hence Proposition 1 gives $\mathcal{H}^{N-1}(\partial^{\mathrm{M}} E_1 \cap \partial^{\mathrm{M}} E_2) = 0$. The proof of (ii) is analogous. □

We conclude this section with the analysis of the relation between indecomposability of a set and the indecomposability of its boundary, in the sense of H. Federer. To do this, we will adopt the notations of [24]; since this comparison is not really needed in the following, the reader unfamiliar with the theory of currents can simply skip this part.

Remark 4 (Indecomposability in the sense of Federer). Let us consider the collection of all k-dimensional normal integer currents T, denoted by $\mathbf{I}^k(\mathbb{R}^N)$. A current $T \in \mathbf{I}^k(\mathbb{R}^N)$ is said to be indecomposable if $T = T_1 + T_2$, $M(T) = M(T_1) + M(T_2)$ and $M(\partial T) = M(\partial T_1) + M(\partial T_2)$ with $T_i \in \mathbf{I}^k(\mathbb{R}^N)$ implies that either T_1 or T_2 are zero (here M denotes the mass, i.e. the area with multiplicities). Using Proposition 1, it is easy to show that the canonical N-current $[\![E]\!] \in \mathbf{I}^N(\mathbb{R}^N)$ associated to a set of finite perimeter E is indecomposable if and only if E is indecomposable; however, notice that the indecomposability of E is not equivalent to the indecomposability of its boundary (it suffices to consider as E an annulus).

In 4.2.25 of [24] it is stated that any $T \in \mathbf{I}^k(\mathbb{R}^N)$ admits a decomposition in finitely or countably many indecomposable components; the proof (suggested and not explicitly given) again relies on the isoperimetric inequality and could be obtained mimicking our one, i.e. maximizing $\sum_i [M(T_i)]^{1/\alpha}$, with $\alpha \in (1, k/(k-1))$, among all possible decompositions T_i. However, no uniqueness theorem for the decomposition holds for $k < N$.

5. Holes, saturation, simple sets

In this section we see how the decomposition theorem leads to reasonably good definitions of "hole" and "saturation" for a set of finite perimeter. These concepts will be used in the next section to recover a canonical decomposition of the measure theoretic boundary.

Definition 2 (Holes, saturation). *Let E be an indecomposable set. We call* hole *of E any M-connected component of $\mathbb{R}^N \setminus E$ with finite measure. We define the saturation of E, denoted by* $\mathrm{sat}(E)$, *as the union of E and its holes. In the general case when E has finite perimeter, we define*

$$\mathrm{sat}(E) := \bigcup_{i \in I} \mathrm{sat}(E_i) \qquad \textit{where} \qquad \mathcal{CC}^{\mathrm{M}}(E) = \{E_i\}_{i \in I}.$$

We call E saturated if $\mathrm{sat}(E) = E$.

We first investigate the saturation operator on indecomposable sets and later we extend this analysis to any set of finite perimeter.

Proposition 6. *Let $E \subseteq \mathbb{R}^N$ be an indecomposable set.*

(i) *Any hole of E is saturated.*
(ii) $\mathrm{sat}(E)$ *is indecomposable, saturated,* $\partial^{\mathrm{M}} \mathrm{sat}(E) \subset \partial^{\mathrm{M}} E \pmod{\mathcal{H}^{N-1}}$ *and* $\mathrm{sat}(E)$ *has finite measure if $|E| < \infty$. In particular* $P(\mathrm{sat}(E)) \leq P(E)$.
(iii) *If* $E \subset \mathrm{sat}(F)$ *then* $\mathrm{sat}(E) \subset \mathrm{sat}(F)$.

(iv) If F is indecomposable and $|F \cap E| = 0$, then the sets $\mathrm{sat}(E)$, $\mathrm{sat}(F)$ *are either one a subset of the other, or are disjoint.*

Proof. (i) Let Y be an hole of E and let $\mathcal{CC}^{\mathrm{M}}(\mathbb{R}^{\mathrm{N}} \setminus E) = \{Y\} \cup \{Y_j\}_{j \in J}$. Then

$$\mathbb{R}^{\mathrm{N}} \setminus Y = E \cup \bigcup_{j \in J} Y_j .$$

Since by (15) $\partial^{\mathrm{M}} Y_j \subset \partial^{\mathrm{M}} E \pmod{\mathcal{H}^{N-1}}$, Proposition 5(i) gives that $E \cup \cup_{j \in J'} Y_{j'}$ is indecomposable for any finite set $J' \subseteq J$. By Proposition 5(ii) we conclude that $\mathbb{R}^{\mathrm{N}} \setminus Y$ is indecomposable, i.e. Y has no hole.

(ii) We can assume with no loss of generality that $|E| < \infty$ (otherwise $\mathrm{sat}(E) = \mathbb{R}^{\mathrm{N}}$) and denote by Y_0 the M-connected component of $\mathbb{R}^{\mathrm{N}} \setminus E$ with infinite measure. The proof that $\mathrm{sat}(E)$ is indecomposable relies, as the one of (i), on Proposition 5. Since $\mathrm{sat}(E) = \mathbb{R}^{\mathrm{N}} \setminus Y_0$, $\mathrm{sat}(E)$ is saturated. Finally, the inclusion $\partial^{\mathrm{M}} \mathrm{sat}(E) \subset \partial^{\mathrm{M}} E \pmod{\mathcal{H}^{N-1}}$ follows by (15).

(iii) Without loss of generality we can assume that $|F| < \infty$. Then $\mathbb{R}^{\mathrm{N}} \setminus \mathrm{sat}(F)$, being indecomposable, is contained in a M-connected component of $\mathbb{R}^{\mathrm{N}} \setminus E$; since $|\mathbb{R}^{\mathrm{N}} \setminus \mathrm{sat}(F)| = \infty$ we conclude that $\mathbb{R}^{\mathrm{N}} \setminus \mathrm{sat}(F) \subseteq \mathbb{R}^{\mathrm{N}} \setminus \mathrm{sat}(E)$.

(iv) We may assume that both sets are nontrivial and that their saturations are not $\mathbb{R}^{\mathrm{N}}$; we denote by E_0, F_0 the M-connected components with infinite measure of $\mathbb{R}^{\mathrm{N}} \setminus E$, $\mathbb{R}^{\mathrm{N}} \setminus F$ respectively. Since $|E \cap F| = 0$, we know that E is contained either in a hole of F or in F_0. If E is contained in a hole of F, then $E \subseteq \mathrm{sat}(F)$ and therefore $\mathrm{sat}(E) \subseteq \mathrm{sat}(F)$. Analogously, if $E \subseteq F_0$ and F is contained in a hole of E, then $\mathrm{sat}(F) \subseteq \mathrm{sat}(E)$. Thus we may assume that $E \subseteq F_0$ and $F \subseteq E_0$, hence

$$|E \cap \mathrm{sat}(F)| = 0 \qquad \text{and} \qquad |F \cap \mathrm{sat}(E)| = 0. \tag{16}$$

Under this assumption, let us prove that $|\mathrm{sat}(E) \cap \mathrm{sat}(F)| = 0$. To this aim, by (16), it suffices to show that $|Y \cap \mathrm{sat}(F)| = 0$ for any hole Y of E. Since, by (16) again, $Y \subset \mathbb{R}^{\mathrm{N}} \setminus F$, Y is contained in a M-connected component of $\mathbb{R}^{\mathrm{N}} \setminus F$. If $Y \subseteq F_0$ the proof is finished, otherwise $Y \subseteq Y'$ for some hole Y' of F which, in turn, is contained in some M-connected component Y'' of $\mathbb{R}^{\mathrm{N}} \setminus E$. But then $Y'' = Y$ and therefore $Y' = Y$. Since by (15) $\partial^{\mathrm{M}} Y \subset (\partial^{\mathrm{M}} E \cap \partial^{\mathrm{M}} F) \pmod{\mathcal{H}^{N-1}}$, if we choose $x \in Y^{1/2} \cap E^{1/2} \cap F^{1/2}$ we find that $|E \cap F \cap B(x,r)| > 0$ for $r > 0$ sufficiently small; this contradiction proves that $Y \subseteq F_0$. □

Definition 3 (Simple sets). *Any indecomposable and saturated subset of* $\mathbb{R}^{\mathrm{N}}$ *will be called simple.*

Notice that the only simple set with infinite measure is $\mathbb{R}^{\mathrm{N}}$ and that, according to Proposition 6, the saturation of any indecomposable set E is simple (actually, the smallest simple set containing E). In order to show coincidence with simple sets we will often use the following proposition.

Proposition 7. *Let E be a simple set and let $F \subseteq \mathbb{R}^{\mathrm{N}}$ be a set with finite perimeter, such that* $\partial^{\mathrm{M}} F \subseteq \partial^{\mathrm{M}} E \pmod{\mathcal{H}^{N-1}}$ *and* $|F| \in (0, \infty)$. *Then* $F = E$.

Proof. It suffices to apply Proposition 4 to E and $F \cap E$ and to $\mathbb{R}^N \setminus E$ and $F \setminus E$. $\square$

The property stated in Proposition 7 actually characterizes simple sets with finite measure; we also give another nice characterization of these sets due to W.H. Fleming.

Proposition 8 (Characterizations of simple sets). *Let $E \subseteq \mathbb{R}^N$ be a set with finite perimeter such that $|E| \in (0, \infty)$. Then, the following conditions are equivalent:*

(i) *E is simple;*
(ii) *E satisfies the property stated in Proposition 7;*
(iii) *$\chi_E / P(E)$ is an extreme point of the convex set*

$$\left\{u \in \mathrm{BV}(\mathbb{R}^N) : \ |Du|(\mathbb{R}^N) \le 1\right\}.$$

Proof. The implication (i)$\Longrightarrow$(ii) is Proposition 7. The converse implication can be proved by noticing that any hole Y of E satisfies $\partial^M Y \subset \partial^M E \pmod{\mathcal{H}^{N-1}}$ and hence coincides with E. This contradiction proves that E has no hole, i.e. $\mathrm{sat}(E) = E$. The equivalence of (ii) and (iii) is proved (in a slightly different setting, since a bound on the supports of the functions is required) in [25]. $\square$

We close this section with the following result, showing that the M-connected components of $\mathrm{sat}(E)$ are contained in the family of saturations of M-connected components of E.

Theorem 3 (M-connected components and saturation). *Let $E \subseteq \mathbb{R}^N$ be a set of finite perimeter. Then*

$$\mathcal{CC}^M(\mathrm{sat}(E)) \subset \{\mathrm{sat}(E_i)\}_{i\in I} \qquad \textit{where} \qquad \mathcal{CC}^M(E) = \{E_i\}_{i\in I}.$$

In particular $\partial^M \mathrm{sat}(E) \subset \partial^M E \pmod{\mathcal{H}^{N-1}}$ and the operator sat *is idempotent, i.e.* $\mathrm{sat}(\mathrm{sat}(E)) = \mathrm{sat}(E)$.

Proof. Let $\mathcal{CC}^M(E) = \{E_i\}_{i\in I}$ and assume with no loss of generality that $|E| < \infty$; we know by Proposition 6 and the isoperimetric inequality that $\mathrm{sat}(E_i)$ are indecomposable sets satisfying the conditions of Lemma 2 below. Hence, $\{\mathrm{sat}(E_j)\}_{j\in J}$ provides a disjoint partition of $\mathrm{sat}(E)$ in indecomposable sets.

Finally, (15) and Proposition 6(ii) give

$$\partial^M \mathrm{sat}(E) \subset \bigcup_{j\in J} \partial^M \mathrm{sat}(E_j) \subset \bigcup_{i\in I} \partial^M E_i \subset \partial^M E$$

where all inclusions are understood $\pmod{\mathcal{H}^{N-1}}$. $\square$

Lemma 2. *Let $I \subset \mathbb{N}$ and let $\{F_i\}_{i\in I}$ be a family of sets such that for any $i, j \in I$ either $F_i \subseteq F_j$ or $F_j \subseteq F_i$ or $F_i \cap F_j = \emptyset \pmod{\mathcal{H}^N}$. Then, assuming that $|F_i| \to 0$ as $i \to \infty$ if I is countable, there exists $J \subseteq I$ such that $\{F_j\}_{j\in J}$ are pairwise disjoint $\pmod{\mathcal{H}^N}$ and $|\cup_i F_i \setminus \cup_j F_j| = 0$.*

Proof. It suffices to consider the partial order $i \preceq j$ if $|F_j \setminus F_i| = 0$ and to take its maximal elements. If I is countable, the existence of maximal elements follows easily by the assumption that $|F_i| \to 0$ as $i \to \infty$. $\square$

6. Description of sets of finite perimeter in terms of their boundary

In general a decomposition in M-connected components does not lead directly to a canonical decomposition of the boundary. The aim of this section is to show that this goal can be achieved by looking to the saturations and to the holes of all M-connected components of E.

Definition 4 (Exterior). *If $E \subseteq \mathbb{R}^N$ has finite perimeter and $|E| < \infty$, we call* exterior *of E the unique* (mod $\mathcal{H}^N$) *M-component of* $\mathbb{R}^N \setminus E$ *with infinite measure. The exterior of E will be denoted by* $\mathrm{ext}(E)$.

Notice that the notion of exterior makes sense only if $|E| < \infty$, due to the fact that $\mathbb{R}^N \setminus E$ has finite measure if $P(E) < \infty$ and $|E| = \infty$.

Definition 5 (Jordan boundary). *We say that a set J is a Jordan boundary if there is a simple set E such that $J = \partial^M E$* (mod $\mathcal{H}^{N-1}$).

By Proposition 7, the simple set E associated to a Jordan boundary J is unique. In this sense, J can also be thought as an *oriented* set, with the orientation induced by the generalized inner normal to E. Our terminology is motivated by the results of the following section concerning sets in the plane, see in particular Theorem 7. We shall write $\mathrm{int}(J) = E$ and $\mathrm{ext}(J) = \mathbb{R}^N \setminus E$; notice that $\mathrm{ext}(J) = \mathrm{ext}(E)$.

Proposition 9. *Let E be indecomposable and let $\{Y_i\}_{i\in I}$ be its holes. Then*

$$E = \mathrm{sat}(E) \setminus \bigcup_{i\in I} Y_i = \mathrm{sat}(E) \cap \bigcap_{i\in I} \mathrm{ext}(Y_i) \tag{17}$$

and

$$P(E) = P(\mathrm{sat}(E)) + \sum_{i\in I} P(Y_i). \tag{18}$$

Conversely, let F be simple and let $\{G_i\}_{i\in I}$ be indecomposable sets such that

$$E = F \setminus \bigcup_{i\in I} G_i \tag{19}$$

and

$$P(E) = P(F) + \sum_{i\in I} P(G_i). \tag{20}$$

Then $F = \mathrm{sat}(E)$ and $\{G_i\}_{i\in I}$ are the holes of E.

Proof. The first equality in (17) is a consequence of Definition 2. The second identity is a consequence of Proposition 6(i). In order to prove (18) we recall that the perimeter and the measure theoretic boundary are invariant under complement and notice that

$$\mathbb{R}^N \setminus E = \big(\mathbb{R}^N \setminus \mathrm{sat}(E)\big) \cup \bigcup_{i\in I} Y_i.$$

Since both $\partial^M \operatorname{sat}(E)$ and $\partial^M Y_i$ are contained in $\partial^M E$ up to $\mathcal{H}^{N-1}$-negligible sets, by Proposition 3 we infer (18).

Let us now prove the uniqueness of the decomposition given in (17). For that, let F be simple and let $\{G_i\}_{i\in I}$ be indecomposable sets satisfying (19) and (20). Assume first that $|E| < \infty$, set $G_\infty = \mathbb{R}^N \setminus F$ and observe that

$$\mathbb{R}^N \setminus E = \bigcup_{i\in I'} G_i$$

with $I' = I \cup \{\infty\}$. Then, Proposition 3 gives that $\{G_i\}_{i\in I'}$ are pairwise disjoint and $\partial^M G_i \cap \partial^M G_j = \emptyset \pmod{\mathcal{H}^{N-1}}$ whenever $i \neq j$.

Note that G_∞ is indecomposable, since F is a simple set. Thus $\{G_i\}_{i\in I'}$ is a partition of $\mathbb{R}^N \setminus E$ into indecomposable sets satisfying (20). By the uniqueness of the decomposition of $\mathbb{R}^N \setminus E$ in M-connected components we conclude that $G_\infty = \mathbb{R}^N \setminus \operatorname{sat}(E)$ (i.e. $F = \operatorname{sat}(E)$) and $\{G_i\}_{i\in I}$ coincides with the family of holes of E. In case that E has infinite measure, $\mathbb{R}^N = \operatorname{sat}(E) \subseteq \operatorname{sat}(F) = F$, i.e. $F = \mathbb{R}^N$ and the proof follows the same steps of the previous one. $\square$

In order to simplify the following statements we enlarge the class of Jordan boundaries by introducing a *formal* Jordan boundary J_∞ whose interior is $\mathbb{R}^N$ and a *formal* Jordan boundary J_o whose interior is empty; we also set $\mathcal{H}^{N-1}(J_\infty) = \mathcal{H}^{N-1}(J_o) = 0$ and denote by $\mathcal{S}$ this extended class of Jordan boundaries. In this way we are able to consider at the same time sets with finite and infinite measure and we can always assume that the list of components (or holes of the components) is infinite, possibly adding to it infinitely many $\operatorname{int}(J_o)$.

In the following theorem we describe $\partial^M E$ by a collection of "external Jordan boundaries" J_i^+ and "internal Jordan boundaries" J_i^- satisfying some inclusion properties; these properties provide an axiomatic characterization of them. However, we emphasize (see Fig. 2 in Sect. 7) that in general this description is not invariant under complementation, i.e. the external (internal) boundaries of a set are not the internal (external) boundaries of the complement; for this reason we give a different definition of these concepts the next section.

Theorem 4 (Decomposition of $\partial^M E$ in Jordan boundaries). *Let $E \subseteq \mathbb{R}^N$ be a set of finite perimeter. Then, there is a unique decomposition of $\partial^M E$ into Jordan boundaries $\left\{J_i^+, J_k^- : i, k \in \mathbb{N}\right\} \subseteq \mathcal{S}$, such that*

(i) *Given* $\operatorname{int}(J_i^+)$, $\operatorname{int}(J_k^+)$, *$i \neq k$, they are either disjoint or one is contained in the other; given* $\operatorname{int}(J_i^-)$, $\operatorname{int}(J_k^-)$, *$i \neq k$, they are either disjoint or one is contained in the other. Each* $\operatorname{int}(J_i^-)$ *is contained in one of the* $\operatorname{int}(J_k^+)$.

(ii) $P(E) = \sum_i \mathcal{H}^{N-1}(J_i^+) + \sum_k \mathcal{H}^{N-1}(J_k^-)$.

(iii) *If* $\operatorname{int}(J_i^+) \subseteq \operatorname{int}(J_j^+)$, *$i \neq j$, then there is some Jordan boundary J_k^- such that* $\operatorname{int}(J_i^+) \subseteq \operatorname{int}(J_k^-) \subseteq \operatorname{int}(J_j^+)$. *Similarly, if* $\operatorname{int}(J_i^-) \subseteq \operatorname{int}(J_j^-)$, *$i \neq j$, then there is some Jordan boundary J_k^+ such that* $\operatorname{int}(J_i^-) \subseteq \operatorname{int}(J_k^+) \subseteq \operatorname{int}(J_j^-)$.

(iv) *Setting* $L_j = \{i : \operatorname{int}(J_i^-) \subseteq \operatorname{int}(J_j^+)\}$, *the sets* $Y_j = \operatorname{int}(J_j^+) \setminus \cup_{i\in L_j} \operatorname{int}(J_i^-)$ *are pairwise disjoint, indecomposable and* $E = \cup_j Y_j$.

Proof. (Existence) Let Y_i be the M-connected components of E. According to Proposition 9, let $J_i^+ = \partial^{\mathrm{M}} \operatorname{sat}(Y_i)$ be the external Jordan boundary of Y_i and let $J_{i,n}^-$, $n = 1, 2, \ldots$, be the family of the internal Jordan boundaries of Y_i, given by the boundaries of the holes of Y_i. Taking into account Proposition 6 and the fact that holes are saturated, we obtain that (i) is satisfied.

Using (18) we immediately obtain (ii). To prove (iii), suppose that $\operatorname{int}(J_i^+) \subseteq \operatorname{int}(J_j^+)$, with $i \neq j$. Since $|Y_i \cap Y_j| = 0$, Y_i is contained in a hole of Y_j. Then there is some Jordan boundary $J_{j,k}^-$ such that $\operatorname{int}(J_i^+) \subseteq \operatorname{int}(J_{j,k}^-) \subseteq \operatorname{int}(J_j^+)$. The other statement included in (iii) follows from the observation that two different holes of the same M-connected component are disjoint. To prove (iv) we observe that

$$
\begin{aligned}
Y_j &= \operatorname{int}\left(J_j^+\right) \setminus \left\{ \operatorname{int}\left(J_{j,n}^-\right) : n \in \mathbb{N} \right\} \\
&= \operatorname{int}\left(J_j^+\right) \setminus \left\{ \operatorname{int}\left(J_{i,n}^-\right) : \operatorname{int}\left(J_{i,n}^-\right) \subseteq \operatorname{int}\left(J_j^+\right) \right\}
\end{aligned}
$$

because any hole $\operatorname{int}(J_{i,n}^-)$ of Y_i contained in $\operatorname{int}(J_j^+)$, being disjoint with Y_j, is contained in a hole of Y_j.

(Uniqueness) Let C_i^+, C_k^-, $i, k \in \mathbb{N}$, be a family of Jordan boundaries satisfying (i), (ii), (iii), (iv). Let $K_j = \operatorname{int}(C_j^+) \setminus \cup_{i \in L_j} \operatorname{int}(C_i^-)$, $j \geq 0$. By assumption, the sets K_j are indecomposable and $E = \cup_j K_j$. Let us prove that

$$
P(E) = \sum_{j=0}^{\infty} P(K_j).
$$

We say that an index i is j-maximal if $\operatorname{int}(C_i^-) \subseteq \operatorname{int}(C_j^+)$ and there is no other $\operatorname{int}(C_k^-)$ such that $\operatorname{int}(C_i^-) \subseteq \operatorname{int}(C_k^-) \subseteq \operatorname{int}(C_j^+)$. Analogously, we say that an index j is i-minimal if $\operatorname{int}(C_i^-) \subseteq \operatorname{int}(C_j^+)$ and there is no other $\operatorname{int}(C_k^+)$ such that $\operatorname{int}(C_i^-) \subseteq \operatorname{int}(C_k^+) \subseteq \operatorname{int}(C_j^+)$.

Let $\Psi_j = \{i : i \text{ is } j\text{-maximal}\}$; we observe that if $\operatorname{int}(C_l^-) \subseteq \operatorname{int}(C_j^+)$, then there exist a j-maximal index i such that $\operatorname{int}(C_l^-) \subseteq \operatorname{int}(C_i^-)$ and a l-minimal index k such that $\operatorname{int}(C_k^+) \subseteq \operatorname{int}(C_j^+)$. Indeed, if there were an increasing chain of sets $\operatorname{int}(C_i^-)$, then, by the isoperimetric inequality we would get that the sum of their perimeters is infinite, a contradiction with (ii). Similarly, there is no decreasing sequence of sets $\operatorname{int}(C_k^+)$ containing $\operatorname{int}(C_l^-)$. As a consequence, we obtain

$$
K_j = \operatorname{int}\left(C_j^+\right) \setminus \bigcup_{i \in \Psi_j} \operatorname{int}\left(C_i^-\right). \tag{21}
$$

Now, observe that the sets Ψ_j are a partition of $\mathbb{N}$. First we observe that they are disjoint. Indeed, let $i \in \Psi_j \cap \Psi_k$, $j \neq k$. Then $\operatorname{int}(C_i^-) \subseteq \operatorname{int}(C_j^+)$ and $\operatorname{int}(C_i^-) \subseteq \operatorname{int}(C_k^+)$. Thus, either $\operatorname{int}(C_j^+) \subseteq \operatorname{int}(C_k^+)$, or $\operatorname{int}(C_k^+) \subseteq \operatorname{int}(C_j^+)$. If we are in the first case, then (iii) proves that the index i cannot be k-maximal. If we are in the second case, then (iii) proves that the index i cannot be j-maximal. Next, let $i \in \mathbb{N}$ and let j such that j is i-minimal. Then, using (iii), we have that i is j-maximal, i.e. $i \in \Psi_j$.

By Theorem 5 below we know that

$$P(K_j) = \mathcal{H}^{N-1}\big(C_j^+\big) + \sum_{i\in\Psi_j} \mathcal{H}^{N-1}(C_i^-)$$

Adding both sides with respect to j we obtain that $P(E) = \sum_j P(K_j)$.

By the uniqueness of the decomposition of E into M-connected components we obtain that, up to a permutation of indexes, $K_j = Y_j$ for all $j \in \mathbb{N}$. Now, the uniqueness result of Proposition 9 proves that $\mathrm{int}(C_j^+) = \mathrm{int}(J_j^+)$ and that $\mathrm{int}(C_i^-)$, $i \in \Psi_j$, coincide with the system of holes of Y_j. □

Theorem 5. *Let $\{J_i^+, J_k^- : i, k \in \mathbb{N}\} \subset \mathcal{S}$ be satisfying the conditions (i), (iii) of Theorem 4 and*

(ii′) *Each two different Jordan boundaries of the system $\{J_i^+, J_k^- : i, k \geq 0\}$ are disjoint* $(\mathrm{mod}\ \mathcal{H}^{N-1})$.
(iv′) $\sum_i P(J_i^+) + \sum_k P(J_k^-) < \infty$.

Let $E = \cup_j Y_j$, where

$$Y_j := \mathrm{int}\left(J_j^+\right) \setminus \bigcup_{i\in L_j} \mathrm{int}\left(J_i^-\right).$$

Then E is a set of finite perimeter and $\partial^{\mathrm{M}} E = \cup_i J_i^+ \cup \cup_k J_k^-$ $(\mathrm{mod}\ \mathcal{H}^{N-1})$.

Proof. Let

$$\Phi_j := \left\{i : \mathrm{int}\left(J_j^+\right) \text{ is the minimal set } \mathrm{int}\left(J_k^+\right) \text{ containing } \mathrm{int}\left(J_i^-\right)\right\}.$$

By definition the sets Φ_j are pairwise disjoint and the axiom (i) provides for any i a minimal set $\mathrm{int}(J_j^+)$ containing $\mathrm{int}(J_i^-)$, so that $\cup_j \Phi_j = \mathbb{N}$. We also notice that

$$Y_j = \mathrm{int}\left(J_j^+\right) \setminus \bigcup_{i\in\Phi_j} \mathrm{int}\left(J_i^-\right).$$

because, whenever $\mathrm{int}(J_i^-) \subseteq \mathrm{int}(J_j^+)$, the maximal set $\mathrm{int}(J_k^-)$ containing $\mathrm{int}(J_i^-)$ and contained in $\mathrm{int}(J_j^+)$ satisfies $k \in \Phi_j$, by the axiom (iii).

Finally, the sets Y_j are pairwise disjoint because if $\mathrm{int}(J_j^+)$ and $\mathrm{int}(J_k^+)$ have a nonempty intersection, then one (say the first) is contained in the other; since there exists $i \in L_k$ such that $\mathrm{int}(J_j^+) \subseteq \mathrm{int}(J_i^-)$ we obtain that $Y_j \subset \mathrm{int}(J_i^-) \subset \mathbb{R}^{\mathrm{N}} \setminus Y_k$, a contradiction.

In view of Proposition 3 and (ii'), (iv'), the proof will be complete if we show that

$$\partial^{\mathrm{M}} Y_j = J_j^+ \cup \bigcup_{i\in\Phi_j} J_i^- \quad (\mathrm{mod}\ \mathcal{H}^{N-1})$$

for any $j \in \mathbb{N}$. To this aim, we notice that $\mathbb{R}^{\mathrm{N}} \setminus Y_j$ is the disjoint union of $\mathrm{ext}(J_j^+)$ and $\mathrm{int}(J_i^-)$, $i \in \Phi_j$; in fact, if $|\,\mathrm{int}(J_i^-) \cap \mathrm{int}(J_l^-)| > 0$ for $i, l \in \Phi_j$, $i \neq j$, then one set (say the first) is contained in the other, hence there is a set $\mathrm{int}(J_k^+)$ contained in $\mathrm{int}(J_l^-)$ and containing $\mathrm{int}(J_i^-)$, contradicting the fact that $i \in \Phi_j$.

By applying Proposition 3 and (ii') again the identity above follows. □

7. Topographic function and internal/external boundaries of sets

The representation of the boundary of a set of finite perimeter by a family of nested Jordan boundaries $J_i^{\pm}$ has the advantage of being easily obtained by the family of saturations and holes of the M-connected components of E, but has the drawback of being not invariant under complementation, as Fig. 2 shows. Another drawback of the $J_i^{\pm}$ representation is the absence of a natural order structure on them, despite conditions (i) and (iii) in Theorem 4.

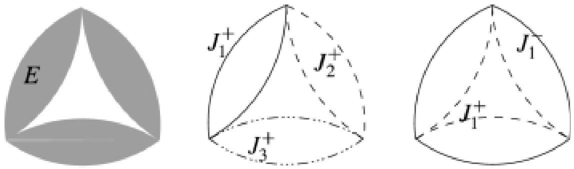

Fig. 2. The set E (in grey), its boundaries $J^{\pm}$ and the boundaries of its complement. The last figure illustrates as well the internal and external boundaries obtained by the topographic function

In this section we prove the existence of a family of nested boundaries which is invariant under complementation; the family is given by $\partial^M\{u \le k\}$ (k even for the external boundaries, k odd for the internal ones), where $u : \mathbb{R}^N \to \mathbb{N}$ is the BV_{loc} function characterized by the following theorem. Heuristically, $u(x)$ measures how "deep" is x inside E, i.e., it counts how many boundaries must be crossed to reach the exterior of E. This is illustrated in Fig. 3 where E is the gray set.

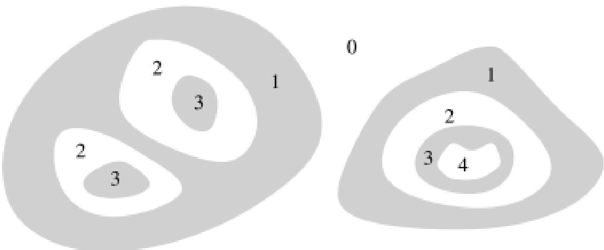

Fig. 3. The topographic function associated with the gray set E counts how many boundaries must be crossed to reach the exterior of E

Theorem 6. *Let $E \subseteq \mathbb{R}^N$ be a set of finite perimeter. Then there exists a unique map $u \in BV_{loc}(\mathbb{R}^N, \mathbb{N})$ such that*

(i) $u = \chi_E \mod 2$ *and all sets $\{u \le k\}$ are indecomposable;*
(ii) $|Du| = \mathcal{H}^{N-1} \llcorner \partial^M E$;
(iii) $u = \chi_E$ *in the M-connected component of E or $\mathbb{R}^N \setminus E$ with infinite measure.*

Proof. We denote by $\{E_i\}_{i\in I}$ the M-connected components of E and by $\{F_j\}_{j\in J}$ the M-connected components of $\mathbb{R}^N \setminus E$. Being the statement invariant under complementation we can assume that $|E| < \infty$ and denote by j_0 the index in J such that $|F_{j_0}| = \infty$. Recall that Proposition 3 gives

$$\partial^M \left(\bigcup_{i \in I_1} E_i \right) = \bigcup_{i \in I_1} \partial^M E_i, \quad \partial^M \left(\bigcup_{j \in J_1} F_j \right) = \bigcup_{j \in J_1} \partial^M F_j \qquad (\mathrm{mod}\ \mathcal{H}^{N-1})$$

whenever $I_1 \subseteq I$, $J_1 \subseteq J$.

(Existence) We define recursively sets $U_k \subseteq \mathbb{R}^N$ and subsets $\Phi_k \subset I$, for k odd, and $\Phi_k \subset J$ for k even as follows: first we set $U_0 = F_{j_0}$ and $\Phi_0 = \{j_0\}$ and then, assuming that all sets U_l and Φ_l have been defined for $l < k$, we define:

$$\Phi_k := \left\{ i \in I : \mathcal{H}^{N-1}\left(\partial^M U_{k-1} \cap \partial^M E_i\right) > 0 \right\}, \quad U_k := U_{k-1} \cup \bigcup_{i \in \Phi_k} E_i \ \text{ if } k \text{ is odd}$$

and

$$\Phi_k := \left\{ j \in J : \mathcal{H}^{N-1}\left(\partial^M U_{k-1} \cap \partial^M F_j\right) > 0 \right\}, \quad U_k := U_{k-1} \cup \bigcup_{j \in \Phi_k} F_j \ \text{ if } k \text{ is even.}$$

Let us prove by induction that all sets U_k are indecomposable. This property is clearly satisfied for $k = 0$, so let us assume it true for $k - 1 \geq 0$ and let us prove it for k. Assuming, to fix the ideas, that k is odd, for any finite set $R \subset \Phi_k$ and any $i \in \Phi_k \setminus R$ we have

$$\partial^M E_i \cap \partial^M \left(U_{k-1} \cup \bigcup_{i \in R} E_i \right) \supseteq \partial^M E_i \cap \partial^M U_{k-1} \neq \emptyset \quad (\mathrm{mod}\ \mathcal{H}^{N-1})$$

because $\partial^M E_i$ are pairwise disjoint $(\mathrm{mod}\ \mathcal{H}^{N-1})$. Hence, by applying inductively Proposition 5(i), we obtain that $U_{k-1} \cup \bigcup_{i \in R} E_i$ is indecomposable for any finite set $R \subseteq \Phi_k$. By Proposition 5(ii) we obtain that U_k is indecomposable. An analogous argument also proves that

$$\partial^M U_k \subseteq \partial^M E \quad (\mathrm{mod}\ \mathcal{H}^{N-1}) \qquad \forall k \in \mathbb{N}. \tag{22}$$

Denoting by I' (respectively J') the subset of I (resp. of J) obtained by taking the union of all sets Φ_{2k+1} (resp. Φ_{2k}), let us prove the following two upper and lower bounds on $\partial^M U_k$, which both will be useful in the following:

$$\begin{aligned} \partial^M U_{2k} \cup \partial^M U_{2k+1} \subseteq \bigcup_{i \in \Phi_{2k+1}} \partial^M E_i, \quad & \partial^M U_{2k-1} \cup \partial^M U_{2k} \\ & \subseteq \bigcup_{j \in \Phi_{2k}} \partial^M F_j \ \ (\mathrm{mod}\ \mathcal{H}^{N-1}) \end{aligned} \tag{23}$$

and

$$\begin{aligned} \bigcup_{i \in \Phi_{2k+1}} \partial^M E_i \setminus \partial^M U_{2k+1} \subseteq \bigcup_{j \in J'} \partial^M F_j, \quad & \bigcup_{j \in \Phi_{2k}} \partial^M F_j \setminus \partial^M U_{2k} \\ & \subseteq \bigcup_{i \in I'} \partial^M E_i \ \ (\mathrm{mod}\ \mathcal{H}^{N-1}). \end{aligned} \tag{24}$$

The first inclusion in (23) follows by

$$\partial^{\mathrm{M}} U_{2k+1} \subset \partial^{\mathrm{M}} U_{2k} \cup \partial^{\mathrm{M}} \left(\bigcup_{i \in \Phi_{2k+1}} E_i \right)$$

using (22) and the definition of Φ_{2k+1}; the second inclusion can be proved in a similar way. The first inclusion in (24) follows by the second one in (23), noticing that

$$\partial^{\mathrm{M}} \left(\bigcup_{i \in \Phi_{2k+1}} E_i \right) \subset \partial^{\mathrm{M}} U_{2k+1} \cup \partial^{\mathrm{M}} U_{2k}. \tag{25}$$

The proof of the second inclusions in (24) is analogous.

Now we prove that $\Omega = \cup_k U_k$ is $\mathbb{R}^{\mathrm{N}}$ (mod $\mathcal{H}^N$) (hence $I' = I$ and $J' = J$). To this aim, we argue by contradiction: since

$$\partial^{\mathrm{M}} \Omega \subset \partial^{\mathrm{M}} \left(\bigcup_{i \in I'} E_i \right) \cup \partial^{\mathrm{M}} \left(\bigcup_{j \in J'} F_j \right) = \bigcup_{i \in I'} \partial^{\mathrm{M}} E_i \cup \bigcup_{j \in J'} \partial^{\mathrm{M}} F_j \quad (\mathrm{mod}\ \mathcal{H}^{N-1})$$

and an analogous property holds for $\mathbb{R}^{\mathrm{N}} \setminus \Omega$ and $I \setminus I'$, $J \setminus J'$, taking into account that $\forall i \in I \setminus I'$, $\forall j \in J \setminus J'$, $\partial^{\mathrm{M}} E_i$ and $\partial^{\mathrm{M}} F_j$ are pairwise disjoint (mod $\mathcal{H}^{N-1}$), assuming that $P(\Omega) > 0$ we can find either $i \in I \setminus I'$ and $j \in J'$ or $i \in I'$ and $j \in J \setminus J'$ such that $\partial^{\mathrm{M}} E_i \cap \partial^{\mathrm{M}} F_j \neq \emptyset$ (mod $\mathcal{H}^{N-1}$). Assume, to fix the ideas, that $i \in I \setminus I'$ and $j \in J'$ and let k such that $j \in \Phi_{2k}$. Then, by (23) and (24) we obtain that

$$\partial^{\mathrm{M}} U_{2k} \cap \partial^{\mathrm{M}} E_i \neq \emptyset \quad (\mathrm{mod}\ \mathcal{H}^{N-1}).$$

This proves that $i \in \Phi_{2k+1} \subseteq I'$ and gives a contradiction.

Finally, we define u equal to k on $U_k \setminus U_{k-1}$ (with $U_{-1} = \emptyset$). By construction $\{u \leq k\} = U_k$ is indecomposable and $u = \chi_E$ mod 2. Let us prove that condition (ii) holds; to this aim, we first prove that all sets Φ_{2k+1} are pairwise disjoint. Assume by contradiction that $i \in \Phi_{2l+1} \cap \Phi_{2k+1}$ with $l < k$; then $E_i \subseteq U_{2l+1} \subseteq U_{2k}$ and the inclusions

$$\partial^{\mathrm{M}} U_{2k} \cap \partial^{\mathrm{M}} E_i \neq \emptyset, \quad \partial^{\mathrm{M}} U_{2k} \subseteq \bigcup_{j \in \Phi_{2k}} \partial^{\mathrm{M}} F_j \qquad (\mathrm{mod}\ \mathcal{H}^{N-1})$$

imply the existence of $j \in \Phi_{2k}$ and $x \in (E_i)^{1/2} \cap \partial^{\mathrm{M}} U_{2k} \cap (F_j)^{1/2}$. Since U_{2k} contains both E_i and F_j we obtain that $x \in \mathring{U}^{\mathrm{M}}_{2k}$ and this is a contradiction.

Now, since the sets Φ_{2k+1} are pairwise disjoint, the first inclusion in (23) implies that $\mathcal{H}^{N-1}(\partial^{\mathrm{M}} U_k \cap \partial^{\mathrm{M}} U_l) = 0$ whenever $k \neq l$. Moreover, (22) and (25) imply that $\cup_k \partial^{\mathrm{M}} U_k = \partial^{\mathrm{M}} E$ (mod $\mathcal{H}^{N-1}$). Since $u = \sum_k \chi_{\mathbb{R}^{\mathrm{N}} \setminus U_k}$ we obtain

$$|Du| = |\sum_k D\chi_{\mathbb{R}^{\mathrm{N}} \setminus U_k}| = |\sum_k D\chi_{U_k}| = \mathcal{H}^{N-1} \llcorner \cup_k \partial^{\mathrm{M}} U_k = \mathcal{H}^{N-1} \llcorner \partial^{\mathrm{M}} E.$$

(Uniqueness) Let v be satisfying (i), (ii), (iii) and let us prove that v coincides with the function u constructed above. First of all, notice that condition (ii) implies that v is (equivalent to) a constant in any M-connected component of E or $\mathbb{R}^N \setminus E$, by the constancy theorem (see Remark 2). Moreover, $\partial^M E$ coincides (mod $\mathcal{H}^{N-1}$) with the jump set of v and $|v^+ - v^-|$ (i.e., the width of the jump) is 1 $\mathcal{H}^{N-1}$-a.e. in $\mathbb{R}^N$ (see Sect. 3).

By condition (iii) the two functions are both 0 on F_{j_0}. Let $i \in \Phi_1$; since

$$\partial^M E \supseteq \partial^M E_i \cap \partial^M F_{j_0} \neq \emptyset \quad (\mathrm{mod}\ \mathcal{H}^{N-1})$$

we obtain that v must be equal to 1 on E_i. Being i arbitrary, this proves that v coincides with u on U_1. Consider now $j \in \Phi_2$; the same argument exploited before proves that either v is a.e. equal to 2 or v is a.e. equal to 0 in F_j. The second possibility can be excluded noticing that in this case the set $\{v \leq 0\}$ would be decomposable: indeed, by (23) we get

$$\partial^M U_0 \subseteq \bigcup_{i \in \Phi_1} \partial^M E_i \cap \partial^M U_0 \subseteq \partial^M \{v \leq 0\} \quad (\mathrm{mod}\ \mathcal{H}^{N-1})$$

and, passing to the complementary sets $\partial^M(\{v \leq 0\} \setminus U_0) \subset \partial^M\{v \leq 0\}$ (mod $\mathcal{H}^{N-1}$), so that Proposition 3(iv) gives

$$P(\{v \leq 0\}) = P(U_0) + P(\{v \leq 0\} \setminus U_0).$$

Continuing by induction in this way and using the inclusions (mod $\mathcal{H}^{N-1}$) (the first for k even, the second for k odd, coming from (23) and the inductive assumption)

$$\partial^M U_{k-2} \subseteq \bigcup_{i \in \Phi_{k-1}} \partial^M E_i \cap \partial^M U_{k-2} \subseteq \partial^M \{v \leq k-2\},$$

$$\partial^M U_{k-2} \subseteq \bigcup_{j \in \Phi_{k-1}} \partial^M F_j \cap \partial^M U_{k-2} \subseteq \partial^M \{v \leq k-2\}$$

we obtain that v coincides with u on U_k. Since k is arbitrary, this proves that $v = u$. □

Definition 6 (Topographic function). *We call the function given by the previous theorem the* topographic function *of E, and denote it by u_E. We also call the sets*

$$\partial^M\{u_E \leq 2k\}, \qquad \partial^M\{u_E \leq 2k+1\} \qquad\qquad k \in \mathbb{N}$$

respectively the external and the internal boundaries of E.

Notice that

$$u_E + 1 = u_{\mathbb{R}^N \setminus E} \qquad \text{whenever } |E| < \infty$$

because it is easy to check that $u_E + 1$ fulfils (i), (ii), (iii) with $\mathbb{R}^N \setminus E$ in place of E. As a consequence, complementation maps internal (external) boundaries into external (internal) boundaries. Passing to the complementary sets, the identity above can also be written as $u_E = u_{\mathbb{R}^N \setminus E} + 1$ whenever $|E| = \infty$. In particular, in this case the topographic function achieves its minimum, equal to 1, on the component of E with infinite measure (if $|E| < \infty$ the minimum is 0, by condition (iii)).

8. Indecomposability and Jordan curves in the plane

The aim of this section is a closer characterization of the M-connected components and of the essential boundary for plane sets of finite perimeter. In particular we prove that $\partial^M E$ can be represented (mod $\mathcal{H}^1$) as a disjoint union of rectifiable Jordan curves; this result has been proved first for simple sets by W.H. Fleming in [25] (see also [26]) and later extended to the general case by H. Federer (see [24], 4.2.25). We also prove that membership to the same M-connected component can be characterized in terms of existence of arcs joining the points and not touching (in a suitable sense) the boundary.

We say that $\Gamma \subseteq \mathbb{R}^2$ is a *Jordan curve* if $\Gamma = \gamma([a, b])$ for some $a, b \in \mathbb{R}$ (with $a < b$) and some continuous map γ, one-to-one on $[a, b)$ and such that $\gamma(a) = \gamma(b)$. In a more geometric language, Γ can be viewed as the image of a continuous and one-to-one map defined on the unit circle $\mathbf{S}^1$. According to the celebrated Jordan curve theorem (see for instance [35]), any Jordan curve Γ splits $\mathbb{R}^2 \setminus \Gamma$ in exactly two connected components, a bounded one and an unbounded one, whose common boundary is Γ. As for Jordan boundaries, these components will be respectively denoted by $\mathrm{int}(\Gamma)$ and $\mathrm{ext}(\Gamma)$. We will also use the *signed distance function* $\mathrm{sdist}(x, \Gamma)$, defined by

$$\mathrm{sdist}(x,\Gamma) := \begin{cases} -\mathrm{dist}(x,\Gamma) \text{ if } x \in \mathrm{int}(\Gamma) \cup \Gamma; \\ \\ \ \ \mathrm{dist}(x,\Gamma) \ \text{ if } x \in \mathrm{ext}(\Gamma) \cup \Gamma. \end{cases} \tag{26}$$

In our context, we are more interested in Lipschitz parameterizations rather than continuous ones; the main tool for providing them is the following well known lemma.

Lemma 3 (Connectedness by arcs). *Let $C \subset \mathbb{R}^N$ be a compact connected set with $\mathcal{H}^1(C) < \infty$. Then for any pair of distinct points $x, y \in C$ there exists a Lipschitz one-to-one map $\gamma : [0, 1] \to C$ such that $\gamma(0) = x$ and $\gamma(1) = y$.*

Proof. The existence of a Lipschitz map (not necessarily one-to-one) joining x to y is proved in [23]. In order to obtain a one-to-one map it suffices to look for solutions of the problem

$$\min\left\{\int_a^b |\gamma'(t)|\,dt : [a,b] \subseteq \mathbb{R},\ \ \gamma \in \mathrm{Lip}([a,b],C),\ \ \gamma(a) = x,\ \ \gamma(b) = y\right\}.$$

Existence of minimizers is a straightforward consequence of Ascoli–Arzelá theorem and of a classical reparameterization argument. Clearly any minimizer γ_0, when parameterized by arc length, is one-to-one. A final reparameterization gives $\gamma : [0, 1] \to C$. □

A first consequence of Lemma 3 is the fact that any Jordan curve Γ with $\mathcal{H}^1(\Gamma) < \infty$ admits a *Lipschitz* reparameterization. In fact, let $x, y \in \Gamma$ with $x \neq y$, let $\gamma : [0, 1] \to \Gamma$ be given by Lemma 3 and let $\tilde{\Gamma} = \Gamma \setminus \gamma\,((0, 1))$. Since $\tilde{\Gamma}$ is homeomorphic to a closed segment, Lemma 3 again gives a Lipschitz homeomorphism $\tilde{\gamma} : [1,2] \to \tilde{\Gamma}$ with $\tilde{\gamma}(1) = y$ and $\tilde{\gamma}(2) = x$. Joining γ and $\tilde{\gamma}$ we obtain the desired Lipschitz parameterization of Γ. In the following we call *rectifiable* the Jordan curves such that $\mathcal{H}^1(\Gamma) < \infty$. More generally, any $\Gamma = \gamma([a,b])$ with γ Lipschitz function in $[a,b]$ will be called rectifiable curve.

In the following lemma we point out some mild regularity properties of rectifiable Jordan curves which will be used in the following.

Lemma 4. *Let $\Gamma \subset \mathbb{R}^2$ be a rectifiable Jordan curve. Then*

$$\mathcal{H}^1\left(\Gamma \cap B(x, r/2)\right) \geq r \qquad \forall x \in \Gamma,\ r \in (0, \mathrm{diam}(\Gamma)), \tag{27}$$

$$\mathcal{H}^1(\Gamma) = P\left(\mathrm{int}(\Gamma)\right) = P\left(\mathrm{ext}(\Gamma)\right) \tag{28}$$

and

$$\liminf_{r \to 0^{\pm}} \mathcal{H}^1\big(\{x \in \mathbb{R}^2 :\ \mathrm{sdist}(x, \Gamma) = r\}\big) = \mathcal{H}^1(\Gamma). \tag{29}$$

Proof. The first property can be easily proved by a projection argument, see for instance Lemma 3.4 of [23], taking into account that Γ intersects at least twice $\partial B(x, r/2)$.

In order to prove the second one, let us represent Γ as $\gamma([0,1])$ with $\gamma : [0,1] \to \mathbb{R}^2$ satisfying $|\gamma'(t)| = 1$ for a.e. $t \in [0,1]$ and let $x_0 \in \Gamma$ such that

$$\limsup_{\rho \to 0^+} \frac{\mathcal{H}^1(\Gamma \cap B_\rho(x_0))}{2\rho} \leq 1$$

and, for $t_0 = \gamma^{-1}(x_0)$, γ is differentiable at t_0 and $|\gamma'(t_0)| = 1$; notice that $\mathcal{H}^1$-a.e. $x_0 \in \Gamma$ has these properties. The coarea formula (see 3.2.3 of [24]) gives

$$\int_0^\rho \mathrm{card}\left(\Gamma \cap \partial B_r(x_0)\right)\, dr \leq \mathcal{H}^1(\Gamma \cap B_\rho(x_0)) \qquad \forall \rho > 0$$

and hence we can find arbitrarily small $r > 0$ such that $\Gamma \cap \partial B_r(x_0)$ contains two points $x_r,\ y_r$; by the differentiability of γ at t_0 we have also that $|x_r - y_r|/2r$ tends to 1 as $r \to 0^+$. Denoting by $J_r^{\pm} \subset \partial B_r(x_0)$ the circular arcs joining x_r and y_r, we obtain that $J_r^{\pm} \cup (\Gamma \cap B_r(x_0))$ are Jordan curves, whose interiors are the connected components of $B_r(x_0) \setminus \Gamma$. It follows that one of these components is contained in $\mathrm{int}(\Gamma)$ and the other one in $\mathrm{ext}(\Gamma)$, and since the angle between x_r and y_r tends to π as $r \to 0^+$ we obtain that x_0 is a point of density $1/2$ for $\mathrm{int}(\Gamma)$ and $\mathrm{ext}(\Gamma)$. This proves that

$$\mathcal{H}^1(\Gamma) \leq \mathcal{H}^1\left(\partial^{\mathrm{M}}\, \mathrm{int}(\Gamma)\right) = P\left(\mathrm{int}(\Gamma)\right), \quad \mathcal{H}^1(\Gamma) \leq \mathcal{H}^1\left(\partial^{\mathrm{M}}\, \mathrm{ext}(\Gamma)\right) = P\left(\mathrm{ext}(\Gamma)\right).$$

The opposite inequalities follow by the inclusions $\partial\, \mathrm{int}(\Gamma) \subset \Gamma$, $\partial\, \mathrm{ext}(\Gamma) \subset \Gamma$.

In order to prove the third property we set $\phi(x) = \mathrm{sdist}(x, \Gamma)$ and recall (see for instance [24], 3.2.11, 3.2.34) that $|\nabla\phi| = 1$ a.e. in $\mathbb{R}^2$, so that the coarea formula gives

$$|\phi^{-1}(-r,r)| = \int_{\phi^{-1}(-r,r)} |\nabla\phi|\, dx = \int_{-r}^{r} \mathcal{H}^1\big(\{x \in \mathbb{R}^2 :\ \phi(x) = t\}\big)\, dt \quad \forall r > 0.$$

On the other hand, it can be proved (see 3.2.39 of [24] or Theorem 2.106 of [5]) that $|\phi^{-1}(-r,r)|/(2r)$ tends to $\mathcal{H}^1(\Gamma)$ as $r \to 0^+$. Hence we can find an infinitesimal sequence of positive numbers r_i such that

$$\limsup_{i\to\infty} \mathcal{H}^1\big(\{x \in \mathbb{R}^2 : |\phi(x)| = r_i\}\big) \le 2\mathcal{H}^1(\Gamma).$$

On the other hand, the lower semicontinuity of perimeter and (28) give

$$\liminf_{i\to\infty} \mathcal{H}^1\big(\{x \in \mathbb{R}^2 :\ \phi(x) = r_i\}\big) \ge \liminf_{i\to\infty} P(\{\phi < r_i\}) \ge P(\phi < 0\}) = \mathcal{H}^1(\Gamma)$$

and, analogously, $\liminf_i \mathcal{H}^1\left(\{x \in \mathbb{R}^2 :\ \phi(x) = -r_i\}\right) \ge \mathcal{H}^1(\Gamma)$. These inequalities imply that both $\mathcal{H}^1(\{\phi = r_i\})$ and $\mathcal{H}^1(\{\phi = -r_i\})$ converge to $\mathcal{H}^1(\Gamma)$ as $i \to \infty$. □

In order to represent the essential boundary of a simple set by a rectifiable Jordan curve we need the following lemma.

Lemma 5. *Let $\gamma : [0, L] \to \mathbb{R}^2$ be a Lipschitz map, let $C = \gamma([0, L])$ and assume that $\gamma(0) = \gamma(L)$ and $\int_0^L |\gamma'|\, dt = \mathcal{H}^1(C) > 0$. Then C contains a rectifiable Jordan curve Γ.*

Proof. After reparameterization we can assume with no loss of generality that $L = \mathcal{H}^1(C)$ and $|\gamma'| = 1$ a.e. in $[0, L]$. By the area formula (see for instance [22])

$$|\gamma^{-1}(A)| = \int_{\gamma^{-1}(A)} |\gamma'(t)|\, dt = \int_A \mathrm{card}(\gamma^{-1}(x))\, d\mathcal{H}^1(x) \qquad \forall A \subseteq C,\ \ A \text{ Borel}$$

with $A = C$ we obtain

$$\int_C \big(\mathrm{card}(\gamma^{-1}(x)) - 1\big)\, d\mathcal{H}^1(x) = \int_0^1 |\gamma'(t)|\, dt - \mathcal{H}^1(C) = 0$$

hence the set $B = \big\{x : \mathrm{card}(\gamma^{-1}(x)) > 1\big\}$ is $\mathcal{H}^1$-negligible, and so is (again by the area formula with $A = B$) the set $S = \gamma^{-1}(B)$.

We now claim that $\overline{S}$ is still Lebesgue negligible. In fact, let $(t_h) \subseteq S$ be converging to t and let $s_h \ne t_h$ such that $\gamma(t_h) = \gamma(s_h)$; assuming with no loss of generality that s_h converge to s, if $s \ne t$ we conclude that $t \in S$, otherwise if $s = t$ we obtain that either γ is not differentiable at t or $\gamma'(t) = 0$. This proves that $\overline{S}$ is Lebesgue negligible.

Take now a connected component (a, b) of $(0, 1) \setminus \overline{S}$ and consider the simple arc $C' = \gamma\,((a, b))$. Since $C \setminus C'$ is connected (being γ a closed curve), by Lemma 3 we can connect $\gamma(b)$ to $\gamma(a)$ by a simple path $\eta : [b, c] \to C \setminus C'$. If $\gamma(a) = \gamma(b)$, then C' is a Jordan curve. If $\gamma(a) \ne \gamma(b)$, then a Jordan curve contained in C can be obtained joining the paths $\gamma|_{[a,b]}$ and $\eta|_{[b,c]}$. □

Theorem 7 (Boundary of simple plane sets). *Let $E \subset \mathbb{R}^2$ be a simple set with $|E| \in (0, \infty)$. Then E is (essentially) bounded and $\partial^{\mathrm{M}} E$ is equivalent* (mod $\mathcal{H}^1$) *to a rectifiable Jordan curve. Conversely,* $\mathrm{int}(\Gamma)$ *is a simple set for any rectifiable Jordan curve Γ.*

Proof. By a rescaling argument we also assume that $P(E) < 1$. Let (E_h) be a sequence of bounded open sets with smooth boundary locally converging in measure to E and such that $P(E_h) \to P(E)$ as $h \to \infty$. Since ∂E_h is smooth and compact, we can represent it by a disjoint union of Jordan curves $\Gamma_{i,h}$, for $1 \le i \le N(h)$, whose length decreases as i increases; we parameterize $\Gamma_{i,h} = \gamma_{i,h}([0,1])$ for some 1-Lipschitz maps $\gamma_{i,h}$, one-to-one on $[0,1)$, and notice that

$$\sum_{i=1}^{N(h)} \int_0^1 |\gamma'_{i,h}(t)|\,dt = \sum_{i=1}^{N(h)} \mathcal{H}^1(\Gamma_{i,h}) = P(E_h) < 1 \tag{30}$$

for h large enough. In the following we assume, to fix the ideas, that $N(h) \to \infty$ as $h \to \infty$, the proof being much simpler if $N(h) \le C$ for infinitely many h. We assume, possibly extracting a subsequence, that for any $i \in \mathbb{N}$ either $\gamma_{i,h}$ uniformly converge in $[0,1]$ to γ_i or $\max|\gamma_{i,h}| \to \infty$. In the latter case we set $\gamma_i \equiv 0$. Setting $\Gamma_i = \gamma_i([0,1])$ and $\Gamma_\infty = \cup_i \Gamma_i$, we will prove that there exists i such that Γ_i is a Jordan curve and Γ_j are points for any $j \ne i$.

Step 1. We claim that $\partial^{\mathrm{M}} E \subset \Gamma_\infty \pmod{\mathcal{H}^1}$. Given an integer $p \ge 1$, we denote by E_h^p the sets obtained from E_h by removing from it the connected components with area smaller than $1/p$ and adding to it all holes with area smaller than $1/p$. By the isoperimetric inequality, the perimeter of any connected component of E_h^p is at least $\sqrt{4\pi/p}$, hence ∂E_h^p is contained in the first $M_p = [\sqrt{p/(4\pi)}] + 1$ curves $\Gamma_{i,h}$. Moreover, we have

$$\left|E_h^p \Delta E_h\right| \le \sum_{j\in J} |Y_j| \le \frac{1}{\sqrt{p}} \sum_{j\in J} |Y_j|^{1/2} \le \frac{1}{\sqrt{4\pi p}} \sum_{j\in J} P(Y_j) \le \frac{1}{\sqrt{4\pi p}}$$

where $\{Y_j\}_{j\in J}$ are the components added or removed. We assume, without loss of generality, that E_h^p locally converge in measure in $\mathbb{R}^{\mathrm{N}}$ to suitable sets E^p as $h \to \infty$ such that $|E^p \Delta E| \le 1/\sqrt{4\pi p}$. Since

$$\partial E_h^p \subset \bigcup_{i=1}^{M_p} \Gamma_{i,h}$$

and since $D\chi_{E_h^p}$ weakly converge as measures to $D\chi_{E^p}$, by the definition of Γ_i we easily obtain that

$$|D\chi_{E^p}| \le \mathcal{H}^1 \llcorner \bigcup_{i=1}^{M_p} \Gamma_i$$

because any closed ball disjoint from the set in the right side does not intersect $\Gamma_{i,h}$, $1 \le i \le M_p$, for h large enough. Hence, $|D\chi_{E^p}| \le \mathcal{H}^1 \llcorner \Gamma_\infty$ for any p. Letting $p \to \infty$ and using the weak convergence of E_p to E we get $|D\chi_E| \le \mathcal{H}^1 \llcorner \Gamma_\infty$. The claim follows by evaluating both measures at $\partial^{\mathrm{M}} E \setminus \Gamma_\infty$.

Step 2. Passing to the limit as $h \to \infty$ in (30) we get

$$\sum_{i=1}^{\infty} \mathcal{H}^1(\Gamma_i) \le \sum_{i=1}^{\infty} \int_0^1 \left|\gamma_i'\right| dt \le P(E) = \mathcal{H}^1(\partial^{\mathrm{M}} E).$$

On the other hand, Step 1 gives

$$\mathcal{H}^1(\partial^{\mathrm{M}} E) \le \mathcal{H}^1\left(\cup_i \Gamma_i\right) \le \sum_{i=1}^{\infty} \mathcal{H}^1(\Gamma_i)$$

Hence, we conclude that $\int_0^1 |\gamma_i'(t)|\, dt = \mathcal{H}^1(\Gamma_i)$ for any $i \ge 1$ and $\mathcal{H}^1(\Gamma_i \cap \Gamma_j) = 0$ whenever $i \ne j$.

Step 3. Let $i \ge 1$ such that $\mathcal{H}^1(\Gamma_i) > 0$ and let $\Gamma \subset \Gamma_i$ be a Jordan curve given by Lemma 5. Then, $F = \operatorname{int} \Gamma$ satisfies

$$\partial^{\mathrm{M}} F \subset \Gamma \subset \Gamma_i \subset \partial^{\mathrm{M}} E \quad (\operatorname{mod} \mathcal{H}^1)$$

so that, being E simple, we conclude from Proposition 7 that $E = F$ and $\partial^{\mathrm{M}} E = \Gamma = \Gamma_i \pmod{\mathcal{H}^1}$. This also proves that $\mathcal{H}^1(\Gamma_j) = 0$ for any $j \ne i$. Since $\operatorname{diam} \Gamma \le \mathcal{H}^1(\Gamma)$ for any rectifiable Jordan curve Γ we obtain that E is bounded.

Finally, the fact that any rectifiable Jordan curve induces a simple set follows by Proposition 2 and by the Jordan curve theorem. □

By Theorem 4, since Jordan boundaries essentially coincide with rectifiable Jordan curves, we obtain the following decomposition result for the boundary of a set of finite perimeter in the plane. As in Theorem 4 we allow the Jordan curves to be also J_∞ and J_o to simplify the statement and to allow sets E with infinite measure.

Corollary 1. *Let E be a subset of $\mathbb{R}^2$ of finite perimeter. Then, there is a unique decomposition of $\partial^{\mathrm{M}} E$ into rectifiable Jordan curves $\left\{C_i^+, C_k^- : i, k \in \mathbb{N}\right\} \subset \mathcal{S}$, such that*

(i) *Given* $\operatorname{int}(C_i^+)$, $\operatorname{int}(C_k^+)$, *$i \ne k$, they are either disjoint or one is contained in the other; given* $\operatorname{int}(C_i^-)$, $\operatorname{int}(C_k^-)$, *$i \ne k$, they are either disjoint or one is contained in the other. Each* $\operatorname{int}(C_i^-)$ *is contained in one of the* $\operatorname{int}(C_k^+)$.
(ii) $P(E) = \sum_i \mathcal{H}^1(C_i^+) + \sum_k \mathcal{H}^1(C_k^-)$.
(iii) *If* $\operatorname{int}(C_i^+) \subseteq \operatorname{int}(C_j^+)$, *$i \ne j$, then there is some rectifiable Jordan curve C_k^- such that* $\operatorname{int}(C_i^+) \subseteq \operatorname{int}(C_k^-) \subseteq \operatorname{int}(C_j^+)$. *Similarly, if* $\operatorname{int}(C_i^-) \subseteq \operatorname{int}(C_j^-)$, *$i \ne j$, then there is some rectifiable Jordan curve C_k^+ such that* $\operatorname{int}(C_i^-) \subseteq \operatorname{int}(C_k^+) \subseteq \operatorname{int}(C_j^-)$.
(iv) *Setting* $L_j = \{i : \operatorname{int}(C_i^-) \subseteq \operatorname{int}(C_j^+)\}$, *the sets* $Y_j = \operatorname{int}(C_j^+) \setminus \cup_{i \in L_j} \operatorname{int}(C_i^-)$ *are pairwise disjoint, indecomposable and $E = \cup_j Y_j$.*

In the remaining part of this section we want to characterize the M-connected components (or, better, suitable representatives in the equivalence class (mod $\mathcal{H}^2$)), by the classical topological property of connectedness by arcs.

To this aim, we need another definition of boundary which, more than ∂^{M}, is suitable for the analysis of connected components. For any set E with finite perimeter in $\mathbb{R}^{\mathrm{N}}$ we define

$$\partial^S E := \left\{ x \in \mathbb{R}^{\mathrm{N}} : \limsup_{r \to 0^+} \frac{\mathcal{H}^{N-1}(\partial^{\mathrm{M}} E \cap B(x, r))}{r^{N-1}} > 0 \right\}.$$

Notice that the relative isoperimetric inequality, together with a continuity argument, easily gives (see (9)) that $\partial^{\mathrm{M}} E \subset \partial^{\mathrm{S}} E$; however (4) guarantees that $\mathcal{H}^{N-1}(\partial^{\mathrm{S}} E \setminus \partial^{\mathrm{M}} E) = 0$, hence $P(E) = \mathcal{H}^{N-1}(\partial^{\mathrm{S}} E)$ still holds.

With this notation we can prove the following result:

Theorem 8 (Indecomposability and connectedness by arcs). *Let $E \subset \mathbb{R}^2$ be a set of finite perimeter and let $\{E_i\}_{i \in I} = \mathcal{CC}^{\mathrm{M}}(E)$. Then $\mathring{E}^{\mathrm{M}} \setminus \partial^{\mathrm{S}} E$ is the disjoint union of $\mathring{E}_i^{\mathrm{M}} \setminus \partial^{\mathrm{S}} E$ and $x,\ y \in \mathring{E}^{\mathrm{M}} \setminus \partial^{\mathrm{S}} E$ belong to the same M-connected component E_i of E if and only if there exists a rectifiable curve Γ joining x to y contained in $\mathring{E}^{\mathrm{M}} \setminus \partial^{\mathrm{S}} E$. Moreover, for any $\delta > 0$, Γ can be chosen so that*

$$\mathcal{H}^1(\Gamma) \le |x - y| + P(E_i) + \delta.$$

In particular the sets $\mathring{E}_i^{\mathrm{M}} \setminus \partial^{\mathrm{S}} E$ are connected.

Our proof of this result actually gives a slightly stronger statement: the sets $\mathring{E}_i^{\mathrm{M}} \setminus (\partial^{\mathrm{S}} E_i \cup L)$ are connected by arcs for any $\mathcal{H}^1$-negligible set $L \subseteq \mathbb{R}^2$; Theorem 8 is a particular case with $L = \mathring{E}_i^{\mathrm{M}} \cap \partial^{\mathrm{S}} E$. In order to show this result, our first lemma proves that points in the same M-connected component can be joined by curves lying in $\mathring{E}^{\mathrm{M}} \cup \partial^{\mathrm{S}} E$.

Lemma 6. *Let $E \subseteq \mathbb{R}^2$ be an indecomposable set and let $x,\ y \in \mathring{E}^{\mathrm{M}} \setminus \partial^{\mathrm{S}} E$. Then there exists a rectifiable curve Γ joining x to y contained in $\mathring{E}^{\mathrm{M}} \cup \partial^{\mathrm{S}} E$. Moreover, the curve can be chosen so that $\Gamma \subset \partial^{\mathrm{S}} E \cup L$, where L is the segment joining x to y.*

Proof. Let J_0 be the rectifiable Jordan curve corresponding to the simple set $\mathrm{sat}(E)$ and let J_i, $1 \le i < p$ with $p \in [2, \infty]$, be the rectifiable Jordan curves corresponding to the holes of E. Since $x,\ y \notin \partial^{\mathrm{S}} E$ and $\cup_i J_i \subset \partial^{\mathrm{M}} E$ (mod $\mathcal{H}^1$), by (27) we obtain that x and y belong to $\mathrm{int}(J_0)$, the topological interior of J_0, and to $\mathrm{ext}(J_i)$, the topological exterior of J_i, for $i \ge 1$. If L crosses an hole $\mathrm{int}(J_i)$ we can replace, using Lemma 3, $L \cap \overline{\mathrm{int}(J_i)}$ by a curve contained in J_i, and similarly we can argue if L crosses $\mathrm{ext}(J_0)$. In this way we obtain a rectifiable curve Γ fully contained in $\mathring{E}^{\mathrm{M}} \cup \cup_{i \ge 1} J_i \subset \mathring{E}^{\mathrm{M}} \cup \partial^{\mathrm{S}} E$. □

In order to improve Lemma 6, proving existence of curves contained in $\mathring{E}^{\mathrm{M}} \setminus \partial^{\mathrm{S}} E$, the natural idea is to enlarge a little bit the holes of E and to shrink a little bit the boundary of $\mathrm{sat}(E)$, to produce a new set whose boundary is "inside" E.

However, this perturbation could not preserve the property that x and y are in the same M-connected component, unless we assume that small balls centered at x and y are contained in E.

Lemma 7. *Let $E \subseteq \mathbb{R}^2$ be an indecomposable set, let $x,\ y \in \mathbb{R}^2$ and assume that $B(x,r) \cup B(y,r) \subseteq E$ (mod $\mathcal{H}^2$) for some $r > 0$. Then, for any $\mathcal{H}^1$-negligible set $N \subseteq \mathbb{R}^2 \setminus (B(x,r) \cup B(y,r))$ there exists an open set $A \subseteq E$ with finite perimeter such that $N \cup \partial^{\mathrm{S}} E \subseteq A$ and $x,\ y$ belong to the same M-connected component of $E \setminus A$. Moreover, given any $\delta > 0$ and any open set S such that $\mathcal{H}^1(\partial^{\mathrm{M}} E \cap \partial S) = 0$, we can choose A so that*

$$P(E \setminus A, S) \le P(E, S) + \delta.$$

Proof. Assuming with no loss of generality that $r < |x-y|$, we will first build a sequence of open sets A_h not intersecting $B(x, r/2) \cup B(y, r/2)$, such that $|E \cap A_h| \to 0$, $P(E \setminus A_h) \to P(E)$ and $N \cup \partial^{\mathrm{S}} E \subseteq A_h$.

Let $J_0,\ J_i$ be as in Lemma 6 and let us denote by L the $\mathcal{H}^1$-negligible set $N \cup \partial^{\mathrm{S}} E \setminus \cup_i J_i$. Given $\epsilon > 0$, by (29) we can find $r_0 < 0$ and positive numbers r_i such that

$$\begin{aligned}
&\left|\mathcal{H}^1\big(\{x \in \mathbb{R}^2 :\ \mathrm{sdist}(x,\ J_0) = r_0\}\big) - \mathcal{H}^1(J_0)\right| \le \epsilon\\
&\left|\mathcal{H}^1\big(\{x \in \mathbb{R}^2 :\ \mathrm{sdist}(x,\ J_i) = r_i\}\big) - \mathcal{H}^1(J_i)\right| \le 2^{-i}\epsilon \quad \forall i \in [1,\ p).
\end{aligned}$$

We also choose balls $B(x_j, \eta_j)$ such that their union contains L and $\sum_j \eta_j < \epsilon$. Choosing $\epsilon = 1/h$, we define

$$\begin{aligned}
A_h := \{x \in \mathbb{R}^2 :\ \mathrm{sdist}(x,\ J_0) > r_0\} &\cup \bigcup_{1 \le i < p} \{x \in \mathbb{R}^2 :\ \mathrm{sdist}(x,\ J_i) < r_i\}\\
&\cup \bigcup_{j=1}^{\infty} B(x_j, \eta_j).
\end{aligned}$$

By construction A_h contains $\partial^{\mathrm{S}} E$ and does not intersect $B(x, r/2) \cup B(y, r/2)$ for h large enough. Moreover, since $\sum_j \pi\eta_j^2 \le \pi\epsilon^2$ and

$$E \cap A_h \subseteq \bigcup_{0 \le i < p} \{x \in \mathbb{R}^2 :\ |\mathrm{sdist}(x,\ J_i)| \le |r_i|\} \cup \bigcup_{j=0}^{\infty} B(x_j, \eta_j)$$

choosing smaller r_i if necessary (again, this is possible due to (29)) we obtain that $|E \cap A_h| \to 0$. In order to prove that $P(E \setminus A_h)$ converge to $P(E)$ it suffices, by the lower semicontinuity of perimeter, to estimate $P(E \setminus A_h)$ from above. Since $\partial^{\mathrm{M}}(E \setminus A_h) \subset \partial^{\mathrm{M}} E \cup \partial^{\mathrm{M}} A_h$ and $\partial^{\mathrm{M}} E \subset \partial^{\mathrm{S}} E \subseteq A_h$ we obtain

$$\begin{aligned}
P(E \setminus A_h) &\le \mathcal{H}^1\big(\partial^{\mathrm{M}} A_h\big) = P(A_h)\\
&\le \sum_{0 \le i < p} \mathcal{H}^1\big(\{x \in \mathbb{R}^2 :\ \mathrm{sdist}(x,\ J_i) = r_i\}\big) + \sum_{j=0}^{\infty} 2\pi\eta_j\\
&\le \sum_{0 \le i < p} \mathcal{H}^1(J_i) + \frac{2\pi+2}{h} = P(E) + \frac{2\pi+2}{h}.
\end{aligned}$$

Now we claim that for h large enough both x and y belong to the same M-connected component of $E \setminus A_h$; indeed, if this is not true we can find partitions (A_h^1, A_h^2) of $E \setminus A_h$ (union of suitable M-connected components of $E \setminus A_h$, see (10)) such that $B(x, r/2) \subseteq A_h^1$, $B(y, r/2) \subseteq A_h^2$, $P(E \setminus A_h) \geq P(A_h^1) + P(A_h^2)$ and $|A_h^1 \cap A_h^2| = 0$. Possibly passing to a subsequence, we can assume that A_h^i locally converge in measure to disjoint sets A^i whose union is E; the lower semicontinuity of perimeter gives

$$P(E) \geq P(A^1) + P(A^2)$$

and, since both A^1 and A^2 contain a ball and E is indecomposable, this gives a contradiction.

The final claim follows noticing that the convergence of perimeters implies that $P(E \setminus A_h, S)$ converge to $P(E \setminus A, S)$ as $h \to \infty$ (see for instance [29], Appendix A). □

Finally, we need the following lemma, showing that many circles centered at points in $\mathring{E}^{\mathrm{M}} \setminus \partial^{\mathrm{S}} E$ are fully contained in $\mathring{E}^{\mathrm{M}}$.

Lemma 8. *Let E be a set of finite perimeter, let $x \in \mathring{E}^{\mathrm{M}} \setminus \partial^{\mathrm{S}} E$ and define*

$$R := \left\{ t > 0 : \ \partial B(x, t) \subset \mathring{E}^{\mathrm{M}} \right\}.$$

Then $|R \cap (0, r)|/r$ tends to 1 as $r \to 0^+$.

Proof. Let us define ϕ equal to 1 on $\mathring{E}^{\mathrm{M}}$, equal to $1/2$ on $E^{1/2}$ and equal to 0 on $\overbrace{\mathbb{R}^2 \setminus \mathring{E}}^{\mathrm{M}}$. Notice that ϕ is undefined only on the $\mathcal{H}^1$-negligible set $\partial^{\mathrm{M}} E \setminus E^{1/2}$, and hence is everywhere defined on almost every circle $\partial B(x, t)$.

Since $x \in \mathring{E}^{\mathrm{M}}$, a simple application of Fubini theorem shows that the set

$$R_1 := \left\{ t > 0 : \ \mathcal{H}^1\left(\partial B(x, t) \cap \mathring{E}^{\mathrm{M}}\right) > 0 \right\}$$

satisfies $|R_1 \cap (0, r)|/r \to 1$ as $r \to 0^+$.

Let $\phi_t(\theta) = \phi(x_1 + t \cos\theta, x_2 + t \sin\theta)$ and let $\mathrm{Var}(\phi_t)$ be its pointwise variation. The statement would be proved if we show that also the set

$$R_2 := \{ t > 0 : \ \mathrm{Var}(\phi_t) = 0 \}$$

satisfies $|R_2 \cap (0, r)|/r \to 1$ as $r \to 0^+$, because any $t \in R_1 \cap R_2$ belongs to R. To this aim, notice that $\mathrm{Var}(\phi_t) \geq 1/2$ for any $t \in (0, \infty) \setminus R_2$, hence the density property of R_2 follows by the inequality

$$\frac{1}{2} |(0, r) \setminus R_2| \leq \int_0^r \mathrm{Var}(\phi_t)\, dt$$

if we prove that $\int_0^r \mathrm{Var}(\phi_t)\, dt/r$ is infinitesimal as $r \to 0^+$. Eventually, this fact follows by the assumption that $x \notin \partial^{\mathrm{S}} E$ and the inequality

$$\int_0^r \mathrm{Var}(\phi_t)\, dt \leq \mathcal{H}^1\left(\partial^{\mathrm{M}} E \cap \overline{B}(x, r)\right) \qquad \forall r > 0. \tag{31}$$

In order to prove (31) we first notice that a polar change of coordinates gives

$$\int_0^r \mathrm{Var}(\varphi_t)\,dt = \int_0^r \int_0^{2\pi} \left|\frac{\partial\varphi_t}{\partial\theta}\right| d\theta dt \le \int_{B(x,r)} |\nabla\varphi|\,dy$$

for any $\varphi \in C^\infty(\mathbb{R}^{\mathrm{N}})$. Now we choose a radial convolution kernel ρ and apply the identity above to the mollified functions $\varphi_\epsilon = \phi * \rho_\epsilon$; taking into account that φ_ϵ pointwise converge to ϕ in its domain (see for instance Theorem 4.5.9(24) in [24]), the lower semicontinuity of the variation under pointwise convergence and the inequality (see for instance Proposition 1.15 in [29])

$$\limsup_{\epsilon\to 0^+} \int_{B(x,r)} |\nabla\varphi_\epsilon|\,dx \le P\left(E, \overline{B}(x,r)\right) = \mathcal{H}^1\left(\partial^{\mathrm{M}} E \cap \overline{B}(x,r)\right)$$

we obtain

$$\begin{aligned}\int_0^r \mathrm{Var}(\phi_t)\,dt &\le \int_0^r \liminf_{\epsilon\to 0^+} \mathrm{Var}(\varphi_{\epsilon t})\,dt \le \liminf_{\epsilon\to 0^+} \int_0^r \mathrm{Var}(\varphi_{\epsilon t})\,dt \\ &\le \liminf_{\epsilon\to 0^+} \int_{B(x,r)} |\nabla\varphi_\epsilon|\,dy \le \mathcal{H}^1\left(\partial^{\mathrm{M}} E \cap \overline{B}(x,r)\right).\end{aligned}$$

This proves (31) and the lemma. □

Proof of Theorem 8. We have proved in (9) that any $x \in \mathring{E}^{\mathrm{M}} \setminus \partial^{\mathrm{S}} E$ is a point of density 1 for some set E_i.

Let now $x \in \mathring{E}_i^{\mathrm{M}} \setminus \partial^{\mathrm{S}} E$, $y \in \mathring{E}_j^{\mathrm{M}} \setminus \partial^{\mathrm{S}} E$, with $i \ne j$. By (27) we obtain that x does not belong neither to the Jordan curve J_0 corresponding to $\mathrm{sat}(E_i)$ nor to the Jordan curves J_k corresponding to the holes of E_i, and the same holds for y. Hence, if $\mathrm{sat}(E_i)$ and $\mathrm{sat}(E_j)$ are disjoint, we conclude that $x \in \mathrm{int}(J_0)$ and $y \in \mathrm{ext}(J_0)$, so that they cannot be connected by a continuous curve not intersecting $J_0 \subset \partial^{\mathrm{S}} E$. If $\mathrm{sat}(E_j) \subset \mathrm{sat}(E_i)$ then E_j is contained in some hole of E_i and the same argument applies for some curve J_k. If $\mathrm{sat}(E_i) \subset \mathrm{sat}(E_j)$ the argument is similar, reversing the roles of i and j.

Conversely, given a M-connected component E_i of E, we will prove that any pair of points $x,\ y \in \mathring{E}_i^{\mathrm{M}} \setminus \partial^{\mathrm{S}} E$ can be connected by a rectifiable curve contained in $\mathring{E}_i^{\mathrm{M}} \cup \partial^{\mathrm{S}} E$. To this aim, we first choose, according to Lemma 8, strictly decreasing sequences of positive numbers $\eta_h,\ \gamma_h$ such that $\partial B(x,\eta_h) \cup \partial B(y,\gamma_h) \subset \mathring{E}_i^{\mathrm{M}} \setminus \partial^{\mathrm{S}} E$ (recall that $\mathring{E}_i^{\mathrm{M}} \cap \partial^{\mathrm{S}} E$ is $\mathcal{H}^1$-negligible), and $2\pi \sum_h (\eta_h + \gamma_h) < \delta/2$. For any integer $h \ge 1$ we define

$$S_h := \left[B(x,\eta_{h-1}) \setminus \overline{B}(x,\eta_h)\right] \cup \left[B(y,\gamma_{h-1}) \setminus \overline{B}(y,\gamma_h)\right],$$

and

$$S_0 = \mathbb{R}^2 \setminus (B(x,\eta_0) \cup B(x,\gamma_0)).$$

Setting $F_h = E_i \cup B(x,\eta_h) \cup B(y,\gamma_h)$, the sets F_h are still indecomposable (see Proposition 5(i)), hence we can apply Lemma 7 with $N_h = (\partial^{\mathrm{S}} E \cap \mathring{E}_i^{\mathrm{M}}) \setminus (B(x,\eta_h) \cup B(y,\gamma_h))$ to obtain open sets $A_h \supset N_h \cup \partial^{\mathrm{S}} F_h$ such that $x,\ y$ belong to the same M-connected component G_h of $F_h \setminus A_h$. Moreover, since $\partial S_h \subset \mathring{F}_h^{\mathrm{M}}$, we can also assume that

$$P(F_h \setminus A_h, S_h) < P(F_h, S_h) + 2^{-h-2}\delta. \tag{32}$$

Finally, we can apply Lemma 6 to G_h to obtain a rectifiable curve Γ_h joining x to y, contained in $\mathring{G}_h^{\mathrm{M}} \cup \partial^{\mathrm{S}} G_h$ and also in $L \cup \partial^{\mathrm{S}} G_h$, where L is the segment joining x to y. Since $\partial^{\mathrm{S}} G_h \subset \partial^{\mathrm{S}}(F_h \setminus A_h)$ we have

$$\Gamma_h \cap S_h \subset (L \cap S_h) \cup \left(\partial^S G_h \cap S_h\right) \subset (L \cap S_h) \cup \left(\partial^S (F_h \setminus A_h) \cap S_h\right).$$

Since $\partial^{\mathrm{S}} F_h \subset \partial^{\mathrm{S}} E_i$, using (32), we obtain

$$\begin{aligned}\mathcal{H}^1(\Gamma_h \cap S_h) &\le \mathcal{H}^1\left(\partial^{\mathrm{M}}(F_h \setminus A_h) \cap S_h\right) + (\eta_{h-1} - \eta_h) + (\gamma_{h-1} - \gamma_h)\\ &\le \mathcal{H}^1\left(\partial^{\mathrm{M}} E_i \cap S_h\right) + (\eta_{h-1} - \eta_h) + (\gamma_{h-1} - \gamma_h) + 2^{-h-2}\delta \end{aligned} \tag{33}$$

for any $h \ge 1$. For $h = 0$, we have

$$\begin{aligned}\mathcal{H}^1(\Gamma_0 \cap S_0) &\le \mathcal{H}^1\left(\partial^{\mathrm{M}}(F_0 \setminus A_0) \cap S_0\right) + \mathcal{H}^1(L \cap S_0)\\ &\le \mathcal{H}^1(\partial^{\mathrm{M}} E_i \cap S_0) + |x - y| - (\eta_0 + \gamma_0) + \frac{1}{2^2}\delta. \end{aligned} \tag{34}$$

Since $\partial^{\mathrm{S}} E \cap \mathring{E}_i^{\mathrm{M}} \subseteq A_h$ and $\partial^{\mathrm{S}} G_h \subset \partial^{\mathrm{S}}(F_h \setminus A_h) \subset \mathbb{R}^2 \setminus A_h$, we have $\partial^{\mathrm{S}} G_h \subset \mathbb{R}^2 \setminus (\partial^{\mathrm{S}} E \cap \mathring{E}_i^{\mathrm{M}})$, and by our choice of η_h and γ_h the curves Γ_h are contained in $\mathring{F}_h^{\mathrm{M}} \setminus (\partial^{\mathrm{S}} E \cap \mathring{E}_i^{\mathrm{M}})$ and hence in $\mathring{E}_i^{\mathrm{M}} \setminus \partial^{\mathrm{S}} E$ out of $\overline{B}(x, \eta_h) \cup \overline{B}(y, \gamma_h)$. Using again our choice of η_h and γ_h we can build from Γ_h a locally rectifiable curve Γ contained in $\mathring{E}_i^{\mathrm{M}} \setminus \partial^{\mathrm{S}} E$ as in Fig. 4 (we have drawn for simplicity the construction only near to x).

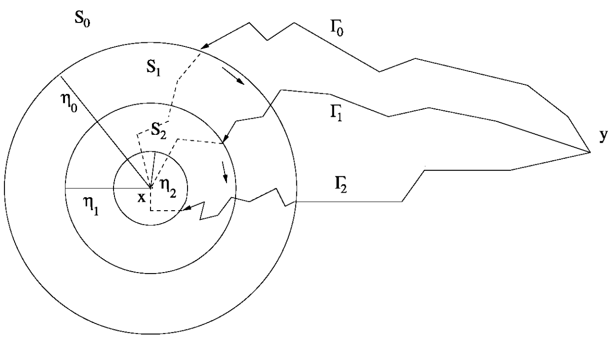

Fig. 4. Recursive construction of Γ near to x

The estimate on $\mathcal{H}^1(\Gamma)$ follows by (33), (34) and by the inclusion

$$\Gamma \setminus \{x, y\} \subset \bigcup_{h=0}^{\infty} (\Gamma_h \cap S_h) \cup \bigcup_{h=0}^{\infty} \partial B(x, \eta_h) \cup \partial B(y, \gamma_h).$$

□

9. Connected operators for image denoising

We call "connected operator" any contrast-invariant operator acting on the connected components of level sets. These operators could be defined on BV but we actually do not need neither the finiteness of the total variation nor the summability property. We need only to know that almost every level set has finite perimeter, so that its M-connected components can be defined. We therefore introduce a new space of functions that we shall call functions of weakly bounded variation.

Definition 7. *We say that a Borel function $u : \Omega \to [-\infty, +\infty]$ has weakly bounded variation in Ω if*

$$P(\{u > t\}, \Omega) < \infty \qquad \text{for a.e. } t \in \mathbb{R}.$$

The space of such functions will be denoted by $\mathrm{WBV}(\Omega)$. *We call total variation of u and denote by $|Du|$ the measure defined on every Borel subset $B \subseteq \Omega$ as*

$$|Du|(B) := \int_{-\infty}^{+\infty} P(\{u > t\}, B)\, dt.$$

It follows from the properties of the perimeter that $|Du|$ is a σ-additive measure on $\mathcal{B}(\Omega)$. Remark that, by Lemma 1, $\mathrm{BV}(\Omega) \subseteq \mathrm{GBV}(\Omega) \subseteq \mathrm{WBV}(\Omega)$ as soon as Ω is bounded. Furthermore, if Ω is bounded, connected and with Lipschitz boundary, $u \in \mathrm{WBV}(\Omega)$ and $|Du|(\Omega) < \infty$ then, by Lemma 1, $u \in \mathrm{BV}(\Omega)$ and, by the coarea formula, $|Du|$ coincides with the total variation of u.

It must be emphasized that WBV is a lattice (because sets of finite perimeter are closed under union and intersection) but is *not* a vector space. Take indeed the two functions $u(x) = 1/x$ and $v(x) = 1/x - \sin(1/x)$ defined on $(-1, 1)$. Then, clearly, $u,\ v \in \mathrm{WBV}(-1, 1)$ whereas $u - v \notin \mathrm{WBV}(-1, 1)$ since $\sin(1/x)$ assumes infinitely many times any value $t \in [-1, 1]$. However, a strong motivation for the introduction of $\mathrm{WBV}(\Omega)$ is the following result, showing that $\mathrm{WBV}(\Omega)$ is the smallest space containing $\mathrm{BV}(\Omega)$ and invariant under *any* continuous and strictly increasing contrast change; notice that, by Vol'pert chain rule for distributional derivatives, $\mathrm{BV}(\Omega)$ is stable only under *Lipschitz* contrast changes.

Theorem 9. *Assume that Ω is bounded, connected and with Lipschitz boundary. For any $u \in \mathrm{WBV}(\Omega)$ there exists a bounded, continuous and strictly increasing function $\phi : [-\infty, +\infty] \to \mathbb{R}$ such that $\phi \circ u \in \mathrm{BV}(\Omega)$.*

Proof. Let ϕ be the primitive of $\exp(-s^2)/(1 + m_u(s))$ such that $\phi(-\infty) = 0$. Then, since $\phi \circ u$ is bounded and takes its values in $[0, \phi(+\infty)]$,

$$\begin{aligned}\int_{-\infty}^{+\infty} m_{\phi\circ u}(t)\, dt = \int_{0}^{\phi(+\infty)} m_{\phi\circ u}(t)\, dt &= \int_{-\infty}^{+\infty} m_u(s)\phi'(s)\, ds \\ &\leq \int_{-\infty}^{+\infty} \exp(-s^2)\, ds < \infty,\end{aligned}$$

hence $\phi \circ u \in \mathrm{BV}(\Omega)$ by Lemma 1(ii). $\square$

Notice that Theorem 9 could be used to extend to WBV(Ω) many results of Sect. 3, as for instance the existence of the approximate differential ∇u, the rectifiability of the approximate discontinuity set S_u, the fact that $\mathcal{H}^{N-1}$-a.e. $x \in S_u$ is an approximate jump point, the structure of Du and so on. However, this analysis goes beyond the main goals of this paper and it will not be pursued here.

The space WBV(Ω) can be endowed with the following distance (identifying as usual the functions which coincide almost everywhere in Ω):

$$d(u_1, u_2) := \int_{\mathbb{R}} e^{-t^2} |\arctan m_{u_1} - \arctan m_{u_2}|\, dt + \int_{\Omega} e^{-|x|^2} |\arctan u_1 - \arctan u_2|\, dx.$$

Since arctan is a homeomorphism between $[-\infty, +\infty]$ and $[-\pi/2, \pi/2]$, it is easy to prove that the convergence with respect to d is equivalent to local convergence in measure of both u and m_u, hence (WBV(Ω), d) is a complete metric space.

L. Vincent's filters

Luc Vincent introduced in [64] a class of connected operators for denoising an image corrupted by a noise that creates small spots, like for instance impulse noise. Our motivation for the study of such filters is, in addition to the fact that they may be considered as the reference connected operators, their great ability to remove impulse noise. The key idea is to remove connected components of level sets whose Lebesgue measure does not exceed some threshold θ. Luc Vincent defined his filters as operators acting on the space of upper semicontinuous functions, in the framework of Mathematical Morphology. We shall now propose a definition adapted to the space WBV which involves the notion of M-connected components. We shall derive new properties of Vincent's filters, regarding in particular the behavior of the total variation. In addition, we shall prove that these filters map SBV onto SBV, Sobolev spaces onto Sobolev spaces and Lipschitz functions onto Lipschitz functions.

First remark that we shall from now assume Ω bounded with Lipschitz boundary. This is motivated by the fact that an image is generally given on a bounded domain. However, all the definitions and results stated above remain valid since any set $E \subset \Omega$ of finite perimeter in Ω has finite perimeter in $\mathbb{R}^{\mathrm{N}}$ (see for instance Remark 2.14 in [29]). For the sake of simplicity, we shall write $\partial^{\mathrm{M}} E$ instead of $\partial^{\mathrm{M}} E \cap \Omega$. We start now by defining the action of Vincent's filters on sets of finite perimeter.

Definition 8. *Let $E \subset \Omega$ be a set of finite perimeter in Ω and $\theta \geq 0$. We define $\mathrm{T}_\theta E$ as the union of the M-connected components E_i of E such that $|E_i| > \theta$.*

Note that $T_0 E = E$ and that $\mathrm{T}_\theta E$ is well defined up to Lebesgue negligible sets. Moreover, by Proposition 3, it follows that

$$P(\mathrm{T}_\theta E, \Omega) \leq P(E, \Omega) \tag{35}$$

with equality only if $\mathrm{T}_\theta E = E \pmod{\mathcal{H}^N}$.

Proposition 10. *Let E, $F \subset \Omega$ be two sets of finite perimeter in Ω. If $E \subseteq F$ (mod $\mathcal{H}^N$), then $\mathrm{T}_\theta E \subseteq \mathrm{T}_\theta F$ (mod $\mathcal{H}^N$).*

Proof. If E_i is a M-connected component of E with $|E_i| > \theta$, then by Theorem 1 there is a M-connected component F_j of F such that $E_i \subseteq F_j$ (mod H^N). Since $|F_j| > \theta$, we conclude that $\mathrm{T}_\theta E \subseteq \mathrm{T}_\theta F$ (mod $\mathcal{H}^N$). □

Now we want to extend T_θ to WBV functions; to this aim, the following lemma will be useful.

Lemma 9. *For any monotone family of sets X_λ, $\lambda \in \mathbb{R}$, there exists a countable set $D \subseteq \mathbb{R}$ such that*

$$\lim_{\mu\to\lambda} X_\mu = X_\lambda \qquad \textit{for all } \lambda \in \mathbb{R} \setminus D,$$

where convergence means convergence with respect to the finite measure $\mu = e^{-|x|^2}\mathcal{L}^N$ (or, equivalently, local convergence in measure in $\mathbb{R}^N$).

Proof. First remark that the map $\lambda \to \mu(X_\lambda)$ is real-valued since $\mu(\Omega) = \displaystyle\int_\Omega e^{-|x|^2} dx < \infty$. Then it is enough to note that this map is monotone, thus has at most countably many discontinuity points, and to choose D as the set of those discontinuity points. We call D the set of discontinuity points of X_λ. □

Theorem 10. *Let $u \in \mathrm{WBV}(\Omega)$ and $\theta \geq 0$. Then there exists a function $\mathrm{S}_\theta u \in \mathrm{WBV}(\Omega)$ (resp. $\mathrm{I}_\theta u \in \mathrm{WBV}(\Omega)$) such that*

$$\{\mathrm{S}_\theta u > \lambda\} = \mathrm{T}_\theta\{u > \lambda\} \quad \textit{(resp. } \{\mathrm{I}_\theta u < \lambda\} = \mathrm{T}_\theta\{u < \lambda\}\textit{)} \qquad \textit{(mod } H^N\textit{)}$$

with at most countably many exceptions. Any other measurable function v with the same property coincides with $\mathrm{S}_\theta u$ (resp. $\mathrm{I}_\theta u$) almost everywhere in Ω. In addition,

$$|D\mathrm{S}_\theta u|(B) \leq |Du|(B) \quad \textit{and} \quad |D\mathrm{I}_\theta u|(B) \leq |Du|(B) \quad \textit{for any Borel set } B \subset \Omega$$

Proof. Let $X_\lambda = \{u > \lambda\}$. By definition of WBV, for almost every $\lambda \in \mathbb{R}$, X_λ has finite perimeter and we can define $Y_\lambda = \mathrm{T}_\theta X_\lambda$. Since $\lambda < \lambda'$ implies that $X_\lambda \supseteq X_{\lambda'}$, we infer from Proposition 10 that (Y_λ) is a decreasing family. Let D be the set of discontinuity points of Y_λ. Let $D^* \subseteq \mathbb{R}$ be countable and dense and define

$$\mathrm{S}_\theta u(x) = \sup\{\lambda \in D^* : x \in Y_\lambda\}.$$

We now prove that $\{\mathrm{S}_\theta u > \lambda\} = Y_\lambda$ (mod H^N) for any $\lambda \notin D$. In fact, we clearly have

$$Y_\eta \subseteq \{\mathrm{S}_\theta u > \lambda\} \subseteq Y_\rho$$

for any $\eta, \rho \in D^*$, $\rho < \lambda < \eta$. If we choose sequences $\eta_k \to \lambda$ and $\rho_k \to \lambda$ in D^*, Lemma 9 proves that Y_λ coincides with $\{\mathrm{S}_\theta u > \lambda\}$ (mod H^N). In particular,

$\{S_\theta u > \lambda\}$ is measurable for any $\lambda \notin D$. By approximation, the same is true for any $\lambda \in \mathbb{R}$. Hence, $S_\theta u$ is measurable.

The uniqueness of $S_\theta u$ can be proved by checking, with a similar argument, that if u_1, u_2 are two measurable functions such that $\{u_1 > \lambda\} = \{u_2 > \lambda\}$ (mod H^N) for a dense set of λ, then $u_1 = u_2$ almost everywhere in Ω.

Remark now that, by assumption, $\{u > \lambda\}$ is a set of finite perimeter in Ω for almost every $\lambda \in \mathbb{R}$, thus $P(\{u > \lambda\}, B) < +\infty$ for any Borel set $B \subseteq \Omega$. Since $\mathcal{CC}^{\mathrm{M}}\{S_\theta u > \lambda\} \subseteq \mathcal{CC}^{\mathrm{M}}\{u > \lambda\}$ we deduce by Proposition 3 that $\partial^{\mathrm{M}}\{S_\theta u > \lambda\} \subseteq \partial^{\mathrm{M}}\{u > \lambda\}$ (mod $\mathcal{H}^{N-1}$). Recalling that $P(E, B) = \mathcal{H}^{N-1}(B \cap \partial^{\mathrm{M}} E)$ whenever E has finite perimeter in B, it follows that $P(\{S_\theta u > \lambda\}, B) \le P(\{u > \lambda\}, B) < \infty$ for every Borel subset $B \subseteq \Omega$ and for almost every $\lambda \in \mathbb{R}$. Thus $S_\theta u \in \mathrm{WBV}(\Omega)$ and $|DS_\theta u|(B) \le |Du|(B)$ for any Borel set $B \subseteq \Omega$.

The proof of the existence and the uniqueness of $I_\theta u$ is analogous to the one for $S_\theta u$, by noting that the sets $X_\lambda = \{u < \lambda\}$, hence also $Y_\lambda = T_\theta X_\lambda$, form an increasing family and defining $I_\theta u(x) = \inf\{\lambda \in D^* : x \in Y_\lambda\}$. Remark now that $\{u > \lambda\} = \{-u < -\lambda\}$, thus

$$S_\theta u = -I_\theta(-u) \qquad \text{a.e. in } \Omega. \tag{36}$$

and it follows that $I_\theta u \in \mathrm{WBV}(\Omega)$ and $|DI_\theta u|(B) \le |Du|(B)$ for any Borel set $B \subseteq \Omega$. □

Remark 11. Recall that, since u, $S_\theta u$ and $I_\theta u$ are measurable, it is equivalent in the previous theorem to deal with upper level sets instead of strictly upper level sets for both essentially coincide except for at most countably many exceptions.

Since $T_\theta\{u > \lambda\} \subset \{u > \lambda\}$ and $T_\theta\{u < \lambda\} \subset \{u < \lambda\}$ we infer that $\{S_\theta u > \lambda\} \subset \{u > \lambda\}$ and $\{I_\theta u < \lambda\} \subset \{u < \lambda\}$ for almost every λ, hence

$$S_\theta u \le u \le I_\theta u \qquad \text{a.e. in } \Omega. \tag{37}$$

In order to study the properties of S_θ and I_θ in the classical functions spaces BV and $W^{1,p}$ the following lemma will be useful.

Lemma 10. *Let* $u, v \in \mathrm{BV}(\Omega)$ *such that* $|Du|(B) \le |Dv|(B)$ *for every Borel set* $B \subset \Omega$. *Then*

(i) $|\nabla u| \le |\nabla v|$ *a.e. in* Ω;

(ii) $S_u \subseteq S_v$ *(mod* $\mathcal{H}^{N-1}$*)*;

(iii) $|u^+ - u^-| \le |v^+ - v^-|$ $\mathcal{H}^{N-1}$*-a.e. in* Ω;

(iv) $|D^c u| \le |D^c v|$.

Proof. Recall that $|Du| = |\nabla u|\,\mathcal{L}^N + |u^+ - u^-|\mathcal{H}^{N-1} \llcorner J_u + |D^c u|$. More precisely, setting

$$\mathcal{N}_u := \left\{x \in \Omega : \lim_{r\downarrow 0} r^{-N}|Du|(B_r(x)) = \infty\right\}$$
$$\text{and } \Theta_u := \left\{x \in \Omega : \liminf_{r\downarrow 0} r^{1-N}|Du|(B_r(x)) > 0\right\}$$

then (see for instance [5]) $\Theta_u \subseteq \mathcal{N}_u$, $|\mathcal{N}_u| = 0$, Θ_u is σ-finite with respect to $\mathcal{H}^{N-1}$ and

$$D^a u = Du \llcorner (\Omega \setminus \mathcal{N}_u), \quad D^j u = Du \llcorner S_u = Du \llcorner \Theta_u \ \text{ and } \ D^c u = Du \llcorner (\mathcal{N}_u \setminus \Theta_u).$$

Let $\mathcal{N} = \mathcal{N}_u \cup \mathcal{N}_v$. Then $|\mathcal{N}| = 0$ and for every Borel set $B \subset \Omega \setminus \mathcal{N}$, $|Du|(B) = |D^a u|(B)$ and $|Dv|(B) = |D^a v|(B)$. Therefore

$$|D^a u|(B) = \int_B |\nabla u| dx \le |D^a v|(B) = \int_B |\nabla v| dx$$

and (i) follows since the inequality is true for every Borel set $B \subset \Omega \setminus \mathcal{N}$.

(ii) Let $B = S_u \setminus S_v$. Then $|Du|(B) \le |Dv|(B)$, $|B| = 0$ and $B \subset \Omega \setminus S_v$ is σ-finite with respect to $\mathcal{H}^{N-1}$ so that $D^a v|(B) = 0$ and $|D^c v|(B) = 0$ (see [5]). Thus $|Du|(B) = |Dv|(B) = 0$ and, therefore,

$$\int_B |u^+ - u^-| d\mathcal{H}^{N-1} = 0.$$

Since $|u^+ - u^-| > 0$ on J_u and $\mathcal{H}^{N-1}(S_u \setminus J_u) = 0$ we deduce that $\mathcal{H}^{N-1}(B) = 0$ thus

$$S_u \subseteq S_v \qquad (\text{mod } \mathcal{H}^{N-1})$$

(iii) For every Borel set $B \subseteq J_u$

$$\int_B |u^+ - u^-| d\mathcal{H}^{N-1} \le \int_B |v^+ - v^-| d\mathcal{H}^{N-1}$$

and we deduce that

$$|u^+ - u^-| \le |v^+ - v^-| \qquad \mathcal{H}^{N-1}\text{-a.e. in } J_u.$$

The result follows by simply remarking that $|u^+ - u^-| = 0$ for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega \setminus J_u$.

(iv) Let $\tilde{\mathcal{N}} = \mathcal{N}_u \setminus (\Theta_u \cup \Theta_v)$. Since Θ_v is σ-finite with respect to $\mathcal{H}^{N-1}$ we deduce that $D^c u = Du \llcorner \tilde{\mathcal{N}}$. It is a straightforward consequence of the definitions that $\mathcal{N}_u \subset \mathcal{N}_v$ thus $Dv \llcorner \tilde{\mathcal{N}} = D^c v \llcorner \tilde{\mathcal{N}}$. For every Borel subset $B \subset \tilde{\mathcal{N}}$ we get

$$|D^c u|(B) = |Du|(B) \le |Dv|(B) = |D^c v|(B)$$

and (iv) follows. □

The following proposition is a straightforward consequence of the previous lemma and Theorem 10.

Proposition 11. *Let* $u \in \mathrm{BV}(\Omega)$ *and* $\theta \ge 0$. *Let* T_θ *denote any of the operator* S_θ *or* I_θ. *Then*

(i) $|\nabla \mathrm{T}_\theta u| \le |\nabla u|$ *a.e. in* Ω;
(ii) $S_{\mathrm{T}_\theta u} \subseteq S_u$ *(mod* $\mathcal{H}^{N-1}$*);*
(iii) $|\mathrm{T}_\theta u^+ - \mathrm{T}_\theta u^-| \le |u^+ - u^-|$ $\mathcal{H}^{N-1}$*-a.e. in* Ω;
(iv) $|D^c \mathrm{T}_\theta u| \le |D^c u|$.

Remark 6. An interesting consequence of this proposition is that S_θ and I_θ map $\mathrm{SBV}(\Omega)$ onto itself in such a way that the jump set is reduced as well as the "height" of the jumps. It is therefore easily seen that any Sobolev space $\mathrm{W}^{1,p}(\Omega)$, $1 \le p \le \infty$, is mapped onto itself by I_θ and S_θ with a decay of the gradient norm at almost every point. Analogously, any Lipschitz function is mapped onto a Lipschitz function with the same Lipschitz constant.

Finally, we conclude this section with some additional properties of the filters S_θ and I_θ.

Proposition 12. *Let* $\theta \ge 0$. *Then* S_θ, I_θ, $\mathrm{I}_\theta\mathrm{S}_\theta$, $\mathrm{S}_\theta\mathrm{I}_\theta$ *are monotone and idempotent operators acting on* $\mathrm{WBV}(\Omega)$. *Moreover, they are covariant with respect to any real continuous and strictly increasing contrast change.*

Proof. The monotonicity of the operators is a simple application of the monotonicity of T_θ on level sets. Observe that if E is a set of finite perimeter in Ω, then $\mathrm{T}_\theta(\mathrm{T}_\theta E) = \mathrm{T}_\theta E$. Therefore, if $u \in \mathrm{WBV}(\Omega)$, then, for almost every $\lambda \in \mathbb{R}$, $\{u > \lambda\}$ has finite perimeter in Ω and we have $\mathrm{T}_\theta(\mathrm{T}_\theta\{u > \lambda\}) = \mathrm{T}_\theta\{u > \lambda\}$. By the uniqueness property stated in Theorem 10, we deduce that $\mathrm{S}_\theta(\mathrm{S}_\theta u) = \mathrm{S}_\theta u$ almost everywhere in Ω. Equation (36) implies that I_θ is idempotent as well. Now, let us prove that

$$\mathrm{S}_\theta\mathrm{I}_\theta\mathrm{S}_\theta u = \mathrm{I}_\theta\mathrm{S}_\theta u. \tag{38}$$

Indeed, let $\lambda \in \mathbb{R}$ be such that $\{u > \lambda\}$ is a set of finite perimeter in Ω, $\{\mathrm{S}_\theta u \le \lambda\} = \{\mathrm{S}_\theta u < \lambda\}$, and $\{\mathrm{I}_\theta\mathrm{S}_\theta u \le \lambda\} = \{\mathrm{I}_\theta\mathrm{S}_\theta u < \lambda\}$ (mod H^N). By Theorem 10, $\{\mathrm{S}_\theta u > \lambda\} = \mathrm{T}_\theta\{u > \lambda\}$, $\{\mathrm{I}_\theta\mathrm{S}_\theta u < \lambda\} = \mathrm{T}_\theta\{\mathrm{S}_\theta u < \lambda\}$, $\{\mathrm{S}_\theta\mathrm{I}_\theta\mathrm{S}_\theta u > \lambda\} = \mathrm{T}_\theta\{\mathrm{I}_\theta\mathrm{S}_\theta u > \lambda\}$ (mod H^N). Then we prove that

$$\{\mathrm{S}_\theta\mathrm{I}_\theta\mathrm{S}_\theta u > \lambda\} = \{\mathrm{I}_\theta\mathrm{S}_\theta u > \lambda\} \qquad (\mathrm{mod}\ H^N). \tag{39}$$

Otherwise, there exists a M-connected component Q of $\{\mathrm{I}_\theta\mathrm{S}_\theta u > \lambda\}$ with $0 < |Q| \le \theta$. Thus Q is a M-connected component of $\mathbb{R}^N \setminus \{\mathrm{I}_\theta\mathrm{S}_\theta u \le \lambda\} = \mathbb{R}^N \setminus \{\mathrm{I}_\theta\mathrm{S}_\theta u < \lambda\} = \mathbb{R}^N \setminus \mathrm{T}_\theta\{\mathrm{S}_\theta < \lambda\} = \mathbb{R}^N \setminus \mathrm{T}_\theta\{\mathrm{S}_\theta \le \lambda\}$ and, according to Theorem 1, we may write

$$\partial^M Q = \cup_{k=1}^{p} \partial^M F_k \qquad (\mathrm{mod}\ H^{N-1}),$$

where F_k, $k = 1, \ldots, p$, denote the M-connected components of $\mathrm{T}_\theta\{\mathrm{S}_\theta u \le \lambda\}$ such that $\partial^M F_k \cap \partial^M Q \ne \emptyset$ (mod H^{N-1}). In particular, F_k, $k = 1, \ldots, p$, are M-connected components of $\{\mathrm{S}_\theta u \le \lambda\}$ such that $|F_k| > \theta$. It follows that Q cannot be contained in $\{\mathrm{S}_\theta u \le \lambda\}$. Hence, Q contains at least a M-connected component of $\{\mathrm{S}_\theta u > \lambda\}$ and, therefore, $|Q| \ge \theta$. This contradiction proves (39) and, as a consequence, (38). Since I_θ is idempotent, we obtain

$$\mathrm{I}_\theta\mathrm{S}_\theta\mathrm{I}_\theta\mathrm{S}_\theta u = \mathrm{I}_\theta\mathrm{I}_\theta\mathrm{S}_\theta u = \mathrm{I}_\theta\mathrm{S}_\theta u.$$

Let us prove the covariance of S_θ with respect to any real continuous increasing contrast change. This is due to the fact the family of level sets is globally invariant by such a contrast change. Let $u \in \mathrm{WBV}(\Omega)$ and let $g : \mathbb{R} \to \mathbb{R}$ be a real continuous increasing function. Then, for almost every $\lambda \in \mathbb{R}$, $\{g(u) > g(\lambda)\} = \{u > \lambda\}$, hence, $\mathrm{T}_\theta\{g(u) > g(\lambda)\} = \mathrm{T}_\theta\{u > \lambda\}$ and, by definition, $\{\mathrm{S}_\theta g(u) > g(\lambda)\} = \{\mathrm{S}_\theta u > \lambda\}$. Thus $\{g^{-1}\mathrm{S}_\theta g(u) > \lambda\} = \{\mathrm{S}_\theta u > \lambda\}$. From the uniqueness statement of Theorem 10, we conclude that $\mathrm{S}_\theta g(u) = g(\mathrm{S}_\theta u)$ a.e. in Ω. The corresponding statements for $\mathrm{I}_\theta, \mathrm{I}_\theta\mathrm{S}_\theta, \mathrm{S}_\theta\mathrm{I}_\theta$ are proved in the same way. The monotonicity assertion is straightforward and we shall omit the details. $\square$

Experiments

First recall that an image can be naturally represented as a piecewise constant function, each pixel being considered as a square with measure one. We have illustrated in Fig. 5 the internal and external boundaries of some level sets of an image (see Sect. 7). For the sake of simplicity, we shall also use the terms *topographic map* to refer to this representation. It is a straightforward consequence of Theorem 6 and the reconstruction formula $u(x) = \sup\{t : x \in \{u > t\}\} = \inf\{t : x \in \{u < t\}\}$ that the topographic map is a complete and contrast-invariant representation of the image. Remark that, for the sake of readability, we have actually illustrated in Fig. 5 the partial topographic map obtained by taking into account only those level sets separated by at least 10 grey levels.

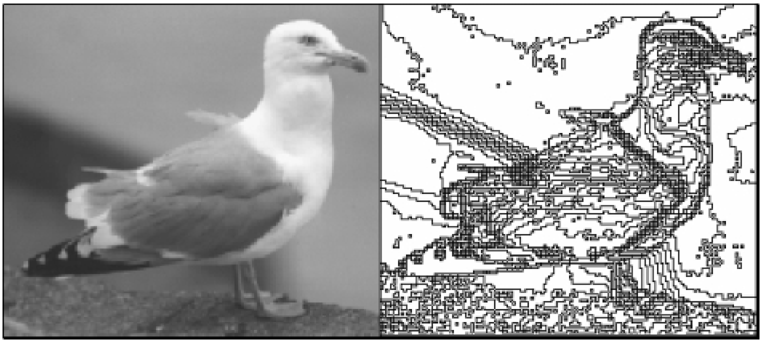

Fig. 5. An image and its partial topographic map (grey level step = 10)

Figure 6 illustrates the ability of the Vincent's filter $\mathrm{I}_\theta\mathrm{S}_\theta$ to remove impulse noise in an image. Recall that impulse noise replaces the value of a prescribed number of pixels, uniformly distributed in the image, by a random value taken between 0 and 255, according to a uniform distribution law. The algorithm for computing the action of I_θ is the following: let x_0 be a pixel where the image, denoted by u, assumes a local minimum and $\lambda = u(x_0)$. Adding progressively pixels in the neighborhood of x_0, one can construct the connected component $I(\lambda)$ containing x_0 of the set $\{x,\ u(x) \le \lambda\}$. Then, setting $\lambda := \lambda + 1$, the process is iterated until $|I(\lambda)| \ge \theta$. Finally, each pixel in $I(\lambda)$ is given the value λ. The whole process is performed for each local minimum of u.

The algorithm for S_θ is stricly analogous, starting from a local maximum and computing iteratively the connected component $S(\lambda)$ containing x_0 of the set

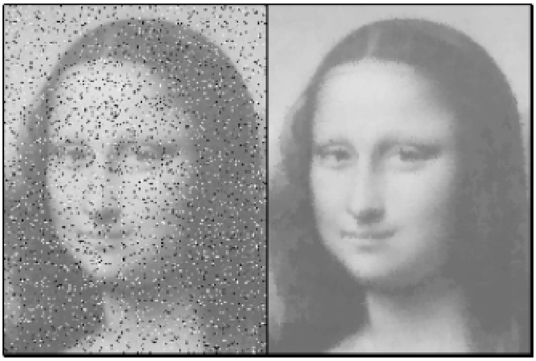

Fig. 6. An image corrupted by an impulse noise with frequency 15% and the result of the denoising performed by $\mathrm{I}_{10}\mathrm{S}_{10}$

$\{x,\ u(x) \geq \lambda\}$, where λ is initially given the value $u(x_0)$ and is lowered until $|S(\lambda)| \geq \theta$. Again, each pixel in the ultimate $S(\lambda)$ is given the value λ.

We shall not address here the problem of the consistency of these algorithms, that is the question whether they converge to the operator $\mathrm{I}_\theta\mathrm{S}_\theta$ as defined for functions, when the discrete grid tends to the continuous plane. This question is obviously far beyond the scope of this paper.

Three properties of $\mathrm{I}_\theta\mathrm{S}_\theta$ are particularly relevant in view of an automated denoising: the idempotence, which prevents from caring about the number of iterations, the dependence on a single parameter θ, which makes the filter much easier to handle with and, finally, the ability of $\mathrm{I}_\theta\mathrm{S}_\theta$ to preserve the unnoisy parts of the image (see Fig. 7) which ensures that only noise is processed.

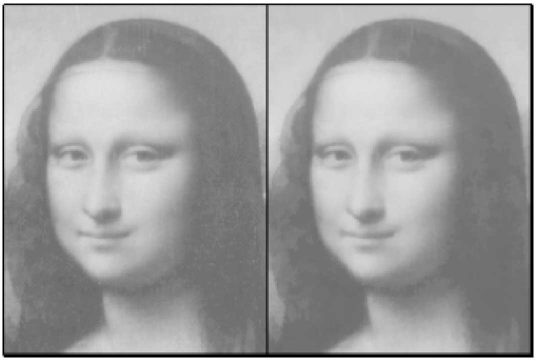

Fig. 7. An uncorrupted image and the result of the filtering by $\mathrm{I}_{10}\mathrm{S}_{10}$. This experiment illustrates the ability of Vincent's filter to preserve uncorrupted parts of an image

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