Image Interpolation

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Abstract

We discuss possible algorithms for interpolating data given in a set of curves and/or points in the plane. We propose a set of basic assumptions to be satisfied by the interpolation algorithms which lead to a set of models in terms of possibly degenerate elliptic partial differential equations. The Absolute Minimal Lipschitz Extension model (AMLE) is singled out and studied in more detail. We show experiments suggesting a possible application, the restoration of images with poor dynamic range. We also analyse the problem of unsmooth interpolation and show how it permits a subsidiary variational method.

1 Introduction

Our purpose in this paper will be to discuss possible algorithms for interpolating scalar data given on a set of points and/or curves in the plane. Our main motivation comes from the field of image processing. A number of different approaches using interpolation techniques have been proposed in the literature for "perceptually motivated" coding applications. The image is assumed to be made mainly of areas of constant or smoothly changing intensity separated by discontinuities represented by strong edges. The coded information, also known as *sketch data*, consists of the geometric structure of the discontinuities and their amplitudes. In very low bit rate applications, the decoder has to reconstruct the smooth areas in between by using the edge information. This can be posed as a scattered data interpolation problem from an arbitrary initial set (the sketch data) under certain smoothness constraints [14]. In the following we assume that a set of curves and points is given and we want to construct a function interpolating these data. Several interpolation techniques using implicitly or explicitly the solution of a partial differential equation have been used in the engineering literature. In the spirit of [1], our approach to the problem will be based on a set of formal requirements that any interpolation operator in the plane should satisfy. Then we show that any operator which interpolates continuous data given on a set of curves can be given as the viscosity solution of a degenerate elliptic partial differential equation of a certain type. The examples include the Laplacian operator and the minimal Lipschitz extension operator [2, 10] which is related to the work of J. Casas [4]. We shall prove mainly two facts :

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• If we want a smooth (e.g. Lipschitz) interpolant of points and curves, the most invariant and reliable interpolate must satisfy $D^2u(Du, Du) = 0$ and is therefore solution of the Absolutely Minimizing Lipschitz Extension (AMLE) model introduced by Aronsson.

• If we want to interpolate discontinuity lines as well, our analysis proves that the interpolant must satisfy $\operatorname{div}(\frac{Du}{|Du|}) = 0$. In other words, the interpolant has all level lines straight ! This solution is much more satisfactory (as experiments will prove) than foreseen, provided we use a subsidiary variational criterion, because of the high multiplicity of solutions for $\operatorname{div}(\frac{Du}{|Du|}) = 0$. We shall therefore introduce a level lines based variational method. We also discuss extensions of this method yielding curved level lines as well.

Our plan is as follows :
In Section 2, we present several axioms for image interpolation and show how they lead to equations $D^2u(Du, Du) = 0$ and $\operatorname{div}(\frac{Du}{|Du|}) = 0$. In Section 3, we give some hints on the properties of solutions of $D^2u(Du, Du) = 0$. In Section 4, we discuss the variational subsidiary method for singular interpolation. Section 5 is devoted to two different applications, the restoration of missing level lines (smooth interpolation) on one hand and the disocclusion, or spots removal (singular interpolation), on the other hand.

2 Axiomatic analysis of smooth interpolation operators

In order to classify interpolation operators, we shall first restrict our discussion to the particular case where the interpolated data is a function defined on a Jordan curve Γ.

Let $\mathcal{C}$ be the set of continuous simple Jordan curves in $\mathbb{R}^2$. For each $\Gamma \in \mathcal{C}$, let $\mathcal{F}(\Gamma)$ be the set of continuous functions defined on Γ. We shall consider an interpolation operator as a transformation E which associates with each $\Gamma \in \mathcal{C}$ and each $\varphi \in \mathcal{F}(\Gamma)$ a unique function $E(\varphi, \Gamma)$ defined in the region $D(\Gamma)$ inside Γ satisfying the following axioms:

$(A1)$ Comparison principle:

$$E(\varphi, \Gamma) \leq E(\psi, \Gamma) \quad \text{for any } \Gamma \in \mathcal{C} \quad \text{and any } \varphi, \psi \in \mathcal{F}(\Gamma) \quad \text{with } \varphi \leq \psi$$

$(A2)$ Stability principle:

$$E(E(\varphi, \Gamma) \mid_{\Gamma'}, \Gamma') = E(\varphi, \Gamma) \mid_{D(\Gamma')}$$

for any $\Gamma \in \mathcal{C}$, any $\varphi \in \mathcal{F}(\Gamma)$ and $\Gamma' \in \mathcal{C}$ such that $D(\Gamma') \subseteq D(\Gamma)$. This principle means that no

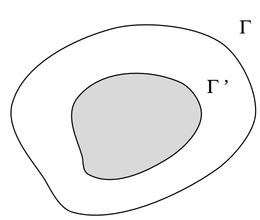

Figure 1: Stability principle

new application of the interpolation can improve a given interpolant. If this were not the case, we should iterate the interpolation operator indefinitely until a limit interpolant satisfying $(A2)$ is attained.

For the next principle, we denote by $SM(2)$ the set of symmetric two-dimensional matrices.
$(A3)$ Regularity principle: Let $A \in SM(2)$, $p \in \mathbb{R}^2 - \{0\}$, $c \in \mathbb{R}$ and

$$Q(y) = \frac{A(y-x, y-x)}{2} + < p, y-x > + c.$$

(where $< x, y >= \sum_{i=1}^{2} x_i y_i$). Let $D(x,r) = \{y \in \mathbb{R}^2 : \|y-x\| \leq r\}$ and $\partial D(x,r)$ its boundary. Then

$$\frac{E(Q\mid_{\partial D(x,r)}, \partial D(x,r))(x) - Q(x)}{r^2/2} \rightarrow F(A,p,c,x) \quad \text{as } r \rightarrow 0+ \tag{1}$$

where $F : SM(2) \times \mathbb{R}^2 - \{0\} \times \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function.
This assumption is much weaker than what it appears to be. Indeed, assume only that given A, p, c, x, we can find a C^2 function u such that $D^2u(x) = A$, $Du(x) = p$, $u(x) = c$, such that the differentiability assumption (1) holds (with u instead of Q). Then, arguing as in Theorem 1 in [6] it is easily proven that (1) holds for all C^2 functions and in particular for Q.

Together with these basic axioms, let us consider the following axioms which express obvious independence properties of the interpolation process with respect to the observer's standpoint and the grey level encoding scale.

$(A4)$ Translation invariance:

$$E(\tau_h \varphi, \Gamma - h) = \tau_h E(\varphi, \Gamma)$$

where $\tau_h \varphi(x) = \varphi(x+h)$, $h \in \mathbb{R}^2$, $\varphi \in \mathcal{F}(\Gamma)$, $\Gamma \in \mathcal{C}$. The interpolant of a translated image is the translated of the interpolant.

$(A5)$ Rotation invariance:

$$E(R\varphi, R\Gamma) = RE(\varphi, \Gamma)$$

where $R\varphi(x) = \varphi(R^t x)$, R being an orthogonal map in $\mathbb{R}^2$, $\varphi \in \mathcal{F}(\Gamma)$, $\Gamma \in \mathcal{C}$. The interpolant of a rotated image is the rotated of the interpolant.

$(A6)$ Grey scale shift invariance:

$$E(\varphi + c, \Gamma) = E(\varphi, \Gamma) + c$$

for any $\Gamma \in \mathcal{C}$, any $\varphi \in \mathcal{F}(\Gamma)$, $c \in \mathbb{R}$.

$(A7)$ Linear grey scale invariance:

$$E(\lambda\varphi, \Gamma) = \lambda E(\varphi, \Gamma) \quad \text{for any } \lambda \in \mathbb{R}$$

$(A8)$ Zoom invariance:

$$E(\delta_\lambda \varphi, \lambda^{-1}\Gamma) = \delta_\lambda E(\varphi, \Gamma)$$

where $\delta_\lambda \varphi(x) = \varphi(\lambda x)$, $\lambda > 0$. The interpolant of a zoomed image is the zoomed interpolant.

Axioms $(A1)$, $(A3)$ and $(A4)$ to $(A8)$ are obvious adaptations from the axiomatic developed in [1]. Let us write $G(A) = F(A, e_1)$, $A \in SM(2)$, $e_1 = (1,0)$. Then G is a continuous function of A. Given a matrix

$$A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix},$$

let us write for simplicity $G(\alpha, \beta, \gamma)$ instead of $G(A)$. Then using an argument similar to the one developed in [1] we prove

Theorem 2.1 [6] *Assume that E is an interpolation operator satisfying $(A1)-(A8)$. Let $\varphi \in \mathcal{C}(\Gamma)$, $u = E(\varphi, \Gamma)$. Then u is a viscosity solution of*

$$\begin{aligned} & G\left(D^2u\left(\frac{Du}{|Du|}, \frac{Du}{|Du|}\right), D^2u\left(\frac{Du}{|Du|}, \frac{Du^\perp}{|Du|}\right), D^2u\left(\frac{Du^\perp}{|Du|}, \frac{Du^\perp}{|Du|}\right)\right) = 0 \quad \text{in } D(\Gamma) \\ & u_{|\Gamma} = \varphi \end{aligned} \tag{2}$$

In addition, $G(A)$ is a nondecreasing function of A satisfying $G(\lambda A) = \lambda G(A)$ for all $\lambda \in \mathbb{R}$

We do not give explicitly the notion of viscosity solution for the general model (2) and we refer to [6, 7]. From now on, we shall assume that the interpolation operator E satisfies $(A1) - (A8)$. Using the monotonicity of G, we can reduce the number of involved arguments inside G.

Proposition 2.1 *i) If G does not depend upon its first or its last argument, then it only depends on its last (resp. its first) argument. In other terms,*

$$\text{If} \quad G(\alpha, \beta, \gamma) = \hat{G}(\alpha, \beta), \quad \text{then} \quad G = \hat{G}(\alpha) = \alpha \hat{G}(1),$$

$$\text{If} \quad G(\alpha, \beta, \gamma) = \hat{G}(\beta, \gamma), \quad \text{then} \quad G = \hat{G}(\gamma) = \gamma \hat{G}(1).$$

$\alpha, \beta, \gamma \in \mathbb{R}$.
ii) If G is differentiable at 0 then G may be written as $G(A) = Tr(BA)$ where B is a nonnegative matrix.

Thus if we assume that G is differentiable at $(0,0,0)$ and set

$$B = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

then we may write Equation (2) as

$$aD^2u\left(\frac{Du}{|Du|}, \frac{Du}{|Du|}\right) + 2bD^2u\left(\frac{Du}{|Du|}, \frac{Du^\perp}{|Du|}\right) + cD^2u\left(\frac{Du^\perp}{|Du|}, \frac{Du^\perp}{|Du|}\right) = 0. \tag{3}$$

where $a, c \geq 0$, $ac - b^2 \geq 0$ which is equivalent to say that the matrix B is nonnegative [6]. Let us explore which of these operators can be used to interpolate data given on a set of points and/or curves. For that we consider $D = B((0,0),1)$ the ball of center $(0,0)$ and radius 1 and look for a solution of (3) on $D \setminus \{(0,0)\}$ such that $U(0,0) = 1$ and $U(x_1, x_2) = 0$ for $(x_1, x_2) \in \partial D$. Assume that we have existence and uniqueness of solutions of (3). Since the equation and the data are rotation invariant then we may look for a radial solution $U = f(r)$ of (3) with $r = \sqrt{x_1^2 + x_2^2}$. If u satisfies (3) then f is a solution of

$$arf'' + cf' = 0 \quad 0 < r < 1 \tag{4}$$

such that $f(0) = 1, f(1) = 0$. In terms of the values of a, b, c we have

i) If $a = 0$, then $b = 0$. If $c = 0$ then we have no equation. If $c > 0$ then $f' = 0$ and the only solution of (4) is $f = constant$. The boundary conditions cannot be satisfied. There are no interpolation operators in this case.

ii) Consider now $a > 0$. Since (4) is an Euler equation the solutions are of the form 1, r^z or $\log r$. If $0 \leq c < a$ then $z = 1 - c/a$ and $f(r) = 1 - r^z$. Notice that ∇U is bounded if and only if $z = 1$, i.e. $c = 0$. In that case also $b = 0$ and the equation is

$$D^2 u \left(\frac{Du}{|Du|}, \frac{Du}{|Du|} \right) = 0 \tag{5}$$

When $0 < c < a$ the solution exists but the gradient is unbounded at $(0,0)$. If $c = a$ then the general solution of (4) is $f(r) = \alpha + \beta \log r$, $\alpha, \beta \in R$ and we cannot match the boundary conditions. Similarly if $c > a$, $f(r) = \alpha + \beta r^z$, $z = 1 - c/a < 0$, $\alpha, \beta \in R$ and again we cannot match the boundary conditions.

This discussion proves that

1) if we require that the interpolation operators described by a smooth function G be able to interpolate data given on curves and/or points, we are forced to assume model (5). As discussed above there are other possibilities with $0 < c < a$ but the gradient may become unbounded even for smooth data at the boundary which means that we have less regularity than in model (5). This model, as we shall see, always keeps a bound on the gradient if the boundary data have a bounded gradient.

2) we can retain the case $a = 0$, which yields the equation $D^2 u(\frac{Du^\perp}{|Du|}, \frac{Du^\perp}{|Du|}) = 0$, as interesting for it allows singular interpolants when the boundary data are unsmooth. This possibility, which is not taken into account in our axiomatics, will be developed in Section 4.

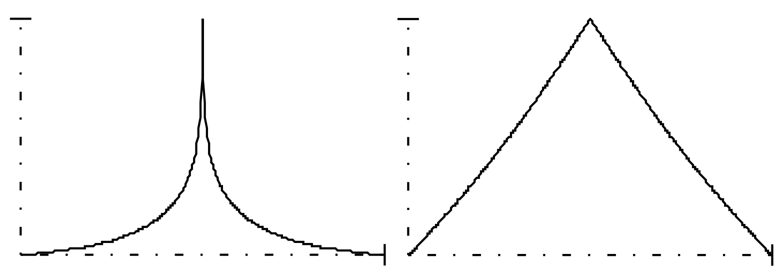

Figure 2: Sections of the radial solutions for a discretized Laplacian and model (5) respectively.

3 The AMLE model

Given a domain Ω with $\partial\Omega \in \mathcal{C}$ and $\varphi \in \mathcal{F}(\partial\Omega)$ we consider $E(\varphi, \partial\Omega)$ to be the viscosity solution of

$$\begin{aligned} & D^2u\left(\frac{Du}{|Du|}, \frac{Du}{|Du|}\right) = 0 \quad \text{in } \Omega \\ & u_{|\partial\Omega} = \varphi. \end{aligned} \tag{6}$$

We consider equation (6) in the viscosity sense. Given $u \in C(\Omega)$ we say that u is a viscosity subsolution (supersolution) of (6) if for any $\psi \in C^2(\Omega)$ and any x_0 local maximum (minimum) of $u - \psi$ in Ω such that $D\psi(x_0) \neq 0$

$$D^2\psi(x_0)\left(\frac{D\psi(x_0)}{|D\psi(x_0)|}, \frac{D\psi(x_0)}{|D\psi(x_0)|}\right) \geq 0 \quad (\leq 0).$$

A viscosity solution is a function which is a viscosity sub- and supersolution [7].

Equation (6) was introduced by G. Aronsson in [2] and recently studied by R. Jensen [10]. In [2] the author considered the following problem:

Given a domain Ω in $\mathbb{R}^n$, does a Lipschitz function u in Ω exist such that

$$\sup_{x\in\tilde{\Omega}} |Du(x)| \leq \sup_{x\in\tilde{\Omega}} |Dw(x)|,$$

for all $\tilde{\Omega} \subseteq \Omega$ and w such that $u - w$ is Lipschitz in $\tilde{\Omega}$ and $u = w$ on $\partial\tilde{\Omega}$. If it exists, such a function will be called an absolutely minimizing Lipschitz extension (AMLE) of $w|_{\partial\Omega}$ inside Ω. Notice that the above definition, if it defines uniquely u, immediately implies the stability of AMLE in the sense of $(A2)$. Then it was proved in [2] that if u is an AMLE and is C^2 in Ω, then u is a classical solution of

$$D^2u(Du, Du) = 0 \quad \text{in } \Omega. \tag{7}$$

Later Jensen [10] proved that if u is an AMLE, then u solves (7) in the viscosity sense. Moreover, the viscosity solution is unique. We shall use the viscosity solution formulation of Equation (7). Given $u \in C(\Omega)$ we say that u is a viscosity subsolution (supersolution) of (7) if for any $\psi \in C^2(\Omega)$ and any x_0 local maximum (minimum) of $u - \psi$ in Ω

$$D^2\psi(x_0)(D\psi(x_0), D\psi(x_0)) \geq 0 \quad (\leq 0).$$

A viscosity solution is a function which is a viscosity sub- and supersolution. Then Jensen proved [10] a comparison principle between sub- and supersolutions of Equation (7) together with an existence result for boundary data in the space of functions $Lip_\partial(\Omega)$ which are Lipschitz continuous with respect to the distance $d_\Omega(x, y)$. We denote by $d_\Omega(x, y)$ the geodesic distance between x and y, i.e., the minimal length of all possible paths joining x and y and contained in Ω [10]. Observe that u is a viscosity subsolution (supersolution, solution) of (6) if and only if u is a viscosity subsolution (supersolution, solution) of (7). From this follows the corresponding comparison principle for solutions of (6).

Theorem 3.1 [10] *Assume that v is a subsolution and w a supersolution of* (6) *(equivalently of* (7)*). If $v\mid_{\partial\Omega}$, $w\mid_{\partial\Omega} \in Lip_\partial(\Omega)$ then*

$$\sup_{x\in\Omega}(v - w) = \sup_{x\in\partial\Omega}(v - w). \tag{8}$$

Theorem 3.2 [10] *Given $g \in Lip_\partial(\Omega)$, u is the AMLE of g into Ω if and only if u is the solution of* (7) *with $u\mid_{\partial\Omega}= g$.*

Theorem 3.3 [6] *The AMLE interpolation model satisfies axioms $(A1)-(A8)$.*

4 Singular interpolation or "disocclusion"

Disocclusion or "amodal completion" is a very common process in human vision. In a natural scene, an object is seldom totally visible. It is generally partially hidden by other objects. But our perception is under certain geometric conditions able to "reconstruct" the whole object by interpolating the missing part. In the example illustrated in Figure 3, one generally "sees" the same black rectangle in both drawings, despite the fact that this rectangle is never totally visible.

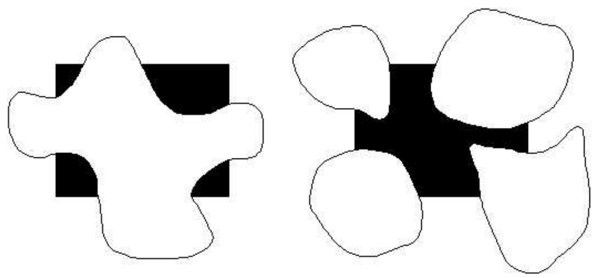

Figure 3: The same rectangle can be seen in both figures by amodal completion.

This ability of human vision has been widely studied by psychophysicists, particularly Gaetano Kanizsa [11]. It appears that continuation of objects boundaries plays a central role in the disocclusion process. This continuation is performed between T-junctions, which are points where image edges form a "T". We call "amodal completion" the process by which our perception extends visible edges "behind" occluding objects. According to psychophysicists, the continuation process is such that restored edges must be as smooth and straight as possible and the shapes as convex as possible.

Recent works [5, 9] have emphasized the importance of level lines for image understanding and representation. Let $u(x)$ denotes the gray level of an image u at point x. We define level lines as boundaries of upper level sets, defined at each gray level λ by $X_\lambda u = \{x,\ u(x) \geq \lambda\}$. In contrast to edge representation, the family of level lines is a complete representation of u, from which u can be reconstructed [5, 9]. In addition, this representation is invariant with respect to any increasing contrast change. This point is crucial since, according to Gestalt school (Wertheimer), human vision is essentially sensitive to the only ordering of gray levels in an image. The intensity difference between two pixels is not a reliable characterization of an image since it arbitrarily depends on the used sensor as well as illumination conditions. To be reliable, any natural image processing must involve the only ordering of gray levels, which remains identical by increasing contrast change.

In the following, we denote by Ω the part of the image plane occupied by the occluding object. We shall assume that Ω is simply connected and we denote by $\partial\Omega$ its boundary which we assume to be a Jordan curve. Our problem is to find some interpolant u of the original image u_0 which coincides with u_0 on Ω^c and satisfies the only interpolation equation $D^2u(\frac{Du^\perp}{|Du|}, \frac{Du^\perp}{|Du|}) = 0$ allowing unsmooth solutions according to Section 2. Remark that

$$D^2u(\frac{Du^\perp}{|Du|}, \frac{Du^\perp}{|Du|}) = |Du|\mathrm{div}(\frac{Du}{|Du|}) = |Du|\mathrm{curv}u,$$

and that the solution u to $|Du|\mathrm{curv}u = 0$ has all level lines straight. It is easily seen that it can have many different solutions. Thus a subsidiary variational criterion is required.

The method we present here allows to recover functions with strong discontinuities. Let us describe an image as a function u of bounded variation (BV) such that $\int_{\mathbb{R}^2}|u|dx < \infty$ and the total variation of u

$$\mathrm{TV}_{\mathbb{R}^2}(u) = \int_{\mathbb{R}^2}|Du| = \sup\{\int_{\mathbb{R}^2} u\,\mathrm{div}\phi\,dx\,:\,\phi\in \mathrm{C}^1_\mathrm{c}(\mathbb{R}^2,\mathbb{R}^2), |\phi|\leq 1\}$$

is finite. The reader may wish to refer to [8] for more details. A Lebesgue measurable subset $E\subset\mathbb{R}^2$ is said of finite perimeter if the quantity $P(E) = TV_{\mathbb{R}^2}(\mathbb{1}_E)$ is finite, where $\mathbb{1}_E$ denotes the characteristic function of E. If E has Lipschitz boundary then $P(E) = \mathrm{H}^1(\partial E)$ is interpreted as the length of the boundary of E. The coarea formula states that

$$TV_{\mathbb{R}^2}(u) = \int_{-\infty}^{\infty} P(X_\lambda u)d\lambda$$

and thus establishes a connection between the total variation of u and the length of its level lines. The next lemmas ensure that there exists a simple rectifiable curve Γ arbitrarily close to Ω such that the one-dimensional restriction of u to Γ has bounded variation and such that the level lines of u are transverse to Γ.

Lemma 4.1 [Trace of BV-functions]
Let $u\in\mathrm{BV}(\mathbb{R}^2,\mathbb{R})$ and Ω an occlusion. Define for every $h>0$

$$\Gamma_h = \{x\in\mathbb{R}^2\,:\,d(x,\,\Omega) = h\},$$

where d is some Lipschitz distance function on $\mathbb{R}^2$. Then for almost every $h>0$,

$$TV_{[0,\mathcal{L}(\Gamma_h)]}\tilde{u} = \int_0^{\mathcal{L}(\Gamma_h)} |\tilde{u}'| < \infty$$

where $\tilde{u}$ is the one-dimensional restriction of u to Γ_h.

Lemma 4.2 and Definition *There exists a Jordan curve Γ with finite length and arbitrarily close to Ω such that, denoting by $\tilde{u}$ the restriction of u to Γ, we have*

$$\begin{array}{ll}\exists\mathcal{R}\subset\mathbb{R}, & \mathcal{H}^1(\mathbb{R}\backslash\mathcal{R}) = 0\\ \forall\lambda\in\mathcal{R}, & \bullet\ \mathcal{H}^1(\partial_* X_\lambda u) < +\infty\\ & \bullet\ \mathcal{H}^0(\partial_* X_\lambda \tilde{u}) < +\infty\\ & \bullet\ \textit{set } E_\lambda = \partial_*(X_\lambda\tilde{u}).\ \textit{Then } E_\lambda = \partial_*(X_\lambda u)\cap\Gamma,\end{array}$$

where $\partial_* A$ is the measure-theoretic boundary [8] of a Lebesgue measurable set A.
If $x\in E_\lambda$ for some λ, we say that x is admissible.

We call admissible occlusion the domain enclosed by a curve Γ satisfying Lemmas 4.1 and 4.2. From the coarea formula almost every level set $X_\lambda u$ has finite perimeter. Since $u(x) = \sup\{\lambda : x \in X_\lambda u\}$ for almost every $x \in \mathbb{R}^2$, there is no loss of generality in previous lemma to concentrate on the only level sets of finite perimeter.

Let x be an admissible point in Γ. In view of previous lemmas we can define for almost every $\lambda \in [\tilde{u}(x-), \tilde{u}(x+)]$ an average direction

$$\nu_\lambda(B, x) = \int_B g.\nu \, d||\partial X_\lambda|| = \int_{\partial^* X_\lambda \cap B} g.\nu \, d\mathcal{H}^1$$

where $B = B(x, r_0)$, r_0 is such that $d(\Gamma, \Omega) > r_0$, ν denotes the normal at every point of the reduced boundary $\partial^* X_\lambda$ [8] and $g \in C^1_c(B, \mathbb{R})$ with $0 \leq g \leq 1$. Without loss of generality, we can assume that the sum goes over the $\mathcal{H}^1$–arcwise connected component of x within $B \cap \partial^* X_\lambda$. By $\mathcal{H}^1$–arcwise connected component we mean an arcwise connected component up to a set of $\mathcal{H}^1$–measure zero. Consequently, each admissible point x is associated for almost every $\lambda \in [\tilde{u}(x-), \tilde{u}(x+)]$ with an average direction $\nu_\lambda(B, x)$ and the orientation $o_\lambda(x) = \pm 1$, which refers to the orientation of the normal along the reduced boundary $\partial^* X_\lambda$ in the vicinity of x.

A logical variational criterion for the interpolant u is

$$E(u) = \int |Du|(1 + |\mathrm{curv} u|^p), \quad p \geq 1,$$

as proposed in [13] in another related framework, the segmentation of images having occlusions. Now, as shown in Bellettini *et al* [3], this criterion is not lower semicontinuous. Bellettini *et al* studied relaxed versions of E. We shall propose another relaxed version of E, which is compatible with Kanizsa's amodal completion theory. According to this theory, *an amodal completion is not a function*, but a set of lines or contours extending the contours of the image below the occluded part. These contours may even cross, but we shall exclude this possibility here, since we want a disocclusion as close as possible to a function. Indeed, from a noncrossing set of contours interpolating level lines, we can easily reconstruct a single function u whose level lines coincide almost everywhere with the contours (see Figure 4). It must be emphasized that the solution to the equation $|Du|\mathrm{curv} u = 0$ can be obtained in the particular case where $p = 1$. Whenever $p > 1$ the interpolating level lines are smooth curves and no more straight in general.

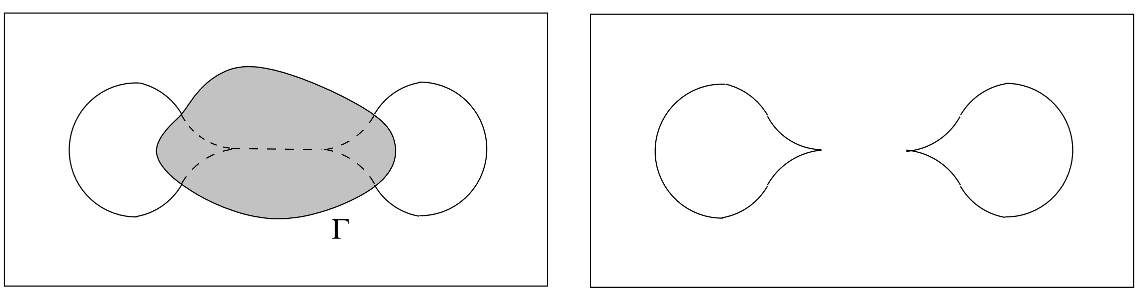

Figure 4: Left : disocclusion as a set of contours, $E(\mathcal{D}) < \infty$
Right : the associated solution u, $E(u) = +\infty$ if $p > 1$.

For every $p \geq 1$, we define the following space of curves with respect to all values of $\tilde{u}$ along Γ,

$$\begin{aligned}\mathcal{M} = \quad & \{\gamma_\lambda : [0,1] \to \Omega : \gamma_\lambda(0), \gamma_\lambda(1) \text{ admissible points in } \Gamma,\ o_\lambda(\gamma_\lambda(0)) = o_\lambda(\gamma_\lambda(1)), \\ & \gamma_\lambda \text{ is a simple curve with } \int_0^{\mathcal{L}(\gamma)} (1 + |\gamma_\lambda''(s)|^p) ds < \infty\},\end{aligned}$$

where γ_λ'' is the second derivative of γ_λ in the sense of distributions and s denotes the arc-length. Denoting by γ' the first derivative in the sense of distributions of a curve γ in $\mathcal{M}$, we can associate γ with an energy

$$E(\gamma) = \int_0^{\mathcal{L}(\gamma)} (1 + |\gamma''(s)|^p) ds + (\widehat{\tau(0), \gamma'(0_+)}) + (\widehat{\tau(1), \gamma'(1-)})$$

where $\tau(0) = \nu(B, \gamma(0))^\perp$ and $\tau(1) = \nu(B, \gamma(1))^\perp$. By $\widehat{(v, w)}$ we denote the angle (modulo 2π) between two vectors v and w in $\mathbb{R}^2$.

We call disocclusion a maximal set of curves in $\mathcal{M}$ connecting the admissible points of Γ two by two – or eventually with themselves – and such that two different curves do not cross. The total energy of a disocclusion $\mathcal{D}$ is

$$E(\mathcal{D}) = \int_{\mathcal{R}} \sum_{x \in E_\lambda} E(\gamma_{\lambda,x}) d\lambda$$

There exists at least a trivial (non optimal) disocclusion which can be obtain by regularizing Γ and simply giving to the domain enclosed by Γ a constant value.

Theorem 4.1 *Let Ω be an admissible occlusion. Then Ω admits a disocclusion with minimal energy.*

5 Experimental results

We display first some experiments illustrating the smooth interpolation with the AMLE model. Figures 5 and 6 show experiments with synthetic images. Figure 5a displays the original image, a single white point inside a rectangle. We impose $u = 0$ on the boundary of the rectangle. Figure 5b shows the result of the interpolation algorithm with Dirichlet boundary conditions. As one would expect, the result is a pyramid whose levels lines are displayed in Figure 5c. Figure 6a displays a synthetic image where we combine open curves, closed curves and points. Figure 6b shows the interpolant and Figure 6c shows the level lines of the interpolant.

Figure 7 shows how one can interpolate an image from the quantized level curves, obtaining a better result than the corresponding quantized image. Figure 7a displays the original image u which takes integer values between 0 and 255. Then we quantize it by giving the grey levels between $r\delta \leq u < (r+1)\delta$ the value $r\delta$, $r = 0, ..., M$, $M = [255/\delta]$. Figures 7c and 7e display the result of this operation applied on Figure 7a for values $\delta = 20$ and $\delta = 30$. Figure 7b displays the boundaries of the level sets $[u \geq r\delta]$ at the corresponding grey level $r\delta$ (for the sake of simplicity, the only level sets for $\delta = 30$ are displayed). We define the boundary values on the pixels belonging to the boundaries of the level sets B and the neighbouring pixels belonging to the boundary of the complement B^c. The solutions to AMLE model with these boundary data are displayed in Figures 7d and 7f.

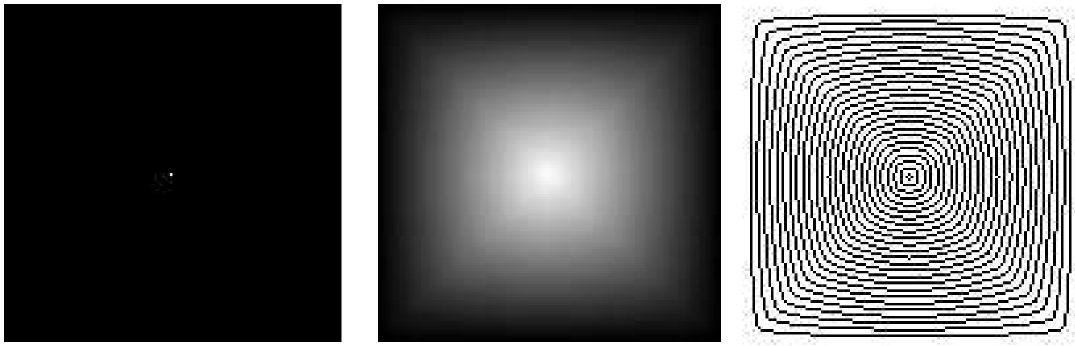

Figure 5: Left (a): original image – Middle (b): interpolant.
Right (c): level lines of the interpolant.

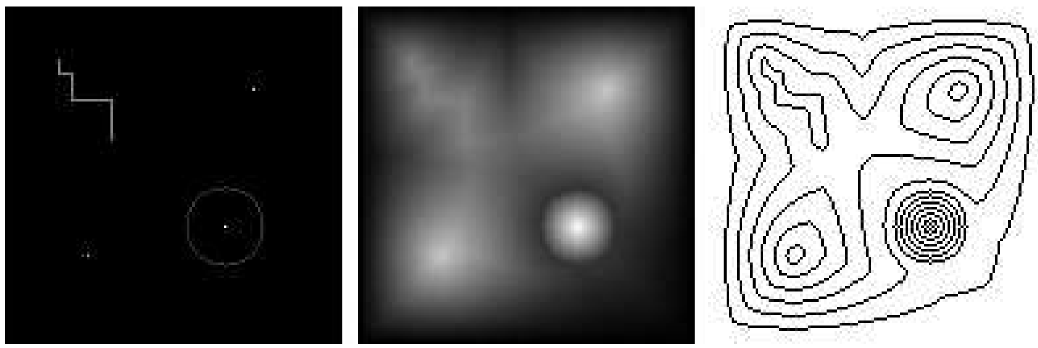

Figure 6: Left (a): original image – Middle (b): interpolant with $u = 0$ on the boundary.
Right (c): level lines of the interpolant.

In practice, the interpolation must keep smooth the regular regions of the image. So if we quantize the image at levels multiple of 30 (e.g.), the jump across the level line after quantization is either 0 or $30, 60$, etc. The behaviour of the algorithm is following: if the jump is just 0, it is likely that the region around is perceptually smooth, so our interpolation maintains it by giving a Lipschitz interpolation. If the jump across the level line is larger (e.g.) than 20, 30, etc., our decision is to maintain the jump because we consider that there must be an edge here. Since a jump larger than 20 is perceptible as edge, we maintain the existing edge by this choice, without significant attenuation or enhancement.

The other experiments are related to the level lines based singular interpolation solving the equation $|Du|\text{curv}u = 0$ with a subsidiary variational criterion (see the case $p = 1$ of the criterion described in Section 4). The reader may wish to refer to [12] for a description of the algorithm. Figure 8 shows that in contrast to regular interpolation, the level lines based disocclusion allows to recover image singularities in a way compatible with Kanizsa's perception theory. Figure 9 illustrates that this method can be used for old photographs restoration. Finally, Figure 10 illustrates the singular restoration of an image where only one pixel every six has been kept on each line and each column.

Figure 7: Left top (a): original image – Right top (b): level lines for $\delta = 30$.
Left middle (c): quantized image for $\delta = 20$ – Right middle (d): the interpolant for $\delta = 20$.
Left bottom (e): quantized image for $\delta = 30$ – Right bottom (f): the interpolant for $\delta = 30$.

Figure 8:

Above Original image where occlusions are in white.

Below-left Disocclusion performed by solving equation $D^2u(\frac{Du}{|Du|}, \frac{Du}{|Du|}) = 0$. Singularities cannot be restored but regular parts of image are well recovered.

Below-right Disocclusion performed by the level lines based algorithm solving equation $D^2u(\frac{Du^\perp}{|Du|}, \frac{Du^\perp}{|Du|}) = 0$. The singularities are well restored.

Figure 9: Left – An old damaged photograph (occlusion is in white).
Right – Result of image disocclusion by the level lines based algorithm.

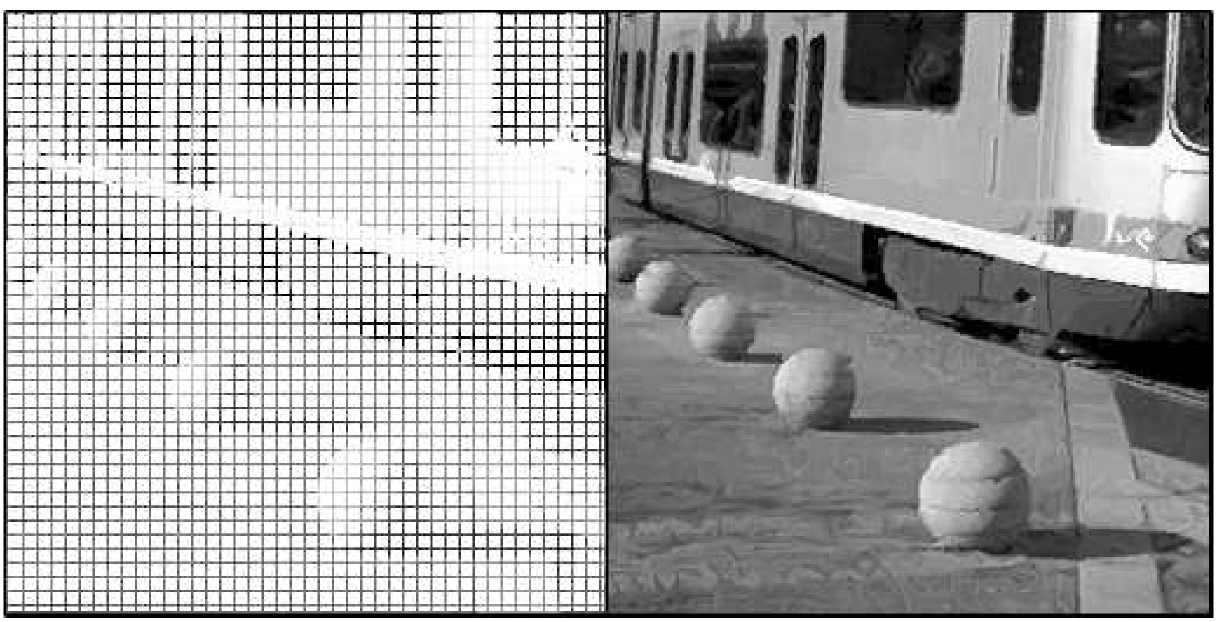

Figure 10: Left – An image with only one pixel every six on each line and each column.
(Occlusions are the white blocks).
Right – Result of image disocclusion.

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