A direct variational approach to a problem arising in image reconstruction

Luigi Ambrosio[1]
Simon Masnou[2]

Abstract

We consider a variational approach to the problem of recovering missing parts in a panchromatic digital image. Representing the image by a scalar function u, we propose a model based on the relaxation of the energy

$$\int |\nabla u|(\alpha + \beta\,|\operatorname{div}\frac{\nabla u}{|\nabla u|}|^p), \qquad \alpha, \beta > 0, \quad p \geq 1$$

which takes into account the perimeter of the level sets of u as well as the L^p norm of the mean curvature along their boundaries. We investigate the properties of this variational model and the existence of minimizing functions in BV. We also address related issues for integral varifolds with generalized mean curvature in L^p.

Keywords: Image processing; image reconstruction; BV; mean curvature; varifolds; relaxation.

1 Introduction

Many problems in digital image processing require the ability to recover missing parts of an image or to remove spurious or undesired objects. One can mention for instance the removal of scratches in old photographs and films, the recovery of pixels blocks corrupted during a binary transmission (or analogously the removal of impulse noise) or the removal of undesired publicity, text or subtitles from a photograph. One can also think to special effects for movie postproduction, e.g. the removal of a microphone appearing in a scene.

A digital image is usually modeled as a function u from a bounded domain of $\mathbb{R}^N$ ($N = 2$ for usual snapshots, $N = 3$ for medical images or movies, $N = 4$ for moving medical images) onto $\mathbb{R}^M$ ($M = 1$ for a grey-level image, $M = 3$ for colour images). Since it is now well

[1]Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy, luigi@ambrosio.sns.it
[2]Lab. J.-L. Lions, B.C. 187, Univ. Pierre et Marie Curie, 75252 Paris Cedex 05, France, masnou@ann.jussieu.fr

admitted that the essential features of any natural image are contained in its grey level representation, we shall concentrate on the panchromatic case $M = 1$. To extend to the colour case an operator designed for grey level images, it is generally enough to process separately each channel in the colour representation, e.g. the red-green-blue representation or, more appropriately, any representation with two channels for the chromaticity and one channel for the luminosity (see [9] and the references herein).

After the work of L. Rudin and S. Osher [34], the usual representation of a panchromatic image is a sum of two components $u_1 \in \mathrm{BV}(\mathbb{R}^{\mathrm{N}})$ and $u_2 \in \mathrm{L}^2(\mathbb{R}^{\mathrm{N}})$. The component u_1 is supposed to describe the *geometry* of the image, i.e. its objects and their boundaries, while u_2 contains all information about *texture* and *additive noise*. The assumption that the geometry of the image can be described by a function of bounded variation sounds quite natural, for it means that there can be discontinuities in the image but supported on rectifiable curves. The necessity of another component that does not necessarily belongs to BV can be corroborated by an experimental procedure that seems to indicate that, given a digital image, the subjacent "real" image may be often too oscillating to belong to BV (see [2] for the details and [11] for connected theoretic issues). The reader may refer to [4, 20] for a detailed survey of the space BV.

Among the large literature that has been published in recent years on the recovery of missing parts in a digital image, one can basically distinguish between two approaches and each of them corresponds in some way to the processing of one component in the decomposition above:

- the stochastic approach, which is based on the modeling of an image as a realization of a random process. Usually, it is assumed that the image intensity derives from a Markov Random Field and, therefore, satisfies properties of locality and stationarity, i.e. each pixel is only related to a small set of neighboring pixels and different regions of the image are perceived similar. This modeling is particularly adapted for texture images (thus to the processing or the component u_2 in the previous decomposition) and has motivated numerous works on texture analysis and synthesis [5, 14, 15, 25, 32, 33, 42, 44],

- the deterministic approach, whose main purpose is to recover the geometry of the image. The model we shall discuss in this paper belongs to this category.

A pioneering work on the recovery of plane image geometry is due to D. Mumford, M. Nitzberg and T. Shiota [31]. They did not directly address the problem of recovering missing parts in an image but rather tried to identify occluding and occluded objects in order to compute the image depth map. Their algorithm starts with the detection of the boundaries of image objects. The next step is the identification of occluded and occluding objects. To this aim, Nitzberg, Mumford and Shiota had the luminous idea to mimic a natural ability of human vision to complete partially occluded objects, the so-called *amodal completion* process described and studied by the Gestalt school of psychology and particularly G. Kanizsa [23]. From a series of perceptual experiments, Kanizsa found out that our vision system detects occlusion at a very low level, actually as soon as it detects *T-junctions*, which are points where an object outline abruptly abuts against the outline of another object and forms a junction in the shape of the letter "T". In particular, our perception of occlusion has nothing to do with a prior recognition of the objects. Being the T-junction detected, our brain performs a continuation of objects boundaries between T-junctions (see figure 1).

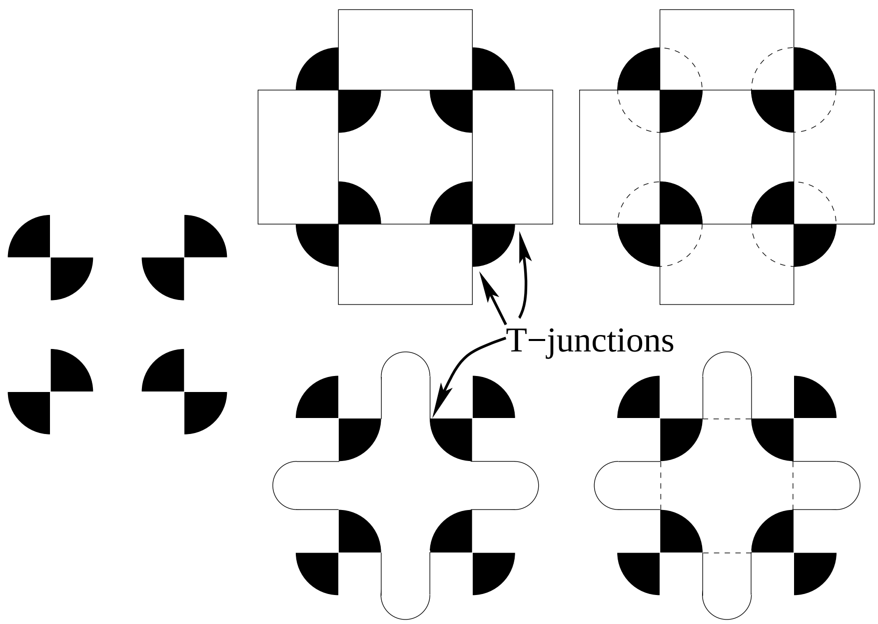

Figure 1: This example, due to G. Kanizsa [23], illustrates the amodal completion process. Starting from the four objects on the left column, the addition of either four white rectangles or a white cross produces T-junctions (middle column), that conduce our brain to perceive occlusions that, in reality, do not exist. This illustrates perfectly the link between the presence of T-junctions and the perception of occlusions. Then, our visual system recovers the virtually occluded objects (four black disks in one case and a black square in the other) by connecting T-junctions with completion curves, following a *good continuation* principle. We have represented those curves with dash lines on the right column.

As pointed out by Kanizsa, this continuation process relies on many different laws [23] and there is actually no obvious may to model it, even in relatively simple situations [18]. Again, it seems that no process of recognition be involved (see figure 2). The idea of

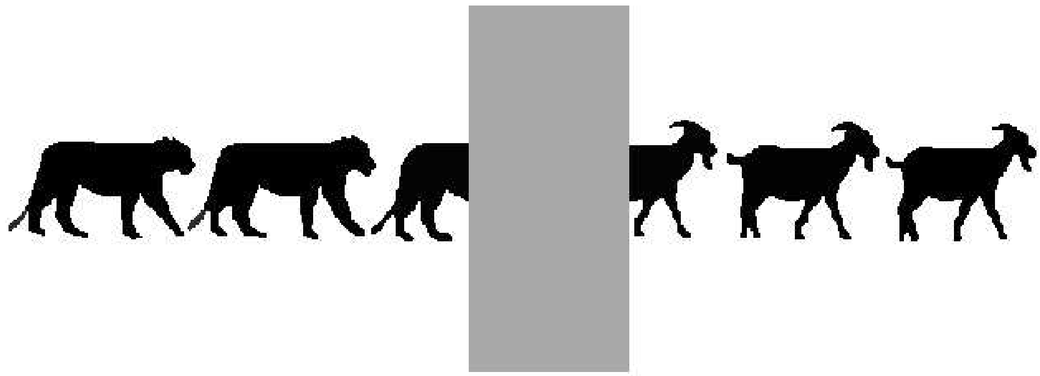

Figure 2: No recognition process seems to be involved in amodal completion. In this figure (from G. Kanisza [23]), our brain creates an hybrid animal that obviously contradicts reality.

Mumford, Nitzberg and Shiota was to adapt the theory of Kanizsa to their framework. Given the objects boundaries, it is easy to detect T-junctions. Hereafter, the main problem is the completion of objects boundaries between T-junctions. As we said, there is no simple model for amodal completion. However, it can be proved that completion curves are in general as short as possible while respecting a principle of *good continuation* with respect to the edges being completed. Thus, the model proposed by Mumford *et al* is the following: given two T-junctions p_1, p_2 and the tangents t_1, t_2 of the respective terminating edges, the continuation curve is the Euler elastica Γ, that is the curve minimizing the energy

$$\int_\Gamma (\alpha + \beta\,\kappa^2) d\mathcal{H}^1,$$

subject to the boundary conditions of beginning at (p_1, t_1) and ending at (p_2, t_2). Here, $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure, κ the curvature on Γ and α, β are positive reals. Of course, this model is far from being fully satisfactory and, in particular, does not allow to create corners. However, it sounds reasonable in a first approximation, particularly when the angle between t_1 and t_2 is small, and offers a good compromise between shortness and good continuation.

The energy above has in fact a long history. It has been initially studied by Euler [16] in 1744, who investigated the bending of a thin rod - $\int \kappa^2 d\mathcal{H}^1$ is the total bending energy - by forces and couples applied at its ends. Then, it was first applied to visual completion by Ullman [39] and Horn [21] and has more recently motivated numerous works (see [24, 37, 40, 41, 43] and the very interesting justification of the model in [30]).

In their paper, Mumford, Nitzberg and Shiota recover partially occluded objects in the following way: among all possible T-junctions pairings, the algorithm first disqualifies those for which intensities are too different, based on the subjacent reasonable assumption that intensity cannot vary too much along an edge. Then, the algorithm finds the pairings of minimal energy, from which the complete objects can be easily deduced.

Though it was not initially designed for, it is not difficult to adapt this approach to the problem of recovering missing parts of an image. If A denotes a hole that we want to fill, we can consider ∂A as an edge, compute all T-junctions on ∂A and try to find optimal pairings between them. The result would be a family of overlapping objects given by their boundaries. The grey level at each point of an object O is known only outside A but one can easily imagine a way to define it also on $O \cap A$, e.g. simply putting the average value computed over $O \setminus A$, and such strategy applied to each object would finally give an image where A has been filled.

It is easily seen that such strategy has however a major drawback: its dependence on a prior edge detection process. It is well known indeed that edges are not reliable features in the sense that they cannot be defined in a reliable way. Actually, each edge detector provides a particular definition of edges and, consequently, the image resulting from the strategy above depends as much on the image itself as on the edge detector ! In addition, edges furnish a very poor representation of the original image, actually a coarse approximation to the component u_1 that we defined previously. The image reconstructed with a strategy *à la* Mumford *et al* is therefore rather incomplete since all the information outside the missing zone A is not taken into consideration.

To remedy these drawbacks, it was proposed in [26] (see also [28]) to adapt Mumford *et al* strategy to the *level lines* framework (see figure 3). Level lines have many advantages in our setting:

- they provide a *complete* representation of any Borel function u: given the *upper level sets* $X_\lambda u = \{x,\, u(x) \geq \lambda\}$, the image can be easily reconstructed with the formula

$$u(x) = \sup\{\lambda,\, x \in X_\lambda u\} \tag{1}$$

  which holds almost everywhere;

- they are perfectly adapted to the description of image geometry. In particular, the family of level sets is globally invariant with respect to any increasing contrast change, exactly like image objects (the shape of a bird remains the shape of a bird after a contrast change). In contrast, edges are fully contrast-dependent features.

- they are well suited with the BV setting for mainly three reasons:

  – almost every level set of a function of bounded variation has finite perimeter (see the next section). In addition to all properties that it implies, the notion of finite perimeter is compatible with a weak notion of connectedness [3] which can be particularly useful for the description of image shapes;

– by the Cavalieri formula, the L^1 norm of a measurable function on $\mathbb{R}^N$ depends on the N-dimensional measure of its level sets;

– by the coarea formula, the total variation of a BV function on $\mathbb{R}^{\mathrm{N}}$ depends on the $(N-1)$-dimensional Hausdorff measure of its level lines.

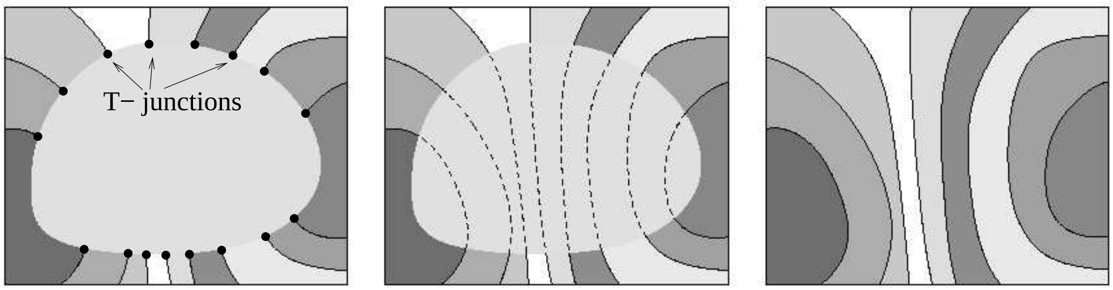

Figure 3: The algorithm in [26] first detects T-junctions as the intersection points between the occlusion's boundary and the outer level lines (left). Then it computes optimal pairings between compatible T-junctions and draws the corresponding completion curves (middle). Finally, the occlusion is removed by simply filling with the appropriate grey levels (right).

In [26], the authors call *disocclusion* their method for recovering missing parts of a grey level two-dimensional image, since missing parts can obviously be considered as occlusions "hiding" some information one wants to recover. It is assumed that u is a BV function on the plane known everywhere except on a bounded simply connected open set A with smooth boundary. T-junctions are defined as those points where ∂A intersects the level lines of u. The algorithm presented in [26] tries to find optimal pairings between compatible T-junctions, i.e. associated to the same level set, such that the orientation of Du is the same at both points and the optimal associated curve do not cross another completion curve. Both conditions ensure that the new sets obtained by the addition of the completion curves still are level sets. Given two compatible T-junctions j_1 and j_2 on ∂A and θ_1, θ_2 the corresponding orientations of Du (computed for instance as an average over some neighborhood), the optimal completion curve proposed in [26] is a curve Γ that lives in A and minimizes the criterion

$$\int_\Gamma (\alpha + \beta\,|\kappa|^p) d\mathcal{H}^1 + (\theta_1, n_1) + (\theta_2, n_2).$$

Here, α, β are positive reals, $p \geq 1$ is a real parameter introduced to generalize the elastica energy and the last two terms denote the angles between θ_1, θ_2 and the normals to Γ at j_1 and j_2 respectively. These terms guarantee that, at least in a first approximation, the good continuation principle is satisfied. The global energy to minimize is finally of the form

$$\int_{-\infty}^{+\infty} \sum_{\Gamma \in F_\lambda} \left(\int_\Gamma (\alpha + \beta\,|\kappa|^p) d\mathcal{H}^1 + (\theta_1, n_1) + (\theta_2, n_2) \right) d\lambda, \tag{2}$$

with F_λ denoting the family of completion curves associated to the level set $\{u \geq \lambda\}$. It must be emphasized that F_λ is generically finite for almost every λ, which explains the finite sum [27].

Given an initial BV function outside A, the existence of an optimal solution with respect to criterion (2) has been proven in [27] for any $p \geq 1$, with the additional assumption in the case $p > 1$ that the restriction of u to ∂A takes finitely many values. In contrast with most variational problems, it is not proven directly the existence of an optimal function interpolating the image in A but rather the existence of an optimal family of interpolating level lines from which a function can be recovered.

Recall now that the angle terms were introduced to guarantee the good continuation principle. Another way, more restrictive, to guarantee this principle is to replace the angle constraint with a higher order constraint. This can be done in a very logical way by computing the criterion $\int (\alpha + \beta\,|\kappa|^p) d\mathcal{H}^1$ not only on the completion curve but also on a small piece of the associated level lines outside A. If $\tilde{A}$ denotes a set slightly bigger than A, our criterion (2) becomes

$$\int_{-\infty}^{+\infty} \sum_{\Gamma \in F_\lambda} \int_\Gamma (\alpha + \beta\,|\kappa|^p) d\mathcal{H}^1, \tag{3}$$

where, now, the elements of F_λ are union of a completion curve and the restrictions to $\tilde{A} \setminus A$ of the associated level lines. Of course, this criterion makes sense under the assumption that the level lines of the initial BV function are essentially $W^{2,p}$ in $\tilde{A} \setminus A$. In a forthcoming paper [29], the existence of an optimal solution with respect to this new criterion is proved for any $p > 1$ without the assumption of finiteness required in [27]. Again, the minimization is performed over a family of curves rather than on a function. Roughly speaking, the existence of optimal curves is proven for a dense family of λ using martingale arguments, then a density argument and a series of diagonal extractions gives an optimal family of completion curves, from which a solution can be deduced.

Our initial motivation in this paper was precisely to study the disocclusion problem from the viewpoint of the direct method of the calculus of variations. To this aim, we first need to rewrite the criterion (3) according to a function rather than a family of curves. Assuming for a moment that the curves Γ in (3) are the level lines of a smooth function u, it is easily seen that the criterion becomes

$$\int_{-\infty}^{+\infty} \left(\int_{\partial\{u \geq \lambda\} \cap \tilde{A}} (\alpha + \beta\,|\kappa|^p) d\mathcal{H}^1 \right) d\lambda.$$

This criterion can be easily generalized to higher dimensions; if u now denotes a function on $\mathbb{R}^N$ $(N \geq 2)$, the curvature κ can be replaced with the mean curvature vector $\mathbf{H}$ of the hypersurface $\partial\{u \geq \lambda\} \cap \tilde{A}$ and the criterion becomes

$$\int_{-\infty}^{+\infty} \left(\int_{\partial\{u\geq\lambda\}\cap\tilde{A}} (\alpha + \beta\,|\mathbf{H}|^p) d\mathcal{H}^{N-1} \right) d\lambda, \tag{4}$$

with $\mathcal{H}^{N-1}$ the $(N-1)$-dimensional Hausdorff measure. Analogously to the two-dimensional case, the minimization of this criterion is equivalent to seeking optimal interpolation hypersurfaces with respect to the energy $\int (\alpha + \beta\,|\mathbf{H}|^p) d\mathcal{H}^{N-1}$. Then, it is very easy to formulate the problem according to the function u rather than its level sets by applying equality (7) below, observing that $\nabla u/|\nabla u|$ is orthogonal to the hypersurface $\partial\{u \geq \lambda\}$ at every point where $|\nabla u| > 0$, and using the change of variables formula. One finally gets the new criterion:

$$F(u) = \int_{\tilde{A}} |\nabla u|(\alpha + \beta\,|\operatorname{div} \frac{\nabla u}{|\nabla u|}|^p) dx \tag{5}$$

with the convention that the integrand is 0 wherever $|\nabla u| = 0$.

Of course, this criterion makes sense only for a certain class of smooth functions and requires to be relaxed in order to deal with more general functions. As usual in the direct method of the calculus of variations, F is first extended to the whole space $\mathrm{L}^1(\mathbb{R}^N)$ then the relaxed functional associated with F is defined as:

$$\overline{F}(u) = \inf\{\liminf_{h\to\infty} F(u_h),\ u_h \to u \in \mathrm{L}^1\}$$

As we will see in this paper, this relaxed criterion is well adapted to the study of our minimization problem.

Another approach by relaxation, taken by C. Ballester, M. Bertalmio, V. Caselles, G. Sapiro and J. Verdera [6], is based on the functional

$$G(u,\nu) = \int_{\tilde{A}} |\operatorname{div} \nu|^p (\alpha + \beta|\nabla(k \star u)|) dx + \lambda \int_{\tilde{A}} (|\nabla u| - \nabla u \cdot \nu) dx$$

where ν is a vector field such that $|\nu| \leq 1$ and k is a smoothing kernel introduced for technical reasons. The advantage of this formulation is the cancellation of the difficulty due to the term $\operatorname{div} \frac{\nabla u}{|\nabla u|}$ in F. Intuitively, the vector field ν plays the role of $\nabla u/|\nabla u|$ but may remain well defined even when $|\nabla u|$ vanishes. The existence of a minimizing couple (u, ν) is proved in [6] but it remains unclear whether this approach and ours are equivalent.

We did not mention until now any numerical implementation of the disocclusion model. A practical algorithm for the *global* minimization of criterion (2) in the case $N = 2$, $p = 1$,

Figure 4: Left : original image where occlusions are in white. Right : disocclusion performed by the algorithm proposed in [26].

based on dynamic programming for finding an optimal set of completion curves, has been proposed in [26]. Its performances are illustrated on figure 4.

In [10], T. Chan and J. Shen derive the Euler-Lagrange equation associated with criterion (5), in the case $N = 2$, $p > 1$. It is a fourth-order equation that raises many problems of unstability and computational time. In addition, the solutions are only *local*.

Finally, the algorithm proposed in [6] computes *local* solutions to the minimum problem associated with the functional G defined above. These solutions are obtained through evolutionary equations of order three, thus much handier from a numerical viewpoint than the fourth-order equation in [10]. This approach gives actually very convincing results.

The Euler-Lagrange equation proposed in [10] is obtained through a formal derivation of criterion (5) but is in fact ill-posed. The usual method in such situation consists in approximating $\overline{F}$ by a Γ-converging family of more regular functionals F_ϵ, i.e. a family satisfying:

$$\begin{array}{ll} \overline{F}(u) \leq \liminf\limits_{\epsilon \to 0} F_\epsilon(u_\epsilon) \quad \text{for every } u_\epsilon \to u & (\Gamma\text{-}\liminf \text{ property}) \\ \exists u_\epsilon \to u \text{ such that } \quad \overline{F}(u) \geq \limsup_{\epsilon \to 0} F_\epsilon(u) & (\Gamma\text{-}\limsup \text{ property}) \end{array}$$

A crucial fact on Γ-convergence is that limits of sequences of minimizers of the F_ϵ's are minimizers of F. Thus, the solutions to the well posed Euler-Lagrange equations derived from the functionals F_ϵ can be considered as good approximations of local minimizers of $\overline{F}$, which is particularly interesting from the numerical point of view. Before we introduce the appropriate regular functionals for our problem, let us recall that, in a different context, it has been proven in [17] that an approximation of the solution to

$$u_t = |\nabla u| \operatorname{div}(\frac{\nabla u}{|\nabla u|}) \text{ in } (0,\infty)\times \mathrm{I\!R}^{\mathrm{N}}, \quad u(0,\cdot) = u_0$$

is given by the solutions u^ϵ of

$$u_t = \sqrt{\epsilon^2 + |\nabla u|^2}\,\operatorname{div}\left(\frac{\nabla u}{\sqrt{\epsilon^2 + |\nabla u|^2}}\right) \tag{6}$$

if the initial function u_0 is $\mathrm{C}^{1,1}$ and constant at infinity (see [17] for details). To understand better this result, it suffices to remark that if $v : (0,\infty)\times \mathrm{I\!R}^{\mathrm{N+1}} \to \mathrm{I\!R}$ is defined by $v(t,x,z) = u(t,x) + \epsilon z$ then (6) rewrites

$$v_t = |\nabla v| \operatorname{div}(\frac{\nabla v}{|\nabla v|}),$$

which coincides with the initial equation.

In the same spirit, let us consider the family of functionals

$$F_\epsilon(u) = \int_{\tilde{A}} \sqrt{|\nabla u|^2 + \epsilon^2}(\alpha + \beta\,|\operatorname{div}(\frac{\nabla u}{\sqrt{|\nabla u|^2+\epsilon^2}})|^p)dx$$

which take finite values for any $u \in \mathrm{C}^2(\mathrm{I\!R}^{\mathrm{N}})$. Considering $v_\epsilon : \mathrm{I\!R}^{\mathrm{N+1}} \to \mathrm{I\!R}$ defined by $v_\epsilon(x,z) = u(x) + \epsilon z$, it is easily seen that

$$\begin{aligned} F_\epsilon(u) &= \int_{\tilde{A}} |\nabla v_\epsilon|(\alpha + \beta\,|\operatorname{div}(\frac{\nabla v_\epsilon}{|\nabla v_\epsilon|})|^p)dx \\ &= \int_{\tilde{A}}\int_{-\frac{1}{2}}^{\frac{1}{2}} |\nabla v_\epsilon|(\alpha + \beta\,|\operatorname{div}(\frac{\nabla v_\epsilon}{|\nabla v_\epsilon|})|^p)dxdz \end{aligned}$$

which is exactly the $(N+1)$-dimensional version of F. This observation combined with Theorem 6 in Section 4 shows that the Γ- lim inf property is satisfied by $\overline{F}$ and the family $(F_\epsilon)_{\epsilon>0}$. We were unfortunately unable to prove more than that and can only state the following

Conjecture *For $\epsilon > 0$, let $F_\epsilon : \mathrm{L}^1(\mathrm{I\!R}^{\mathrm{N}}) \to [0,\infty]$ defined by*

$$F_\epsilon(u) = \begin{cases} \displaystyle\int_{\tilde{A}} \sqrt{|\nabla u|^2+\epsilon^2}\left(\alpha + \beta\left|\operatorname{div}\frac{\nabla u}{\sqrt{|\nabla u|^2+\epsilon^2}}\right|^p\right)dx & \text{if } u \in \mathrm{C}^2_{\mathrm{c}}(\mathrm{I\!R}^{\mathrm{N}}), \\ +\infty & \text{otherwise} \end{cases}$$

Then, for $N < p < \infty$ and for every $(\epsilon_h)_{h\in\mathbb{N}} \to 0$,

$$\overline{F} = \Gamma\text{-}\lim_{h\to\infty} F_{\epsilon_h}$$

This paper is organized as follows. Section 2 introduces some notations and basic facts about tools from geometric measure theory that we shall need. In Section 3, we prove a locality result for the mean curvature vector $\mathbf{H}$ of integral $(N-1)$-varifolds when $\mathbf{H} \in \mathrm{L}^p$, $p > N-1$, $p \geq 2$ (Theorem 2). A direct consequence of this result is the lower semicontinuity (Theorem 4) of the functional

$$\int_{\partial E} (1 + |\mathbf{H_E}|^p) d\mathcal{H}^{\mathrm{N}-1}, \quad p > N-1,\, N \geq 3$$

with respect to convergence in L^1 in the class of sets $E \subset \mathbb{R}^{\mathrm{N}}$ with $\partial E \in \mathrm{C}^2$. This result extends to higher dimensions a previous result due to G. Bellettini, G. Dal Maso and M. Paolini [7] for $N = 2$, $p > 1$.

Section 4 is devoted to the study of the disocclusion problem in dimension N. We prove the existence of an optimal solution (Theorem 5), the coincidence between F and the associated relaxed functional $\overline{F}$ for smooth functions (Theorem 6) and give some results on the regularity of the optimal solution in the particular case $N = 2$ (Corollary 1). For the sake of simplicity and with absolutely no loss of generality, we shall assume in what follows that $\alpha = \beta = 1$.

2 Notations and main facts about varifolds

We collect below, for the reader's convenience, the main facts about varifolds (see for instance [4, 19, 38]).

We let $\mathcal{L}(\mathbb{R}^{\mathrm{n+k}})$ denote the space of linear maps from $\mathbb{R}^{\mathrm{n+k}}$ onto itself, equipped with the usual scalar product $A \bullet B = \mathrm{trace}(A^* B)$. $G(n+k, n)$ denotes the space of n-dimensional unoriented subspaces of $\mathbb{R}^{\mathrm{n+k}}$ and we shall often identify in the sequel a n-subspace $S \in G(n+k, n)$ with the associated orthogonal projection $p_S \in \mathcal{L}(\mathbb{R}^{\mathrm{n+k}})$ given by the matrix $p_S^{ij} = e_i \cdot p_S(e_j)$ with respect to the standard orthonormal basis $e_1, \ldots, e_{n+k}$ for $\mathbb{R}^{\mathrm{n+k}}$. $G(n+k, n)$ is equipped with the metric

$$\|p_S - p_T\| := \left(\sum_{i,j=1}^{n+k} (p_S^{ij} - p_T^{ij})^2 \right)^{\frac{1}{2}}$$

induced by the scalar product $\bullet$ on $\mathcal{L}(\mathbb{R}^{\mathrm{n+k}})$. The tensor product $v \otimes w$ of two vectors $v, w \in \mathbb{R}^{\mathrm{n+k}}$ is in $\mathcal{L}(\mathbb{R}^{\mathrm{n+k}})$ and satisfies for any $S \in G(n+k, n)$

$$v \otimes w \bullet S = S(v) \bullet w = v \bullet S(w) = S(v) \bullet S(w).$$

For a subset $A \subset \mathrm{I\!R}^{\mathrm{n+k}}$ we define the Grassmannian

$$G_n(A) = A \times G(n+k, n)$$

equipped with the product metric. By an n-varifold on an open subset U of $\mathrm{I\!R}^{\mathrm{n+k}}$ we mean any Radon measure V on $G_n(U)$. It is associated with a Radon measure μ_V on U (called the *weight* of V) defined by

$$\mu_V(A) = V(\pi^{-1}(A)), \qquad A \subset U \text{ Borel},$$

where π is the projection $(x, S) \mapsto x$ of $G_n(U)$ onto U.

Given M, a countably $\mathcal{H}^{\mathrm{n}}$-rectifiable subset of $\mathrm{I\!R}^{\mathrm{n+k}}$, and θ, a positive and locally $\mathcal{H}^{\mathrm{n}}$-integrable function on M, we define the associated n-rectifiable varifold $V \equiv \underline{\underline{\mathbf{v}}}(M, \theta)$ by

$$V(A) = \mu_V(\pi(TM \cap A)), \qquad A \subset G_n(U) \text{ Borel},$$

where $\mu_V := \mathcal{H}^{\mathrm{n}} \llcorner \theta$ is the weight of V, $TM = \{(x, T_x M) : x \in M^*\}$ and M^* stands for the set of all $x \in M$ such that M has an approximate tangent space $T_x M$ with respect to θ at x, i.e.

$$\lim_{\lambda \downarrow 0} \lambda^{-n} \int_M f(\lambda^{-1}(z-x))\theta(z) d\mathcal{H}^{\mathrm{n}}(z) = \theta(x) \int_{T_x M} f(y) d\mathcal{H}^{\mathrm{n}}(y), \qquad \forall f \in \mathrm{C}^0_{\mathrm{c}}(\mathrm{I\!R}^{\mathrm{n+k}}).$$

We say that $V = \underline{\underline{\mathbf{v}}}(M, \theta)$ is an *integral* varifold if the function θ is integer valued. Remark that $\mathcal{H}^{\mathrm{n}}(M \setminus M^*) = 0$ and that the approximate tangent spaces of M with respect to two different positive $\mathcal{H}^{\mathrm{n}}$-integrable functions θ, $\tilde{\theta}$ coincide $\mathcal{H}^{\mathrm{n}}$-a.e. in M.

The *first variation* of the n-varifold V, denoted by δV, is the linear functional on $\mathrm{C}^1_{\mathrm{c}}(U, \mathrm{I\!R}^{\mathrm{n+k}})$ defined by

$$\delta V(X) := \int_{G_n(U)} \operatorname{div}_S X dV(x, S),$$

where, for any $S \in G(n+k, n)$,

$$\operatorname{div}_S X := \sum_{i=1}^{n+k} \nabla_i^S X^i = \sum_{i=1}^{n} < \tau_i, D_{\tau_i} X >,$$

where $\tau_1, \ldots, \tau_n$ is an orthonormal basis for S and $\nabla_i^S = e_i . \nabla^S$ with $\nabla^S f(x) = S(\nabla f(x))$, $f \in \mathrm{C}^1(U)$.

A varifold V is said to have locally bounded first variation in U if for each $W \subset\subset U$ there is a constant $c < \infty$ such that $|\delta V(x)| \le c \sup_U |X|$ for any $X \in \mathrm{C}^1_{\mathrm{c}}(U, \mathrm{I\!R}^{\mathrm{n+k}})$ with $\operatorname{spt} |X| \subset W$. By the Riesz representation theorem, it follows that there exist a Radon measure $\|\delta V\|$ on U - the total variation measure of δV - and a $\|\delta V\|$-measurable function ν with $|\nu| = 1$ $\|\delta V\|$-a.e. in U satisfying

$$\delta V(X) = -\int_U \nu \cdot X d\|\delta V\| \qquad \forall X \in \mathrm{C}^1_\mathrm{c}(U, \mathrm{I\!R}^{\mathrm{n+k}}).$$

A varifold V is said to have mean curvature in L^p if $\|\delta V\|$ is absolutely continuous with respect to μ_V and its density belongs to L^p. The density will be denoted by $\mathbf{H_V}$ and it will be called the *generalized mean curvature* of V.

In the case when M is a smooth n-dimensional submanifold of $\mathrm{I\!R}^{\mathrm{n+k}}$, with $\overline{M}\backslash M \cap U = \emptyset$, the divergence theorem on manifolds implies that the generalized mean curvature of the varifold $\underline{\underline{\mathbf{v}}}(M, 1)$ is exactly the classical mean curvature of M. When $k = 1$ (i.e. codimension 1 manifolds) the mean curvature vector can be locally computed by

$$\mathbf{H} = -\mathrm{div}(\nu)\nu \tag{7}$$

where ν is a unit vector field orthogonal to M.

3 Locality of the mean curvature and applications

The main result of this section is stated in Theorem 2, whose proof is based on the quadratic decay of the tilt-excess established by R. Schätzle in [36]. We recall his result below.

Theorem 1 (Quadratic tilt-excess decay [36, Thm. 5.1]) *Let $U = \underline{\underline{\mathbf{v}}}(M, \theta)$ be an integral $N-1$-varifold in an open set $\Omega \subseteq \mathrm{I\!R}^{\mathrm{N}}$ with $\mathbf{H_U} \in \mathrm{L}^p_{\mathrm{loc}}(\mu_U)$, $p > N-1$, $p \geq 2$. Then for μ_U-almost all $x \in \mathrm{spt}\,\mu_U$, the tilt-excess*

$$\mathrm{tiltex}_\mu(x, \rho) := \rho^{1-N} \int_{B^N_\rho(x)} \|T_y M - T_x M\|^2 \, d\mu_U(y)$$

decays quadratically, that is

$$\mathrm{tiltex}_\mu(x, \rho) = O_x(\rho^2).$$

Theorem 2 (Locality of the mean curvature) *Let $U = \underline{\underline{\mathbf{v}}}(M, \theta_U)$, $V = \underline{\underline{\mathbf{v}}}(N, \theta_V)$ be integral $(N-1)$-varifolds in $\Omega \subseteq \mathrm{I\!R}^{\mathrm{N}}$. If $\mathbf{H_U} \in \mathrm{L}^p_{\mathrm{loc}}(\mu_U)$ and $\mathbf{H_V} \in \mathrm{L}^p_{\mathrm{loc}}(\mu_V)$ for some $p > N-1$, $p \geq 2$ then*

$$\mathbf{H_U}(x) = \mathbf{H_V}(x)$$

for $\mathcal{H}^{\mathrm{N}-1}$-almost all $x \in M \cap N$.

Proof Given nonzero integers θ_0, θ_1, we call $x \in M \cap N$ a *generic point* of order (θ_0, θ_1) if

(i) $\theta^{N-1}(M \cap N, x) = 1$ and $\theta_0 = \theta^{N-1}(\mu_U, x)$, $\theta_1 = \theta^{N-1}(\mu_V, x)$;

(ii) x is a Lebesgue point of $\mathbf{H_U}$ and $\mathbf{H_V}$;

(iii) μ_U, $\mathbf{H_U}\mu_U$, μ_V and $\mathbf{H_V}\mu_V$ have the same approximate tangent plane at x (with multiplicities θ_0, $\theta_0\mathbf{H_U}(x)$, θ_1 and $\theta_1\mathbf{H_V}(x)$ respectively) which in turn coincide with the approximate tangent plane $T = T_xM = T_xN$.

(iv) $\mathbf{H_U}(x)$ and $\mathbf{H_V}(x)$ are orthogonal to T.

The theory of rectifiable sets and of rectifiable measures (see for instance [4, 38]) ensures that $\mathcal{H}^{\mathrm{N}-1}$-almost all points in $x \in M \cap N$ have properties (i), (ii), (iii). The proof that also condition (iv) holds $\mathcal{H}^{\mathrm{N}-1}$-a.e. is much harder, see [8, Thm 5.8]. Therefore $\mathcal{H}^{\mathrm{N}-1}$-almost every point of $M \cap N$ is generic of order (θ_0, θ_1) for some θ_0, θ_1. We fix θ_0, θ_1 and a generic point x of the corresponding order.

Following the proof of Lemma 6.3 in [35], we choose $\chi \in \mathrm{C}_0^\infty(B_1^N(0))$ rotationally symmetric with $0 \le \chi \le 1$ and $\chi \equiv 1$ on $B_{\frac{1}{2}}^N(0)$. Setting $\chi_r(y) := \chi(r^{-1}(y - x))$, using (iii) we have

$$\lim_{r\to 0^+} r^{1-N}\delta U(\chi_r) = -\lim_{r\to 0^+} r^{1-N}\int_{B_r^N(x)} \chi_r \mathbf{H_U} d\mu_U = \theta_0\mathbf{H_U}(x)\int_{T_xM\cap B_1^N(0)} \chi\, d\mathcal{H}^{N-1},$$

and

$$\lim_{r\to 0^+} r^{1-N}\delta V(\chi_r) = -\lim_{r\to 0^+} r^{1-N}\int_{B_r^N(x)} \chi_r \mathbf{H_V} d\mu_V = \theta_1\mathbf{H_V}(x)\int_{T_xM'\cap B_1^N(0)} \chi\, d\mathcal{H}^{N-1}.$$

Now we choose $\nu(x)$ normal to $T_xM = T_xN$ and we deduce by (iv) that $\mathbf{H_U}(x)$, $\mathbf{H_V}(x) \in \mathrm{span}\{\nu(x)\}$. Hence, in order to show that they actually coincide, it suffices to prove that

$$\lim_{r\to 0^+} r^{1-N}[\theta_1\delta U(\chi_r) - \theta_0\delta V(\chi_r)]\nu(x) = 0. \tag{8}$$

Let now denote by A the collection of all generic points y of order (θ_0, θ_1). We assume, in addition, that

(v) $\lim_{r\to 0^+} r^{1-N}\mathcal{H}^{\mathrm{N}-1}(B_r^N(x) \setminus A) = 0$.

It is a consequence of Theorem 2.9.11 in [19] that $\mathcal{H}^{\mathrm{N}-1}$-almost every generic point of order (θ_0, θ_1) has this property. Then, we notice for $W = U$ or $W = V$ that

$$\int_{B_r^N(x)} \chi_r \mathbf{H_W} d\mu_W = -\int_{B_r^N(x)} T_yW(D\chi_r)d\mu_W(y),$$

thus

$$\left[\int_{B_r^N(x)} \chi_r \theta_1 \mathbf{H_U} d\mu_U - \int_{B_r^N(x)} \chi_r \theta_0 \mathbf{H_V} \mu_V\right] =$$
$$= -\left[\int_{B_r^N(x)} T_y M (D\chi_r) \theta_1 d\mu_U(y) - \int_{B_r^N(x)} T_y N (D\chi_r) \theta_0 d\mu_V(y)\right].$$

Then, by looking to the $\nu(x)$ component, we get

$$\left[\int_{B_r^N(x)} \chi_r \theta_1 \mathbf{H_U} d\mu_U - \int_{B_r^N(x)} \chi_r \theta_0 \mathbf{H_V} \mu_V\right] \cdot \nu(x) =$$
$$= -\left[\int_{B_r^N(x)} (D\chi_r(y) \otimes \nu(x)) T_y M \, \theta_1 d\mu_U(y) - \int_{B_r^N(x)} (D\chi_r(y) \otimes \nu(x)) T_y N \, \theta_0 \mu_V(y)(y)\right].$$

Now we use the fact that $\nu(x)$ is normal to T, that $T_y M = T_y N$, $\theta_U(y) = \theta_0$ and $\theta_V(y) = \theta_1$ on A to obtain

$$\left[\int_{B_r^N(x)} \chi_r \theta_1 \mathbf{H_U} d\mu_U - \int_{B_r^N(x)} \chi_r \theta_0 \mathbf{H_V} \mu_V\right] \cdot \nu(x) =$$
$$= -\left[\int_{B_r^N(x)} (D\chi_r(y) \otimes \nu(x))(T_y M - T) \theta_1 d\mu_U(y) \right.$$
$$\left. - \int_{B_r^N(x)} (D\chi_r(y) \otimes \nu(x))(T_y N - T) \theta_0 d\mu_V(y)\right]$$
$$= -\left[\theta_1 \int_{B_r^N(x) \setminus A} (D\chi_r(y) \otimes \nu(x))(T_y M - T) d\mu_U(y) \right.$$
$$\left. - \theta_0 \int_{B_r^N(x) \setminus A} (D\chi_r(y) \otimes \nu(x))(T_y N - T) d\mu_V(y)\right].$$

Denoting

$$R_{r,W} = r^{1-N} \int_{B_r^N(x) \setminus A} (D\chi_r(y) \otimes \nu(x))(T_y W - T) d\mu_W(y)$$

for $W = U, V$ we obtain

$$\lim_{r \to 0^+} r^{1-N} [\theta_1 \delta U(\chi_r) - \theta_0 \delta V(\chi_r)] \nu(x) = \lim_{r \to 0^+} [\theta_1 R_{r,U} - \theta_0 R_{r,V}].$$

Then we estimate, for $W = U, V$,

$$|R_{r,W}| \le C_\chi r^{-N} \int_{B_r^N(x)\setminus A} ||T_y W - T|| d\mu_W(y) \le$$
$$\le C_\chi \Big(r^{1-N}\mu_W(B_r^N(x)\setminus A)\Big)^{\frac{1}{2}} \Big(r^{-1-N}\int_{B_r^N(x)} ||T_y W - T||^2 d\mu_W(y)\Big)^{\frac{1}{2}}, \tag{9}$$

where C_χ is such that $\sup_{B_1^N(0)} |D\chi| \le C_\chi$. By conditions (iv), (v) and by Theorem 1 the first factor is infinitesimal as $r \to 0^+$ and the second factor is bounded, hence (8) holds and the proof is achieved. □

Theorem 2 is a key point in the proof of the lower semicontinuity of the mean curvature's L^p norm, stated in Theorem 4 below, which is a generalization to the higher dimensional case of a previous result obtained in dimension two by G. Bellettini, G. Dal Maso and M. Paolini in [7]. We recall their result below.

Theorem 3 ([7, Thm 7.1]) *Let Ω be an open subset of $\mathrm{I\!R}^2$, let $p > 1$ and let E be an open bounded subset of $\mathrm{I\!R}^2$ such that $\partial E \cap \Omega \in \mathrm{C}^2$. Then*

$$\int_{\partial E\cap\Omega} (1 + |\kappa|^p) d\mathcal{H}^1 \le \liminf_{h\to\infty} \int_{\partial E_h \cap \Omega} (1 + |\kappa_h|^p) d\mathcal{H}^1$$

for any sequence $(E_h)_{h\in\mathrm{I\!N}}$ of bounded open sets such that $\partial E_h \cap \Omega \in \mathrm{C}^2$ and $E_h \to E$ in $\mathrm{L}^1(\Omega)$ as $h \to \infty$.

Theorem 4 (Lower semicontinuity of the mean curvature's L^p norm)
Let Ω be an open subset of $\mathrm{I\!R}^\mathrm{N}$, $N \ge 3$, and $p > N-1$. Let $\{E_h\}_{h\in\mathrm{I\!N}} \subset \mathrm{I\!R}^\mathrm{N}$ be converging in $L^1(\Omega)$ to E, with $\partial E_h \cap \Omega \in \mathrm{C}^2$ and $\partial E \cap \Omega \in \mathrm{C}^2$. Then

$$\int_{\partial E\cap\Omega} (1 + |\mathbf{H_E}|^p) d\mathcal{H}^{\mathrm{N}-1} \le \liminf_{h\to\infty} \int_{\partial E_h\cap\Omega} (1 + |\mathbf{H_{E_h}}|^p) d\mathcal{H}^{\mathrm{N}-1},$$

where $\mathbf{H_E}$ (resp. $\mathbf{H_{E_h}}$) denotes the mean curvature vector on $\partial E \cap \Omega$ (resp. $\partial E_h \cap \Omega$).

Proof Due to the lower semicontinuity of the perimeter, it is clearly enough to prove the part of the claim that involves curvature. We can assume that the right hand side of the inequality to prove is finite, otherwise the result is trivial. In addition, possibly taking a subsequence, there is no loss of generality if we assume that

$$\sup_{h\in\mathrm{I\!N}} \int_{\partial E_h\cap\Omega} (1 + |\mathbf{H_{E_h}}|^p) d\mathcal{H}^{\mathrm{N}-1} \le C < \infty.$$

Let $V_h = \underline{\underline{\mathbf{v}}}(\partial E_h \cap \Omega, 1)$ be the unit-density rectifiable $(N-1)$-varifolds associated with the sets $E_h \cap \Omega$ and let $\mu_{V_h} = \mathcal{H}^{\mathrm{N}-1} \llcorner \partial E_h \cap \Omega$ be the corresponding weights. By the divergence theorem, the first variation of the V_h's in Ω can be written as

$$\delta V_h(X) = -\int_\Omega X \cdot \mathbf{H}_{\mathbf{E_h}} d\mu_{V_h}, \quad \forall X \in \mathrm{C}^1_{\mathrm{c}}(\Omega, \mathrm{I\!R}^{\mathrm{N}})$$

hence the L^p norms of δV_h with respect to μ_{V_h} are uniformly bounded.

By Allard's compactness theorem (see [1] or Theorem 42.7 in [38]) we obtain, possibly passing to a subsequence, that there exists a limit integral $(N-1)$-varifold V in Ω such that $V_h \rightharpoonup V$ and $V = \underline{\underline{\mathbf{v}}}(M, \theta_V)$ with M a countably $\mathcal{H}^{\mathrm{N}-1}$-rectifiable set and θ_V a positive integer-valued and locally $\mathcal{H}^{\mathrm{N}-1}$-integrable function on Ω. As $\delta V_h = \mathbf{H}_{\mathbf{E_h}} d\mu_{V_h} \rightharpoonup \delta V$, a well-known lower semicontinuity theorem (see for instance Example 2.36 in [4]) yields that $\delta V = \mathbf{H}_{\mathbf{V}} \mu_V$ with $\mathbf{H}_{\mathbf{V}} \in L^p(\mu_V)$ and

$$\int_M \theta_V |\mathbf{H}_{\mathbf{V}}|^p d\mathcal{H}^{\mathrm{N}-1} \leq \liminf_{h\to\infty} \int_{\partial E_h \cap \Omega} |\mathbf{H}_{\mathbf{E_h}}|^p d\mathcal{H}^{\mathrm{N}-1}. \tag{10}$$

Notice that so far we used only the fact that $p > 1$.

Now we show that $\mathcal{H}^{\mathrm{N}-1}$-almost all points in $\partial E \cap \Omega$ belong to M and $\mathbf{H}_{\mathbf{V}}$ coincides with the classical mean curvature $\mathbf{H}_{\mathbf{E}}$ for $\mathcal{H}^{\mathrm{N}-1}$-almost every point of $\partial E \cap \Omega$ whenever $p > N-1$.

Let $x \in \partial E \cap \Omega$. Since E has finite perimeter, for all $r > 0$ except possibly for a countable set, $\mu_V(\partial B^N_r(x)) = 0$, hence $\mu_V(B^N_r(x)) = \lim_{h\to\infty} \mu_{V_h}(B^N_r(x)) = \lim_{h\to\infty} \mathcal{H}^{\mathrm{N}-1}(B^N_r(x) \cap \partial E_h \cap \Omega) \geq \mathcal{H}^{\mathrm{N}-1}(B^N_r(x) \cap \partial E \cap \Omega)$, using the lower semicontinuity of the perimeter. It follows that x is a point where the lower $(N-1)$-dimensional density of μ_V is strictly positive. As $\mu_V = \theta_V \mathcal{H}^{\mathrm{N}-1} \llcorner M$, $\mathcal{H}^{\mathrm{N}-1}$-almost any point with this property belongs to M.

Let $U := \underline{\underline{\mathbf{v}}}(\partial E \cap \Omega, 1)$ be the unit-density rectifiable $(N-1)$-varifold in Ω associated with $\partial E \cap \Omega$. By the divergence theorem we have

$$\mathbf{H}_{\mathbf{U}}(x) = \mathbf{H}_{\mathbf{E}}(x)$$

for $\mathcal{H}^{\mathrm{N}-1}$-almost all $x \in \partial E \cap \Omega$. Since $\partial E \cap \Omega \in \mathrm{C}^2$, it is easily seen that $\mathbf{H}_{\mathbf{U}} \in \mathrm{L}^p_{\mathrm{loc}}(\mu_U)$. By Theorem 2 we obtain

$$\mathbf{H}_{\mathbf{E}}(x) = \mathbf{H}_{\mathbf{U}}(x) = \mathbf{H}_{\mathbf{V}}(x)$$

for $\mathcal{H}^{\mathrm{N}-1}$-almost all $x \in M \cap \partial E \cap \Omega$ and therefore for $\mathcal{H}^{\mathrm{N}-1}$-almost all $x \in \partial E \cap \Omega$.

Plugging this into (10) and using $\theta_V(x) \geq 1$ for $\mathcal{H}^{\mathrm{N}-1}$-almost every $x \in M$, we finally obtain

$$\int_{\partial E \cap \Omega} |\mathbf{H}_{\mathbf{E}}|^p d\mathcal{H}^{\mathrm{N}-1} \leq \int_{\partial E \cap \Omega} |\mathbf{H}_{\mathbf{V}}|^p \theta_V d\mathcal{H}^{\mathrm{N}-1} \leq \liminf_{h\to\infty} \int_{\partial E_h} |\mathbf{H}_{\mathbf{E_h}}|^p d\mathcal{H}^{\mathrm{N}-1}$$

and the theorem ensues. □

Remark 1 The varifold arguments we use require the technical assumption $p \geq 2$, which prevents the result by Bellettini *et al* from being a particular case of Theorem 4 whenever $N = 2$ and $1 < p < 2$.

Remark 2 Whenever $\mathcal{H}^{N-1}(\partial E \cap \Omega) = \lim_{h\to\infty} \mathcal{H}^{N-1}(\partial E_h \cap \Omega)$, the lower semicontinuity is true for any $p \geq 1$. This is an easy consequence of (10) and of Reshetnyak continuity Theorem (see for instance [4, Thm. 2.39]), which implies that $V \equiv \underline{\underline{\mathbf{v}}}(\partial E \cap \Omega, 1)$.

4 Analysis of the disocclusion problem

Let Ω be a bounded open set in $\mathbb{R}^N$ ($N \geq 2$) with Lipschitz boundary, representing the image domain, let $A \subset\subset \Omega$ be an open, connected set with Lipschitz boundary representing the occlusion and let $u_0 \in \mathrm{BV}(\Omega \setminus \bar{A})$ be the original image. By Theorem 3.87 in [4], for any $\lambda \in \mathbb{R}$, the function given by $u|_{\Omega\setminus\bar{A}} = u_0$ and $u|_A = \lambda$ belongs to $\mathrm{BV}(\Omega)$. This ensures that the occlusion can always be filled in.

Let $\mathcal{O}(\Omega)$ denote the family of open subsets of Ω. We consider the functional mapping $\mathrm{L}^1(\Omega) \times \mathcal{O}(\Omega)$ onto $[0, \infty]$ and defined for every $(u, B) \in \mathrm{L}^1(\Omega) \times \mathcal{O}(\Omega)$ by

$$F_p(u, B) := \begin{cases} \displaystyle\int_B |\nabla u| \Big(1 + \Big|\operatorname{div} \frac{\nabla u}{|\nabla u|}\Big|^p\Big)\, dx & \text{if } u \in C^2(B) \\ +\infty & \text{otherwise} \end{cases},$$

with the convention that the integrand is 0 wherever $|\nabla u| = 0$.

The relaxed functional associated with F is defined for every $(u, B) \in \mathrm{L}^1(\Omega) \times \mathcal{O}(\Omega)$ by

$$\overline{F_p}(u, B) := \inf\{\liminf_{h\to\infty} F_p(u_h, B) : \ u_h \xrightarrow{\mathrm{L}^1(B)} u\}.$$

Since $F_p(u, B) \geq \int_B |\nabla u|\, dx$ whenever $u \in C^2(B)$ the lower semicontinuity of the total variation yields

$$\overline{F_p}(u, B) \geq |Du|(B) \qquad \forall (u, B) \in \mathrm{L}^1(\Omega) \times \mathcal{O}(\Omega). \tag{11}$$

In the following we assume that there exist an open set $\tilde{A} \subset \Omega$ such that $\tilde{A} \supset\supset A$ and a function $u \in \mathrm{L}^1(\Omega)$ such that $u = u_0$ on $\Omega \setminus \bar{A}$ and $\overline{F_p}(u, \tilde{A}) < \infty$. This could be considered as a mild regularity and compatibility condition between the image and the occlusion.

Theorem 5 *The problem*

$$\mathrm{Min}\, \{\overline{F_p}(u, \tilde{A}) : \ u = u_0 \ \text{on } \Omega \setminus \bar{A}\} \tag{12}$$

has at least one solution $u \in \mathrm{BV}(\Omega)$.

Proof Let $(v_h)_{h\in\mathbb{N}} \subset \mathrm{L}^1(\Omega)$ be a minimizing sequence. Without loss of generality we may assume that $\sup_{h\in\mathbb{N}} \overline{F_p}(v_h, \tilde{A}) < \infty$. Then (11) yields $\sup_{h\in\mathbb{N}} |Dv_h|(\tilde{A}) < \infty$, and therefore $\sup_{h\in\mathbb{N}} |Dv_h|(\Omega) < \infty$ because $v_h = u_0$ on $\Omega \setminus \bar{A}$. Since the values of v_h are fixed on $\Omega \setminus \bar{A}$, the generalized Poincaré inequality given in Theorem 5.11.1 of [45] gives that the $L^1(\Omega)$ norms of v_h are uniformly bounded. Hence $\sup_{h\in\mathbb{N}} \|v_h\|_{\mathrm{BV}(\Omega)} < \infty$ and there exists a subsequence, still denoted by $(v_h)_{h\in\mathbb{N}}$, converging in $\mathrm{L}^1(\Omega)$ to a function $u \in \mathrm{BV}(\Omega)$. Obviously, $u = u_0$ on $\Omega \setminus \bar{A}$. From the lower semicontinuity of $\overline{F_p}$ we finally obtain that

$$\overline{F_p}(u, \tilde{A}) \leq \liminf_{h\to\infty} \overline{F_p}(v_h, \tilde{A}) = \inf\{\overline{F_p}(v, \tilde{A}) : \ v = u_0 \text{ on } \Omega \setminus \bar{A}\},$$

and the theorem ensues. □

Now we can show that the relaxed functional coincides with F_p on C^2 functions. The proof is based on the geometric lower semicontinuity results of the previous section and on the identity

$$F_p(u, B) = \int_{\mathbb{R}} \int_{\partial\{u\geq t\}} (1 + |\mathbf{H}_{\{\mathbf{u}\geq\mathbf{t}\}}|^p)\, d\mathcal{H}^{\mathrm{N}-1}\, dt \qquad \forall u \in C^2(B). \tag{13}$$

The identity is a straightforward consequence of the coarea formula and of (7) with $\nu = \nabla u/|\nabla u|$.

Theorem 6 *Let $B \subset \Omega$ be an open set and assume that $N \geq 2$ and $p > N - 1$. The functional $F_p(\cdot, B)$ is lower semicontinuous on $L^1(\Omega)\cap C^2(B)$ with respect to the L^1 topology. In particular*

$$F_p(u, B) = \overline{F_p}(u, B) \qquad \forall u \in C^2(B).$$

Proof Let $(u_h)_{h\in\mathbb{N}} \subset L^1(\Omega) \cap C^2(B)$ be converging in $L^1(B)$ to $u \in C^2(B)$ and set $L := \liminf_{h\to\infty} F_p(u_h, \tilde{A})$, assuming with no loss of generality that $L < \infty$, that the lim inf is a limit and that u_h converge a.e. to u. By the dominated convergence theorem, $\chi_{\{u_h\geq t\}} \to \chi_{\{u\geq t\}}$ in $L^1(B)$ whenever $\{u = t\}$ is Lebesgue negligible, hence for a.e. $t \in \mathbb{R}$. In addition, by Morse Theorem, for almost every $t \in \mathbb{R}$, $\{u_h \geq t\}$, $h \in \mathbb{N}$, and $\{u \geq t\}$ have smooth boundaries. Therefore, by applying either Theorem 3 or Theorem 4 we obtain that

$$\int_{\partial\{u\geq t\}\cap B} (1 + |\mathbf{H}_{\{\mathbf{u}\geq\mathbf{t}\}}|^p) d\mathcal{H}^{\mathrm{N}-1} \leq \liminf_{h\to\infty} \int_{\partial\{u_h\geq t\}\cap B} (1 + |\mathbf{H}_{\{\mathbf{u_h}\geq\mathbf{t}\}}|^p) d\mathcal{H}^{\mathrm{N}-1}$$

for a.e. $t \in \mathbb{R}$. Integrating over $\mathbb{R}$ and using Fatou's lemma, (13) yields

$$F_p(u, B) \leq \liminf_{h\to\infty} F_p(u_h, B).$$

□

In the two-dimensional case we can say something more about the structure of the solutions. The following theorem is easily deduced from the proofs of Theorems 4.1 and 7.1 in [7]. In particular, it suffices to replace ∂E by $\partial^* E$ in the last part of the proof of Theorem 7.1, page 292.

Theorem 7 ([7]) *Let $\Omega \subset \mathbb{R}^2$ be an open set and $p > 1$. Let E be a Borel set such that there exists a sequence $\{E_h\}_{h\in\mathbb{N}}$ of bounded open sets of class $\mathrm{C}^2(\Omega)$ converging to E in $\mathrm{L}^1(\Omega)$ and satisfying*

$$\sup_{h\in\mathbb{N}} \int_{\partial E_h \cap \Omega} (1 + |\kappa|^p)\, d\mathcal{H}^1 < \infty.$$

Then E has finite perimeter in Ω and there exists a locally finite family $\Gamma = \{\gamma_i\}_{i\in I}$ of regular curves of class $\mathrm{W}^{2,p}$ such that

1. *$\partial^* E \cap \Omega \subset \bigcup_{i\in I} (\gamma_i)$;*
2. *Γ is without crossings, i.e. $\frac{d\gamma_i(t_1)}{dt} // \frac{d\gamma_j(t_2)}{dt}$ whenever $\gamma_i(t_1) = \gamma_j(t_2) \in \Omega$ and $t_1, t_2 \in [0,1]$.*

Corollary 1 *Let $B \subset \Omega \subset \mathbb{R}^2$ be an open set and $u \in \mathrm{L}^1(\Omega)$ such that $\overline{F_p}(u,B) < \infty$, with $p > 1$. Then, for almost every $t \in \mathbb{R}$, there exists a locally finite family $\Gamma^t = \{\gamma_i^t\}_{i\in I_t}$ of regular curves of class $\mathrm{W}^{2,p}$ such that $\partial^*\{u > t\} \cap B \subset \bigcup_{i\in I_t} (\gamma_i^t)$ and Γ^t is without crossings.*

Proof Let $\{u_h\}_{h\in\mathbb{N}} \subset C^2(B)$ be converging to u in $\mathrm{L}^1(\Omega)$ and a.e. and satisfying $L := \lim_{h\to\infty} F_p(u_h, B) = \overline{F_p}(u,B) < \infty$. Using Fatou's Lemma and (13) we get

$$\int_{\mathbb{R}} \liminf_{h\to\infty} \int_{\partial\{u_h > t\}\cap B} (1 + |\mathbf{H}_{\{\mathbf{u_h}>\mathbf{t}\}}|^p)\, d\mathcal{H}^{\mathrm{N}-1} \leq$$
$$\leq \liminf_{h\to\infty} \int_{\mathbb{R}} \int_{\partial\{u_h > t\}\cap B} (1 + |\mathbf{H}_{\{\mathbf{u_h}>\mathbf{t}\}}|^p)\, d\mathcal{H}^{\mathrm{N}-1} = L < \infty,$$

thus $\liminf_{h\to\infty} \int_{\partial\{u_h>t\}\cap B} (1+|\mathbf{H}_{\{\mathbf{u_h}>\mathbf{t}\}}|^p) d\mathcal{H}^{\mathrm{N}-1}$ is finite for almost every $t \in \mathbb{R}$. The conclusion follows by the application for almost every $t \in \mathbb{R}$ of Theorem 7, possibly passing to a subsequence (depending on t). □

An obvious consequence of this result is that *the same regularity holds* for the boundaries $\partial^*\{u > t\} \cap \tilde{A}$ (for almost every $t \in \mathbb{R}$) of any solution of the disocclusion problem in $\mathbb{R}^2$. In addition, since this regularity holds in $\tilde{A}$ and not only within the occlusion A, it gives a necessary condition for the existence of a solution, namely that the level lines of the initial function u_0 must satisfy this regularity property, at least near the boundary of A. As a consequence, the only way - essentially - for these level lines to intersect on ∂A is to form a cusp point.

Remark This regularity result cannot be extended to higher dimensions, due to the fact that controlling the mean curvature does not necessarily guarantee the regularity of a hypersurface. By Allard's regularity theorem (see [1] or Theorem 23.1 in [38]), a $(N-1)$-varifold with density 1 and generalized mean curvature in L^p, $p > N-1$, is supported on a set that can be represented locally as the graph of a $\mathrm{C}^{1,1-\frac{N-1}{p}}$ function (J. Duggan [13] showed that a $\mathrm{W}^{2,p}$ regularity actually holds). Unfortunately, such regularity does not apply anymore in the multiple density case. An example is given in [8] of a varifold V with *bounded* mean curvature whose support contains a set A of strictly positive measure such that if $a \in A$ then $\operatorname{spt} V$ does not correspond to the graph of even a multiple-valued function in any neighborhood of a. Thus, controlling only the mean curvature is not enough.

On the other hand, it has been shown by J. Hutchinson [22] that if the *second fundamental form* of a varifold V is in L^p, $p > N-1$, then V is locally supported on the graph of a multiple-valued $\mathrm{C}^{1,1-\frac{N-1}{p}}$ function.

In our disocclusion problem, we can neither ensure that the varifolds supported on the sets $\partial\{u_h > t\}$ converge to unit-density varifolds, nor that the second fundamental form is uniformly bounded in L^p, except in the particular case $N = 2$ where the mean curvature coincides with the second fundamental form. This explains why our regularity result is stated only for $N = 2$.

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