On a variational theory of image amodal completion

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Abstract

We study a variational model for image *amodal completion*, i.e., the recovery of missing or damaged portions of a digital image by technics inspired by the well-known *amodal completion* process in human vision. Representing the image by a real-valued function and following an idea initially proposed in [32], our approach consists in finding a set of interpolating level lines which is optimal with respect to an appropriate criterion. We prove that this method is theoretically well-founded and we show the equivalence with a more classical approach based on a direct interpolation of the function.

1 Introduction

Digital images can be represented as gray level functions $u(x, y)$ defined on a simple open subset of $\mathbb{R}^2$ (usually a rectangle) called "image domain". Of course, digital images are given as a discrete set of samples, but there are standard interpolation methods to get back to a smooth image, e.g. a trigonometric polynomial by Shannon interpolation (also called zero-padding [38]). There is no substantial difference between digital images and what we know of retinal images as rough data: in both cases, images are band-limited by an optical device and then sampled on a grid. So most questions in visual perception theory are easily translated into "computer vision" problems. This opens the way to a mathematical formalization and numerical experiments.

We shall deal in this paper with the counterpart in image processing of the "amodal completion" phenomenon that arises in human vision. This phenomenon has been widely studied by the phenomenologist Gaetano Kanizsa [28], who tried to give a consistent answer to one of the major enigmas of visual perception. Its understanding starts with the straightforward observation that the objects that we see in all day life are partially occulting each other, so that we only see parts of them. Georges Matheron [33] actually proved that, under a simple and realistic stochastic model of object occultation, the so called "dead leaves model", we only see half of the objects in sight. To be more explicit, in any all day life image or photograph, whenever we distinguish some object, we only see, on the average, half of it. Mathematically precise versions of this statement can be found in the aforementioned book and in [25, 1]. So we only see (significant) pieces of all shapes we perceive, but these pieces change constantly according to our position with respect to all objects present in the scene. Our perception, however, does not even notice this problem: we perceive objects as though they were complete. The mechanism of this visual illusion was formalized by Kanizsa who formulated two geometric laws, under the names of "amodal completion" and "good continuation".

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The *amodal completion* principle applies when a perceived curve stops on another perceived curve, entailing the perception of a "T-junction". The stopped curve is the leg of the T and the other curve is represented by the horizontal bar of the T. In such a situation, our perception tends to interpret the interrupted curve as the boundary of some object undergoing an occlusion. The leg of the T is then mentally extrapolated and, whenever possible, connected to another leg in front. This fact is illustrated in figure 1 and is called "amodal completion". The connection of two T-legs in front obeys the "good continuation" principle, according to which the reconstructed amodal curve must be similar to the pieces of curve it interpolates (same direction, curvature, etc.)

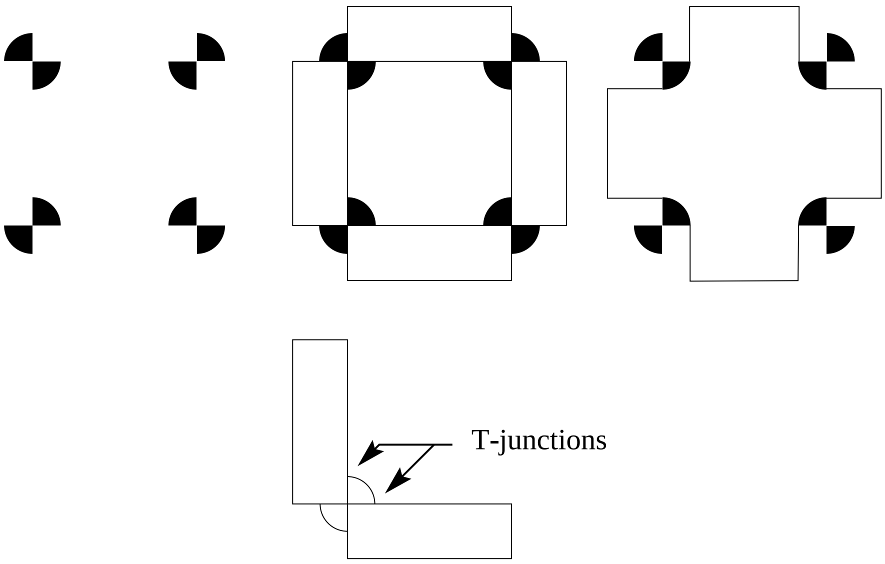

Figure 1: T-junctions entail an amodal completion and a completely different interpretation of the image.

In figure 1 we see first four black butterfly-like shapes. By superposing to them four rectangles, the butterflies get amodally completed into disks. By adding instead to the butterflies a central white cross, the butterflies contribute to the perception of an amodal black big square. In all cases, the reconstructed amodal boundaries obey the *good continuation principle*, namely they are as homogeneous as possible to the visible parts (circles in one case, straight segments in the other case).

The work of Kanizsa and his collaborators was directed at proving that this completion mechanism can be fully formalized as an automatic geometric mechanism that we will call "amodal completion algorithm". An amodal completion algorithm proceeds as follows: given a homogeneous gray level or color region Ω bounded by a smooth Jordan curve, it detects "T-junctions", namely points of $\partial\Omega$ at which other contours stop, thus forming a typical T-shaped singularity. The leg of the T is understood as the boundary of some object in back, while the upper bar of the T is understood as the occulting contour. For instance, in the case of figure 2, the boundary of the disk stops on the boundary of the square, thus forming two T-junctions. The amodal completion phenomenon happens whenever two T-junctions turn out to face each other: in that case, both legs of both junctions tend to be perceptually connected by a "good", smooth curve. How to decide the shape of this interpolating curve ? Several models have been proposed (see an exhaustive review

Figure 2: A square above a disk seen by amodal completion: Kanizsa showed that the T-junctions were crucial for the perception of a full circle where only an arc of circle is actually present in the image.

in [24]); most suggest more or less explicitly that the interpolating curve must offer a compromise between the good continuation of the visible edges and the length minimality. In other words, the curve must be as smooth and as short as possible. A generic enough model proposed in [35, 36] defines the completion curves as minimizers of the Euler elastica energy

$$E(\gamma) = \int_0^{\mathcal{L}(\gamma)} (\alpha + |\gamma''|^2(s))\, ds,$$

given the positions of the extremities and the associated tangent vectors. Here, α is a positive parameter and γ is a parameterization by length of the curve so that $|\gamma'(s)| = 1$ a.e and $\gamma''(s)$ coincides with the curvature. By extension, one can define the completion curves as minimizers of the more general energy

$$E(\gamma) = \int_0^{\mathcal{L}(\gamma)} (\alpha + |\gamma''|^p(s))\, ds, \tag{1}$$

where $\alpha > 0$ and $p \geq 1$. There is no particular reason to choose one value or another for the parameters α, p because they are highly context-dependent, i.e., they depend on the position of T-junctions, the edge orientation, the convexity of the shape, etc., see [24, 37]. Since all results below are valid for any $\alpha > 0$, there is no loss of generality to let $\alpha = 1$.

Let us see now how Kanizsa's amodal completion principles can be translated into an image processing framework – we shall speak of image *amodal completion* – and can be used to tackle the problem of recovering missing parts in an image. This approach was initially developed in [32, 29, 30], following a previous work of Mumford and Nitzberg in a different context [36]. We shall present here a mathematical analysis of the image *amodal completion* problem that completes the results obtained in [32, 29, 30, 4].

Let us first proceed to some mathematical notation. The occlusion shall be represented as an open, bounded and simply connected subset $\Omega \subset \mathbb{R}^2$ with C^∞ boundary. For the sake of simplicity, the original image u_0 is supposed to be known on $\mathbb{R}^2 \setminus \Omega$ but one could as well assume that it is known only on $\tilde{\Omega} \setminus \Omega$, where $\tilde{\Omega} \supset\supset \Omega$ is open, bounded and has Lipschitz boundary. In addition, let us assume that u_0 is the trace on $\mathbb{R}^2 \setminus \Omega$ of an *analytic* function U_0 of $\mathrm{BV}(\mathbb{R}^2)$. This regularity assumption finds a rather natural justification in Shannon interpolation theory but is of course much stronger than the only $U_0 \in \mathrm{BV}(\mathbb{R}^2)$ that has been used in recent years to model the image geometry (see the excellent discussion on this topic in [34]). We actually make this assumption to simplify the proofs but it is worth noticing that the existence of an optimal solution to the image completion problem, as stated in Theorem 2, can be proved as well under the only hypothesis that $U_0 \in \mathrm{BV}(\mathbb{R}^2)$.

Our second assumption on U_0 is $F(U_0) < +\infty$ where the functional F, whose link with the mean curvature of sets has already been examined in [6] in the context of Γ-convergence, is defined as :

$$F(u) := \begin{cases} \displaystyle\int_\Omega |\nabla u|(1 + |\operatorname{div}(\frac{\nabla u}{|\nabla u|})|^p)dx & \text{if } u \in \mathrm{C}^2(\mathbb{R}^2) \\ +\infty & \text{if } u \in \mathrm{L}^1(\mathbb{R}^2) \setminus \mathrm{C}^2(\mathbb{R}^2), \end{cases}$$

with the convention that the integrand is zero wherever $|\nabla u| = 0$. Before justifying the use of this energy, recall a well-known property of the curvature along level lines, namely that for almost every $t \in \mathbb{R}$ and for every $x \in \{u = t\}$, the curvature $\kappa(x)$ of the level line $\{u = t\}$ at x satisfies

$$\kappa(x) = -(\operatorname{div} \frac{\nabla u}{|\nabla u|}) \frac{\nabla u}{|\nabla u|}(x).$$

From this and a call to the change of variables formula, one gets that for any $u \in \mathrm{C}^2(\mathbb{R}^2)$

$$F(u) \equiv \int_{-\infty}^{+\infty} \int_{\Omega \cap \partial\{u \geq \lambda\}} (1 + |\kappa|^p) d\mathcal{H}^1 \, d\lambda$$

when both terms are finite. This leads to define a broader version of F as

$$\mathcal{F}(u) = \int_{-\infty}^{+\infty} \int_{\Omega \cap \partial\{u \geq \lambda\}} (1 + |\kappa|^p) d\mathcal{H}^1 \, d\lambda, \tag{2}$$

this definition making of course sense only when, for almost every level λ, the restriction to Ω of the level lines are a countable set of smooth enough curves.

The regularity assumption $F(U_0) = \mathcal{F}(U_0) < \infty$ implies that for almost every $\lambda \in \mathbb{R}$, $E(\partial\{U_0 \geq \lambda\} \cap \Omega) < \infty$, thus the level lines of U_0 have a "good continuation" behavior.

Following the model proposed in [32, 30], we can reinterpret Kanizsa's amodal completion in a functional framework, where *all missing level lines of the image* $u_0 = U_0|_{\mathbb{R}^2 \setminus \Omega}$ *have to be interpolated inside* Ω *according to the good continuation principle.* To this aim, let us call "T-junction" every point $x \in \partial\Omega$ where ∇U_0 does not vanish, which means that there is a level line passing by x. Let us parameterize the trace of this line on $\mathbb{R}^2 \setminus \Omega$ near x as $\gamma^1(t)$, $t \in [-\varepsilon, 0]$, with $\gamma^1(0) = x$. This is a first T-junction leg. This leg has to be matched to another one of the same level and arriving elsewhere at some $y \in \partial U_0$. Let us denote as $\gamma^2(t)$, $t \in [1, 1+\varepsilon]$, with $\gamma^2(1) = y$ this second one and assume that both T-junctions are compatible, i.e. $\det(\nabla U_0(x), (\gamma^1)'(0))$ and $\det(\nabla U_0(y), (\gamma^2)'(0))$ have the same sign (see figure 3). This compatibility condition is necessary to ensure that we will not reconstruct a "'twisted"' level line that could not be considered as the level line of a function. Our problem is to connect γ^1 with γ^2 by a smooth curve γ: $[0, 1] \to \Omega$, with the condition that the concatenated curve $\tilde{\gamma} : t \in [-\varepsilon,\ 1 + \varepsilon] \to \tilde{\gamma}(t)$ coinciding with γ^1 on $[-\epsilon, 0]$, with γ on $[0, 1]$ and with γ^2 on $[1, 1 + \varepsilon]$ is in $W^{2,p}(-\epsilon, 1 + \epsilon)$ or, equivalently, that $E(\tilde{\gamma}) < \infty$.

We finally define an *amodal completion* as *a set of interpolating curves* γ_x *associated with almost every T-junction* x *on* $\partial\Omega$. Each γ_x joins a junction x to a junction y (so that $U_0(x) = U_0(y)$ and, up to a reparameterization, $\gamma_x = \gamma_y$). The interpolating curves must fill some requirements making them fit to become level lines, namely

- $\nabla U_0(x)$ and $\nabla U_0(y)$ have the same orientation along the curve γ_x (see figure 3);

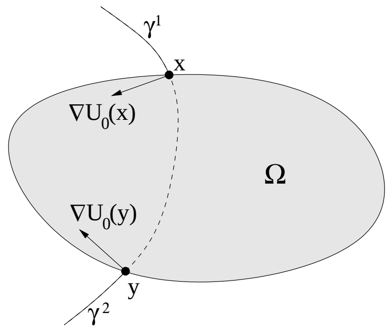

Figure 3: Two T-junctions and a possible amodal completion.

- if γ_x arrives at y, the curves γ_x and γ_y coincide up to reparameterization;
- two curves γ_x and γ_y can meet only tangentially and never cross each other, i.e., at every point of intersection there exists a neighborhood in which γ_x and γ_y form an upper graph and a lower graph (see figure 4);
- a curve γ_x may have self-intersections but only tangentially and without crossing; in addition, γ_x may touch $\partial\Omega$ out from x and y but only tangentially (see figure 4).

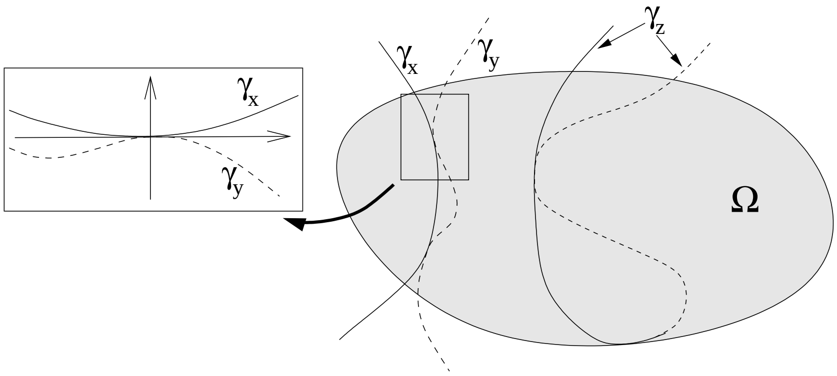

Figure 4: γ_x and γ_y intersect tangentially without crossing; γ_z self intersects tangentially without crossing, and also intersects $\partial\Omega$ tangentially. For clarity, γ_z is shown decomposed into two arcs.

We call $\mathcal{D}$ the set of all amodal completions of U_0 inside Ω. With each curve γ_x of an amodal completion is associated a gray level $U_0(x)$ and the non crossing constraint makes the curves γ_x fit to be level lines of a function u_γ that shall be called the *amodal completion of* U_0 *inside* Ω. There is a standard way to construct such a function u_γ inside Ω from γ, so that all level lines of u are contained in a countable or finite union of curves γ_x (see Theorem 1). The fact that there is not necessarily identity between level curves of the reconstructed u_γ and the γ_x is illustrated in figure 5, where two level lines of the same level coincide on some interval. Since the piece of curve where they coincide shows no contrast, the reconstructed function u_γ loses this part of the level curve. This possibility that a singularity is created was pointed out in [7] and shall be called the *curve gluing phenomenon*.

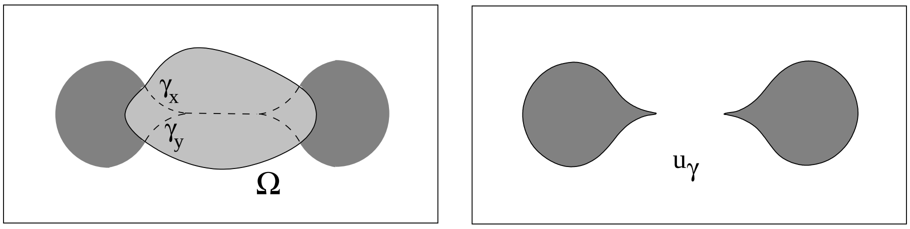

Figure 5: The *gluing phenomenon*: the curves γ_x, γ_y of an amodal completion and the level lines of the associated function u_γ may have totally different structures.

Introducing the measure $\mu := |\nabla U_0|\mathcal{H}^1 \llcorner \partial\Omega$, the energy of the amodal completion is defined as the sum of all energies of all interpolated level lines, namely,

$$\mathcal{E}(\gamma) = \frac{1}{2}\int_{\partial\Omega} E(\gamma_x)d\mu(x),$$

where $E(\gamma_x)$ has been defined above in (1). The factor $\frac{1}{2}$ recalls that we count the energy twice, since $E(\gamma_x) = E(\gamma_y)$ when x and y are two matching T-junctions. A numerical theory and experiments for minimizing $\mathcal{E}$ when $p = 1$ was developed in [29]. In that case, an absolute minimum was theoretically and computationally attained. Actually there are two kinds of numerical theories dealing with the same problem, namely the ones which minimize either $F(u)$ or $\mathcal{F}(u)$ and the ones which minimize $\mathcal{E}(\gamma)$. Now, the *gluing phenomenon* explains why it may be expected that sometimes

$$\mathcal{E}(\gamma) \neq \mathcal{F}(u_\gamma). \tag{3}$$

We shall prove, however, that with any amodal completion γ and for every $h \in \mathbb{N}^\star$ we can associate a function $u_{\gamma,h}$ so that $u_{\gamma,h} = U_0$ outside Ω, $\|u_\gamma - u_{\gamma,h}\|_{L^1(\Omega)} \leq 1/h$ and $|\mathcal{E}(\gamma) - \mathcal{F}(u_{\gamma,h})| \leq 1/h$ (see Lemma 7).

All the same, (3) suggests that we cannot just solve the amodal completion problem by looking for u minimizing $\mathcal{F}(u)$ with the constraint $u = U_0$ on $\mathbb{R}^2 \setminus \Omega$. Indeed, there is not necessarily a solution to either problems

$$\min_{u=U_0 \text{ on } \mathbb{R}^2\setminus\Omega} \mathcal{F}(u). \tag{4}$$

or

$$\min_{u=U_0 \text{ on } \mathbb{R}^2\setminus\Omega} F(u). \tag{5}$$

We shall instead prove that there is a solution to

$$\min_{\gamma\in\mathcal{D}} \mathcal{E}(\gamma) \tag*{(P_1)}$$

The fact that (4) and (5) are ill-posed led the authors of [4] to adopt a slightly different strategy which is very classical in the calculus of variations. First, in order to incorporate an explicit reference to the good continuation requirement, they define the energy on a domain slightly bigger than Ω.

More precisely, given an open and smooth subset $\tilde{\Omega}$ such that $\tilde{\Omega} \supset\supset \Omega$, the authors consider the energy $F^{\rm e}$ defined by

$$F^{\rm e}(u) = \begin{cases} \int_{\tilde{\Omega}} |\nabla u|(1 + |\operatorname{div}(\frac{\nabla u}{|\nabla u|})|^p)dx & \text{if } u \in \mathrm{C}^2(\mathbb{R}^2) \\ +\infty & \text{if } u \in \mathrm{L}^1(\mathbb{R}^2) \setminus \mathrm{C}^2(\mathbb{R}^2) \end{cases}$$

with the convention that the integrand is zero wherever $|\nabla u| = 0$. The minimization process is not performed directly on $F^{\rm e}$, for the same reason why (5) is ill-posed, but rather on the lower semicontinuous envelope $\overline{F^{\rm e}}$ of $F^{\rm e}$ whose sequential definition is (see [19])

$$\overline{F^{\rm e}}(u) := \inf\{\liminf_{h\to\infty} F^{\rm e}(u_h) : \ u_h \to u \text{ in } \mathrm{L}^1(\mathbb{R}^2)\}.$$

Then it is proved in [4] that the problem

$$\min_{u \equiv U_0 \text{ on } \mathbb{R}^2\setminus\Omega} \overline{F^{\rm e}}(u)$$

is well-posed.

We should work with a different definition of the relaxed functional associated with F in order to reintroduce the good continuation requirement that does not appear in F. Given a function $u \in \mathrm{L}^1(\mathbb{R}^2)$ that coincides with U_0 outside Ω, we define

$$\overline{F}(u) := \inf\{\liminf_{h\to\infty} F(u_h) : \ u_h \to u \text{ in } \mathrm{L}^1(\mathbb{R}^2), \ u_h \equiv U_0 \text{ on } \mathbb{R}^2 \setminus \Omega\}$$

Of course, this relaxed functional is still the largest lower semicontinuous functional less than F, *when restricted to the class of functions that coincide with* U_0 *outside* Ω. Under the crucial assumptions that U_0 is smooth and $F(U_0) < \infty$, all results of [4] remain true when particularized to the class of functions that coincide with U_0 outside Ω and one gets that

$$\min_{u \equiv U_0 \text{ on } \mathbb{R}^2\setminus\Omega} \overline{F}(u) \qquad (P_2')$$

is well posed.

For every $u \in \mathrm{L}^1(\mathbb{R}^2)$ such that $u = U_0$ on $\mathbb{R}^2 \setminus \Omega$, we can also define the relaxed functional associated with $\mathcal{F}$ as

$$\overline{\mathcal{F}}(u) = \inf\{\liminf_{h\to\infty} \mathcal{F}(u_h) : \ u_h \to u \text{ in } \mathrm{L}^1(\mathbb{R}^2), \, u_h = U_0 \text{ on } \mathbb{R}^2 \setminus \Omega\}$$

It will be established in Theorem 3 that

$$\min_{u \equiv U_0 \text{ on } \mathbb{R}^2\setminus\Omega} \overline{\mathcal{F}}(u) \qquad (P_2)$$

is also well posed.

Our main results in this paper are, first, that problem (P_1) is well posed (Theorem 2) and, second, that problems (P_1) and (P_2) (Theorem 4) have the same minimal energies. We also prove that there is a solution u of (P_2) satisfying $u = u_\gamma$, where γ is a solution of (P_1). Conversely, given any u minimizing (P_2), there is an amodal completion minimizing $\mathcal{E}$ whose curves contain all level lines of u. The amodal completion problem therefore yields a very intuitive geometric interpretation ($\mathcal{E}$) of a relaxed functional ($\overline{\mathcal{F}}$)

We were not able to determine whether (P_2) and (P_2') also have the same minimizers because we actually do not know whether $\mathcal{F}(u) = \overline{F}(u)$ for any $u \in \mathcal{S}$, a class of – non necessarily smooth – functions in the domain of $\mathcal{F}$ (see section 2).

To end this section, let us briefly describe the state-of-art relative to the subject of this paper.

The first adaptation of amodal completion's principles to image processing can be found in [36]: in order to reconstruct partially occulted objects, the authors propose to interpolate their boundaries *below* the occlusions using curves that minimize the Euler elastica energy $\int(\alpha + \kappa^2)ds$, where κ is the curvature.

This idea was adapted in [32, 29, 30] to the level lines framework in order to solve the problem of recovering missing areas in an image, following the strategy that we previously described. Figure 1 illustrates the kind of results that can be obtained with this approach. The bottom left image is the result of a *global* minimization of E by dynamic programming with $p = 1$ (see [32, 29, 30]) whereas the bottom right image is obtained by a *global* minimization of E with $p = 2$, still by dynamic programming, among the collection of all amodal completions made of Euler spirals, i.e. curves whose curvature depends linearly on the arc-length [31].

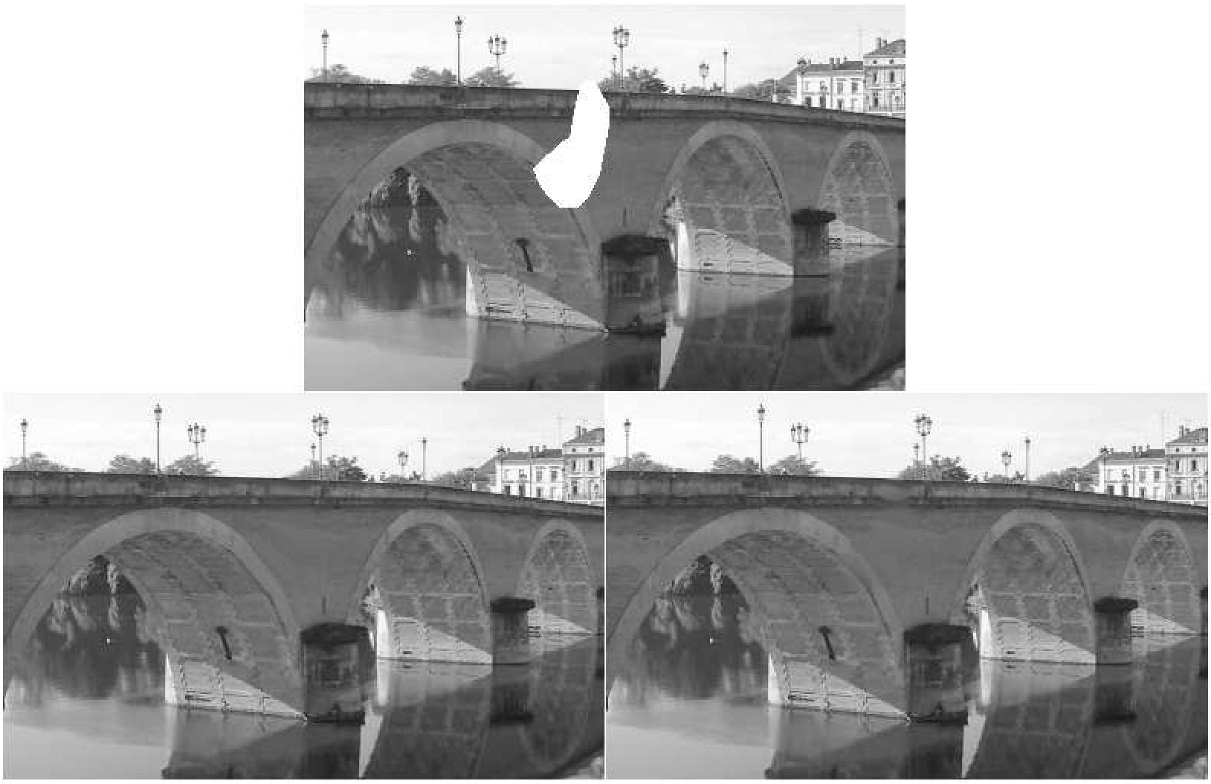

Figure 6: Top: original image with missing area shown in white. Bottom left: after amodal completion by minimization of E in the case $p = 1$ (see the description of the algorithm in [32, 30]). Bottom right: after amodal completion by minimizing E for $p = 2$ in the class of amodal completions made of Euler spirals [31].

The problem of recovering missing areas in an image is addressed in a completely different way in [11]. The proposed method is inspired by the technics employed by professionals for the restoration of old paintings. It consists in a progressive diffusion of the information from the boundary of the domain towards the interior by means of a partial differential equation that aims at transporting along the isophotes a specific criterion of image smoothness. The connections of this model – the so-called *inpainting* model – with the classical Navier-Stokes equation of fluid dynamics are shown in [10].

In [15], the authors propose a denoising/interpolation model based on the joint minimization of a quadratic fidelity term outside the occulting domain and a total variation criterion within a domain slightly bigger than the occlusion (see also a variant of the equation associated with this model in [16]). The model proposed in [14] aims at recovering a piecewise smooth function inside the occlusion by minimizing the classical Mumford-Shah functional with the additional constraint that the discontinuity set, whenever it exists, has minimal Euler elastica energy – concerning the minimization of the elastica energy, see also the recent approach of [22] based on a convolution-thresholding scheme for the Willmore flow proposed in [26]. Finally, the authors of [23] introduce a numerical scheme for the fourth order nonlinear flow associated with F and perform image completion by computing local minimizers of F.

A more sophisticated criterion is derived in [5], where the authors propose a joint interpolation of image intensity and level lines directions using a functional that can be seen as a clever relaxation of F. The resulting model offers many advantages both from a theoretical and a practical viewpoint.

This is also the case of the approach followed in [17, 27], where a geometrical model of the functional architecture of the primary visual cortex is proposed after the work of [37]. This approach amounts to replacing the minimization of the Euler elastica's energy in the Euclidean space with the minimization of the horizontal perimeter of surfaces in the roto-translation group endowed with an appropriate graded differentiable structure.

All these methods are essentially dedicated to the reconstruction of the geometric information and usually perform badly for the interpolation of texture. Very recently, a new class of methods have appeared that perform very well in many situations. All these methods rely on a very simple "copy-paste" procedure that was introduced for the first time in [20] in the context of texture synthesis. The first adaptations to image interpolation can be found in [13, 18]. They perform remarkably well in most situations, except when the information to recover requires some large scale interpretation, which indicates that these methods could be advantageously combined with the approach of this paper.

Let us finally mention two recent variational models based on a linear decomposition of the image into a geometric component and a texture component and the use of two different reconstruction methods, one for each component. The decomposition/reconstruction process is performed either independently [12] or, more interestingly, jointly [21].

1.1 Anterior work, novelties

Whatever is being done here can be derived from anterior works in the particular case where u is the characteristic function of a set A. In that case, G. Bellettini, G. Dal Maso and M. Paolini in [7] and G. Bellettini and L. Mugnai in [8, 9] studied the relaxation of

$$\mathcal{F}(\chi_A) = \int_{\partial A} (1 + |\kappa|^p) d\mathcal{H}^1,$$

where χ_A denotes the characteristic function of $A \subset \mathbb{R}^2$. In particular, it is shown in [8] that if A satisfies $\overline{\mathcal{F}}(\chi_A) < \infty$ then A essentially coincides with the interior set of a limit system of curves $(\Gamma_i)_{0\leq i\leq m}$ and $\overline{\mathcal{F}}(\chi_A) = \sum_{i=0}^{m} E(\Gamma_i)$. In the particular case where ∂A is piecewise $\mathrm{W}^{2,p}$ with finitely many cusps then Γ consists in adding to ∂A an appropriate collection of smooth curves that connect the cusps pairwise (see also a representation with varifolds in [9]) and give them a "good continuation", thus realizing a kind of "amodal completion" like in figure 5.

What we added to these arguments:

- the slight changes necessary to deal with the amodal completion problem (which amounts to treating the kind of Dirichlet condition given by T-junctions);
- as in the mentioned papers, we obtain a geometric characterization of the relaxation of $\mathcal{F}$, i.e. the abstract $\overline{\mathcal{F}}(u)$ is translated into the intuitive $\mathcal{E}(\gamma)$;
- now, these functionals deal with all level sets $A_\lambda = \{u \geq \lambda\}$ together instead of just one. The extension is not trivial as one can judge from section 4;
- the existence of a minimal amodal completion is proven for every $p > 1$. This completes [29, 30] where the existence was proven in general for $p = 1$ but, for every $p > 1$, with the additional constraint that the trace of the function on $\partial\Omega$ takes finitely many values. The extension to the general case as stated in Theorem 2 is not straightforward; taking a minimizing sequence of amodal completions, it is indeed not too difficult to prove the existence of limit curves for countably many points by an extensive use of diagonal extraction. But the treatment of the remaining points requires a control of the energy for sequences of amodal completion curves that we were able to prove only by a specific averaging process and a call to the theory of martingales.
- we show the equivalence between the minimization of $\mathcal{E}$ on curves and the minimization of $\overline{\mathcal{F}}$ on functions, i.e., between a model designed to imitate the physiological amodal completion process and a derived model obtained by mathematical interpretation.

1.2 Reader's guide

The definition of T-junctions is given in section 2.1. We precisely introduce in section 2.2 amodal completions and the amodal energy $\mathcal{E}$ (Definition 5). The main point is to impose the non intersection constraint on the curves of the amodal completion. This permits to uniquely define from an amodal completion γ a function u_γ so that, if $\mathcal{E}(\gamma)$ is finite and γ has no contact, $\mathcal{F}(u_\gamma) = \mathcal{E}(\gamma)$ (Theorem 1). In section 4, Theorem 2, we prove the first main result of the paper, namely the existence of a solution to the amodal completion problem

$$\min \{\mathcal{E}(\gamma) : \gamma \in \mathcal{D}\}. \qquad (P_1).$$

The solution γ to this problem yields an amodal completion image u_γ with bounded variation. In order to identify the bridges between the functional viewpoint and the amodal completion viewpoint, it is proven in Lemma 6 that every amodal completion can be approximated by a sequence of amodal completions without contact, from which we construct a sequence of continuous functions (u_h) converging to u_γ and whose energy $\mathcal{F}(u_h)$ is arbitrarily close to $\mathcal{E}(\gamma)$ (Lemma 7).

Conversely, from any $u \in \mathrm{C}^2$ one can define an amodal completion γ, obtained by a selection of the level lines of u, such that $\mathcal{E}(\gamma) \leq \mathcal{F}(u) = F(u)$ (Lemma 8).

Our main second result is the equivalence of (P_1) with (P_2). We prove it in section 5, Theorem 4 and show the close relationships between the minimizers of (P_1) and those of (P_2). In particular, if γ minimizes (P_1) then u_γ minimizes (P_2) and $\overline{\mathcal{F}}(u_\gamma) = \mathcal{E}(\gamma)$. Conversely, if u is a minimizer of (P_2) then there exists an amodal completion γ_u such that γ_u is a minimizer of (P_1) and its associated function u_γ coincides with u almost everywhere.

2 Notations and definitions

It is assumed once for all that $p > 1$. As mentioned in the introduction, the occlusion will be represented as an open, bounded and simply connected subset $\Omega \subset \mathbb{R}^2$ with C^∞ boundary. The original image is supposed to be known only outside Ω. We assume that it is the trace on $\mathbb{R}^2 \setminus \Omega$ of an analytic function U_0 such that $U_0 \in \mathrm{BV}(\mathbb{R}^2)$ and $F(U_0) < \infty$. The interpolation within Ω being trivial if U_0 is constant on $\partial\Omega$, we can exclude this case. Then it is a straightforward consequence of Sard Lemma and the coarea formula that there exists a non empty subset $\Lambda \subset \mathbb{R}$ with $\mathcal{H}^1(U_0(\Omega) \setminus \Lambda) = 0$ such that, for all $\lambda \in \Lambda$,

(H_1) $\{U_0 = \lambda\}$ is an analytic curve of finite length;

(H_2) $\mathcal{H}^0(\{U_0|_{\partial\Omega} = \lambda\}) < \infty$;

Observing that the function $\lambda \mapsto |\{U_0|_{\partial\Omega} \geq \lambda\}|$ is monotone and therefore admits countably many discontinuities, Λ can be chosen so that for all $\lambda \in \Lambda$,

(H_3) $\{U_0|_{\partial\Omega} \geq \lambda\} = \lim_{\mu \to \lambda} \{U_0|_{\partial\Omega} \geq \mu\}$;

where the convergence is meant as the convergence in measure.

Definition 1 *A pair (U_0, Λ) satisfying conditions $(H_1) - (H_3)$ is called an admissible occlusion.*

We recall that, given a function u of bounded variation (see [3] for a survey on BV functions), its level sets $\{u \geq \lambda\}$ are sets of finite perimeter for almost every $\lambda \in \mathbb{R}$ and we can define their reduced boundary as the set

$$\partial^*\{u \geq \lambda\} := \begin{array}{c}\{x \in \mathbb{R}^2 : \nu_{\{u\geq\lambda\}}(x) := \lim_{r\downarrow 0} \frac{D\chi_{\{u\geq\lambda\}}}{|D\chi_{\{u\geq\lambda\}}|}(B_r(x)) \text{ exists} \\ \text{and satisfies } |\nu_{\{u\geq\lambda\}}(x)| = 1\},\end{array}$$

i.e. the set of all points where a generalized inner normal to $\{u \geq \lambda\}$ exists.

The space $\mathcal{S}$ defined below is the set of all functions u of bounded variation in $\mathbb{R}^2$ that coincide with U_0 outside Ω and such that, for almost every λ, $\partial^*\{u \geq \lambda\} \cap \Omega$ essentially coincides with a finite union of curves that all join two points of $\partial\Omega$ and properly extend outside Ω. Clearly, $\mathcal{S}$ is the space of the functions that follow Kanizsa's good continuation principle.

Definition 2 *We call $\mathcal{S}$ the space of all functions $u \in \mathrm{BV}(\mathbb{R}^2)$ such that $u = U_0$ on $\mathbb{R}^2 \setminus \Omega$ and for almost every $\lambda \in U_0(\partial\Omega)$, $\partial^*\{u \geq \lambda\} \cap \overline{\Omega}$ coincides, up to a $\mathcal{H}^1$-negligible set, with the trace of a finite union of curves $\gamma_i^\lambda : [0,1] \to \overline{\Omega}$, $i = 0, \ldots, n_\lambda$ with the following properties:*

- $\gamma_i^\lambda(0), \gamma_i^\lambda(1) \in \partial\Omega$;
- $\gamma_i^\lambda \in \mathrm{W}^{2,p}(0,1)$ *and* $|d\gamma_i^\lambda/dt|$ *is constant almost everywhere on* $[0,1]$;
- *there exists an extension* $\gamma_i^{\lambda,\epsilon} \in \mathrm{W}^{2,p}(-\epsilon, 1+\epsilon)$ *of* γ_i^λ *such that* $\gamma_i^{\lambda,\epsilon}([-\epsilon,0])$ *and* $\gamma_i^{\lambda,\epsilon}([1,1+\epsilon])$ *are (possibly overlapping) subsets of* $\{x : U_0(x) = \lambda\} \cap (\mathbb{R}^2 \setminus \Omega)$ *with positive length;*
- $\forall i,j$, γ_i^λ *and* γ_j^λ *may intersect but only tangentially and without crossing each other.*

Then one defines the functional $\mathcal{F}$ acting on $\mathrm{L}^1(\mathbb{R}^2)$ by

$$\mathcal{F}(u) = \begin{cases} \displaystyle\int_{-\infty}^{+\infty} \int_{\Omega\cap\partial^*\{u\geq\lambda\}} (1+|\kappa|^p)d\mathcal{H}^1\, d\lambda & \text{if } u \in \mathcal{S} \\ +\infty & \text{if } u \in \mathrm{L}^1(\mathbb{R}^2)\setminus\mathcal{S}. \end{cases}$$

where, for almost every $\lambda \in U_0(\partial\Omega)$, it is meant

$$\int_{\Omega\cap\partial^*\{u\geq\lambda\}} (1+|\kappa|^p)d\mathcal{H}^1 = \sum_{i=0}^{n_\lambda} \int_{\gamma_i^\lambda} (1+|\kappa|^p)d\mathcal{H}^1.$$

In figure 7 below, we show an example of a piecewise constant element u of $\mathcal{S}$ such that $\mathcal{F}(u) < \infty$ but $F(u) = +\infty$.

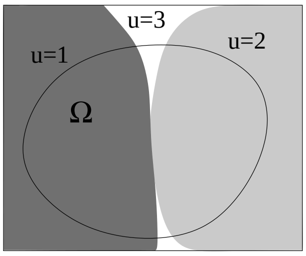

Figure 7: A piecewise constant function $u \in \mathcal{S}$ such that $\mathcal{F}(u) < \infty$ but $F(u) = +\infty$.

2.1 Defining T-junctions

Definition 3 (Occlusion's boundary measure) *An admissible occlusion* Ω *is endowed with a boundary measure*

$$\mu := |\nabla U_0|\mathcal{H}^1 \llcorner \partial\Omega.$$

Definition 4 (T-junctions) *We call T-junction any element of the set*

$$\mathcal{T} := \{x \in \partial\Omega :\ \exists \lambda \in \Lambda,\ x \in \partial\{U_0 \geq \lambda\},$$

and we shall denote by $\mathcal{T}_\lambda$ *the set of all T-junctions associated with the level* $\lambda \in \mathbb{R}$, *i.e.*

$$\mathcal{T}_\lambda := \{x \in \mathcal{T} : U_0(x) = \lambda\}.$$

Proposition 1 *μ-almost every $x \in \partial\Omega$ is a T-junction, i.e. $\mu(\partial\Omega \backslash \mathcal{T}) = 0$.*

PROOF: By the coarea formula for Lipschitz functions,

$$|DU_0|(\partial\Omega\backslash\mathcal{T}) = \int_{-\infty}^{+\infty} \mathcal{H}^0((\partial\Omega\backslash\mathcal{T}) \cap \{U_0 = t\})dt.$$

Since, for almost all t,

$$\mathcal{H}^0((\partial\Omega\backslash\mathcal{T}) \cap \{U_0 = t\}) = 0$$

by definition of $\mathcal{T}$, we conclude that $|DU_0|(\partial\Omega\backslash\mathcal{T}) = 0$ and the proposition follows. □

2.2 Defining amodal completions and their "amodal energy"

Definition 5 (Amodal completion on Ω associated with U_0) *We call amodal completion of class $W^{2,p}$ associated with U_0 a map γ from $\mathcal{T}$ to $W^{2,p}([0,1],\mathbb{R}^2)$ that associates with every $x \in \mathcal{T}$ a function γ_x describing a curve in $\overline{\Omega}$ with distinct endpoints on $\partial\Omega$. In addition:*

- *for μ-almost every $x \in \mathcal{T}$, there exists a $W^{2,p}$ parameterization of γ_x (still denoted by γ_x) on $[0,1]$ with positive and constant velocity. The endpoints conditions rewrite $\gamma_x(0)$, $\gamma_x(1) \in \mathcal{T}$ and $\gamma_x(0) \neq \gamma_x(1)$;*
- *if two curves γ_x and γ_y have a common endpoint then $\gamma_x(s) = \gamma_y(s)$ for every $s \in [0,1]$, or $\gamma_x(s) = \gamma_y(1-s)$ for every $s \in [0,1]$;*
- *each curve may have tangential self-contacts but without crossing;*
- *two curves may intersect tangentially but without crossing;*
- *for μ-almost every $x \in \mathcal{T}$ there exists an extension $\gamma_x^\varepsilon \in W^{2,p}[-\varepsilon, 1+\varepsilon]$ of γ_x such that $\gamma_x^\varepsilon([-\varepsilon,0])$ and $\gamma_x^\varepsilon([1,1+\varepsilon])$ are (possibly overlapping) subsets of $\{y : U_0(y) = U_0(x)\} \cap (\mathbb{R}^2\backslash\Omega)$ with positive length and the orientation of ∇U_0 on these subsets can be continuously extended to the whole curve γ_x (see figure 3).*

The set of all such amodal completions will be denoted by $\mathcal{D}$.

Remark 1 Since, for every $(x,\lambda) \in \mathcal{T}$, $|\gamma_x'(t)|$ is assumed to be constant almost everywhere on $[0,1]$, the arc-length parameter of the curve is $s(t) = t\mathcal{L}(\gamma_x)$, where $\mathcal{L}(\gamma_x)$ denotes the curve total length. We let $\tilde{\gamma}_x$ represent the curve by arc-length. Clearly, for every $s \in [0, \mathcal{L}(\gamma_x)]$, $\tilde{\gamma}_x(s) = \gamma_x(s/\mathcal{L}(\gamma_x))$. Therefore

$$\tilde{\gamma}_x''(s) = \frac{\gamma_x''(t)}{[\mathcal{L}(\gamma_x)]^2}$$

Now, it is well known that the curvature along the curve satisfies, as a function of arc-length,

$$\kappa(s) = \tilde{\gamma}_x''(s)$$

and we deduce

$$\int_0^{\mathcal{L}(\gamma_x)} (1 + |\tilde{\gamma}_x''(s)|^p)ds = \int_0^{\mathcal{L}(\gamma_x)} (1 + |\kappa|^p)ds = \int_0^1 \left(|\gamma_x'(t)| + [\mathcal{L}(\gamma_x)]^{1-2p}|\gamma_x''(t)|^p\right) dt \qquad (6)$$

Assuming that $\gamma_x \in W^{2,p}(0,1)$ is therefore equivalent to saying that $\tilde{\gamma}_x$ belongs to $W^{2,p}(0, \mathcal{L}(\gamma_x))$ and that $\tilde{\gamma}_x$ has finite E energy. For the seek of simplicity, we shall in the sequel also denote by γ_x the representation of the curve by arc-length.

Definition 6 (Amodal completion without contact) *An amodal completion $\gamma \in \mathcal{D}$ is said to be without contact if:*

- *for μ-almost every $x \in \partial\Omega$, γ_x is simple and $(\gamma_x) \cap \partial\Omega = \{\gamma_x(0), \gamma_x(1)\}$;*
- *for μ-almost every $x, y \in \partial\Omega$ such that $x \neq y$, $(\gamma_x) \cap (\gamma_y) = \emptyset$.*

Definition 7 *We define the amodal energy of an amodal completion γ as*

$$\mathcal{E}(\gamma) = \frac{1}{2}\int_{\partial\Omega} E(\gamma_x) d\mu(x)$$

where

$$E(\gamma_x) = \int_0^{\mathcal{L}(\gamma_x)} (1 + |\gamma_x''|^p(s))\, ds$$

3 From amodal completions to functions, and back

Theorem 1 (Function associated with an amodal completion) *Let (U_0, Λ) be an admissible occlusion. Any amodal completion γ on Ω of class $\mathrm{W}^{2,p}$ satisfying $\mathcal{E}(\gamma) < \infty$ can be associated with a function $u_\gamma \in \mathrm{BV}(\mathbb{R}^2)$ such that $u_\gamma \equiv U_0$ on $\mathbb{R}^2 \setminus \Omega$ and, for almost every $\lambda \in \mathbb{R}$, $\partial^*\{u_\gamma \geq \lambda\} \cap \overline{\Omega} \subset \bigcup_{x \in \mathcal{T}_\lambda} \gamma_x$ up to a $\mathcal{H}^1$-negligible set. If, in addition, γ has no contact, then $u_\gamma \in \mathcal{S}$ and $\mathcal{F}(u_\gamma) = \mathcal{E}(\gamma)$.*

Proof: The proof is essentially based on a straightforward filling up algorithm, permitting to define uniquely from the amodal curves at level λ a set A_λ bounded by them and $\partial\Omega$. This set will be the λ-upper level set of u_λ inside Ω. A consistency check must then be performed, namely that $\mu > \lambda \Rightarrow A_\mu \subset A_\lambda$.

Step 1. From amodal completion curves at level λ to a level set $\mathbf{A}_\lambda$

$\partial\Omega$ is provided with an orientation so that we can talk of arc intervals $[x, y] \subset \partial\Omega$ without ambiguity. Let $\lambda \in \Lambda$ such that $\mathcal{T}_\lambda$ be not empty. By definition of Λ, it is also finite. The upper level set of U_0 on $\partial\Omega$, $\{x \in \partial\Omega,\ U_0(x) \geq \lambda\}$ is a finite union of disjoint arcs of $\partial\Omega$. Let us call $[x_1, x_2]$ any of these arcs and set (see figure 8)

$$x_3 = \begin{cases} \gamma_{x_2}(1) & \text{if } x_2 = \gamma_{x_2}(0), \\ \gamma_{x_2}(0) & \text{if } x_2 = \gamma_{x_2}(1). \end{cases}$$

Take for $x_4 \in \mathcal{T}_\lambda$ the unique point such that $[x_3, x_4]$ is a connected component of the upper level set $\{x \in \partial\Omega,\ U_0(x) \geq \lambda\}$. This construction can be iterated and, after a finite number of steps, one gets a series of intervals $[x_{2i+1}, x_{2i+2}]$, $i = 0, \ldots j$ such that $x_{2j+1} = x_1$. Since each T-junction is the tip of a single amodal curve, no shorter cycle is possible in the mentioned sequence. Besides, since the arcs $[x_{2i+1}, x_{2i+2}]$ are disjoint and the curves $\gamma_{x_{2i+2}}$ cannot cross, we can concatenate them all into a rectifiable image of the circle into the plane, with no crossing but possibly self-contacts. We call Γ_1 this generalized Jordan curve. Using the usual index with respect to a curve, one defines the interior Ω_1 of Γ_1 as the set of points with index 1. Notice that, by construction, Ω_1 is contained in Ω and Ω_1 is not empty due to our assumption (H_3) at the beginning of section 2. In addition, due to the regularity of $\partial\Omega$ and all amodal curves, every point of $[x_1, x_2]$ is the limit of a sequence of points in Ω with index one with respect to Ω_1.

This construction can be iterated until all T-junctions at level λ have been exhausted. The successive generalized Jordan curves Γ_1, ..., Γ_k thus obtained do not cross. Thus, the sets $\Omega_1, \cdots, \Omega_k$ are disjoint.

We finally define

$$A_\lambda = \bigcup_{h=1}^{k} \Omega_h \tag{7}$$

and remark that, by construction,

$$\partial A_\lambda \subset \bigcup_{h=1}^{k} \Gamma_h \subset \partial\Omega \cup \bigcup_{x\in\mathcal{T}_\lambda} \gamma_x, \tag{8}$$

and

$$\mathcal{H}^1(\{x \in \partial\Omega : U_0(x) \geq \lambda\} \setminus \partial A_\lambda) = 0.$$

Besides, since each curve Γ_h is a finite union of C^1 curves that do not cross each other, if follows that

$$\partial^* A_\lambda = \partial A_\lambda \quad \text{up to a } \mathcal{H}^1\text{-negligible set.} \tag{9}$$

The same construction is performed for every $\lambda \in \Lambda$ (recall that $\mathcal{H}^1(U_0(\partial\Omega) \setminus \Lambda) = 0$). Then let $A_\lambda = \Omega$ for every $\lambda < \min_{x\in\partial\Omega} U_0(x)$ and $A_\lambda = \emptyset$ for every $\lambda > \max_{x\in\partial\Omega} U_0(x)$, in order to ensure that a set A_λ is associated with almost every $\lambda \in \mathbb{R}$.

Let us prove now that for any $\lambda, \mu \in \Lambda$,

$$\lambda \leq \mu \Rightarrow A_\lambda \supset A_\mu \text{ (up to a Lebesgue negligible set).}$$

Let $\Gamma_1^\mu = \partial\Omega_1^\mu$ be one of the generalized Jordan curve defining A_μ and $[y_1, y_2]$ its first interval. By the inclusion of upper level sets property, this arc is contained in $\{x \in \partial\Omega : U_0(x) \geq \lambda\}$ and therefore in some maximal interval of this set which we denote by $[x_1, x_2]$. Consider Γ_1^λ, the unique generalized Jordan curve in the preceding construction containing $[x_1, x_2]$. The curves Γ_1^λ and Γ_1^μ do not cross. Indeed, their intersections with $\partial\Omega$ are nested and their other parts are amodal completion curves which do not cross each other. Thus, their associated sets Ω_1^λ and Ω_1^μ are either disjoint, or $\Omega_1^\mu \subset \Omega_1^\lambda$. Now, the first possibility is ruled out because $[y_1, y_2] \subset [x_1, x_2]$ and every point of $[y_1, y_2]$ (resp. $[x_1, x_2]$) is the limit of a sequence of points in Ω with index 1 with respect to Ω_1^μ (resp. Ω_1^λ). Therefore, we have proved that $\Omega_1^\mu \subset \Omega_1^\lambda$ and, by extension, that

$$\forall\lambda, \mu \in \Lambda, \quad \lambda \leq \mu \Rightarrow A_\lambda \supset A_\mu.$$

The same result is obviously true whenever λ or μ are either less than $\inf_{x\in\partial\Omega} U_0(x)$ or larger than $\sup_{x\in\partial\Omega} U_0(x)$ and one can conclude that

$$\text{for a.e. } \lambda, \mu \in \mathbb{R}, \quad \lambda \leq \mu \Rightarrow A_\lambda \supset A_\mu.$$

Step 2. From level sets to a function

We now have a nested family of measurable sets $(A_\lambda) \subset \Omega$ defined for almost every $\lambda \in \mathbb{R}$, actually for all $\lambda \in \Lambda \cup (\mathbb{R} \setminus U_0(\partial\Omega))$. Let us see how they can generate an essentially unique

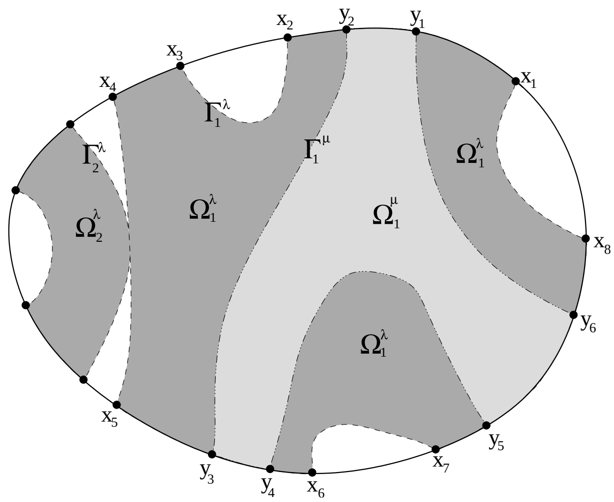

Figure 8: Construction of the function associated with an amodal completion.

function u_γ defined on Ω. Under the notations of Lemma 1 below, let D be the set of discontinuity points of (A_λ). Let $D' \subset \Lambda \cup (\mathbb{R} \setminus U_0(\partial\Omega))$ be countable and dense and define for every $x \in \Omega$

$$u_\gamma^\Omega(x) := \sup\{\lambda \in D' : \ x \in A_\lambda\}.$$

We now prove that $\{u_\gamma^\Omega \geq \lambda\} = A_\lambda$ (up to a Lebesgue-negligible set) for any $\lambda \notin D$, $\lambda \in \Lambda \cup (\mathbb{R} \setminus U_0(\partial\Omega))$. By definition of u_γ^Ω,

$$A_\eta \subset \{u_\gamma^\Omega \geq \lambda\} \subset A_\nu$$

for any η, $\nu \in D'$, $\eta > \lambda > \nu$. Choose sequences $\eta_h \downarrow \lambda$ and $\gamma_h \uparrow \lambda$ in D'. Then, by Lemma 1,

$$A_\lambda = \{u_\gamma^\Omega \geq \lambda\} \quad \text{(up to a Lebesgue negligible set).} \tag{10}$$

In particular, $\{u_\gamma^\Omega \geq \lambda\}$ is measurable for any $\lambda \notin D$, $\lambda \in \Lambda \cup (\mathbb{R} \setminus U_0(\partial\Omega))$. By approximation, the same property extends to any real number λ and u_γ^Ω is measurable.

The uniqueness of u_γ^Ω follows by a similar argument: if two functions u_1, u_2 are such that $\{u_1 \geq \lambda\} = \{u_2 \geq \lambda\}$ (up to a Lebesgue negligible set) for a dense set of λ's, then $u_1 = u_2$ almost everywhere in Ω.

Step 3. Properties of the new function

First remark that it follows from (8) and the finiteness of $\mathcal{T}_\lambda$ that for every $\lambda \in \Lambda$, $\mathcal{H}^1(\partial A_\lambda) \leq \mathcal{H}^1(\partial\Omega) + \frac{1}{2}(\sum_{x \in \mathcal{T}_\lambda} \mathcal{H}^1(\gamma_x)) < \infty$. Thus A_λ has finite perimeter and, by (9) and (10), also $\{u_\gamma^\Omega \geq \lambda\}$ has finite perimeter and its essential boundary satisfies

$$\partial^*\{u_\gamma^\Omega \geq \lambda\} = \partial A_\lambda \subseteq \partial\Omega \cup \bigcup_{x \in \mathcal{T}_\lambda} \gamma_x \quad \text{up to a } \mathcal{H}^1\text{-negligible set.} \tag{11}$$

In particular, $\mathcal{H}^1(\partial^*\{u_\gamma^\Omega \geq \lambda\}) = \mathcal{H}^1(\partial A_\lambda)$. Since $A_\lambda = \emptyset$ for all $\lambda \leq \inf_{x \in \partial\Omega} U_0(x)$ and $A_\lambda = \Omega$ for all $\lambda \geq \sup_{x \in \partial\Omega} U_0(x)$ and because $\mathcal{H}^1(U_0(\partial\Omega) \setminus \Lambda) = 0$ we can conclude that $\{u_\gamma^\Omega \geq \lambda\}$ has finite perimeter in Ω for almost every $\lambda \in \mathbb{R}$. Then,

$$\int_{-\infty}^{+\infty} \mathcal{H}^1(\partial^*\{u_\gamma^\Omega \geq \lambda\} \cap \Omega) d\lambda \leq \frac{1}{2} \int_{-\infty}^{+\infty} (\sum_{x \in \mathcal{T}_\lambda} \mathcal{H}^1(\gamma_x)) d\lambda \leq \frac{1}{2} \int_{\partial\Omega} E(\gamma_x) d\mu(x)$$

It follows from Lemma 2 that $u_\gamma^\Omega \in \mathrm{BV}(\Omega)$. In addition, since $U_0 \in \mathrm{BV}(\mathbb{R}^2 \setminus \Omega)$ and $\partial\Omega$ is smooth, the usual properties of the trace operator in BV (see for instance Corollary 3.89 in [3]) imply that the function u_γ defined by

$$u_\gamma(x) = \begin{cases} u_\gamma^\Omega(x) & \text{on } \Omega \\ U_0(x) & \text{on } \mathbb{R}^2 \setminus \Omega \end{cases}$$

is in $\mathrm{BV}(\mathbb{R}^2)$. Then, it is a direct consequence of (11) that for almost every $\lambda \in \mathbb{R}$,

$$\partial^*\{u_\gamma \geq \lambda\} \cap \Omega \subset \bigcup_{x \in \mathcal{T}_\lambda} \gamma_x \quad \text{up to a } \mathcal{H}^1\text{-negligible set.}$$

Step 4. Case where the original amodal completion has no contact

In this situation, it follows from our construction in Step 1 that (8) rewrites

$$\partial A_\lambda \cap \Omega = \bigcup_{x \in \mathcal{T}_\lambda} \gamma_x \cap \Omega,$$

thus, for almost every $\lambda \in \mathbb{R}$,

$$\partial^*\{u_\gamma \geq \lambda\} \cap \Omega = \bigcup_{x \in \mathcal{T}_\lambda} \gamma_x \cap \Omega \quad \text{up to a } \mathcal{H}^1\text{-negligible set.} \tag{12}$$

It follows that $u_\gamma \in \mathcal{S}$ and, as a direct consequence of the definition of the energies,

$$\mathcal{F}(u_\gamma) = \mathcal{E}(\gamma).$$

□

Lemma 1 *For any monotone family of sets $(X_\lambda)_{\lambda \in \mathbb{R}} \subset \mathbb{R}^2$, there exists a finite or countable set D such that*

$$\lim_{\mu \to \lambda} X_\mu = X_\lambda \qquad \forall \lambda \in \mathbb{R} \setminus D,$$

where convergence means convergence in measure. We shall call D the set of discontinuity points of $(X_\lambda)_{\lambda \in \mathbb{R}}$.

Proof: It is enough to notice that the map $\lambda \mapsto |X_\lambda|$ is monotone, thus has at most countably many discontinuity points, and to choose D as the set of these discontinuity points. □

Lemma 2 *Let $\omega \subset \mathbb{R}^2$ be bounded, connected and with Lipschitz boundary. If $u : \omega \to [-\infty, +\infty]$ is a Borel function such that $u \not\equiv +\infty$ and $u \not\equiv -\infty$ up to a Lebesgue negligible set, then $\lambda \mapsto \mathcal{H}^1(\partial^*\{u \geq \lambda\} \cap \omega)$ is in $\mathrm{L}^1(\mathbb{R})$ if and only if $u \in \mathrm{BV}(\omega)$.*

Proof: See [2, Lemma 1] □

Lemma 3 *Let $u \in \mathcal{S}$. There exists an amodal completion γ_u naturally associated with u such that, in view of Theorem 1, $u = u_{\gamma_u}$ almost everywhere in $\mathbb{R}^2$ and*

$$\mathcal{E}(\gamma_u) = \mathcal{F}(u).$$

PROOF: Recall from the definition of $\mathcal{S}$ that for almost every $\lambda \in U_0(\partial\Omega)$, $\partial^*\{u \geq \lambda\} \cap \overline{\Omega}$ is a finite union of simple curves γ_i^λ, $i = 0, \cdots, n_\lambda$ with good properties. For every $x \in \mathcal{T}$ such that $\partial^*\{u \geq U_0(x)\} \cap \overline{\Omega}$ satisfy this decomposition, let us define $\gamma_{u,x}$ as the unique curve γ_i^λ that passes through x. Clearly, $\{\gamma_{u,x}, x \in \mathcal{T}\}$ satisfies all the properties of an amodal completion. In addition, the map $\gamma_u : x \in \mathcal{T} \mapsto \gamma_{u,x}$ maps $\mathcal{T}$ into $\mathrm{W}^{2,p}([0,1], \mathbb{R}^2)$. It follows from the definition of $\mathcal{S}$ that γ_u is an amodal completion on Ω. Observe now that, up to a $\mathcal{H}^1$-negligible set, $\partial^*(\{u \geq \lambda\} \cap \overline{\Omega}) = \cup_{i=1}^{n_\lambda} \gamma_i^\lambda \bigcup (\{u \geq \lambda\} \cap \partial\Omega)$. Let $(\Gamma_j^\lambda)_{j=1}^{k_\lambda}$ denote the associated family of closed curves as given by Step 1 in the proof of Theorem 1 and let A_λ be the associated set. Then $\partial^* A_\lambda = \partial^*(\{u \geq \lambda\} \cap \overline{\Omega})$ up to a $\mathcal{H}^1$-negligible set thus $A_\lambda = \{u \geq \lambda\} \cap \Omega$ up to a Lebesgue-negligible set. Since we already know that $A_\lambda = \{u_{\gamma_u} \geq \lambda\} \cap \Omega$ up to a Lebesgue-negligible set, the uniqueness of the representation implies that $u = u_{\gamma_u}$ almost everywhere in $\mathbb{R}^2$. The claim about the energy is a direct consequence of the definitions of $\mathcal{S}$ and $\mathcal{F}$. □

4 Minimizing the amodal energy of an amodal completion

We recall our assumption that the data to interpolate within Ω is the trace of an analytic function U_0 such that $\mathcal{F}(U_0) < +\infty$. We assume that (U_0, Λ) is an admissible occlusion. Thus, the set of T-junctions $\mathcal{T}$ is not empty and one can define a canonical amodal completion associated with U_0 in the following way: for every $x \in \mathcal{T}$, let γ_x denote the connected component of $\{U_0 = \lambda\} \cap \overline{\Omega}$ containing x and remark that, by (H_1), γ_x is an analytic curve. It is then easily seen that the map $\gamma_0 : x \in \mathcal{T} \mapsto \gamma_x$ is an amodal completion. Since γ_0 is made of the trace on $\overline{\Omega}$ of all level lines of U_0 that intersect $\partial\Omega$, it is a straightforward consequence of the change of variables formula that

$$\mathcal{E}(\gamma_0) \leq \mathcal{F}(U_0) < \infty.$$

Then the following result can be established.

Theorem 2 *The problem*

$$\min\{\mathcal{E}(\gamma) : \gamma \in \mathcal{D}\} \tag{P_1}$$

has at least one solution $\gamma \in \mathcal{D}$ that can be associated with a function $u_\gamma \in \mathrm{BV}(\mathbb{R}^2)$.

PROOF: The canonical amodal completion associated with U_0 has finite energy thus we may consider a minimizing sequence of amodal completions $(\gamma^\ell)_{\ell \in \mathbb{N}}$ and assume, without loss of generality, that

$$\sup_{\ell \in \mathbb{N}} \mathcal{E}(\gamma^\ell) = C < \infty,$$

so that the functions

$$f_\ell(x) = E(\gamma_x^\ell)$$

are uniformly bounded in $\mathrm{L}^1(\partial\Omega, \mu)$.

Step 1. Convergence of the energies $\mathbf{f}_\ell(\mathbf{x}) = \mathbf{E}(\gamma_{\mathbf{x}}^\ell)$

Since U_0 is assumed to be nonconstant on $\partial\Omega$, one can without loss of generality renormalize the measure μ so that $\mu(\partial\Omega) = 1$. Since μ has no atoms, any point on $\partial\Omega$ can be associated with a unique value in $[0, 1[$. More precisely, given an origin x_0 on $\partial\Omega \cap \mathcal{T}$, we associate with any $x \in \mathcal{T}$ the unique $n \in [0, 1[$ such that $n = \mu([x_0, x])$ and denote $f_\ell(n) := f_\ell(x)$. Conversely, almost every $n \in [0, 1[$ is associated with a unique $x \in \mathcal{T}$ such that $n = \mu([x_0, x])$ and one shall write $x = \mu^{-1}(n)$ for simplicity.

Let us now consider for $k, N \in \mathbb{N}$ the dyadic intervals on $[0, 1[$:

$$I_{N,k} = [k2^{-N}, (k + 1)2^{-N}[,$$

and define the functions

$$f_\ell^N : m \in [0, 1] \mapsto 2^N \int_{I_{N,k}} f_\ell(n)dn = 2^N \int_{\mu^{-1}(I_{N,k})} f_\ell(x)d\mu(x)$$

where $I_{N,k}$ is the unique dyadic interval containing m. Remark that the functions f_ℓ^N are constant on each interval $I_{N,k}$ and, for every $m \in [0, 1[$, $|f_\ell^N(m)| \le 2^N \int_0^1 f_\ell(n)dn \le 2^N C$. Using a diagonal extraction argument, we can find a subsequence of (f_ℓ), still denoted by (f_ℓ), such that

$$\forall (N, k), \quad f_\ell^N(m) \to f^N(m) \quad \text{for every } m \in I_{N,k}$$

For every $N \in \mathbb{N}$, the limit function f^N is positive and piecewise constant on the $I_{N,k}$'s. Moreover, remark that $I_{N,k} = [k2^{-N}, (2k + 1)2^{-N-1}[\cup[(2k + 1)2^{-N-1}, (k + 1)2^{-N}[$ and

$$\begin{aligned} &\forall m \in [k2^{-N}, (2k + 1)2^{-N-1}[\qquad && f_\ell^{N+1}(m) = 2^{N+1} \int_{I_{N+1,2k}} f_\ell(n)dn \\ &\forall m \in [(2k + 1)2^{-N-1}, (k + 1)2^{-N}[&& f_\ell^{N+1}(m) = 2^{N+1} \int_{I_{N+1,2k+1}} f_\ell(n)dn \end{aligned}$$

thus

$$\int_{I_{N,k}} f_\ell^{N+1}(n)dn = \int_{I_{N,k}} f_\ell(n)dn$$

and finally, for every $m \in I_{N,k}$,

$$f_\ell^N(m) = 2^N \int_{I_{N,k}} f_\ell^{N+1}(n)dn$$

By the Dominated Convergence Theorem, it follows that

$$f^N(m) = 2^N \int_{I_{N,k}} f^{N+1}(n)dn,$$

which proves that $(f^N)_{N\in\mathbb{N}}$ is a martingale. In addition, remark that

$$\int_0^1 f_\ell^N(n)dn = \sum_k \int_{I_{N,k}} f_\ell(n)dn = \int_0^1 f_\ell(n)dn \le C,$$

so that, by Fatou's Lemma

$$\int_0^1 f^N(n)dn \le C < \infty.$$

Since, in addition, $f^N \ge 0$, it follows that $(f^N)_{N\in\mathbb{N}}$ is a bounded positive martingale. By Doob's convergence theorem, there exists $f \in \mathrm{L}^1([0,1])$ such that

$$f^N \to f \qquad \text{almost everywhere;}$$

Step 2. Definition of a limit amodal completion

Let $N, k \in \mathbb{N}$ and recall from above that the sequence $(f_\ell^N)_{\ell\in\mathbb{N}}$ satisfies

$$\sup_{n\in I_{N,k}} \sup_{\ell\in\mathbb{N}} f_\ell^N(n) < \infty.$$

Lemma 4 *Let* $I := I_{N,k}$ *and* $A_\ell := \{x \in \mu^{-1}(I) \cap \mathcal{T} : f_\ell(x) < 2^N \int_I f_\ell(n)dn + \frac{1}{N}\}$. *Then there exists some* $\epsilon > 0$ *such that* $\mu(A_\ell) \ge \epsilon$ *for every* $\ell \in \mathbb{N}$.

Proof: By definition and since $\mu(\partial\Omega\backslash\mathcal{T}) = 0$, $\mu^{-1}(I)\backslash A_\ell$ essentially coincides with $\{x \in \mu^{-1}(I) \cap \mathcal{T} : f_\ell(x) \ge 2^N \int_I f_\ell(n)dn + \frac{1}{N}\}$, hence

$$\int_{\mu^{-1}(I)\backslash A_\ell} f_\ell(x)d\mu(x) \ge \mu(\mu^{-1}(I)\backslash A_\ell)\left(2^N \int_I f_\ell(n)dn + \frac{1}{N}\right)$$

Assume that for all $\epsilon > 0$ there exists $\ell \in \mathbb{N}$ such that $\mu(A_\ell) < \epsilon$. Then,

$$\int_I f_\ell(n)dn \ge (2^{-N} - \epsilon)\left(2^N \int_I f_\ell(n)dn + \frac{1}{N}\right),$$

and therefore

$$2^N \int_I f_\ell(n)dn \ge \frac{1}{N}\left(\frac{1}{\epsilon 2^N} - 1\right),$$

which gives a contradiction for ϵ small enough since $\sup_{\ell\in\mathbb{N}} f_\ell^N < +\infty$ on I. □

Lemma 5 *Under the notations above, there exists a T-junction* $x \in \mu^{-1}(I) \cap \mathcal{T}$ *such that, possibly passing to a subsequence of* $(A_\ell)_{\ell\in\mathbb{N}}$,

$$x \in A_\ell, \quad \forall \ell \in \mathbb{N}.$$

Proof: $\{\bigcup_{k\ge n} A_k\}_{n\in\mathbb{N}}$ is a decreasing family of sets such that, by Lemma 4, $\mu(\bigcup_{k\ge n} A_k) \ge \epsilon$, $\forall n \in \mathbb{N}$, and therefore

$$\mu\Big(\bigcap_{n\in\mathbb{N}} \bigcup_{k\ge n} A_k\Big) \ge \epsilon,$$

from which the lemma follows. □

We are now in position to finish the proof of Theorem 2. Denoting by x the T-junction given by the previous lemma and choosing the appropriate subsequence, we have

$$E(\gamma_x^\ell) \equiv f_\ell(x) \leq 2^N \int_{I_{N,k}} f_\ell(n)dn + \frac{1}{N} \equiv f_\ell^N(x) + \frac{1}{N} < \infty.$$

By the weak compactness of the unit ball in $\mathrm{W}^{2,p}$ there exists a further subsequence, and a limit arc $\gamma_x^{N,k} \in \mathrm{W}^{2,p}$ such that

- $\gamma_x^\ell \rightharpoonup \gamma_x^{N,k}$ weakly in $\mathrm{W}^{2,p}([0,1],\mathbb{R}^2)$ (thus strongly in C^1).
- $E(\gamma_x^{N,k}) \leq \liminf_{\ell\to\infty} f_\ell(x)$, using (6), the lower semicontinuity of the $\mathrm{W}^{2,p}$ norm and the fact that $\mathcal{L}(\gamma_x^{N,k}) = \lim_{\ell\to\infty} \mathcal{L}(\gamma_x^\ell)$.

Since $\mathcal{H}^0(\partial\{U_0 \geq \lambda\})$ is finite, the limit arc $\gamma_x^{N,k}$ passes through another T-junction $y \in \mathcal{T}_{U_0(x)}$. Thus $\gamma_x^{N,k}$ can be extended outside Ω using arcs of $\{y \in \mathbb{R}^2 \setminus \Omega : U_0(y) = U_0(x)\}$. Let us prove that the extended curve, defined for example on $[-\epsilon, 1+\epsilon]$, is globally of class $\mathrm{W}^{2,p}$. We already know that it is $\mathrm{W}^{2,p}$ on each interval $[-\epsilon, 0]$, $[0,1]$ and $[1, 1+\epsilon]$. In addition, each arc γ_x^ℓ extends outside Ω into a globally C^1 arc. Since U_0 is analytic outside Ω and since the convergence of (γ_x^ℓ) holds also in the strong topology of C^1, we infer that $\gamma_x^{N,k}$ is in $\mathrm{C}^1([0,1],\mathbb{R}^2)$ and admits a globally C^1 extension. By the usual properties of Sobolev functions, it follows that $\gamma_x^{N,k}$ is of class $\mathrm{W}^{2,p}$ in $(-\epsilon, 1+\epsilon)$.

Since the $I_{N,k}$'s are countably many, we can again use a diagonal extraction to get a subsequence, still denoted as f_ℓ, and for each (N,k) a limit arc of class $\mathrm{W}^{2,p}([0,1])$, denoted as $\gamma^{N,k}$, such that

$$E(\gamma^{N,k}) \leq 2^N \int_{I_{N,k}} f^N(n)dn + \frac{1}{N} \quad (= 2^N \lim_{\ell\to\infty} \int_{I_{N,k}} f_\ell(n)dn + \frac{1}{N}).$$

Moreover, the limit curves $(\gamma^{N,k})_{N,k}$ do not cross by construction and can be extended outside Ω into globally $\mathrm{W}^{2,p}$ curves.

Let us now see how a limit curve can be defined for any T-junction. Given $x \in \mathcal{T}$, there exists for every $N \in \mathbb{N}$ some k_N such that $\mu([x_0, x]) \in I_{N,k_N}$. Considering the family of arcs $(\gamma^{N,k})_{N,k}$ defined above, it holds by definition

$$E(\gamma^{N,k_N}) \leq 2^N \int_{I_{N,k_N}} f^N(n)dn + \frac{1}{N} = f^N(x) + \frac{1}{N}$$

which converges - for μ-almost every x - to $f(x)$. Using the weak compactness of $\mathrm{W}^{2,p}$, there exists a subsequence, still denoted by $(\gamma^{N,k_N})_{N\in\mathbb{N}}$, that weakly converges in $\mathrm{W}^{2,p}$, thus uniformly in C^1, to a limit arc γ_x. By the lower semicontinuity of the $\mathrm{W}^{2,p}$ norm and the fact that $\mathcal{L}(\gamma_x) = \lim_{N\to\infty} \mathcal{L}(\gamma^{N,k_N})$, it follows that

$$E(\gamma_x) \leq \liminf_{N\to\infty} E(\gamma^{N,k_N}) \leq \liminf_{N\to\infty} (f^N(x) + \frac{1}{N}) = f(x).$$

This procedure can be applied for μ-almost every T-junction and thus one can define a limit amodal completion γ. In particular, two different arcs γ_{x_1} and γ_{x_2} cannot cross (but may intersect tangentially) since they are uniform limits of arcs $(\gamma^{N,k})_{N,k}$ that do not cross by construction.

There are two technical points that must be checked in this construction process :

1) Given $x \in \mathcal{T}_\lambda$ and its associated limit curve $\gamma_x \in \mathrm{W}^{2,p}(0,1)$, one has $\gamma_x(1) \in \mathcal{T}_\lambda$. Thus γ_x can be extended outside Ω using arcs of $\{y \in \mathbb{R}^2 \setminus \Omega : U_0(y) = U_0(x)\}$ into a curve γ_x^ϵ defined on $[-\epsilon, 1+\epsilon]$. Following the same argument as above, we prove that γ_x^ϵ is of class $\mathrm{W}^{2,p}$ on $(-\epsilon, 1+\epsilon)$;

2) We must control whether the curve γ_x passing through another T-junction $y \in \mathcal{T}_{U_0(x)}$ coincides with γ_y. The answer is positive because there are finitely many T-junctions per level and because the convergence of the curves is meant in the strong topology of C^1.

Finally, we have built a limit amodal completion γ defined for μ-almost every $x \in \mathcal{T}$ and such that each curve γ_x is of class $\mathrm{W}^{2,p}$ and can be extended outside Ω into a globally $\mathrm{W}^{2,p}$ curve whose restriction to $\mathbb{R}^2 \setminus \Omega$ coincides with arcs of $\{y \in \mathbb{R}^2 \setminus \Omega : U_0(y) = U_0(x)\}$. Moreover,

$$E(\gamma_x) \leq f(x), \qquad \mu - \text{a.e. } x \in \mathcal{T}.$$

It follows from Fatou's Lemma that

$$\begin{aligned}
\mathcal{E}(\gamma) &= \int_{\partial\Omega} E(\gamma_x)d\mu(x) \leq \int_{\partial\Omega} f(x)d\mu(x) = \int_0^1 f(n)dn \\
&\leq \liminf_{N\to\infty} \int_0^1 f^N(n)dn \leq \liminf_{N\to\infty} \liminf_{\ell\to\infty} \int_0^1 f_\ell^N(n)dn = \liminf_{\ell\to\infty} \int_0^1 f_\ell(n)dn
\end{aligned}$$

Thus

$$\mathcal{E}(\gamma) \leq \liminf_{\ell\to\infty} \mathcal{E}(\gamma^\ell) = \inf_{\tilde{\gamma}\in\mathcal{D}} \mathcal{E}(\tilde{\gamma})$$

and we have proved that the limit amodal completion is optimal. □

Remark 2 The previous theorem involves a definition of convergence in the class of amodal completions, namely, $(\gamma_h)_{h\in\mathbb{N}} \to \gamma$ if

1. for each dyadic interval $[k_N 2^{-N}, (k_N+1)2^{-N})$, there exists an appropriate point $x_{k_N,N}$ in the interval such that $\gamma_h(x_{k_N,N})$ converges weakly in $\mathrm{W}^{2,p}$ to $\gamma(x_{k_N,N})$ as $h \to \infty$;
2. for μ-almost every $x \in \partial\Omega$, γ_x is the weak limit in $\mathrm{W}^{2,p}$ of a sequence $(\gamma(x_{k_N,N}))_{N\in\mathbb{N}}$ where $x_{k_N,N} \to x$ as $N \to \infty$.

In other words, for μ-almost every $x \in \partial\Omega$, there exists a sequence $(h_M, k_M)_{M\in\mathbb{N}}$ such that

$$x_{k_M,M} \to x, \quad h_M \to \infty \quad \text{and} \quad \gamma_{h_M}(x_{k_M,M}) \underset{\mathrm{W}^{2,p}}{\rightharpoonup} \gamma_x \quad \text{as } M \to \infty.$$

Corollary 1 *Let $(\gamma_h)_{h\in\mathbb{N}}$ be a sequence of amodal completions with uniformly bounded energies, i.e.*

$$\sup_{h\in\mathbb{N}} \mathcal{E}(\gamma_h) < \infty.$$

Then, possibly extracting a subsequence, there exists a limit amodal completion γ such that $(\gamma_h)_{h\in\mathbb{N}}$ converges to γ in the sense of Remark 2 above.

It is easily seen that the function u_γ associated with an amodal completion γ, as defined by Theorem 1, may have level lines that are not curves of γ (see figure 5). Consequently, the relationship between $\overline{\mathcal{F}}(u_\gamma)$ and $\mathcal{E}(\gamma)$ is not clear. The purpose of Lemma 7 is to provide a continuous function $u_h \in \mathcal{S}$ whose level lines are arbitrarily close to the curves of γ and such that $\mathcal{F}(u_h)$ is arbitrarily close to $\mathcal{E}(\gamma)$. We start with a lemma providing a way to separate the curves of an amodal completion.

Lemma 6 *Let γ be an amodal completion with finite energy $\mathcal{E}(\gamma)$. Then for every $\eta > 0$ there is another amodal completion γ^η without contact such that $|\mathcal{E}(\gamma^\eta) - \mathcal{E}(\gamma)| \leq \eta$ and for μ-almost every $x \in \partial\Omega$,*

$$\sup_{s\in[0,1]} |\gamma_x^\eta(s) - \gamma_x(s)| \leq \eta.$$

Proof: The proof is tedious, but not deep. Let us consider a dense set $\{x_n\}_{n\in\mathbb{N}}$ of T-junctions in $\mathcal{T}$ such that all the curves $\gamma_n = \gamma_{x_n}$ have finite energy $E(\gamma_n) < \infty$. The idea of the proof is to move all curves of the amodal completion smoothly and slightly in such a way that they all fall apart from the γ_n's. In other terms, we shall create around each γ_n an open security region – that can be seen as a dilation of γ_n – where no other curve can pass.

Given two curves γ_x and γ_y with $x \neq y$, there always exists a curve γ_n which separates them in wide sense. After the dilation, γ_x and γ_y will not touch anymore. In that way, any two distinct curves γ_x and γ_y will be separated by an open domain and have therefore a positive distance to each other. This argument needs some detail. Indeed, notice that a curve γ_n may meet the boundary and that it may meet itself. Thus, one must be careful to move the curve away from the boundary and to move it apart from itself at points where it is tangent to itself. The dilations of γ_n will be done by smooth diffeomorphisms close enough to identity, which will increase very little the energy of the curves.

Step 1. Dividing all curves γ_n into graphs
Let us start by covering the domain $(0, L_n)$ of γ_n with a finite set of open intervals (s_i^n, t_i^n), $i \in [0, N_n]$ such that

1. $s_i^n < s_{i+1}^n < t_i^n < t_{i+1}^n$, $s_0^n = 0$, $t_{N_n}^n = L_n$,
2. γ_n restricted to $[s_i^n, t_i^n]$ is a graph,
3. the restriction of γ_n to $[s_i^n, t_i^n]$ meets $\partial\Omega$ at most on one side.

For simplicity, let us index by $i \in \mathbb{N}$ all the pieces of curves of all γ_n.

Step 2. Defining a diffeomorphism dilating locally γ_n.
On the interval $[s_i, t_i]$, the curve γ_n is represented as a graph $\Gamma_i = \{(x, f_i(x))\}$, $x \in [0, x_i]$ in local coordinates (x, y). The third condition implies that if Γ_i touches $\partial\Omega$ at several points, then it is only from above or only from below. Assume that it is from above, the other case being similar (the case where Γ_i does not touch $\partial\Omega$ can be treated using indifferently one or the other way). There exists a C^∞ function $y = \psi_i(x)$ such that $\psi_i(x) > f_i(x)$ on $(0, x_i)$, $\psi_i(0) = f_i(0)$, $\psi_i(x_i) = f_i(x_i)$ and the open domain $\mathcal{D}_i = \{(x, y),\ 0 < x < x_i,\ f_i(x) < y < \psi_i(x)\}$ is contained in Ω (see figure 9).

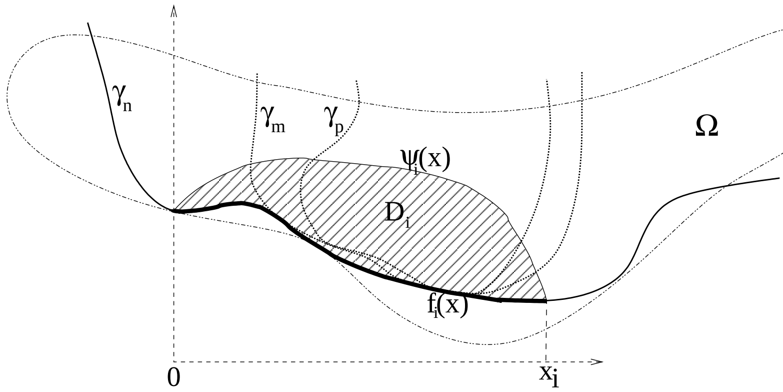

Figure 9: The operator Θ_i^ε differs from the identity in $\mathcal{D}_i$. It is designed to move the curves γ_m and γ apart from γ_n and to move γ_n itself apart from $\partial\Omega$.

We consider the diffeomorphism of $\mathbb{R}^2$ defined by

$$\Theta_i^\epsilon(x,y) = \begin{cases} (x, y + \epsilon M_i e^{-\frac{1}{(\psi_i(x)-y)^+}} \cdot e^{-\frac{1}{x^+}-\frac{1}{(x_i-x)^+}}) & \text{if } 0 \le x \le x_i,\ f_i(x) \le y \le \psi_i(x), \\ (x,y) & \text{otherwise} \end{cases}$$

with

$$M_i \le \inf_{0 \le x \le x_i} \frac{1}{1 + \psi_i'^2(x) + |\psi_i''(x)|}.$$

Obviously, Θ_i^ε is C^∞ and there exists a constant $C > 0$ independent of i and independent of the curve γ_n such that :

- $D\Theta_i^\varepsilon = Id + \varepsilon \Xi_i$, where Ξ_i is C^∞ and uniformly bounded on $[0, x_i]$ by C;
- $D^2\Theta_i^\varepsilon = \varepsilon \Phi_i$, where Φ_i is C^∞ and uniformly bounded on $[0, x_i]$ by C.

This follows immediately from the chain rule and the fact that $s \to e^{-\frac{1}{s^+}}$ is a C^∞ function with all derivatives bounded on $\mathbb{R}$.

Step 3. Using the diffeomorphism to separate all curves from γ_n and γ_n from $\partial\Omega$
Let us define the following operation, indexed by $i \in \mathbb{N}$. For every curve γ_x of the amodal completion, let us consider all maximal intervals (s,t) such that $\gamma_x(s,t) \subset \overline{\mathcal{D}_i}$ and replace γ_x on (s,t) by the new curve $\Theta_i^\varepsilon \circ \gamma_x$. In the particular case of the curve γ_n from which $\mathcal{D}_i$ has been defined, we rather replace γ_n on (s_i, t_i) by $\Theta_i^{\frac{\varepsilon}{2}} \circ \gamma_n$. Now, the curve γ_n may have multiple points on (s_i, t_i), like on figure 10; in this situation, γ_n will be no more a degenerate simple curve (i.e. a curve that becomes simple after an arbitrarily small deformation) if only $\gamma_n(s_i, t_i)$ is moved. So one must define a specific rule.

Let (s,t) be a maximal interval not intersecting (s_i, t_i) and such that $\gamma_n(s,t) \subset \overline{\mathcal{D}_i}$. If $\gamma(s,t) \ne \gamma_n(s_i, t_i)$ (see figure 10), let us consider that this part of γ_n is "above" the restriction to (s_i, t_i) and move it as are moved the other curves γ_x, i.e. replace γ_n on (s,t) by $\Theta_i^\varepsilon \circ \gamma_n$.

If instead $\gamma_n([s,t])$ coincides with $\gamma_n([s_i, t_i])$ (figures 11 and 12), we consider the maximal intervals $[\sigma_i, \tau_i] \supseteq [s_i, t_i]$ and $[\sigma, \tau] \supseteq [s,t]$ on which both arcs coincide. Assume for instance that $\gamma_n(\sigma_i) = \gamma_n(\tau)$ (these pieces of curves can also have same orientation, i.e. $\gamma_n(\sigma_i) = \gamma_n(\sigma)$). Consider the continuous unit normal $n(s)$ along γ_n such that on (s,t), $n(s)$ has an acute angle with the coordinate axis $(0,y)$. Consider two neighborhoods $\mathcal{V}(\sigma_i)$ and $\mathcal{V}(\tau)$ such that $\gamma_n(\mathcal{V}(\sigma_i))$ and $\gamma_n(\mathcal{V}(\tau))$ are graphs with respect to the reference frame $(\gamma_n(\sigma_i), \gamma_n'(\sigma_i), n(\sigma_i))$. If in these coordinates, the graph of γ_n around τ is above the graph of γ_n around σ_i (figure 11), we move $\gamma_n(s,t)$ like the other curves γ_x, i.e. we replace γ_n on (s,t) with $\Theta_i^\epsilon \circ \gamma_n$. If, instead, the graph of γ_n around τ is below the graph of γ_n around σ_i (figure 12), then $\gamma_n(s,t)$ is not moved. Doing this ensures that $\gamma_n(s,t)$ will be properly separated from $\gamma_n(s_i,t_i)$, i.e., without creating any new self-crossing.

An analogous procedure applies when $\gamma_n(\sigma_i) = \gamma_n(\sigma)$.

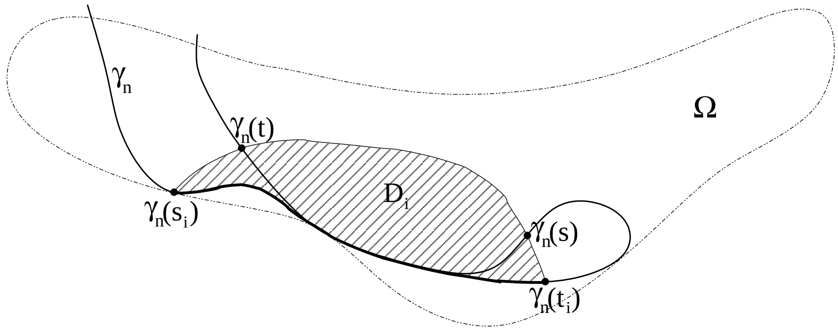

Figure 10: γ_n has autocontact on a maximal proper subset of $\gamma_n([s_i,t_i])$.

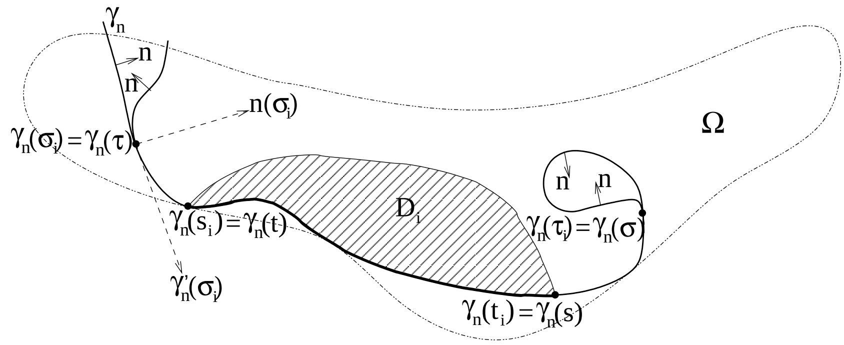

Figure 11: There exists an autocontact set that strictly contains $\gamma_n([s_i,t_i])$ and the curve folds back "from above".

Step 4. Checking that the moving apart does not increase much the energy of the amodal completion

By construction, there exists $C > 0$ such that $\|D\Theta_i^\varepsilon - Id\| \le C\varepsilon$ and $\|D^2\Theta_i^\varepsilon\| \le C\varepsilon$. It is easily checked that the energy of a curve γ_x deformed by Θ_i^ε satisfies

$$(1 - D\varepsilon)E(\gamma_x) \le E(\Theta_i^\varepsilon(\gamma_x)) \le (1 + D\varepsilon)E(\gamma_x)$$

for some constant $D > C$ independent of γ_x. Thus, taking η such that $D\varepsilon = \eta 2^{-i}$ and setting $\Theta_i = \Theta_i^\varepsilon$, we can ensure that the energy of the whole amodal completion, denoted by $\Theta_i(\gamma)$, satisfies

$$|\mathcal{E}(\Theta_i(\gamma)) - \mathcal{E}(\gamma)| \leq \eta 2^{-i}. \tag{13}$$

This also entails that for any pair of points z_1 and z_2 belonging to some curves γ_{x_1} and γ_{x_2},

$$(1 + \eta 2^{-i})|z_1 - z_2| \geq |\Theta_i(z_1) - \Theta_i(z_2)| \geq (1 - \eta 2^{-i})|z_1 - z_2|. \tag{14}$$

One moving apart operation therefore defines a new amodal completion with energy arbitrarily closed to the original and curves arbitrarily closed to the originals. The moving apart operation has then to be performed recursively at step i on the amodal completions resulting from the $i-1$ former operations. To formalize this, we set $T_i = \Theta_{i-1} \circ \Theta_i \circ \cdots \circ \Theta_2 \circ \Theta_1$ and $T(z) = \lim_{i\to\infty} T_i(z)$. So at the i-th step, all operations described in steps 1 to 5 are applied to the curves $T_{i-1}(\gamma_x)$ ($T_0 = Id$). From (14) follows that T is a bilipschitz map for $\eta < \frac{1}{2}$, so that open sets are mapped onto open sets.

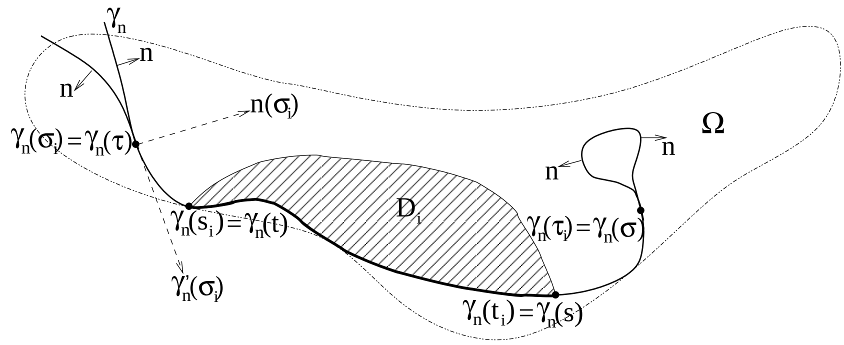

Figure 12: There exists an autocontact set that strictly contains $\gamma_n([s_i, t_i])$ and the curve folds back "from below".

Step 5. The moving apart operation isolates $\gamma_\mathbf{n}$ from all other curves and eliminates its self-contacts on $(\mathbf{s_i}, \mathbf{t_i})$
Indeed, the fact that we move γ_n only halfway at step i implies that $T_i(\gamma_n((s_i, t_i)))$ is contained in the open set $O_i = \mathcal{D}_i \setminus \Theta_i(\mathcal{D}_i)$, which contains no piece of no other curve of the amodal completion. Now, by the same argument as for T_i, $\tilde{T_i} = \lim_{k\to\infty} \Theta_k \circ \ldots \Theta_{i+1}$ also is a bilipschitz map. So the final position of $\gamma_n((s_i, t_i))$ is in the open domain $\tilde{T_i}(O_i)$. This being true for all i, we deduce that every curve $T(\gamma_n)$ is contained (except its endpoints) in an open set $\mathcal{C}_n$ which does not contain any other curve $T(\gamma_x)$.

Step 6. Iteration of the moving apart operation
The image of each curve γ_x at step i is given by $T_i(\gamma_x) = \Theta_i \circ \Theta_{i-1} \circ \cdots \circ \Theta_2 \circ \Theta_1(\gamma_x)$. By (14), the sequence of curves $T_i(\gamma_x)$ converges uniformly to a curve $T(\gamma_x)$ and by (13),

$$\sup_{s\in[0,1]} |T(\gamma_x)(s) - \gamma_x(s)| \leq \eta.$$

Thus, by Fatou's lemma,

$$|\mathcal{E}(T(\gamma)) - \mathcal{E}(\gamma)| \leq \eta.$$

Letting $\gamma^\eta = T(\gamma)$, the theorem ensues if we can prove that $T(\gamma)$ is without contact.

Step 7. The final amodal completion is without contact
Given two curves γ_x and γ_y of the amodal completion, there exists a curve γ_n which separates γ_x and γ_y, namely γ_x and γ_y do not belong to the same connected component of $\Omega \setminus \gamma_n$. Thus $T(\gamma_x)$ and $T(\gamma_y)$ are contained in two different connected components of $\Omega \setminus \mathcal{C}_n$ and therefore stand at a positive distance from each other.

Let us now deal with curves γ_x which have at least one self-meeting. Call loops of γ_x the open connected components of $\Omega \setminus \gamma_x$ whose boundary is fully contained in γ_x. If at least one loop of γ_x does not contain any piece of any other curve γ_n – which means that the previous procedure will let the loop unchanged – this is equivalent to saying that it does not contain any piece of any other curve γ_x. Since loops have positive measure, only a countable set of *curves* γ_x can have such empty loops. So we are allowed to add them up from the start to the curves γ_n. Thus, one may assume from now that all curves γ_x having a loop are such that the loop contains some piece of γ_n. This implies that the self-meeting points of γ_x also are self-meeting points for some γ_n and we conclude that the moving apart operation have moved them apart too. $\square$

Lemma 7 *Let $\gamma \in \mathcal{D}$ be an amodal completion with finite energy and u_γ the associated function in* $\mathrm{BV}(\mathbb{R}^2)$*. For every $h \in \mathbb{N}^\star$ there exists a continuous function $u_h \in \mathcal{S}$ such that $|\mathcal{E}(\gamma) - \mathcal{F}(u_h)| \leq 1/h$. In addition, u_h tends to u_γ in $\mathrm{L}^1(\mathbb{R}^2)$ as $h \to \infty$.*

Proof: In view of Lemma 6, for every $h \in \mathbb{N}^\star$ there exists an amodal completion without contact γ_h such that $|\mathcal{E}(\gamma_h) - \mathcal{E}(\gamma)| \leq 1/h$. By Theorem 1, γ_h can be associated with a function $u_h \in \mathcal{S}$ such that $\mathcal{E}(\gamma_h) = \mathcal{F}(u_h)$ thus $|\mathcal{E}(\gamma) - \mathcal{F}(u_h)| \leq 1/h$. In addition, u_h is continuous because its level lines are disjoint by construction. From the construction procedure of Theorem 1 and the fact that the curves of γ are uniform limits of curves of γ_h, we also deduce that u_h tends to u_γ almost everywhere on Ω. Remark now that, by construction, for every $h \in \mathbb{N}^\star$ and for almost every $x \in \Omega$, $|u_h(x)| \leq \max_{y \in \partial\Omega} |U_0(y)|$. It follows by the Dominated Convergence Theorem that u_h tends to u_γ in $\mathrm{L}^1(\mathbb{R}^2)$ as $h \to \infty$. $\square$

Lemma 8 *Let $u \in \mathrm{C}^2(\mathbb{R}^2)$ such that $F(u) < \infty$ and u coincide with U_0 outside Ω. Then there exists an amodal completion γ whose trace is contained in the topographic map of u, i.e. for almost every $\lambda \in \Lambda$, $\bigcup_{x \in \mathcal{T}_\lambda}(\gamma_x) \subset \{u = \lambda\} \cap \overline{\Omega}$. Consequently,*

$$\mathcal{E}(\gamma) \leq \mathcal{F}(u) = F(u).$$

Proof: This amodal completion will be constructed as a selection of level lines of u inside Ω. By Sard Lemma we can find a set $\tilde{\Lambda} \subset \Lambda$ of full measure such that $\{u = \lambda\} \cap \Omega$ is a union of C^2 curves. Of course we also have $\mu(\{x \in \partial\Omega,\ u(x) \notin \tilde{\Lambda}\}) = 0$. For every T-junction $x \in \mathcal{T}$ such that $u(x) \in \tilde{\Lambda}$, the level line L_x of u passing by x is by definition of $\tilde{\Lambda}$ a C^2 Jordan curve and intersects $\partial\Omega$ at some other T-junction $y \in \mathcal{T}_{u(x)}$. We take γ_x to be a C^2 parameterization on $[0,1]$ of the arc of L_x between x and y. The map $\gamma : x \in \mathcal{T} \cap u^{-1}(\tilde{\Lambda}) \mapsto \gamma_x$ is clearly an amodal completion whose trace is contained in the topographic map of u. The inequality $\mathcal{E}(\gamma) \leq \mathcal{F}(u)$ is then an obvious consequence of the coarea formula, as $\mathcal{E}(\gamma)$ is obtained from $\mathcal{F}$ by a restriction to the levels of $\tilde{\Lambda}$ and a selection of pieces of level lines at these levels. $\square$

5 Comparison with the direct variational approach

This section is devoted to the proof that the problems (P_1) and (P_2) are equivalent. We do not know whether they also are equivalent with (P_2'), which would actually be true if one could prove that for $u \in \mathcal{S}$, $\mathcal{F}(u) = \overline{F}(u)$.

Let us start with the proof that (P_2) is well posed.

Theorem 3 *The problem*

$$\min\{\overline{\mathcal{F}}(u) : \, u = U_0 \text{ on } \mathbb{R}^2 \setminus \Omega\} \tag{P_2}$$

has at least one solution $u \in \mathrm{BV}(\mathbb{R}^2)$. In addition, for almost every $\lambda \in \mathbb{R}$, there exists a finite family $\Gamma^\lambda = \{\gamma_i^\lambda\}_{i \in I_\lambda}$ of regular curves of class $\mathrm{W}^{2,p}$ such that $\partial^*\{u \geq \lambda\} \cap \Omega \subset \bigcup_{i \in I_\lambda}(\gamma_i^\lambda)$ up to a $\mathcal{H}^1$-negligible set and any two curves of Γ^λ may intersect but only tangentially and without crossing each other.

Proof: Let $(u_h)_{h \in \mathbb{N}}$ be a minimizing sequence. Without loss of generality, let us assume that $\sup_{h \in \mathbb{N}} \overline{\mathcal{F}}(u_h) < +\infty$.

Observe that, by the sequential characterization of relaxation [19], every $v \in \mathrm{L}^1(\mathbb{R}^2)$ such that $\overline{\mathcal{F}}(v) < \infty$ and v coincides with U_0 outside Ω is the limit in $\mathrm{L}^1(\mathbb{R}^2)$ of a sequence $(v_k)_{k \in \mathbb{N}}$ in $\mathcal{S}$ such that $\overline{\mathcal{F}}(v) = \lim_{k \to \infty} \mathcal{F}(v_k)$. Since, by the coarea formula, $\mathcal{F}(v_k) \geq |Dv_k|(\Omega)$, it follows from the lower semicontinuity of perimeter that $\overline{\mathcal{F}}(v) \geq |Dv|(\Omega)$.

Thus, $\sup_{h \in \mathbb{N}} \overline{\mathcal{F}}(u_h) < +\infty$ implies that $\sup_{h \in \mathbb{N}} |Du_h|(\Omega) < +\infty$. Since every u_h coincides with $U_0 \in \mathrm{BV}(\mathbb{R}^2)$ outside Ω, it follows that $\sup_{h \in \mathbb{N}} |Du_h|(\mathbb{R}^2) < +\infty$ and the generalized Poincaré inequality in Theorem 5.11.1 of [39] shows that $\sup_{h \in \mathbb{N}} |||u_h\|_{\mathrm{L}^1(\mathbb{R}^2)} < +\infty$. Hence there exists a subsequence, still denoted by $(u_h)_{h \in \mathbb{N}}$, and a limit function $u \in \mathrm{BV}(\mathbb{R}^2)$ such that $(u_h)_{h \in \mathbb{N}}$ converges to u in $\mathrm{L}^1(\mathbb{R}^2)$ and u coincides with U_0 outside Ω. Furthermore, by the lower semicontinuity of relaxed functionals [19],

$$\overline{\mathcal{F}}(u) \leq \liminf_{h \to \infty} \overline{\mathcal{F}}(u_h) = \inf\{\overline{\mathcal{F}}(u) : \, u = U_0 \text{ on } \mathbb{R}^2 \setminus \Omega\},$$

thus u is a solution of (P_2).

Since $\overline{\mathcal{F}}(u) < \infty$ and u coincides with U_0 outside Ω, there exists a sequence of functions $\{v_h\}_{h \in \mathbb{N}} \subset \mathcal{S}$ converging to u in $\mathrm{L}^1(\mathbb{R}^2)$ and such that $\overline{\mathcal{F}}(u) = \lim_{h \to \infty} \mathcal{F}(v_h)$. By definition, for every $h \in \mathbb{N}$ and for almost every $\lambda \in \mathbb{R}$, $\partial^*\{v_h \geq \lambda\} \cap \Omega$ essentially coincides with a finite union of curves of class $\mathrm{W}^{2,p}$. In addition,

$$\mathcal{F}(v_h) = \int_{-\infty}^{+\infty} \int_{\Omega \cap \partial^*\{v_h \geq \lambda\}} (1 + |\kappa|^p) d\mathcal{H}^1 \, d\lambda.$$

By Fatou's lemma we get that

$$\int_{-\infty}^{+\infty} \liminf_{h \to \infty} \int_{\Omega \cap \partial^*\{v_h \geq \lambda\}} (1 + |\kappa|^p) d\mathcal{H}^1 \, d\lambda \leq \liminf_{h \to \infty} \int_{-\infty}^{+\infty} \int_{\Omega \cap \partial^*\{v_h \geq \lambda\}} (1 + |\kappa|^p) d\mathcal{H}^1 \, d\lambda < +\infty.$$

Thus, for almost every $\lambda \in \mathbb{R}$,

$$\liminf_{h \to \infty} \int_{\Omega \cap \partial^*\{v_h \geq \lambda\}} (1 + |\kappa|^p) d\mathcal{H}^1 < \infty.$$

Since $v_h \to u$ in $L^1(\mathbb{R}^2)$, the Cavalieri's principle implies that, possibly taking a subsequence and reindexing by h, the sequence of characteristic functions $(\chi_{\{v_h \geq \lambda\}})_{h\in\mathbb{N}}$ converges to $\chi_{\{u\geq\lambda\}}$ in $\mathrm{L}^1(\mathbb{R}^2)$ for almost every $\lambda \in \mathbb{R}$. Then the lower semicontinuity of $\overline{\mathcal{F}}$ shows that

$$\overline{\mathcal{F}}(\chi_{\{u\geq\lambda\}}) \leq \liminf_{h\to\infty} \mathcal{F}(\chi_{\{v_h\geq\lambda\}}) = \liminf_{h\to\infty} \int_{\Omega\cap\partial^*\{v_h\geq\lambda\}} (1+|\kappa|^p) d\mathcal{H}^1 < \infty.$$

The theorem ensues by a straightforward application of Theorem 4.1 in [7]. □

Remark 3 An interesting consequence of the next result is that it provides an explicit integral formulation for $\overline{\mathcal{F}}(u)$ when u is a minimizer of (P_2). Such a formulation could also be obtained, under a slightly different form, by combining the direct method developed in [4] with Theorem 8.6 in [8]. Indeed, by passing to a subsequence for which there is convergence in L^1 of the characteristic functions of almost every level set, it can be easily proved that almost every limit set has finitely many singularity points. Then Theorem 8.6 in [8] provides an explicit formula for the relaxed energy of the limit set, from which an expression of $\overline{\mathcal{F}}(u)$ is easily deduced when u is a minimizer of (P_2).

Theorem 4 *Problems (P_1) and (P_2) are equivalent, i.e.*

$$\min\{\mathcal{E}(\gamma) : \gamma \in \mathcal{D}\} = \min\{\overline{\mathcal{F}}(u) : u = U_0 \text{ on } \mathbb{R}^2 \setminus \Omega\}$$

In addition, if $\gamma \in \mathcal{D}$ is a minimizer of (P_1) then u_γ is a minimizer of (P_2) and, in particular, $\overline{\mathcal{F}}(u_\gamma) = \mathcal{E}(\gamma)$. Conversely, if u is a minimizer of (P_2) then there exists an amodal completion γ_u that minimizes (P_1) and whose associated function u_γ coincides with u almost everywhere. In particular, $\overline{\mathcal{F}}(u) = \mathcal{E}(\gamma_u) = \overline{\mathcal{F}}(u_\gamma)$ and for almost every $\lambda \in \mathbb{R}$,

$$\partial^*\{u \geq \lambda\} \cap \Omega = \partial^*\{u_\gamma \geq \lambda\} \cap \Omega \subset \bigcup_{x\in\mathcal{T}_\lambda} \gamma_u(x) \quad \text{up to a } \mathcal{H}^1\text{-negligible set.}$$

PROOF: Let u be a minimizer of (P_2). Because $\overline{\mathcal{F}}(u) < \infty$ and u coincides with U_0 outside Ω, there exists a sequence $(u_h)_{h\in\mathbb{N}}$ of functions in $\mathcal{S}$ such that u_h tends to u in $\mathrm{L}^1(\mathbb{R}^2)$ and $\mathcal{F}(u_h)$ converges to $\overline{\mathcal{F}}(u)$. According to Lemma 3, we can associate with each u_h an amodal completion γ_h such that $\mathcal{F}(u_h) = \mathcal{E}(\gamma_h)$. Since $\sup_{h\in\mathbb{N}} \mathcal{E}(\gamma_h) < \infty$, there exists by Corollary 1 a limit amodal completion γ such that (γ_h) converges to γ (in the sense of Remark 2) and $\mathcal{E}(\gamma) \leq \liminf_{h\to\infty} \mathcal{E}(\gamma_h) = \overline{\mathcal{F}}(u)$. It follows that

$$\min\{\mathcal{E}(\gamma) : \gamma \in \mathcal{D}\} \leq \min\{\overline{\mathcal{F}}(u) : u = U_0 \text{ on } \mathbb{R}^2 \setminus \Omega\}.$$

Conversely, let γ be a minimizer of (P_1). By Lemma 7, for each $k \in \mathbb{N}^\star$ there exists a continuous $u_k \in \mathcal{S}$ such that $|\mathcal{E}(\gamma) - \mathcal{F}(u_k)| \leq 1/k$ and, in addition, u_k tends to u_γ in $\mathrm{L}^1(\mathbb{R}^2)$ as $k \to \infty$, where u_γ is associated with γ through Theorem 1. The lower semicontinuity of $\overline{\mathcal{F}}$ shows that $\overline{\mathcal{F}}(u_\gamma) \leq \liminf_{h\to\infty} \mathcal{F}(u_k) = \mathcal{E}(\gamma)$, therefore

$$\min\{\mathcal{E}(\gamma) : \gamma \in \mathcal{D}\} \geq \min\{\overline{\mathcal{F}}(u) : u = U_0 \text{ on } \mathbb{R}^2 \setminus \Omega\}$$

thus

$$\min\{\mathcal{E}(\gamma) : \gamma \in \mathcal{D}\} = \min\{\overline{\mathcal{F}}(u) : u = U_0 \text{ on } \mathbb{R}^2 \setminus \Omega\}. \tag{15}$$

In particular, $\overline{\mathcal{F}}(u_\gamma) = \min\{\overline{\mathcal{F}}(u) : u = U_0 \text{ on } \mathbb{R}^2 \setminus \Omega\} = \mathcal{E}(\gamma)$ and therefore u_γ is a minimizer of (P_2).

Take now again a minimizer u of (P_2). Like above we may find a sequence (u_h) of functions in $\mathcal{S}$ and their associated amodal completions (γ_h) such that $u_h \to u$ in $\mathrm{L}^1(\mathbb{R}^2)$, $\overline{\mathcal{F}}(u) = \lim_{h\to\infty} \mathcal{F}(u_h)$, $\mathcal{F}(u_h) = \mathcal{E}(\gamma_h)$ and (γ_h) converges (in the sense of Remark 2) to a limit amodal completion γ_u such that $\mathcal{E}(\gamma_u) \leq \liminf_{h\to\infty} \mathcal{E}(\gamma_h) = \liminf_{h\to\infty} \mathcal{F}(u_h) = \overline{\mathcal{F}}(u)$. By (15), $\mathcal{E}(\gamma_u) = \overline{\mathcal{F}}(u)$ and γ_u is a minimizer of (P_1).

Let u_γ denote the function associated with γ_u according to Theorem 1. By (12), by the fact that curves of γ are uniform limits of curves of γ_h, that all functions coincide with U_0 on $\partial\Omega$ and by the construction procedure of Theorem 1, it follows that u_h tends to u_γ a.e. on Ω. Since, by definition, (u_h) also tends to u in $\mathrm{L}^1(\mathbb{R}^2)$, it ensues that u and u_γ coincide almost everywhere on Ω, thus on $\mathbb{R}^2$ because both functions coincide with U_0 outside Ω. Finally, (15) shows that $\overline{\mathcal{F}}(u) = \mathcal{E}(\gamma_u) = \overline{\mathcal{F}}(u_\gamma)$ and Theorem 1 yields that for almost every $\lambda \in \mathbb{R}$,

$$\partial^*\{u \geq \lambda\} \cap \Omega = \partial^*\{u_\gamma \geq \lambda\} \cap \Omega \subset \bigcup_{x\in\mathcal{T}_\lambda} \gamma_u(x) \quad \text{up to a } \mathcal{H}^1\text{-negligible set.}$$

□

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