Luca's

Second question

↓

↓

↓

Luca's second question

Hello Again!

Chevalley's thm.

$G \subset GL(V)$ finite subgroup.

Thm Are equivalent:

(1) G is generated by q. reflection

(2) SV is a free SV^G module

(3) SV^G is a polynomial algebra

~~Lucas~~ Proof $(1) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$

$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$

Lucas question:

$(3) \Rightarrow (1)$

$G \supset K$ subgroup

generated by q. reflections.

$K \triangleleft G$.

I used that $(SV)^K$ is a polynomial algebra.

It is because I already proved $(1) \Rightarrow (3)$.

2 answers:

(1) $(SV)^K \simeq SU$ where U is a G/K-module

(2) Indeed of all this is not really necessary.

Proof of (1) | $R := (SV)^K$. It is known to be a a poly. alg.

Nakayama lemma (see this video Hilbert finitud. thm).

Choose any graded vector space $U \subset R^+$ s.t. $U \oplus (R^+)^2 = R^+$ then U generates R as an alg. therefore $R \cong SU$.

R is a G/K-module.
On cane choose U to bea G|K-module.

$$SV^G \underset{\sim}{=} (S\underset{\sim}{U})^{G/K}$$

Reduced the proof to the case where G has no q. reflections.

$R = \mathbb{C}[y_1, \cdots, y_n]$

$R^+ = \Sigma\ y_i R$

$U \subset R^+$ vector space

s.t. $U \oplus (R^+)^2 = R^+$

Example $U = \mathbb{C} \cdot x_1 \oplus \cdots \oplus \mathbb{C} x_n$

In such case $R = SU$

In general U is not as gentle

$SU \twoheadrightarrow \mathbb{C}[x_1, \dots x_n]$

Why injehve.

~~X~~$X_1 \dots X_n$ basis of U.

If $SU \longrightarrow \mathbb{C}[x_1 \dots x_n]$

not inj. then

$H(X_1, \dots X_n) = 0$ for some H.

$\Rightarrow$ dim R_m bounded by a polynomial of degrée $\leq n-2$

---

dim $(\mathbb{C}[x_1 \dots x_n])_m$

polynomial in m of degré $(n-1)$

Atiaygh- McDonald

Hilbert polynomial.

Trick if R $\mathbb{Z}$-graded commutatne generated by R_1, then $\dim R_m$ = polynomial in m for $m >> 0$ and the degree is $\dim \operatorname{Spec} R - 1$

$R = \mathbb{C}[t]$ $\qquad$ $\operatorname{Spc} R = \mathbb{C}$

$\dim R_m = 1$

THM : (Hilbert)

$$\sum_{m=0}^{\infty} (\dim R_m) z^m = \text{Polynome}(z) / (1-z)^n$$

where $n = \dim R_1$

$R = S U$

$U = U_1 \oplus U_2 \oplus \cdots \oplus U_d.$

THM $R = SU / \underset{\text{graded}}{\text{ideal}}$

$$\sum \dim R_m z^m$$

$$= \text{Polynomial}(z) \Big/ \prod (1-z^i)^{d_i}$$

$d_i = \dim U_i$

$\Rightarrow$ R_m is a periodic poly. for $m \gg 0$