Which Nilpotent Groups are Self-Similar? *

Olivier Mathieu [†]

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Abstract

Let Γ be a finitely generated torsion free nilpotent group, and let A^ω be the space of infinite words over a finite alphabet A. We investigate two types of self-similar actions of Γ on A^ω, namely the faithfull actions with dense orbits and the free actions. A criterion for the existence of a self-similar action of each type is established.

Two corollaries about the nilmanifolds are deduced. The first involves the nilmanifolds endowed with an Anosov diffeomorphism, and the second about the existence of an affine structure.

Then we investigate the virtual actions of Γ, i.e. actions of a subgroup Γ' of finite index. A formula, with some number theoretical content, is found for the minimal cardinal of an alphabet A endowed with a virtual self-similar action on A^ω of each type.

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Introduction

1. General introduction

Let A be a finite alphabet and let A^ω be the topological space of infinite words $a_1 a_2 \ldots$ over A, where the topology of $A^\omega = \varprojlim A^n$ is the pro-finite topology.

An *action* of a group Γ on A^ω is called *self-similar* iff for any $\gamma \in \Gamma$ and $a \in A$ there exists $\gamma_a \in \Gamma$ and $b \in A$ such that
$$\gamma(aw) = b\gamma_a(w) \text{ for any } w \in A^\omega.$$
The group Γ is called *self-similar* (respectively *densely self-similar*, respectively *freely self-similar*, respectively *freely densely self-similar*) if Γ admits a faithfull self-similar action (respectively a faithfull self-similar action with dense orbits, respectively a free self-similar action, respectively a free self-similar action with dense orbits) on A^ω for some finite alphabet A.

Self-similar groups appeared in the early eighties, in the works of R. Grigorchuk [10] [11] and in the joint works of N. Gupta and S. Sidki [13] [14]. See also the monography [24] for an extensive account before 2005 and [25] [2] [9] [16] [12] for more recent works. A general question is
    which groups Γ are (merely, or densely ...) self-similar?

This paper brings an answer for finitely generated torsion-free nilpotent groups Γ, called FGTF nilpotent groups in the sequel. Then we will connect the main result with topics involving differential geometry and arithmetic groups.

The systematic study of self-similar actions of nilpotent groups started with [4], and the previous question has been raised in some talks of S. Sidki.

2. The main results

A few definitions are now required. A *grading* of a Lie algebra $\mathfrak{m}$ is a decomposition $\mathfrak{m} = \oplus_{n \in \mathbb{Z}} \mathfrak{m}_n$ such that $[\mathfrak{m}_n, \mathfrak{m}_m] \subset \mathfrak{m}_{n+m}$ for all $n, m \in \mathbb{Z}$. It is called *special* if $\mathfrak{m}_0 \cap \mathfrak{z} = 0$, where $\mathfrak{z}$ is the center of $\mathfrak{m}$. It is called *very special* if $\mathfrak{m}_0 = 0$.

Let's assume now that Γ is a FGTF nilpotent group. By Malcev Theory [18][26], Γ is a cocompact lattice in a unique connected, simply connected (or CSC in what follows) nilpotent Lie group N. Let $\mathfrak{n}^\mathbb{R}$ be the Lie algebra of N and set $\mathfrak{n}^\mathbb{C} = \mathbb{C} \otimes_\mathbb{R} \mathfrak{n}^\mathbb{R}$.

The main results, proved in Section 7, are the following

Theorem 2. *The group Γ is densely self-similar iff the Lie algebra $\mathfrak{n}^\mathbb{C}$ admits a special grading.*

Theorem 3. *The following assertions are equivalent*
    (i) The group Γ is freely self-similar,
    (ii) the group Γ is freely densely self-similar, and
    (iii) the Lie algebra $\mathfrak{n}^\mathbb{C}$ admits a very special grading.

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As a consequence, let's mention

Corollary 4. *Let M be a nilmanifold endowed with an Anosov diffeomorphism. Then there a free self-similar action of $\pi_1(M)$ with dense orbits on A^ω, for some finite A.*

Corollary 8. *Let M be a nilmanifold. If $\pi_1(M)$ is freely self-similar, then M is affine complete.*

Among FGTF nilpotent groups, some of them are self-similar but not densely self-similar. Some of them are not even self-similar, since Theorem 2 implies the next

Corollary 7. *Let M be one of the non-affine nilmanifolds appearing in [3]. Then $\pi_1(M)$ is not self-similar.*

3. A concrete version of Theorems 2 and 3
Let N be a CSC nilpotent Lie group, with Lie algebra $\mathfrak{n}^{\mathbb{R}}$. Let's assume that N contains some cocompact lattices Γ. By definition, the *degree* of a self-similar action of Γ on A^ω is Card A. We ask the following question
    For a given cocompact lattice $\Gamma \subset N$, what is the minimal degree
    degree of a faithfull (or a free) self-similar action with dense orbits?
More notions are now defined. Recall that the *commensurable class* ξ of a cocompact lattice $\Gamma_0 \subset N$ is the set of all cocompact lattices of N which share with Γ_0 a subgroup of finite index. The *complexity* $\mathrm{cp}\,\xi$ (respectively the free complexity $\mathrm{fcp}\,\xi$) of the class ξ is the minimal degree of a self-similar action of Γ with dense orbits (respectively a free self-similar action of Γ), for some $\Gamma \in \xi$.

For any algebraic number $\lambda \neq 0$, set $d(\lambda) = \mathrm{Card}\,\mathcal{O}(\lambda)/\pi_\lambda$, where $\mathcal{O}(\lambda)$ is the ring of integers of $\mathbb{Q}(\lambda)$ and $\pi_\lambda = \{x \in \mathcal{O}(\lambda) | x\lambda \in \mathcal{O}(\lambda)\}$. For any isomorphism h of a finite dimensional vector space over $\mathbb{Q}$, set
$$\mathrm{ht}\,h = \prod\nolimits_{\lambda \in \mathrm{Spec}\,h/\mathrm{Gal}(\mathbb{Q})} d(\lambda)^{m_\lambda},$$
where $\mathrm{Spec}\,h/\mathrm{Gal}(\mathbb{Q})$ is the list of eigenvalues of h up to conjugacy by $\mathrm{Gal}(\mathbb{Q})$ and where m_λ is the multiplicity of the eigenvalue λ.

By Malcev's Theory, the commensurable class ξ determines a canonical $\mathbb{Q}$-form $\mathfrak{n}(\xi)$ of the Lie algebra $\mathfrak{n}^{\mathbb{R}}$. Let $\mathcal{S}(\mathfrak{n}(\xi))$ (respectively $\mathcal{V}(\mathfrak{n}(\xi))$) be the set of all $f \in \mathrm{Aut}\,\mathfrak{n}(\xi)$ such that $\mathrm{Spec}\,f|_{\mathfrak{z}}^{\mathbb{C}}$ (respectively $\mathrm{Spec}\,f$) contains no algebraic integer.

Theorem 9. *We have*

$$\operatorname{cp} \xi = \operatorname{Min}_{h \in \mathcal{S}(\mathfrak{n}(\xi))} \operatorname{ht} h, \; and$$
$$\operatorname{fcp} \xi = \operatorname{Min}_{h \in \mathcal{V}(\mathfrak{n}(\xi))} \operatorname{ht} h.$$

If, in the previous statement, $\mathcal{S}(\mathfrak{n}(\xi))$ is empty, then the equality $\operatorname{cp} \xi = \infty$ means that no $\Gamma \in \xi$ admits a faithfull self-similar action with dense Γ-orbits.

Theorem 9 answers the previous question only for the commensurable classes ξ. For an individual $\Gamma \in \xi$, it provides some ugly estimates for the minimal degree of Γ-actions, and nothing better can be expected.

The framework of nonabelian Galois cohomology shows the concreteness of Theorem 9. Up to conjugacy, the commensurable classes in N are classified by the $\mathbb{Q}$-forms of some classical objects with a prescribed $\mathbb{R}$-form, see Corollary 4 of ch. 9, and their complexity is an invariant of the arithmetic group $\operatorname{Aut} \mathfrak{n}(\xi)$.

As an illustration of the previous vague sentence, we investigate a class $\mathcal{N}$ of CSC nilpotent Lie groups N, with Lie algebra $\mathfrak{n}^{\mathbb{R}}$. The commensurable classes $\xi(q)$ in N are classified, up to conjugacy, by the positive definite quadratic forms q on $\mathbb{Q}^2$. Then, we have

$$\operatorname{cp} \xi(q) = F(d)^{e(N)}$$

where $e(N)$ is an invariant of the special grading of $\mathbb{C} \otimes \mathfrak{n}^{\mathbb{R}}$, where $-d$ is the discriminant of q, and where $F(d)$ is the norm of a specific ideal in $\mathbb{Q}(\sqrt{-d})$, see Theorem 11 and Lemma 28.

In particular, N contains some commensurable classes of arbitrarily high complexity. In a forthcoming paper [22], more complicated examples are investigated, but the formulas are less explicit.

4. About the proofs. The proofs of the paper are based on different ideas. Theorem 1, which is a statement about rational points of algebraic tori, is the key step in the proof of Theorems 2, 3 and 11. It is based on standard results of number theory, including the Cebotarev's Theorem. It is connected with the density of rational points for connected groups proved by A. Borel [6], see also [27].

Also, the proof of Corollary 4 is based on a paper of A. Manning [20] about Anosov diffeomorphisms. The proof of Corollary 7 is based on very difficult computations, which, fortunately, were entirely done in [3].

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1 Self-similar actions and self-similar data

Let Γ be a group. This section explains the correspondence between the faithfull transitive self-similar Γ-actions and some virtual endomorphisms of Γ, called *self-similar data*. Usually self-similar actions are actions on a rooted tree A^*, see [24]. Here the groups are acting on the boundary A^ω of A^*. This equivalent viewpoint is better adapted to our setting.

1.1 Transitive self-similar actions
In addition of the definitions of the introduction, the following technical notion of transitivity will be used.

A self-similar action of Γ on A^ω induces an action of Γ on A. Indeed, for a, $b \in A$ and $\gamma \in \Gamma$, we have $\gamma(a) = b$ if
$$\gamma(aw) = b\gamma_a(w),$$
for all $w \in A^\omega$. A self-similar action is called *transitive* if the induced action on A is transitive. The group Γ is called *transitive self-similar* if it admits a faithfull transitive self-similar action.

Similarly the self-similar action of Γ on A^ω induces an action of Γ on each level set A^n. Such an action is often called *level transitive* if Γ acts transitively on each level A^n. Obviously, the level transitive actions on A^* of [24] corresponds with the actions on A^ω with dense orbits.

1.2 Core and f-core
Let Γ be a group and Γ' be a subgroup. The *core* of Γ' is the biggest normal subgroup $K \lhd G$ with $K \subset \Gamma'$. Equivalently the core is the kernel of the left action of Γ on Γ/Γ'.

Now let $f : \Gamma' \to \Gamma$ be a group morphism. By defintion the *f-core* is the biggest normal subgroup $K \lhd G$ with $K \subset \Gamma'$ and $f(K) \subset K$.

1.3 Self-similar data
Let Γ be a group. A *virtual endomorphism* of Γ is a pair (Γ', f), where Γ' is a subgroup of finite index and $f : \Gamma' \to \Gamma$ is a group morphism. A *self-similar datum* is a virtual endomorphism (Γ', f) with a trivial f-core.

Assume given a faithfull transitive self-similar action of Γ on A^ω. Let $a \in A$, and let Γ' be the stabilizer of a. By definition, for each $\gamma \in \Gamma'$ there is a unique $\gamma_a \in \Gamma$ such that
$$\gamma(aw) = a\gamma_a(w),$$
for any $w \in A^\omega$. Let $f : \Gamma' \to \Gamma$ be the map $\gamma \mapsto \gamma_a$. Since the action is faithfull, γ_a is uniquely determined and f is a group morphism. Also it

follows from Proposition 2.7.4 and 2.7.5 of [24] that the f-core of Γ' is the kernel of the action, therefore it is trivial. Hence (Γ', f) is a self-similar datum.

Conversely, a virtual endomorphism (Γ', f) determines a transitive self-similar action of Γ on A^ω, where $A \simeq \Gamma/\Gamma'$. Moreover the f-core is the kernel of the corresponding action, see ch 2 of [24] for details, especially subsection 2.5.5 of [24]). In conclusion, we have

Lemma 1. *Let Γ be a group. There is a correspondence between the self-similar data (Γ', f) and the faithfull transitive self-similar actions of Γ on A^ω, where $A \simeq \Gamma/\Gamma'$.*

This correspondence is indeed a bijection up to conjugacy, see [24] for a precise statement.

1.4 Good self-similar data
Let Γ be a group, and let (Γ', f) be a virtual endomorphism. Let Γ_n be the subgroups of Γ inductively defined by $\Gamma_0 = \Gamma$, $\Gamma_1 = \Gamma'$ and for $n \geq 2$
$$\Gamma_n = \{\gamma \in \Gamma_{n-1} | f(\gamma) \in \Gamma_{n-1}\}$$

Lemma 2. *The sequence $n \mapsto [\Gamma_n : \Gamma_{n+1}]$ is not increasing.*

Proof. For $n > 0$, the map f induces an injection of the set Γ_n/Γ_{n+1} into Γ_{n-1}/Γ_n, thus we have $[\Gamma_n : \Gamma_{n+1}] \leq [\Gamma_{n-1} : \Gamma_n]$. $\square$

The virtual endomorphism (Γ', f) is called *good* if $[\Gamma_n : \Gamma_{n+1}] = [\Gamma/\Gamma']$ for all n.

Let (Γ', f) be a self-similar datum, and let A^* be the corresponding tree on which Γ acts. If a is the distinguished point in $A \simeq \Gamma/\Gamma'$, then Γ_n is the stabilizer of a^n. If the self-similar datum (Γ', f) is good, then $[\Gamma : \Gamma_n] = \text{Card } A^n$ and therefore Γ acts transitively on A^n. Exactly as before, we have

Lemma 3. *Let Γ be a group. There is a correspondence between the good self-similar data (Γ', f) and the faithfull self-similar actions of Γ on A^ω with dense orbits, where $A \simeq \Gamma/\Gamma'$.*

1.5 Fractal self-similar data
Let Γ be a group. A *self-similar datum* (Γ', f) is called *fractal* (or *recurrent*) if $f(\Gamma') = \Gamma$. A self-similar action of Γ on some A^ω is called *fractal* if it is

transitive and the corresponding self-similar datum is fractal, see [24] section 2.8. Obviously a fractal action has dense orbits.

The group Γ is called *fractal* (respectively *freely fractal*) if Γ admits a faithfull (respectively free) fractal action on some A^ω.

2 Rational points of a torus

We are going to prove Theorem 1, about the rational points of algebraic tori.

For the whole chapter, let $\mathbf{H}$ be an algebraic torus defined over $\mathbb{Q}$ and let $X(\mathbf{H})$ be the group of characters of $\mathbf{H}$. For a number field K, let's denote by $\mathrm{Gal}(K) := \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ its absolute Galois group. The group $X(\mathbf{H})$ is a $\mathrm{Gal}(\mathbb{Q})$-module which is isomorphic to $\mathbb{Z}^r$ as an abelian group, where $r = \dim \mathbf{H}$. The *splitting field* of $\mathbf{H}$ is the smallest Galois extension L of $\mathbb{Q}$ such that $\mathrm{Gal}(L)$ acts trivially on $X(\mathbf{H})$, or, equivalently such that $\mathbf{H}$ is L-isomorphic to $\mathbb{G}_m^r$, where $\mathbb{G}_m$ denotes the multiplicative group. Moreover, we have

$$\chi(h) \in L^*$$

for any $\chi \in X(\mathbf{H})$ and any $h \in \mathbf{H}(\mathbb{Q})$.

Let $\mathcal{O}$ be the ring of integers of L. Recall that a *fractional ideal* is a nonzero finitely generated $\mathcal{O}$-submodule of K. A fractional ideal I is called *integral* if $I \subset \mathcal{O}$. If the fractional ideal I is integral and $I \neq \mathcal{O}$, then I is merely an ideal of $\mathcal{O}$.

Let $\mathcal{I}$ be the set of all fractional ideals and $\mathcal{I}^+$ be the subset of all integral ideals. Given I and J in $\mathcal{I}$, their *product* is the $\mathcal{O}$-module generated by all products ab where $a \in I$ and $b \in J$. Since $\mathcal{O}$ is a Dedekind ring, we have

$$\mathcal{I} \simeq \oplus_{\pi \in \mathcal{P}} \mathbb{Z}\left[\pi\right]$$
$$\mathcal{I}^+ \simeq \oplus_{\pi \in \mathcal{P}} \mathbb{Z}_{\geq 0}\left[\pi\right],$$

where $\mathcal{P}$ is the set of prime ideals of $\mathcal{O}$. Indeed the additive notation is used for for the group $\mathcal{I}$ and the monoid $\mathcal{I}^+$: view as an element of $\mathcal{I}$ the fractional ideal $\pi_1^{m_1} \dots \pi_n^{m_n}$ will be denoted as $m_1[\pi_1] + \dots + m_n[\pi_n]$.

Since $\mathrm{Gal}(L/\mathbb{Q})$ acts by permutation on $\mathcal{P}$, $\mathcal{I}$ is a $\mathbb{Z}\mathrm{Gal}(L/\mathbb{Q})$-module. For $S \subset \mathcal{P}$, set

$$\mathcal{I}_S = \oplus_{\pi \in \mathcal{P} \setminus S} \mathbb{Z}\left[\pi\right].$$

Lemma 4. *Let $S \subset \mathcal{P}$ be a finite subset and let $r > 0$ be an integer.*

The $\mathrm{Gal}(L/\mathbb{Q})$-module $\mathcal{I}$ contains a free $\mathbb{Z}\mathrm{Gal}(L/\mathbb{Q})$-module $M(r)$ of rank r such that

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(i) $M(r) \cap \mathcal{I}^+ = \{0\}$, *and*

(ii) $M(r) \subset \mathcal{I}_S$.

Proof. Let S' be the set of all prime numbers which are divisible by some $\pi \in S$. Let Σ be the set of prime numbers p that are completely split in K, i.e. such that $\mathcal{O}/p\mathcal{O} \simeq \mathbb{F}_p^{[L:\mathbb{Q}]}$. For $p \in \Sigma$, let $\pi \in \mathcal{P}$ be a prime ideal over p. When σ runs over $\mathrm{Gal}(L/\mathbb{Q})$ the ideals π^σ are all distinct, and therefore $[\pi]$ generates a free $\mathbb{Z}\mathrm{Gal}(L/\mathbb{Q})$-submodule of rank one in $\mathcal{I}$.

By Cebotarev theorem, the set Σ is infinite. Choose $r+1$ distinct prime numbers $p_0, \ldots p_r$ in $\Sigma \setminus S'$, and let $\pi_0, \ldots, \pi_r \in \mathcal{P}$ such that $\mathcal{O}/\pi_i = \mathbb{F}_{p_i}$. For $1 \le i \le r$, set $\tau_i = [\pi_i] - [\pi_0]$ and let $M(r)$ be the $\mathbb{Z}\mathrm{Gal}(L/\mathbb{Q})$-module generated by $\tau_1, \ldots, \tau_r$.

Obviously, the $\mathrm{Gal}(L/\mathbb{Q})$-module $M(r)$ is free of rank r and $M(r) \subset \mathcal{I}_S$. It remains to prove that $M(r) \cap \mathcal{I}^+ = \{0\}$. Let
$$A = \sum_{1 \le i \le r, \sigma \in \mathrm{Gal}(L/\mathbb{Q})} m_{i,\sigma}\, \tau_i^\sigma$$
be an element of $M(r) \cap \mathcal{I}^+$. We have $A = B - C$, where
$$B = \sum_{1 \le i \le r, \sigma \in \mathrm{Gal}(L/\mathbb{Q})} m_{i,\sigma}\, [\pi_i^\sigma], \text{ and}$$
$$C = \sum_{\sigma \in \mathrm{Gal}(L/\mathbb{Q})} \Big(\sum_{1 \le i \le r} m_i^\sigma \Big)[\pi_0^\sigma].$$
Thus the condition $A \in \mathcal{I}^+$ implies that
$$m_i^\sigma \ge 0, \text{ for any } 1 \le i \le k \text{ and } \sigma \in \mathrm{Gal}(L/\mathbb{Q}), \text{ and}$$
$$\sum_{1 \le i \le r} m_i^\sigma \le 0, \text{ for any } \sigma \in \mathrm{Gal}(L/\mathbb{Q}).$$
Thus all the integers m_i^σ vanish. Therefore $M(r)$ intersects $\mathcal{I}^+$ trivially.
$\square$

For $\pi \in \mathcal{P}$, let $v_\pi : L^* \to \mathbb{Z}$ be the corresponding valuation.

Lemma 5. *Let $S \subset \mathcal{P}$ be a finite $\mathrm{Gal}(L/\mathbb{Q})$-invariant subset and let $r > 0$ be an integer.*

The $\mathrm{Gal}(L/\mathbb{Q})$-module L^* contains a free $\mathbb{Z}\mathrm{Gal}(L/\mathbb{Q})$-module $N(r)$ of rank r such that

(i) $N(r) \cap \mathcal{O} = \{1\}$, *and*

(ii) $v_\pi(x) = 0$ *for any $x \in N(r)$ and any $\pi \in S$.*

Proof. Set $L_S^* = \{x \in L^* | v_\pi(x) = 0, \forall \pi \in S\}$ and let $\theta : L_S^* \to \mathcal{I}_S$ be the map $x \mapsto \sum_{\pi \in \mathcal{P}} v_\pi(x)\, [\pi]$.

By Lemma 4, $\mathcal{I}_S$ contains a free $\mathbb{Z}\mathrm{Gal}(L/\mathbb{Q})$-module $M(r)$ of rank r such that $M(r) \cap \mathcal{I}^+ = \{0\}$. Let's remark that $\mathrm{Coker}\,\theta$ is a subgroup of the class

group $Cl(L)$ of L. Since, by Dirichelet Theorem, $Cl(L)$ is finite, there is a positive integer d such that $d.M(r)$ lies in the image of θ. Since $M(r)$ is free, there is a free $\mathbb{Z}Gal(K/\mathbb{Q})$-module $N(r) \subset L_S^*$ of rank r which is a lift of $dM(r)$, i.e. such that θ induces an isomorphism $N(r) \simeq d.M(r)$. Since $\theta(\mathcal{O}\backslash 0)$ lies in $\mathcal{I}^+$, we have $\theta(N(r)\cap\mathcal{O}) = \{0\}$. It follows that $N(r)\cap\mathcal{O} = \{1\}$.

The second assertion follows from the fact that $N(r)$ lies in L_S^*. $\qquad\square$

For $\pi \in \mathcal{P}$, let $\mathcal{O}_\pi$ and L_π be the π-adic completions of $\mathcal{O}$ and L. Let $x, y \in L$ and let $n > 0$ be an integer. In what follows, the congruence
$$x \equiv y \text{ modulo } n\mathcal{O}_\pi$$
means $x_\pi \equiv y_\pi \bmod n\mathcal{O}_\pi$, where x_π and y_π are the images of x and y in L_π.

The case $n = 1$ of the next statement will be used in further sections. In such a case, Assertion (ii) is tautological.

Theorem 1. *Let $\mathbf{H}$ be an algebraic torus defined over $\mathbb{Q}$, and let L be its splitting field. Let $n > 0$ be an integer and let $S \subset \mathcal{P}$ be the set of prime divisors of n.*

There exists $h \in \mathbf{H}(\mathbb{Q})$ such that
(i) $\chi(h)$ is not an algebraic integer, for any non-trivial $\chi \in X(\mathbf{H})$, and
(ii) $\chi(h) \equiv 1 \bmod n\mathcal{O}_\pi$ for any $\chi \in X(\mathbf{H})$ and any $\pi \in S$.

Proof. Step 1. First an element $h' \in \mathbf{H}(\mathbb{Q})$ satisfying Assertion (i) and
$\qquad\qquad$ (iii) $v_\pi(\chi(h')) = 0$, for any $\pi \in S$ and any $\chi \in X(\mathbf{H})$
is found.

The abelian group $X(\mathbf{H})$ is free of rank r where $r = \dim \mathbf{H}$. Therefore, the comultiplication $\Delta : X(\mathbf{H}) \to X(\mathbf{H})\otimes\mathbb{Z}Gal(L/\mathbb{Q})$ provides an embedding of $X(\mathbf{H})$ into a free $\mathbb{Z}Gal(L/\mathbb{Q})$-module of rank r. By lemma 5, there a free $\mathbb{Z}Gal(L/\mathbb{Q})$-module $N(r) \subset L_S^*$ of rank r with $N(r)\cap\mathcal{O} = \{1\}$. Let
$$\mu : X(\mathbf{H}) \otimes \mathbb{Z}Gal(L/\mathbb{Q}) \to N(r),$$
be an isomorphism of $\mathbb{Z}Gal(L/\mathbb{Q})$-modules, and set $h' = \mu \circ \Delta$.

Since $\mathbf{H}(\mathbb{Q}) = \text{Hom}_{Gal(L/\mathbb{Q})}(X(\mathbf{H}), L^*)$, the embedding h' is indeed an element of $\mathbf{H}(\mathbb{Q})$. Viewed as a map from $X(\mathbf{H})$ to L^*, h' is the morphism $\chi \in X(\mathbf{H}) \mapsto \chi(h')$.

Since $\text{Im } h'\cap\mathcal{O} = 1$ and h' is injective, $\chi(h')$ is not an algebraic integer if χ is a non-trivial character. Since $\text{Im } h' \subset L_S^*$, we have $v_\pi(\chi(h')) = 0$ for any $\chi \in X(\mathbf{H})$. Therefore h' satisfies Assertions (i) and (iii).

Step 2. Let $\chi_1, \ldots, \chi_r$ be a basis of $X(\mathbf{H})$. Since $v_\pi(\chi_i(h')) = 0$ for any $\pi \in S$, the element $\chi_i(h') \bmod n\mathcal{O}_\pi$ is an inversible element in the finite ring $\mathcal{O}_\pi/n\mathcal{O}_\pi$. Therefore there are positive integers $m_{i,\pi}$ such that

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$$\chi_i(h')^{m_{i,\pi}} \equiv 1 \bmod n\mathcal{O}_\pi,$$
for all $1 \le i \le r$ and all $\pi \in S$. Set $m = \mathrm{lcm}(m_{i,\pi})$ and set $h = h'^m$. Obviously h satisfies Assertion (i). Moreover we have $\chi_i(h) \equiv 1 \bmod n\mathcal{O}_\pi$, for all $\pi \in S$ and all $1 \le i \le r$, and therefore h satisfies Assertion (ii) as well.
$\square$

3   Special Gradings

Let $\mathfrak{n}$ be a finite dimensional Lie algebra defined over $\mathbb{Q}$ and let $\mathfrak{z}$ be its center. The relations between the gradings of $\mathbb{C} \otimes \mathfrak{n}$ and the automorphisms of $\mathfrak{n}$ are investigated now.

The following important definitions will be used in the whole paper. Let $\mathcal{S}(\mathfrak{n})$ (respectively $\mathcal{V}(\mathfrak{n})$) be the set of all $f \in \mathrm{Aut}\,\mathfrak{n}$ such that $\mathrm{Spec}\,f|_{\mathfrak{z}}$ (respectively $\mathrm{Spec}\,f$) contains no algebraic integers. Moreover let $\mathcal{F}(\mathfrak{n})$ be the set of all $f \in \mathcal{S}(\mathfrak{n})$ such that all eigenvalues of f^{-1} are algebraic integers. Also set $\mathcal{F}^+(\mathfrak{n}) = \mathcal{F}(\mathfrak{n}) \cap \mathcal{V}(n)$. Here, by eigenvalues of a $\mathbb{Q}$-linear endomorphism F, we always mean the eigenvalues of F in $\overline{\mathbb{Q}}$.

For any field K of characteristic zero, set $\mathfrak{n}^K = K \otimes \mathfrak{n}$ and $\mathfrak{z}^K = K \otimes \mathfrak{z}$. Let $\mathbf{G} = \mathbf{Aut}\,\mathfrak{n}$ be the algebraic group of automorphisms of $\mathfrak{n}$. By definition, $\mathbf{G}$ is defined over $\mathbb{Q}$, and we have $\mathbf{G}(K) = \mathrm{Aut}\,\mathfrak{n}^K$ for any field K of characteristic zero. The notation $\mathfrak{n}$ underlines that $\mathfrak{n}$ can be viewed as the functor in Lie algebras $K \mapsto \mathfrak{n}^K$. Let $\mathbf{H} \subset \mathbf{G}$ be a maximal torus defined over $\mathbb{Q}$, whose existence is proved in [7], see also [6], Theorem 18.2.

By definition, a K-*grading* of $\mathfrak{n}$ is is a decomposition of $\mathfrak{n}^K$
$$\mathfrak{n}^K = \oplus_{n \in \mathbb{Z}}\, \mathfrak{n}_n^K$$
such that $[\mathfrak{n}_n^K, \mathfrak{n}_m^K] \subset \mathfrak{n}_{n+m}^K$ for all $n, m \in \mathbb{Z}$. A grading is called *special* (respectively *very special*) if $\mathfrak{z}^K \cap \mathfrak{n}_0^K = 0$ (respectively if $\mathfrak{n}_0^K = 0$). A grading is called *non-negative* (respectively *positive*) if $\mathfrak{n}_n^K = 0$ for $n < 0$ (respectively $\mathfrak{n}_n^K = 0$ for $n \le 0$).

For any field K of characteristic zero, a K-grading of $\mathfrak{n}$ can be identified with an algebraic group morphism
$$\rho : \mathbb{G}_m \to \mathbf{G}$$
defined over K, where $\mathbb{G}_m$ denotes the multiplicative group.

Consider the following two hypotheses
$(\mathcal{H}_K)$ The Lie algebra $\mathfrak{n}$ admits a special K-grading,
$(\mathcal{H}_K^0)$ The Lie algebra $\mathfrak{n}$ admits a very special K-grading.

Lemma 6. *Let K be the splitting field of $\mathbf{H}$. Up to conjugacy, any grading of $\mathfrak{n}^{\mathbb{C}}$ is defined over K. In particular*
  (i) The hypotheses $\mathcal{H}_{\mathbb{C}}$ and $\mathcal{H}_{\overline{\mathbb{Q}}}$ are equivalent.
  (ii) The hypotheses $\mathcal{H}_{\mathbb{C}}^0$ and $\mathcal{H}_{\overline{\mathbb{Q}}}^0$ are equivalent.

Proof. Let
$$\mathfrak{n}^{\mathbb{C}} = \oplus_{n \in \mathbb{Z}} \, \mathfrak{n}_n^{\mathbb{C}}$$
be a grading of $\mathfrak{n}^{\mathbb{C}}$ and let $\rho : \mathbb{G}_m \to \mathbf{G}$ be the corresponding algebraic group morphism. Since any maximal torus of $\mathbf{G}$ is $\mathbf{G}(\mathbb{C})$-conjugate to $\mathbf{H}$, it can be assumed that $\rho(\mathbb{G}_m) \subset \mathbf{H}$.

Let $X(\mathbf{H})$ be the character group of $\mathbf{H}$. The group morphism ρ is determined by the dual morphism $L : X(\mathbf{H}) \to \mathbb{Z} = X(\mathbb{G}_m)$. However, $\mathrm{Gal}(K)$ acts trivially on $X(\mathbf{H})$. Thus ρ is automaticaly defined over K. $\qquad\square$

Lemma 7. *Let Λ be a finitely generated abelian group and let $S \subset \Lambda$ be a finite subset containing no element of finite order. Then there exists a morphism $L : \Lambda \to \mathbb{Z}$ such that*
$$L(\lambda) \neq 0 \text{ for any } \lambda \in S.$$

Proof. Let F be the subgroup of finite order elements in Λ. Using Λ/F instead of Λ, it can be assumed that $\Lambda = \mathbb{Z}^d$ for some d an $0 \notin S$. Let's choose a positive integer N such that $S \subset \,] - N, N[^d$ and let $L : \Lambda \to \mathbb{Z}$ be the function defined by
$$L(a_1, \ldots, a_d) = \sum_{1 \leq i \leq d} a_i N^{i-1}.$$
For any $\lambda = (a_1, \ldots, a_d) \in S$, there is a smallest index i with $a_i \neq 0$. We have $L(\lambda) = a_i N^{i-1}$ modulo N^i. Since $|a_i| < N$, it follows that $L(\lambda) \neq 0 \bmod N^i$ and therefore $L(\lambda) \neq 0$. $\qquad\square$

Lemma 8. *Let $f \in \mathbf{G}(\mathbb{Q})$. There is a f-invariant $\mathbb{Z}$-grading of $\mathfrak{n}^{\overline{\mathbb{Q}}}$ such that all eigenvalues of f on $\mathfrak{n}_0^{\overline{\mathbb{Q}}}$ are roots of unity.*

  In particular, if $\mathrm{Spec}\, f$ contains no root of unity, then $\mathfrak{n}^{\overline{\mathbb{Q}}}$ admits a very special grading.

Proof. Let $\Lambda \subset \overline{\mathbb{Q}}^*$ be the subgroup generated by the $\mathrm{Spec}\, f$. For any $\lambda \in \Lambda$ denote by $E_{(\lambda)} \subset \mathfrak{n}^{\overline{\mathbb{Q}}}$ the corresponding generalized eigenspace of f. Let R be the set of all roots of unity in $\mathrm{Spec}\, f$ and set $S = \mathrm{Spec}\, f \setminus R$.

  By Lemma 7, there is a morphism $L : \Lambda \to \mathbb{Z}$ such that $L(\lambda) \neq 0$ for any $\lambda \in S$. Let $\mathcal{G}$ be the decomposition

$$\mathfrak{n}^{\overline{\mathbb{Q}}} = \oplus_{k \in \mathbb{Z}} \mathfrak{n}_k^{\overline{\mathbb{Q}}}$$

of $\mathfrak{n}^{\overline{\mathbb{Q}}}$ defined by $\mathfrak{n}_k^{\overline{\mathbb{Q}}} = \oplus_{L(\lambda)=k} E_{(\lambda)}$. Since $[E_{(\lambda)}, E_{(\mu)}] \subset E_{(\lambda\mu)}$ and $L(\lambda\mu) = L(\lambda) + L(\mu)$ for any λ, $\mu \in \Lambda$, it follows that $\mathcal{G}$ is a grading of the Lie algebra $\mathfrak{n}^{\overline{\mathbb{Q}}}$. Moreover we have

$$\mathfrak{n}_0^{\overline{\mathbb{Q}}} = \oplus_{\lambda \in R} E_{(\lambda)},$$

from which the lemma follows. $\qquad\square$

Lemma 9. *With the previous notations*

(i) the Lie algebra $\mathfrak{n}^{\mathbb{C}}$ admits a special grading iff $\mathcal{S}(\mathfrak{n}) \neq \emptyset$.

(ii) the Lie algebra $\mathfrak{n}^{\mathbb{C}}$ admits a very special grading iff $\mathcal{V}(\mathfrak{n}) \neq \emptyset$.

Proof. In order to prove Assertion (i), let's consider the following assertion

$(\mathcal{A})$ $\qquad\qquad\qquad H^0(\mathbf{H}(\overline{\mathbb{Q}}), \mathfrak{z}^{\overline{\mathbb{Q}}}) = 0.$

The proof is based on the following "cycle" of implications

$\mathfrak{n}^{\mathbb{C}}$ has a special grading $\Rightarrow (\mathcal{A}) \Rightarrow \mathcal{S}(\mathfrak{n}) \neq \emptyset \Rightarrow \mathfrak{n}^{\mathbb{C}}$ has a special grading.

Step 1: the existence of a special grading of $\mathfrak{n}^{\mathbb{C}}$ implies $(\mathcal{A})$. By hypothesis and Lemma 6, $\mathfrak{n}^{\overline{\mathbb{Q}}}$ admits a special grading. Let $\rho : \mathbb{G}_m \to \mathbf{G}$ be the corresponding group morphism. Since all maximal tori of $\mathbf{G}$ are conjugate to $\mathbf{H}$, we can assume that $\rho(\mathbb{G}_m) \subset \mathbf{H}$. Therefore we have

$$H^0(\mathbf{H}(\overline{\mathbb{Q}}), \mathfrak{z}^{\overline{\mathbb{Q}}}) \subset H^0(\rho(\overline{\mathbb{Q}}^*), \mathfrak{z}^{\overline{\mathbb{Q}}}) = 0.$$

Thus Assertion $\mathcal{A}$ is proved.

Step 2: proof that $(\mathcal{A})$ implies $\mathcal{S}(\mathfrak{n}) \neq \emptyset$. By Theorem 1, there exists $f \in \mathbf{H}(\mathbb{Q})$ such that $\chi(f)$ is not an algebraic integer for any non-trivial character $\chi \in X(\mathbf{H})$. If we assume $(\mathcal{A})$, then Spec $f|_{\mathfrak{z}}$ contains no algebraic integers and therefore $\mathcal{S}(\mathfrak{n}) \neq \emptyset$.

Step 3: proof that $\mathcal{S}(\mathfrak{n}) \neq \emptyset$ implies the existence of a special grading. For any $f \in \mathcal{S}(\mathfrak{n})$, Since Spec $f|_{\mathfrak{z}}$ contains no roots of unity. It follows from Lemma 8 that the Lie algebra $\mathfrak{n}^{\overline{\mathbb{Q}}}$ (and therefore $\mathfrak{n}^{\mathbb{C}}$) admits a special grading. Therefore $\mathcal{S}(\mathfrak{n}) \neq \emptyset$ implies the existence of a special grading.

The proof of Assertion (ii) is almost identical. Instead of $(\mathcal{A})$, the "cycle" of implications uses the following assertion

$(\mathcal{A}^0)$ $\qquad\qquad\qquad H^0(\mathbf{H}(\overline{\mathbb{Q}}), \mathfrak{n}^{\overline{\mathbb{Q}}}) = 0.$ $\qquad\square$

Lemma 10. *The following are equivalent:*

(i) the Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ admits a non-negative special grading,

(ii) the Lie algebra $\mathfrak{n}^{\mathbb{C}}$ admits a non-negative special grading, and

(iii) The set $\mathcal{F}(\mathfrak{n})$ is not empty.

Proof. Proof that $(ii) \Rightarrow (iii)$. Let $\mathfrak{n}^{\mathbb{C}} = \oplus_{k \geq 0} \mathfrak{n}_k^{\mathbb{C}}$ be a non-negative special grading of $\mathfrak{n}^{\mathbb{C}}$ and let $\rho : \mathbb{G}_m \to \mathbf{G}$ be the corresponding group morphism. Up to conjugacy, we can assume that $\rho(\mathbb{G}_m) \subset \mathbf{H}$. It follows that the grading is defined over the splitting field K of $\mathbf{H}$.

Let $g_1 \in \mathbf{H}(K)$ be the isomorphism defined by $g_1 x = 2^k x$ if $x \in \mathfrak{n}_k^{\mathbb{C}}$. Set $n = [K : \mathbb{Q}]$ and let $g_1, g_2 \ldots g_n$ be the $\mathrm{Gal}(L/\mathbb{Q})$-conjugates of g_1. Since all g_i belongs to $\mathbf{H}(K)$, the automorphisms g_i commute. Hence the product $g := g_1 \ldots g_n$ is well defined and g belongs to $\mathbf{H}(\mathbb{Q})$. By hypotheses, all eigenvalues of g_i are power of 2, and all eigenvalues of $g_i|_{\mathfrak{z}^{\mathbb{C}}}$ are distinct from 1. Therefore all eigenvalues of g are integers, and all eigenvalues of $g|_{\mathfrak{z}^{\mathbb{C}}}$ are $\neq \pm 1$. It follows that g^{-1} belongs to $\mathcal{F}(\mathfrak{n})$. Therefore $\mathcal{F}(\mathfrak{n}) \neq \emptyset$

Proof that $(iii) \Rightarrow (i)$. Let $f \in \mathcal{F}(\mathfrak{n})$ and set $g = f^{-1}$. Set $K = \mathbb{Q}(\mathrm{Spec}\, g)$ and let $L : K^* \to \mathbb{Z}$ be the map defined by

$$L(x) = \sum_p v_p(N_{K/\mathbb{Q}}(x))$$

where the sum runs over all prime numbers p and where $N_{K/\mathbb{Q}} : K^* \to \mathbb{Q}^*$ denotes the norm map.

For any integer k, set

$$\mathfrak{n}_k^{\overline{\mathbb{Q}}} = \bigoplus_{L(x)=k} E_{(x)}$$

where $E_{(x)} \subset \mathfrak{n}^{\overline{\mathbb{Q}}}$ denotes the generalized eigenspace associated to $x \in \mathrm{Spec}\, g$. We have $[E_{(x)}, E_{(y)}] \subset E_{(xy)}$ and $L(xy) = L(x) + L(y)$, for any $x, y \in K$. Therefore the decomposition

$$\mathfrak{n}^K = \oplus_{k \in \mathbb{Z}} \mathfrak{n}_k^{\overline{\mathbb{Q}}}$$

is a grading $\mathcal{G}$ of the Lie algebra $\mathfrak{n}^{\overline{\mathbb{Q}}}$. Since the function L is $\mathrm{Gal}(\mathbb{Q})$-invariant, the grading $\mathcal{G}$ is indeed defined over $\mathbb{Q}$. It remains to prove that $\mathcal{G}$ is non-negative and special.

Since any $x \in \mathrm{Spec}\, g$ is an algebraic integer, we have $L(x) \geq 0$ and the grading is non-negative. Since no $x \in \mathrm{Spec}\, g|_{\mathfrak{z}}$ is an algebraic unit, we have $N_{K/\mathbb{Q}}(x) \neq \pm 1$ and $L(x) > 0$. Thus the grading is special, what proves that $(iii) \implies (i)$.

$\square$

Lemma 11. *The following are equivalent:*
   (i) the Lie algebra $\mathfrak{n}^{\mathbb{Q}}$ admits a positive grading,
   (ii) the Lie algebra $\mathfrak{n}^{\mathbb{C}}$ admits a positive grading, and
   (iii) The set $\mathcal{F}^+(\mathfrak{n})$ is not empty.

Since the proof is almost identical to the previous proof, it will be skipped. The equivalence $(i) \Leftrightarrow (ii)$ also appears in [8].

4 Height and relative complexity

For the whole chapter, V denotes a finite dimensional vector space over $\mathbb{Q}$. In this section, we define the notion of the *height* of the isomorphisms $h \in GL(V)$ and the notion of a *minimal lattice*.

4.1 Height, complexity and minimality
Let $h \in GL(V)$. Recall that a *lattice* of V is a finitely generated subgroup Λ which contains a basis of V. Let $\mathcal{D}(h)$ be the set of all couple of lattices (Λ, E) of V such that $E \subset \Lambda$ and $h(E) \subset \Lambda$. By definition, the *height* of h, is the integer
$$\mathrm{ht}(h) := \mathrm{Min}_{(\Lambda, E) \in \mathcal{D}(h)} [\Lambda : E].$$
Let $\mathcal{D}_{min}(h)$ be the set of all couples $(\Lambda, E) \in \mathcal{D}(h)$ such that $[\Lambda : E] = \mathrm{ht}(h)$.

Similarly, for a lattice Λ of V, the *h-complexity* of Λ is the integer
$$\mathrm{cp}_h(\Lambda) := \mathrm{Min}_{(\Lambda, E) \in \mathcal{D}(h)} [\Lambda : E].$$
It is clear that $\mathrm{cp}_h(\Lambda) = [\Lambda : E]$, where $E = \Lambda \cap h^{-1}(\Lambda)$. The lattice Λ is called *minimal relative to* h if $\mathrm{cp}_h(\Lambda) = \mathrm{ht}(h)$.

For the proofs, a technical notion of relative height is needed. Let $\mathrm{End}_h(V)$ be the commutant of h and let let $A \subset C(h) \subset \mathrm{End}_h(V)$ be a subring. By definition, an *A-lattice* Λ means a lattice Λ which is an A-module. Let $\mathcal{D}^A(h)$ be the set of all couple of A-lattices (Λ, E) in $\mathcal{D}(h)$. The *A-height* of h is the integer
$$\mathrm{ht}_A(h) := \mathrm{Min}_{(\Lambda, E) \in \mathcal{D}^A(h)} [\Lambda : E].$$
Obviously, we have $\mathrm{ht}_A(h) \geq \mathrm{ht}(h) = \mathrm{ht}_{\mathbb{Z}}(h)$. Let $\mathcal{D}^A_{min}(h))$ be the set of all couples $(\Lambda, E) \in \mathcal{D}^A(h))$ such that $[\Lambda : E] = \mathrm{ht}_A(h)$.

4.2 Height and filtrations
Let V be a finite dimensional vector space over $\mathbb{Q}$ and let $h \in GL(V)$. Let A be a subring of $\mathrm{End}_h(V)$ and let $A[h]$ be the subring of $\mathrm{End}_h(V)$ generated by A and h.

Lemma 12. *Let $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ be a fitration of V, where each vector space V_i is a $A[h]$-submodule. For $i = 1$ to n, set $h_i = h_{V_i/V_{i-1}}$. Then we have*
$$\mathrm{ht}_A(h) \geq \prod_{1 \leq i \leq n} \mathrm{ht}_A(h_i).$$
Moreover if $V \simeq \oplus V_i/V_{i-1}$ as a $A[\bar{h}]$-module, we have
$$\mathrm{ht}_A(h) = \prod_{1 \leq i \leq n} \mathrm{ht}_A(h_i).$$

Proof. Clearly it is enough to prove the lemma for $n = 2$. Let $(\Lambda, E) \in \mathcal{D}_{min}^A(h)$. Set $\Lambda_1 = \Lambda \cap V_1$, $E_1 = E \cap V_1$, $\Lambda_2 = \Lambda/\Lambda_1$ and $E_2 = E/E_1$. We have
$$[\Lambda : E] = [\Lambda_1 : E_1][\Lambda_2 : E_2].$$
Since $(\Lambda_1, E_1) \in \mathcal{D}^A(h_1)$ and $(\Lambda_2, E_2) \in \mathcal{D}^A(h_2)$, we have
$$\mathrm{ht}_A(h) \geq \mathrm{ht}_A(h_1)\,\mathrm{ht}_A(h_2),$$
what proves the first assertion.

Next, we assume that $V \simeq V_1 \oplus V_2$ as a $A[h]$-module. Let $(\Lambda_1, E_1) \in \mathcal{D}_{min}^A(h_1)$, $(\Lambda_2, E_2) \in \mathcal{D}_{min}^A(h_2)$ and set $\Lambda = \Lambda_1 \oplus \Lambda_2$ and $E = E_1 \oplus E_2$. We have
$$[\Lambda : E] = [\Lambda_1 : E_1][\Lambda_2 : E_2] = \mathrm{ht}_A(h_1)\,\mathrm{ht}_A(h_2).$$
Therefore $\mathrm{ht}_A(h) \leq \mathrm{ht}_A(h_1)\,\mathrm{ht}_A(h_2)$. Hence $\mathrm{ht}_A(h) = \mathrm{ht}_A(h_1)\,\mathrm{ht}_A(h_2)$.
$\square$

Let $h \in GL(V)$ as before. Its *Chevalley decomposition* $h = h_s h_u$ is uniquely defined by the following three conditions: h_s and h_u commutes, h_s is semi-simple and h_u is unipotent.

Lemma 13. *We have*
$$\mathrm{ht}(h) = \mathrm{ht}(h_s).$$

Proof. By Lemma 12, it can be assumed that the $\mathbb{Q}[h]$-module V is indecomposable. Therefore there is a vector space V_0, a semi-simple endomorphism $h_0 \in \mathrm{End}(V_0)$ and an isomorphism
$$V \simeq V_0 \otimes \mathbb{Q}[t]/(t^n),$$
relative to which h_s acts as $h_0 \otimes 1$ and h_u acts as $1 \otimes t$. Let $(\Lambda_0, E_0) \in \mathcal{D}_{min}(h_0)$ and set $\Lambda = \Lambda_0 \otimes \mathbb{Z}[t]/(t^n)$ and $E = E_0 \otimes \mathbb{Z}[t]/(t^n)$. By Lemma 12, we have $\mathrm{ht}(h) \geq \mathrm{ht}(h_s) = \mathrm{ht}(h_0)^n$. Since $(\Lambda, E) \in \mathcal{D}(h)$ and
$$[\Lambda : E] = [\Lambda_0 : E_0]^n = \mathrm{ht}(h_0)^n,$$
it follows that $\mathrm{ht}(h) = \mathrm{ht}(h_s)$
$\square$

4.3 Complexity of $\mathcal{O}(h)$-lattices
For any algebraic number λ, let $\mathcal{O}(\lambda)$ be the ring of integers of the number field $\mathbb{Q}(\lambda)$. Set $\pi_\lambda = \{x \in \mathcal{O}(\lambda)|\, x\lambda \in \mathcal{O}(\lambda)\}$. Then π_λ is an integral ideal and its norm is the integer
$$d(\lambda) := \mathrm{N}_{\mathbb{Q}(\lambda)/\mathbb{Q}}(\pi_\lambda) = \mathrm{Card}\,\mathcal{O}(\lambda)/\pi_\lambda.$$
Let $h \in GL(V)$ be semi-simple. Let $P(t)$ be its minimal polynomial, let $P = P_1 \ldots P_k$ be its factorization into irreducible factors. For $1 \leq i \leq k$, set

$K_i = \mathbb{Q}[t]/(P_i(t))$ and let $\mathcal{O}_i$ be the ring of integers of the number field K_i. Set $\mathcal{O}(h) = \oplus_{1 \le i \le k} \mathcal{O}_i$.

For each $\lambda \in \operatorname{Spec} h$, let m_λ be its multiplicity. Note that the functions $\lambda \mapsto m_\lambda$ and $\lambda \mapsto d(\lambda)$ are $\operatorname{Gal}(\mathbb{Q})$-invariant, so they can be viewed as functions defined over $\operatorname{Spec} h/\operatorname{Gal}(\mathbb{Q})$.

Lemma 14. *Let Λ be an $\mathcal{O}(h)$-lattice of V. Then*
$$\operatorname{cp}_h(\Lambda) = \prod d(\lambda)^{m_\lambda},$$
where the product runs over $\operatorname{Spec} h/\operatorname{Gal}(\mathbb{Q})$.

Proof. With the previous notations, let e_i be the unit of $\mathcal{O}_i$ and set $\Lambda_i = e_i \Lambda$. Since $\Lambda = \oplus_{1 \le i \le k} \Lambda_i$, it is enough to prove the lemma for $k = 1$, i.e. when the minimal polynomial of h is irreducible.

Let λ be one eigenvalue of h. With these new hypotheses, we have $\mathbb{Q}[h]/(P(t)) \simeq \mathbb{Q}(\lambda)$, $\mathcal{O}(h) \simeq \mathcal{O}(\lambda)$ and V is a vector space of dimension m_λ over $\mathbb{Q}(\lambda)$, relative to which h is identified with the multiplication by λ. We have
$$r_\lambda \Lambda = \Lambda \cap h^{-1} \Lambda.$$
Since $\Lambda/r_\lambda \mathcal{I} \simeq (\mathcal{O}(\lambda)/r_\lambda)^{m_\lambda}$, it follows that $\operatorname{cp}_h(\Lambda) = d(\lambda)^{m_\lambda}$. $\qquad \square$

4.4 Computation of the height
Let $h \in GL(V)$ be semi-simple.

Lemma 15. *We have*
$$\operatorname{ht}(h) = \prod d(\lambda)^{m_\lambda},$$
where the product runs over $\operatorname{Spec} h/\operatorname{Gal}(\mathbb{Q})$.

Proof. Using Lemmas 13 and Lemma 12, we can be assumed V is a simple $\mathbb{Q}[h]$-module, and let n be its dimension. The eigenvalues $\lambda_1, \ldots, \lambda_n$ of h are conjugate by $\operatorname{Gal}(\mathbb{Q})$. Under these simplifying hypotheses, the formula to be proved is
$$\operatorname{ht}(h) = d(\lambda_1).$$
Step 1: scalar extension. Set $K = \mathbb{Q}(\lambda_1, \ldots, \lambda_n)$, let $U = K \otimes V$, let $\tilde{h} = 1 \otimes h$ be the extension of h to U and let $\{v_1, \ldots, v_n\}$ be a K basis of U such that $\tilde{h}.v_i = \lambda_i v_i$. We have $U = \oplus_{1 \le i \le n} U_i$, where $U_i = K v_i$.

Let $\mathcal{O}$ be the ring of integers of K. For each $1 \le i \le n$, set $\tilde{h}_i = h|_{U_i}$. Since each U_i is a $\mathcal{O}[\tilde{h}]$-module, Lemma 12 shows that
$$\operatorname{ht}_{\mathcal{O}}(\tilde{h}) = \prod_{1 \le i \le n} \operatorname{ht}_{\mathcal{O}}(\tilde{h}_i).$$

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Next, the integers $\operatorname{ht}_{\mathcal{O}}(\tilde{h}_i)$ are computed. Let $\Lambda_i \subset U_i$ be any $\mathcal{O}$-lattice. Since $\mathcal{O}$ contains $\mathcal{O}(\lambda_i) = \mathcal{O}(\tilde{h}_i)$, it follows from Lemma 14 that
$$\operatorname{cp}_{\tilde{h}_i}(\Lambda_i) = d(\lambda_i)^r$$
where $r = \operatorname{rk}_{\mathcal{O}}(\Lambda_i) = [K : \mathbb{Q}(\lambda_i)]$. Hence we have $\operatorname{ht}_{\mathcal{O}}(\tilde{h}_i) = d(\lambda_i)^{[K:\mathbb{Q}(\lambda_i)]}$. It follows that
$$\operatorname{ht}_{\mathcal{O}}(\tilde{h}) = \prod_{1 \leq i \leq n} d(\lambda_i)^{[K:\mathbb{Q}(\lambda_i)]} = d(\lambda_1)^{[K:\mathbb{Q}]}$$
Step 2: end of the proof. Now let $(\Lambda, E) \in \mathcal{D}_{min}(h)$. Set $\tilde{\Lambda} = \mathcal{O} \otimes \Lambda$ and $\tilde{E} = \mathcal{O} \otimes E$. Since $\tilde{E}$ is an $\mathcal{O}$-module, we have $[\tilde{\Lambda} : \tilde{E}] \geq \operatorname{ht}_{\mathcal{O}}(\tilde{h})$. It follows that
$$\operatorname{ht}(h)^{[K:\mathbb{Q}]} = [\Lambda : E]^{[K:\mathbb{Q}]} = [\tilde{\Lambda} : \tilde{E}] \geq \operatorname{ht}_{\mathcal{O}}(\tilde{h}) = d(\lambda)^{[K:\mathbb{Q}]}.$$
Thus we have $d(\lambda_1) \leq \operatorname{ht}(h)$. By Lemma 14, we have $\operatorname{ht}_{\mathcal{O}(h)}(h) = d(\lambda_1)$. It follows that
$$d(\lambda_1) \leq \operatorname{ht}(h) \leq \operatorname{ht}_{\mathcal{O}(\langle\rangle)}(h) = d(\lambda_1),$$
what proves the formula.

$\square$

Remark: In number theory, the Weil height of an algebraic number λ is $H(\lambda) = \theta d(\lambda)^{1/n}$, where θ involves the norms at infinite places. Therefore $\operatorname{ht}(h)$ is essentially the Weil's height of h, up to the factor at infinite places.

4.5 A simple criterion of minimality
An obvious consequence of Lemmas 14 and 15 is

Lemma 16. *Let $h \in GL(V)$ be semi-simple and let Λ be an $\mathcal{O}(h)$-lattice of V. Then Λ is minimal relative to h.*

5   Malcev's Theorem and self-similar data

In this chapter, we recall Malcev's Theorem. Then we collect some related results, which are due to Malcev or viewed as folklore results. Then it is easy to characterize the self-similar data for FGTF nilpotent groups.

5.1 Three types of lattices
Let $\mathfrak{n}$ be a finite dimensional be a nilpotent Lie algebra over $\mathbb{Q}$. The Lie algebra $\mathfrak{n}$ is endowed with two group structures, the addition and the the Campbell-Hausdorff product. To avoid confusion, the Campbell-Hausdorff product is called the *multiplication* and it is denoted accordingly.

A *multiplicative subgroup* Γ of $\mathfrak{n}$ means a subgroup relative to the Campbell-Hausdorff product. In general, a multiplicative subgroup Γ is not an additive subgroup of $\mathfrak{n}$. However, notice that $\mathbb{Z}.x \subset \Gamma$ for any $x \in \Gamma$, because $x^n = nx$ for any $n \in \mathbb{Z}$.

A finitely generated multiplicative subgroup Γ is called a *multiplicative lattice* if $\Gamma \bmod [\mathfrak{n}, \mathfrak{n}]$ generates the $\mathbb{Q}$-vector space $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$, or, equivalently, if Γ generates the Lie algebra $\mathfrak{n}$. Let N be the CSC nilpotent Lie group with Lie algebra $\mathfrak{n}^R = \mathbb{R} \otimes \mathfrak{n}$. A discrete subroup Γ of N is called a *cocompact lattice* if N/Γ is compact.

It should be noted that three distinct notions of lattices will be used in the sequel: the additive lattices, the multiplicative lattices and the cocompact lattices. When it is used alone, a lattice is always an additive lattice. This very commoun terminology could be confusing: the reader should read "multiplicative lattice" or "cocompact lattice" as single words.

5.2 Malcev's Theorem
Any multiplicative lattice Γ of a finite dimensional nilpotent Lie algebra over $\mathbb{Q}$ is a FGTF nilpotent group. Conversely, Malcev proved in [18]

Malcev's Theorem. *Let Γ be a FGTF nilpotent group.*

1. There exists a unique nilpotent Lie algebra $\mathfrak{n}$ over $\mathbb{Q}$ wich contains Γ as a multiplicative lattice.

2. There exists a unique CSC nilpotent Lie group N which contains Γ as a cocompact lattice.

3. The Lie algebra of N is $\mathbb{R} \otimes \mathfrak{n}$.

The Lie algebra $\mathfrak{n}$ of the previous theorem will be called the *Malcev Lie algebra* of Γ.

5.3 The coset index
From now on, let $\mathfrak{n}$ will be a finite dimensional nilpotent Lie algebra. The coset index, which is defined now, generalizes the notions of indices for additive lattices and for multiplicative lattices.

A subset X of $\mathfrak{n}$ is called a *coset union* if X is a finite union of Λ-coset for some additive lattice Λ.

Recall that the *nilpotency index* of $\mathfrak{n}$ is the smallest integer n such that $C^{n+1}\mathfrak{n} = 0$, where $(C^n \mathfrak{n})_{n \geq 0}$ is its descending central series. The following lemma is easily proved by induction on the nilpotency index of $\mathfrak{n}$.

Lemma 17. *Any multiplicative lattice Γ of $\mathfrak{n}$ is a coset union.*

Let $X \supset Y$ be two coset unions in $\mathfrak{n}$. Obviously, there is a lattice Λ such that X and Y are both a finite union of Λ-coset. The *coset index* of Y in X is the number

$$[X : Y]_{coset} = \frac{\text{Card } X/\Lambda}{\text{Card } Y/\Lambda}$$

The numerator and denominator of the previous expression depends on the choice of Λ, but $[X : Y]_{coset}$ is well defined. In general, the coset index is *not* an integer. Obviously if $\Lambda \supset \Lambda'$ are additive lattices in $\mathfrak{n}$, we have

$$[\Lambda : \Lambda']_{coset} = [\Lambda : \Lambda'].$$

Similarly, for multiplicative lattices there is

Lemma 18. *Let $\Gamma \supset \Gamma'$ be multiplicative lattices in $\mathfrak{n}$, we have*

$$[\Gamma : \Gamma']_{coset} = [\Gamma : \Gamma'].$$

The proof, done by induction on the nipotency index of $\mathfrak{n}$, is skipped.

5.4 Morphims of FGTF nilpotent groups

Lemma 19. *Let Γ, $\Gamma' \subset \mathfrak{n}$ be multiplicative lattices in $\mathfrak{n}$ and let $f : \Gamma' \to \Gamma$ be a group morphism. Then f extends uniquely to a Lie algebra morphism*

$$\tilde{f} : \mathfrak{n} \to \mathfrak{n}.$$

Moreover $\tilde{f}$ is an isomorphism if f is injective.

When f is an isomorphism, the result is due to Malcev, see [18], Theorem 5. In general, the lemma is a folklore result and it is implicitely used in Homotopy Theory, see e.g. [1]. Since we did not found a precise reference, a proof, essentially based on Hall's collecting formula (see Theorem 12.3.1 in [15]), is now provided.

Proof. Let $x \in \mathfrak{n}$. Since Γ contains an additive lattice by Lemma 17, we have $m\mathbb{Z}x \subset \Gamma$ for some $m > 0$. Thus there is a unique map $\tilde{f} : \mathfrak{n} \to \mathfrak{n}$ extending f such that $\tilde{f}(nx) = n\tilde{f}(x)$ for any $x \in \mathfrak{n}$ and $n \in \mathbb{Z}$. It remains to prove that

$$\tilde{f}(x + y) = \tilde{f}(x) + \tilde{f}(y), \text{ and } \tilde{f}([x, y]) = [\tilde{f}(x), \tilde{f}(y)],$$

for any x, $y \in \mathfrak{n}$.

Let n be the nilpotency index of $\mathfrak{n}$. Set $\mathcal{L}(2, n) = \mathcal{L}(2)/C^{n+1}\mathcal{L}(2)$, where $\mathcal{L}(2)$ denotes the free Lie algebra over $\mathbb{Q}$ freely generated by X and Y. Let $\Gamma(2, n) \subset \mathcal{L}(2, n)$ be the multiplicative subgroup generated by X and Y.

As before, $m(X + Y)$ and $m[X, Y]$ belongs to $\Gamma(2, n)$ for some $m > 0$. Thus there are w_1, w_2 in the free group over two generators, such that

$$w_1(X, Y) = m(X + Y) \text{ and } w_2(X, Y) = m[X, Y].$$

Since $\mathcal{L}(2, n)$ is a free in the category of nilpotent Lie algebras of nilpotency index $\leq n$, we have
$$w_1(x, y) = m(x + y) \text{ and } w_2(x, y) = m[x, y]$$
for any $x, y \in \mathfrak{n}$. From this it follows easily that $\tilde{f}$ is a Lie algebra morphism.
$\square$

5.6 Self-similar data for FGTF nilpotent groups

Let $\mathfrak{z}$ be the center of $\mathfrak{n}$. Recall that $\mathcal{S}(\mathfrak{n})$ (respectively $\mathcal{V}(\mathfrak{n})$) is the set of all $f \in \operatorname{Aut} \mathfrak{n}$ such that $\operatorname{Spec} f|_{\mathfrak{z}}$ (respectively $\operatorname{Spec} f$) contains no algebraic integers. Let $\Gamma \supset \Gamma'$ be multiplicative lattices of $\mathfrak{n}$, let $f : \Gamma' \to \Gamma$ be a morphism and let $\tilde{f} : \mathfrak{n} \to \mathfrak{n}$ be its extension.

Lemma 20. *Let's assume that f is injective. Then*
(i) (Γ', f) is a self-similar datum iff $\tilde{f}$ belongs to $\mathcal{S}(\mathfrak{n})$,
(ii) (Γ', f) is a free self-similar datum iff $\tilde{f}$ belongs to $\mathcal{V}(\mathfrak{n})$
(iii) if (Γ', f) is a fractal datum, then f belongs to $\mathcal{F}(\mathfrak{n})$.

Proof. Let V be a finite dimensional vector space over $\mathbb{Q}$ and let $f \in GL(V)$. We will repeatedly use the fact that $\operatorname{Spec} f$ contains an algebraic integer iff V contains a finitely generated subgroup $E \neq 0$ such that $f(E) \subset E$.

Proof of Assertion (i). Since Γ' contains a set of generators of $\mathfrak{n}$, the subgroup $Z(\Gamma') := \Gamma' \cap \mathfrak{z}$ is the center of Γ'. Let K be the f-core of the virtual endomorphism (Γ', f).

Let's assume that (Γ', f) is not a self-similar datum. Since $K \neq 1$, the additive group $K \cap Z(\Gamma')$ is non-trivial, finitely generated and $\tilde{f}$-invariant. Therefore $\tilde{f} \notin \mathcal{S}(\mathfrak{n})$.

Conversely let's assume that $\tilde{f} \notin \mathcal{S}(\mathfrak{n})$. Then there is a nonzero finitely generated subgroup $E \subset \mathfrak{z}$ such that $\tilde{f}(E) \subset E$. By Lemma 17, $Z(\Gamma')$ is an additive lattice of $\mathfrak{z}$. Therefore we have $mE \subset Z(\Gamma')$ for some $m > 0$. Since K contains mE, it follows that (Γ', f) is not a self-similar datum.

Proof of Assertion (ii). Let $A \subset \Gamma$ be a set of representatives of Γ/Γ'. Let's consider the action of Γ on A^ω associated with the virtual endomorphism (Γ', f).

Let's assume that $\tilde{f} \notin \mathcal{V}(\mathfrak{n})$. Then there is a nonzero finitely generated abelian subgroup $F \subset \mathfrak{n}$ such that $\tilde{f}(F) \subset F$. As before, it can be assumed F lies in Γ'. Let $e \in A$ be the representative of the trivial coset and let $e^\omega = ee \dots$ be the infinite word over the single letter e. Since $f(F) \subset F$, it follows that $\gamma(e^\omega) = e^\omega$ for any $\gamma \in F$. Hence Γ does not act freely on A^ω.

Conversely, let assume that Γ does not act freely on A^ω. Let's define inductively the subsets $\mathcal{H}(n) \subset \Gamma$ by $\mathcal{H}(1) = \cup_{a \in A} a\Gamma'a^{-1}$ and
$$\mathcal{H}(n+1) = \{\gamma \in \Gamma \mid \exists\, a \in A : a\gamma a^{-1} \in \Gamma' \wedge f(a\gamma a^{-1}) \in \mathcal{H}(n)\},$$
for $n \geq 1$. Indeed $\mathcal{H}(n)$ is the set of all $\gamma \in \Gamma$ which have at least one fixed point on A^n. It follows easily that $\mathcal{H} := \cap_{n \geq 1} \mathcal{H}(n)$ is the set of all $\gamma \in \Gamma$ which have at least one fixed point on A^ω. There is an integer k such that
$$\mathcal{H} \subset C^k\mathfrak{n} \text{ but } \mathcal{H} \not\subset C^{k+1}\mathfrak{n}.$$
Let $\overline{\mathcal{H}}$ be the image of $\mathcal{H}$ in $C^k\mathfrak{n}/C^{k+1}\mathfrak{n}$ and let F be the additive subgroup of $C^k\mathfrak{n}/C^{k+1}\mathfrak{n}$ generated by $\overline{\mathcal{H}}$. Since Γ lies in a lattice, F is finitely generated. Moreover we have $axa^{-1} \equiv x \bmod C^{k+1}\mathfrak{n}$, for any $x \in C^k\mathfrak{n}$ and $a \in A$. It follows that $\tilde{f}_k(\overline{\mathcal{H}}) \subset \overline{\mathcal{H}}$, where $\tilde{f}_k$ is the linear map induced by $\tilde{f}$ on $C^k\mathfrak{n}/C^{k+1}\mathfrak{n}$. Hence $\tilde{f}_k(F) \subset F$ and Spec $\tilde{f}_k$ contains an algebraic integer. Therefore $\tilde{f} \notin \mathcal{V}(\mathfrak{n})$.

Proof of Assertion (iii). Let (Γ', f) be a fractal datum. Let Λ be the additive lattice generated by Γ. Since
$$\tilde{f}^{-1}(\Lambda) \subset \Lambda,$$
all $x \in \operatorname{Spec} \tilde{f}^{-1}$ are algebraic integers. Therefore $\tilde{f}$ belongs to $\mathcal{F}(\mathfrak{n})$. $\square$

6 Relative complexity of multiplicative lattices

This chapter is the mutiplicative analogue of ch. 4. The main result is the refined criterion of minimality. Together with Theorem 1, it is the main ingredient of the proof of Theorem 2 and 3.

Throughout the whole chapter, $\mathfrak{n}$ is finite dimensional nilpotent Lie algebra over $\mathbb{Q}$, and $\mathfrak{z}$ is its center.

6.1 Complexity of multiplicative lattices
Let $f \in \operatorname{Aut}\mathfrak{n}$ and let Γ be a multiplicative lattice of $\mathfrak{n}$. The *complexity* of Γ relative to f is the integer
$$\operatorname{cp}_f(\Gamma) = [\Gamma : \Gamma'],$$
where $\Gamma' = \Gamma \cap f^{-1}(\Gamma)$. The multiplicative lattice Γ is called *minimal* relative to f if $\operatorname{cp}_f(\Gamma) = \operatorname{ht}(f)$. Thanks to Lemma 18 the notation $\operatorname{cp}_f(\Gamma)$ is unambiguous.

Lemma 21. *Let Γ be multiplicative lattices of $\mathfrak{n}$. Then we have*
$$\operatorname{cp}_f(\Gamma) \geq \operatorname{ht}(f).$$

Proof. The proof goes by induction on the nilpotency index of $\mathfrak{n}$.

Let Z be the center of Γ. Set $\Gamma' = \Gamma \cap f^{-1}(\Gamma)$, $Z' = Z \cap f^{-1}(Z)$, $\overline{\Gamma} = \Gamma/Z$, $\overline{\Gamma'} = \Gamma'/Z'$. Also set $\overline{\mathfrak{n}} = \mathfrak{n}/\mathfrak{z}$ and let $\overline{f} : \overline{\mathfrak{n}} \to \overline{\mathfrak{n}}$ and $f_0 : \mathfrak{z} \to \mathfrak{z}$ be the isomorphisms induced by f.

By induction hypothesis, we have $\mathrm{cp}_{\overline{f}}(\overline{\Gamma}) \geq \mathrm{ht}(\overline{f})$ and therefore
$$[\overline{\Gamma} : \overline{\Gamma'}] \geq \mathrm{ht}(\overline{f}).$$
By definition, we have $[Z : Z'] = \mathrm{cp}_{f_0} Z \geq \mathrm{ht}(f_0)$. Moreover by Lemma 15 we have $\mathrm{ht}(f) = \mathrm{ht}(f_0)\mathrm{ht}(\overline{f})$. It follows that
$$\mathrm{cp}_f \Gamma = [\Gamma : \Gamma'] = [Z : Z'] \, [\overline{\Gamma} : \overline{\Gamma'}] \geq \mathrm{ht}(f_0)\mathrm{ht}(\overline{f}) = \mathrm{ht}(f),$$
and the statement is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

6.2 A property of the minimal multiplicative lattices

Let Γ be a multiplicative lattice of $\mathfrak{n}$ and let $h \in \mathrm{Aut}\,\mathfrak{n}$. For simplicity, let's assume that h is semi-simple.

Lemma 22. *The following assertions are equivalent*

(i) Γ is minimal relative to h, and

(ii) the virtual morphism (Γ', h) is good, where $\Gamma' = \Gamma \cap h^{-1}(\Gamma)$.

In particular, there is a multiplicative lattice $\tilde{\Gamma} \subset \Gamma$ which is minimal relative to h.

Proof. By Lemma 17, Γ is a coset union. Any additive lattice contains a $\mathcal{O}(h)$-module of finite index. Therefore there is an $\mathcal{O}(h)$-lattice Λ such that Γ is an union of Λ-cosets.

Let $\Gamma_0, \Gamma_1, \ldots$ be the multiplicative lattices inductively defined by $\Gamma_0 = \Gamma$, $\Gamma_1 = \Gamma'$ and $\Gamma_{n+1} = \Gamma_n \cap h^{-1}(\Gamma_n)$ for $n \geq 1$. Similarly let $\Lambda_0, \Lambda_1, \ldots$ be the additive lattices defined by $\Lambda_0 = \Lambda$, and $\Lambda_{n+1} = \Lambda_n \cap h^{-1}(\Lambda_n)$ for $n \geq 0$.

By Lemma 2, the sequence $[\Gamma_n : \Gamma_{n+1}]$ is not increasing. By Lemma 21, we have $[\Gamma_n : \Gamma_{n+1}] \geq \mathrm{ht}(f)$. Moreover, it follows from Lemma 16 that $[\Lambda_n : \Lambda_{n+1}] = \mathrm{ht}(h)$ for all n.

Let's assume now that Γ is minimal relative to h. We have $[\Gamma_n : \Gamma_{n+1}] = \mathrm{ht}(f)$ for all n, and therefore the virtual morphism (Γ', h) is good.

Conversely, let's assume that the virtual morphism (Γ', h) is good. By hypotheses we have $[\Gamma_0 : \Gamma_n] = [\Gamma_0 : \Gamma_1]^n$ and $[\Lambda_0 : \Lambda_n] = \mathrm{ht}(h)^n$ for all $n \geq 1$. It follows that
$$[\Gamma_0 : \Lambda_n]_{coset} = [\Gamma_0 : \Lambda_0]_{coset} \, \mathrm{ht}(h)^n.$$
Since $\Gamma_n \supset \Lambda_n$, we have $[\Gamma_0 : \Gamma_n] \leq [\Gamma_0 : \Lambda_n]_{coset}$.

and therefore

$$[\Gamma_0 : \Gamma_1]^n \leq [\Gamma_0 : \Lambda_0]_{coset} \operatorname{ht}(h)^n, \text{ for all } n \geq 0.$$
Hence $[\Gamma_0 : \Gamma_1] \leq \operatorname{ht}(f)$. It follows from Lemma 21 that $[\Gamma_0 : \Gamma_1] = \operatorname{ht}(f)$, thus Γ is minimal relative to h.

In order to prove the last assertion, notice that the sequence $[\Gamma_n : \Gamma_{n+1}]$ is stationary for $n \geq N$, for some $N > 0$. Therefore (Γ_{N+1}, h) is a good virtual morphism of Γ_N. Thus the subgroup $\tilde{\Gamma} = \Gamma_N$ is minimal relative to h.

$\square$

6.3 A refined criterion of minimality

A refined version of Lemma 16 is now provided. Let Γ be a multiplicative lattice in $\mathfrak{n}$ and let $h \in \operatorname{Aut}\mathfrak{n}$ be semi-simple. Let L be the field generated by $\operatorname{Spec} h$, let $\mathcal{O}$ be its ring of integers and let $\mathcal{P}$ be the set of prime ideals of $\mathcal{O}$.

Let Λ be an $\mathcal{O}(h)$-lattice and let $n > 0$ be an integer. Let's assume that
$$\Lambda \supset \Gamma \text{ and } \Gamma \text{ is an union of } n\Lambda\text{-cosets.}$$

Lemma 23. *Let S be the set of divisors of n in $\mathcal{P}$. Assume that*
$$\lambda \equiv 1 \bmod n\mathcal{O}_\pi,$$
for any $\lambda \in \operatorname{Spec} h$ and any $\pi \in S$. Then Γ is minimal relative to h.

Proof. Step 1. Since $\operatorname{Spec} h$ lies in $\mathcal{O}_\pi$ for all $\pi \in S$, there exists a positive integer d, which is prime to n, such that $d\lambda \in \mathcal{O}$ for all $\lambda \in \mathcal{O}$. Moreover we can assume that $d \equiv 1 \bmod n$.

Let $\lambda \in \operatorname{Spec} h$. We have $d\lambda \equiv 1 \bmod n\mathcal{O}_\pi$ for all $\pi \in S$. Therefore we have
$$d\lambda \in 1 + n\mathcal{O},$$
for all $\lambda \in \operatorname{Spec} h$. Set $H = dh$. Since $\operatorname{Spec} dH$ and $\operatorname{Spec} (H-1)/n$ lie in $\mathcal{O}$, it follows that
$$H \in \mathcal{O}(h) \text{ and } H \in 1 + n\mathcal{O}(h).$$
Step 2. Set $\Lambda' = \Lambda \cap h^{-1}\Lambda$. Since all eigenvalues of h are units in $\mathcal{O}_\pi$ whenever π divides n, the height of h is prime to n. By Lemma 16, we have $[\Lambda : \Lambda'] = \operatorname{ht}(h)$. Therefore we get
$$\Lambda = \Lambda' + n\Lambda.$$

It follows that
$$\Gamma = \coprod_{1 \leq i \leq k} g_i + n\Lambda$$
for some $g_1, ..., g_k \in \Lambda'$, where $k = [\Gamma : n\Lambda]$ and where $\coprod$ is the symbol of the disjoint union. Since $H(g_i) \equiv g_i \bmod n\Lambda$, we get that $h(g_i) \in g_i + n\Lambda \subset \Gamma$. Therefore we have

$$\Gamma' \supset \coprod_{1 \leq i \leq k} g_i + n\Lambda',$$

Therefore we have $[\Gamma' : n\Lambda'] \geq k = [\Gamma : n\Lambda]$. It follows that
$$[\Gamma : \Gamma'] \leq [n\Lambda : n\Lambda'] = \mathrm{ht}(h).$$
By Lemma 21, we have $[\Gamma : \Gamma'] = \mathrm{ht}(h)$. Thus Γ is minimal relative to h. $\square$

7 Proof of Theorems 2 and 3

7.1 Proof of Theorem 2 and 3.
Let $\mathfrak{n}$ be a finite dimensional nilpotent Lie algebra over $\mathbb{Q}$ and let $\mathfrak{z}$ be its center and let Γ be a multiplicative lattice of $\mathfrak{n}$.

Theorem 2. *The following assertions are equivalent*
    (i) The group Γ is transitive self-similar,
    (ii) the group Γ is densely self-similar, and
    (iii) the Lie algebra $\mathfrak{n}^{\mathbb{C}}$ admits a special grading.

Proof. Let's consider the following assertion
$(\mathcal{A})$ $\mathcal{S}(\mathfrak{n}) \neq \emptyset$.
The implication $(ii) \Rightarrow (i)$ is tautological. Together with the Lemmas 6(i) and 9(i), the following implications are already proved
$$(ii) \Rightarrow (i) \Rightarrow (\mathcal{A}) \Leftrightarrow (iii).$$
Therefore, it is enough to prove that $(\mathcal{A}) \Rightarrow (ii)$.
Step 1. Definition of some $h \in \mathbf{G}(\mathbb{Q})$. Let's assume that $\mathcal{S}(\mathfrak{n}) \neq \emptyset$, and let $f \in \mathcal{S}(\mathfrak{n})$. Since the semi-simple part of f is also in $\mathbf{G}(\mathbb{Q})$, it can be assumed that f is semi-simple. Let $\mathbf{K} \subset \mathbf{G}$ be the Zariski closure of the subgroup generated by f and set $\mathbf{H} = \mathbf{K}^0$.

Let Λ be the $\mathcal{O}(f)$-module generated by Γ. By Lemma 17, Γ is a coset union. Therefore Λ is a lattice and Γ is an union of $n\Lambda$-coset for some positive integer n.

Let $X(\mathbf{H})$ be the group of characters of $\mathbf{H}$, let K be the splitting field of $\mathbf{H}$, let $\mathcal{O}$ be the ring of integers of K, let $\mathcal{P}$ be the set of prime ideals of $\mathcal{O}$ and let S be set set of all $\pi \in \mathcal{P}$ dividing n.

By Theorem 1, there exists $h \in \mathbf{H}(\mathbb{Q})$ such that, for any non-trivial $\chi \in X$ we have
    (i) $\chi(h)$ is not an algebraic integer, and
    (ii) $\chi(h) \equiv 1 \bmod n\mathcal{O}_\pi$ for any $\pi \in S$.

Step 2. Let $\Gamma' = \Gamma \cap h^{-1}(\Gamma)$. We claim that the virtual morphism (Γ', h) is a good self-similar datum.

Since $\mathbf{K} \subset \mathbf{G}$ is the Zariski closure of the subgroup generated by f, we have $\mathbb{Q}[h] \subset \mathbb{Q}[f]$ and therefore Λ is a $\mathcal{O}(h)$-lattice. It follows from Lemma 23 that the virtual endomorphism (Γ', h) is good.

Moreover, let Ω_0 be the set of weights of $\mathbf{H}$ over $\mathfrak{z}^{\overline{\mathbb{Q}}}$. There is an integer l such that $f^l \in \mathbf{K}^0 = \mathbf{H}$. The spectrum of f^l on $\mathfrak{z}^{\overline{\mathbb{Q}}}$ are the numbers $\chi(f^l)$ when χ runs over Ω_0. Thus it follows that Ω_0 does not contain the trivial character, hence h belongs to $\mathcal{S}(\mathfrak{n})$.

Therefore by Lemma 20, the virtual endomorphism (Γ', h) is a good self-similar datum. Thus by Lemma 3, Γ is a densely self-similar group. $\qquad\square$

Theorem 3. *The following assertions are equivalent*
  (i) The group Γ is freely self-similar,
  (ii) the group Γ is freely densely self-similar, and
  (iii) the Lie algebra $\mathfrak{n}^{\mathbb{C}}$ admits a very special grading.

Proof. Let's assume Assertion (i). Let's consider a free self-similar action of Γ on some A^{ω} and let A' be any Γ-orbit in A. Then the action of Γ on A'^{ω} is free transitive self-similar, thus Γ is freely transitive self-similar.

The rest of the proof is identical to the previous proof, except that

1) the assertion $(\mathcal{A})$ is replaced by $(\mathcal{A}')$: $\mathcal{V}(\mathfrak{n}) \neq \emptyset$,

2) the Lemmas 6(ii) and 9(ii) are used instead of Lemmas 6(i) and 9(i) in order to prove that $(ii) \Rightarrow (i) \Rightarrow (\mathcal{A}') \Leftrightarrow (iii)$,

3) the proof that $\mathcal{A}' \Rightarrow (ii)$ uses the weights of $\mathbf{H}$ and the eigenvalues of f on $\mathfrak{n}$ instead of $\mathfrak{z}$. $\qquad\square$

7.2 Manning's Theorem

Let N be a CSC nilpotent Lie group N and let Γ be a cocompact lattice. The manifold $M = N/\Gamma$ is called a *nilmanifold*.

A diffeomorphism $f : M \to M$ is called an *Anosov diffeomorphism* if

(i) there is a continuous splitting of the tangent bundle TM as $TM = E_u \oplus E_s$ which is invariant by df, and

(ii) there is a Riemannian metric relative to which $df|_{E_s}$ and $df^{-1}|_{E_u}$ are contracting.

For any $x \in M$, f induces a group automorphism f_* of $\Gamma \simeq \pi_1(M)$. By Lemma 19, f_* extends to an isomorphism $\tilde{f}_* : \mathfrak{n}^{\mathbb{R}} \to \mathfrak{n}^{\mathbb{R}}$, where $\mathfrak{n}^{\mathbb{R}}$ is

the Lie algebra of N. Strictly speaking, $\tilde{f}_*$ is only defined up to an inner automorphism. Since f_* is well defined modulo the unipotent radical of $\operatorname{Aut} \mathfrak{n}^{\mathbb{R}}$, the set $\operatorname{Spec} f_*$ is unambiguously defined.

Manning's Theorem. *The set* $\operatorname{Spec} f_*$ *contains no root of unity.*

See [18]. Later on, A. Manning proved a much stronger result. Namely $\operatorname{Spec} f_*$ contains no eigenvalues of absolute value 1, and f is topologically conjugated to an Anosov automorphism, see [19].

7.3 A Corollary for nilmanifolds with an Anosov diffeomorphim

Corollary 4. *Let M be a nilmanifold endowed with an Anosov diffeomorphism. Then $\pi_1(M)$ is freely densely self-similar.*

Proof. By definition, we have $M = N/\Gamma$, where N is a CSC nilpotent Lie group and $\Gamma \simeq \pi_1(M)$ is a cocompact lattice. Set $\mathfrak{n}^{\mathbb{R}} = \operatorname{Lie} N$ and $\mathfrak{n}^{\mathbb{C}} = \mathbb{C} \otimes \mathfrak{n}^{\mathbb{R}}$. By Manning's Theorem and Lemma 8, $\mathfrak{n}^{\mathbb{C}}$ has a very special grading. Therefore Γ is freely densely self-similar by Theorem 3. $\square$

7.4 Characterisation of fractal FGTF nilpotent groups
For completeness purpose, we will now investigate the non-negative gradings of $\mathfrak{n}^C$. Unlike Theorems 2 and 3, the proof of Propositions 5 and 6 are quite obvious.

Let $\mathfrak{n}^{\mathbb{Q}}$ be a finite dimensional nilpotent Lie algebra and let Γ be a multiplicative lattice in $\mathfrak{n}$. Set $\mathfrak{n}^{\mathbb{C}} = \mathbb{C} \otimes \mathfrak{n}^{\mathbb{Q}}$.

Proposition 5. *The following assertions are equivalent*
(i) The group Γ is fractal
(ii) $\mathfrak{n}^{\mathbb{C}}$ admits a non-negative special grading.
(iii) $\mathfrak{n}^{\mathbb{Q}}$ admits a non-negative special grading.

Proof. It follows from Lemma 10 that Assertions (ii) and (iii) are equivalent. *Proof that (i) $\Rightarrow$ (ii).* By assumption, there is a fractal datum (Γ', f). Let $g : \Gamma \to \Gamma'$ be the inverse of f and let $\tilde{g} \in \operatorname{Aut} \mathfrak{n}$ be its unique extension.

Let $\Lambda \subset \mathfrak{n}$ be the additive subgroup generated by Γ. By Lemmas 17, Λ is an additive lattice. Since we have $\tilde{g}(\Lambda) \subset \Lambda$, it follows that all eigenvalues of $\tilde{g}$ are algebraic integers.

Moreover (Γ', g^{-1}) is a self-similar datum, thus $\operatorname{Spec} \tilde{g}^{-1}|_{\mathfrak{z}}$ contains no root of unity. Therefore, by Lemma 10, Assertion (ii) holds.

Proof that (iii) ⇒ (i). Let's assume Assertion (iii) and let
$$\mathfrak{n}^{\mathbb{Q}} = \oplus_{k \geq 0} \mathfrak{n}_k^{\mathbb{Q}}$$
be a non-negative special grading of $\mathfrak{n}^{\mathbb{Q}}$.

By Lemma 17, Γ lies in a lattice Λ. Since it is possible to enlarge Λ, we can assume that
$$\Lambda = \oplus_{k \geq 0} \Lambda_k,$$
where $\Lambda_k = \Lambda \cap \mathfrak{n}_k^{\mathbb{Q}}$. Since Γ is a coset union, there is an integer $d \geq 1$ such that Γ is an union of $d\Lambda$-cosets.

Let g be the automorphism of $\mathfrak{n}^{\mathbb{Q}}$ defined by $g(x) = (d+1)^k x$ if $x \in \mathfrak{n}_k^{\mathbb{Q}}$. We claim that $g(\Gamma) \subset \Gamma$. Let $x \in \Gamma$ and let $x = \sum_{k \geq 0} x_k$ be its decomposition into homogenous components. We have
$$g(x) = x + \sum_{k \geq 1} ((d+1)^k - 1)x_k.$$
By hypothesis each homogenous component x_k belongs to Λ. Since $(d+1)^k - 1$ is divisible by d, we have $g(x) \in x + d\Lambda \subset \Gamma$ and the claim is proved.

Set $\Gamma' = g(\Gamma)$ and let $f : \Gamma' \to \Gamma$ be the inverse of g. It is clear that (Γ', f) is a fractal datum for Γ, what proves Assertion (i). $\square$

Proposition 6. *The following assertions are equivalent*
(i) The group Γ is freely fractal
(ii) $\mathfrak{n}^{\mathbb{C}}$ admits a positive grading.
(iii) $\mathfrak{n}^{\mathbb{Q}}$ admits a positive grading.

Since the proof is strictly identical, it will be skipped.

8 Not self-similar FGTF nilpotent groups and affine nilmanifolds

This section provides an example of a FGTF nilpotent group which is not even self-similar, see subsection 8.6. The end of the section is about the Milnor-Scheuneman conjecture.

8.1 FGTF nilpotent groups with rank one center
Let Γ be a FGTF nilpotent group and let $Z(\Gamma)$ be its center.

Lemma 24. *Let's asssume that Γ is self-similar and $Z(\Gamma) \simeq \mathbb{Z}$. Then Γ is transitive self-similar.*

Proof. Assume that Γ admits a faithful self-similar action on some A^ω, where A is a finite alphabet. Let $a_1, \ldots, a_k$ be a set of representatives of A/Γ, where k is the number of Γ-orbits on A. For each $1 \le i \le k$, let Γ_i be the stabilizer of a_i. For any $h \in \Gamma_i$, there is $h_i \in \Gamma$ such that
$$h(a_i w) = a_i h_i(w),$$
for all $w \in A^\omega$. Since the action is faithfull h_i is uniquely determined and the map $f_i : \Gamma_i \to \Gamma, h \mapsto h_i$ is a group morphism.

Let $\mathfrak{n}^\mathbb{Q}$ be the Malcev Lie algebra of Γ, and let $\mathfrak{z}$ be its center, and let $z \ne 0$ be a generator of $\cap_i Z(\Gamma_i)$. By Lemma 19, the group morphism f_i extends to a Lie algebra morphism $\tilde{f}_i : \mathfrak{n}^\mathbb{Q} \to \mathfrak{n}^\mathbb{Q}$. Since $\mathfrak{z} = \mathbb{Q} \otimes Z(\Gamma)$ is one dimensional, it follows that either $\tilde{f}_i$ is an isomorphism or $\tilde{f}_i(\mathfrak{z}) = 0$. In any case, we have $\tilde{f}_i(z) = x_i z$, for some $x_i \in \mathbb{Q}$. However $\mathbb{Z}z$ is not invariant by all $\tilde{f}_i$, otherwise it would be in the kernel of the action. It follows that at least one x_i is not an integer.

For such an index i, the f_i-core of Γ_i is trivial, and the virtual morphism (Γ_i, f_i) is a self-similar datum for Γ. Thus Γ is transitive self-similar. $\square$

8.2 Small representations

Let N be a CSC nilpotent Lie group with Lie algebra $\mathfrak{n}^\mathbb{R}$ and let Γ be a cocompact lattice.

Lemma 25. *If Γ is transitive self-similar, then there exists a faithfull $\mathfrak{n}^\mathbb{R}$-module of dimension $1 + \dim \mathfrak{n}^\mathbb{R}$.*

Proof. By hypothesis, Γ is transitive self-similar. By Theorem 2, $\mathfrak{z}^\mathbb{C}$ admits a special grading
$$\mathfrak{n}^\mathbb{C} = \oplus_{n \in \mathbb{Z}} \mathfrak{n}_n^\mathbb{C}.$$
Let $\delta : \mathfrak{n}^\mathbb{C} \to \mathfrak{n}^\mathbb{C}$ be the derivation defined by $\delta(x) = nx$ if $x \in \mathfrak{n}_n$. Since $\delta|_{\mathfrak{z}^\mathbb{C}}$ is injective, it follows that there is some $\partial \in \mathrm{Der}\,\mathfrak{n}^\mathbb{R}$ such that $\partial|_{\mathfrak{z}^\mathbb{R}}$ is injective.

Set $\mathfrak{m}^\mathbb{R} = \mathbb{R}\partial \ltimes \mathfrak{n}^\mathbb{R}$. Relative to the adjoint action, $\mathfrak{m}^\mathbb{R}$ is a faithfull $\mathfrak{z}^\mathbb{R}$-module. Therefore $\mathfrak{m}^\mathbb{R}$ is a faithfull $\mathfrak{n}^\mathbb{R}$-module with the prescribed dimension. $\square$

8.3 Filiform nilpotent Lie algebras

Let $\mathfrak{n}$ be a nilpotent Lie algebra over $\mathbb{Q}$. Let $C^n\mathfrak{n}$ be the decreasing central series, which is inductively defined by $C^1\mathfrak{n} = \mathfrak{n}$ and $C^{n+1}\mathfrak{n} = [\mathfrak{n}, C^n\mathfrak{n}]$. The

nilpotent Lie algebra $\mathfrak{n}$ is called *filiform* if $\dim C^1\mathfrak{n}/C^2\mathfrak{n} = 2$ and $\dim C^k\mathfrak{n}/C^{k+1}\mathfrak{n} \leq 1$ for any $k > 1$. Set $n = \dim \mathfrak{n}$. It follows from the definition that $\dim C^k\mathfrak{n}/C^{k+1}\mathfrak{n} = 1$ for any $0 < k \leq n-1$ and $C^k\mathfrak{n} = 0$ for any $k \geq n$.

Lemma 26. *Let $\mathfrak{n}$ be a filiform nilpotent Lie algebra over $\mathbb{Q}$, with $\dim \mathfrak{n} \geq 3$. Then its center $\mathfrak{z}$ has dimension one.*

Proof. Let $z \in \mathfrak{n}$ be nonzero. Let k be the integer such that $z \in C^k\mathfrak{n} \setminus C^{k+1}\mathfrak{n}$. Since $C^k\mathfrak{n} = C^{k+1}\mathfrak{n} \oplus \mathbb{Q}z$
$$C^{k+1}\mathfrak{n} = [\mathfrak{n}, C^k\mathfrak{n}] = [\mathfrak{n}, C^{k+1}\mathfrak{n}] + [\mathfrak{n}, z] = C^{k+2}\mathfrak{n}.$$
It follows that $C^{k+1}\mathfrak{n} = 0$. Therefore $\mathfrak{z}$ lies in $C^k\mathfrak{n}$, which is a one dimensional ideal. $\square$

8.4 Benoist Theorem

Benoist's Theorem. *There is a nilpotent Lie algebra $\mathfrak{n}_B^{\mathbb{R}}$ of dimension 11 over $\mathbb{R}$, with the following properties*
  (i) The Lie algebra $\mathfrak{n}_B^{\mathbb{R}}$ has no faithfull representations of dimension 12,
  (ii) the Lie algebra $\mathfrak{n}_B^{\mathbb{R}}$ is defined over $\mathbb{Q}$, and
  (iii) the Lie algebra $\mathfrak{n}_B^{\mathbb{R}}$ is filiform.

The three assertions appear in different places of [3]. Indeed Assertion (i), which is explicitely stated in Theorem 2 of [3], hold for a one-parameter family of eleven dimensional Lie algebras, which are denoted $\mathfrak{a}_{-2,1,t}$ in section 2.1 of [3]. These Lie algebras are filiform by Lemma 4.2.2 of [3]. Moreover, when t is rational, $\mathfrak{a}_{-2,1,t}$ is defined over $\mathbb{Q}$. Therefore the Benoist Theorem holds for the Lie algebras $\mathfrak{n}_B = \mathfrak{a}_{-2,1,t}$ where t is any rational number.

8.5 A FGTF group which is not self-similar
Let N_B the CSC nilpotent Lie group with Lie algebra $\mathfrak{n}_B^{\mathbb{R}}$. Since $\mathfrak{n}_B^{\mathbb{R}}$ is defined over $\mathbb{Q}$, N_B contains some cocompact lattice.

Corollary 7. *Let Γ be any cocompact lattice in N_B. Then Γ is not self-similar.*

Proof. Let's assume otherwise. By Benoist Theorem and Lemma 26, the center of $\mathfrak{n}_B^{\mathbb{R}}$ is one dimensional. Thus the center of Γ has rank one, and by Lemma 24, Γ is transitive self-similar. By Lemma 25, $\mathfrak{n}_B^{\mathbb{R}}$ admits a faithfull representation of dimension 12, which contradicts Benoist Theorem.

Therefore Γ is not self-similar. $\square$

8.6 On the Scheuneman-Milnor conjecture

A smooth manifold M is called *affine* if it admits a torsion-free and flat connection. Scheuneman [28] and Milnor [23] asked the following question

<div style="text-align:center">is any nilmanifold M affine?</div>

The story of the Scheuneman-Milnor conjecture is quite interesting. For many years, there are been a succession of proofs followed by refutations, but there was no doubts that the conjecture should be ultimalely proved... until a counterexample has been found by Benoist [3]. Indeed it is an easy corollary of his previously mentionned Theorem.

The following question is a refinement of the previous conjecture

<div style="text-align:center">if $\pi_1(M)$ is densely self-similar, is the nilmanifold M affine?</div>

A positive result in that direction is

Corollary 8. *Let M be a nilmanifold. If $\pi_1(M)$ is freely self-similar, then M is affine complete.*

Proof. Set $M = N/\Gamma$, where N is a CSC nilpotent Lie group and Γ is a cocompact lattice. Let $\mathfrak{n}^{\mathbb{R}}$ be the Lie algebra of N. By Theorem 3, $\mathbb{C} \otimes \mathfrak{n}^{\mathbb{R}}$ admits a very special grading, what implies that a generic derivation is injective. Therefore there is a derivation δ of $\mathfrak{n}^{\mathbb{R}}$ which is injective. Set $\mathfrak{m}^{\mathbb{R}} = \mathbb{R}\delta \ltimes \mathfrak{n}^{R}$. Then N is equivariantly diffeomorphic to the affine space $\delta + \mathfrak{n}^{\mathbb{R}} \subset \mathfrak{m}^{\mathbb{R}}$. Therefore M is affine complete.

$\square$

9   Absolute Complexities

For the whole chapter, N will be a CSC nilpotent Lie groups, with Lie algebra $\mathfrak{n}^{\mathbb{R}}$. Let's assumethat that N contains some cocompact lattices.

Under the condition of Theorem 2 or 3, any cocompact lattice Γ in N admits a transitive or free self-similar action on some A^{ω}. In this section, we try to determine the minimal degree of these actions.

9.1 Three type of absolute complexities

The *complexity* of a cocompact lattice $\Gamma \subset N$, denoted by $\mathrm{cp}\,\Gamma$, is the smallest degree of a faithfull transitive self-similar action of Γ on some A^{ω}, with the convention that $\mathrm{cp}\,\Gamma = \infty$ if Γ is not transitive self-similar. Similarly, the *free complexity* of Γ, denoted by $\mathrm{fcp}\,\Gamma$, is the smallest degree of a free self-similar action of Γ. Two cocompact lattices are called *commensurable* if they share a

commoun subgroup of finite index. The complexity and the free complexity of a commensurable class ξ are the integers

$$\mathrm{cp}\,\xi = \mathrm{Min}_{\Gamma \in \xi}\,\mathrm{cp}\,\Gamma, \text{ and}$$
$$\mathrm{fcp}\,\xi = \mathrm{Min}_{\Gamma \in \xi}\,\mathrm{fcp}\,\Gamma.$$

Then, the complexity of the nilpotent group N is

$$\mathrm{cp}\,N = \mathrm{Max}_{\xi}\,\mathrm{cp}\,\xi,$$

where ξ runs over all commensurable classes in N. In what follows, we will provide a formula for the complexity of commensurable classes. The question

under which condition $\mathrm{cp}N < \infty$?

is not solved, but it is a deep question. In chapter 10, a class of CSC nilpotent Lie groups of infinite complexity is investigated.

9.2 Theorem 9

Let ξ be a commensurable class of cocompact lattices in N, and let $\Gamma \in \xi$. The Malcev Lie algebra Γ is a $\mathbb{Q}$-form of the Lie algebra $\mathfrak{n}^{\mathbb{R}}$. Since it depends only on ξ, it will be denoted by $\mathfrak{n}(\xi)$.

Theorem 9. *We have*

$$\mathrm{cp}\,\xi = \mathrm{Min}_{h \in \mathcal{S}(\mathfrak{n}(\xi))}\,\mathrm{ht}(h), \text{ and}$$
$$\mathrm{fcp}\,\xi = \mathrm{Min}_{h \in \mathcal{V}(\mathfrak{n}(\xi))}\,\mathrm{ht}(h).$$

Proof. Let $h \in \mathcal{S}(\mathfrak{n}(\xi))$ be an isomorphism of minimal height. In order to show that $\mathrm{cp}\,\xi = \mathrm{ht}(h)$, we can assume that h is semi-simple, by Lemma 13.

Further, let Γ be any cocompact lattice in ξ. By Lemma 20, we have $\mathrm{cp}\,\Gamma = \mathrm{Min}_{f \in \mathcal{S}(\mathfrak{n}(\xi))}\,\mathrm{cp}_f\,\Gamma$. By lemma 21, we have $\mathrm{cp}_f\,\Gamma \geq \mathrm{ht}(f)$, therefore we have $\mathrm{cp}\,\Gamma \geq \mathrm{ht}(h)$. In particular $\mathrm{cp}\,\xi \geq \mathrm{ht}(h)$.

By Lemma 22, Γ contains a finite index subgroup $\tilde{\Gamma}$ which is minimal relative to h. Since $\mathrm{cp}_h\,\tilde{\Gamma} = \mathrm{ht}(h)$, it follows that $\mathrm{cp}\,\xi \leq \mathrm{ht}(h)$.

Therefore $\mathrm{cp}\,\xi = \mathrm{ht}(h)$ and the first assertion is proved.

For the second assertion, let's notice that an free action of minimal degree is automatically transitive, see the proof of Theorem 3. Then the rest of the proof is strictly identical to the previous proof. $\qquad\square$

9.3 Classification of lattices in a CSC nilpotent Lie groups

Obviously Malcev's Theorem implies the following

Malcev's Corollary. *The map* $\xi \mapsto \mathfrak{n}(\xi)$ *establishes a bijection between the commensurable classes of lattices and the $\mathbb{Q}$-forms of the Lie algebra $\mathfrak{n}^{\mathbb{R}}$.*

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For the next chapter, it is interesting to translate this into the framework of non-abelian Galois cohomology. Somehow, it is more concrete, since the non-abelian Galois cohomology classifies $\mathbb{Q}$-forms of classical objects.

Set $\mathbf{G} = \mathrm{Aut}\mathfrak{n}^{\mathbb{C}}$, let $\mathbf{U}$ be its unipotent radical and set $\overline{\mathbf{G}} = \mathbf{G}/\mathbf{U}$. From now on, fix once for all a commensurable class ξ_0 of cocompact lattices. Then $\mathfrak{n}(\xi_0)$ is a $\mathbb{Q}$-form of $\mathfrak{n}^{\mathbb{C}}$, what provides a $\mathbb{Q}$-form of the algebraic groups $\mathbf{G}$ and $\overline{\mathbf{G}}$. It induces an action of $\mathrm{Gal}(\mathbb{Q})$ over $\overline{\mathbf{G}}(\overline{\mathbb{Q}})$.

Set $\overline{\mathbb{Q}}_{re} = \overline{\mathbb{Q}} \cap \mathbb{R}$ and let
$$\overline{\pi} : H^1(\mathrm{Gal}(\mathbb{Q}), \overline{\mathbf{G}}(\overline{\mathbb{Q}})) \to H^1(\mathrm{Gal}(\overline{\mathbb{Q}}_{re}), \overline{\mathbf{G}}(\overline{\mathbb{Q}}))$$
be the natural map. Recall that these two non-abelian cohomologies are pointed sets, where the distinguished point $*$ comes from the given $\mathbb{Q}$-form and the induced $\overline{\mathbb{Q}}_{re}$-form. Denote by $\mathrm{Ker}\,\overline{\pi}$ the kernel of $\overline{\pi}$, i.e. the fiber $\overline{\pi}^{-1}(*)$ of the distinguished point.

Let $\mathcal{L}(N)$ be the set of all commensurable classes classes of lattices of N, up to conjugacy.

Corollary 10. *There is a natural identification*
$$\mathcal{L}(N) \simeq \mathrm{Ker}\,\overline{\pi}.$$

Proof. For any field $K \subset \mathbb{C}$, set $\mathfrak{n}^K = K \otimes \mathfrak{n}(\xi_0)$. For any two fields $K \subset L \subset \mathbb{C}$, let $\mathcal{F}(L/K)$ be the set of K-forms of $\mathfrak{n}^L$, up to conjugacy. Then $\mathcal{F}(L/K)$ is a pointed set, whose distinguished point is the K-form $\mathfrak{n}^K$.

By the Lefschetz principle, the $\mathbb{Q}$-forms of $\mathfrak{n}^{\mathbb{C}}$ (up to conjugacy) are in bijection with the $\mathbb{Q}$-forms of $\mathfrak{n}^{\overline{\mathbb{Q}}}$. Similarly by the Tarski-Seidenberg principle the real forms (up to conjugacy) of $\mathfrak{n}^{\mathbb{C}}$ are in bijection with the $\overline{\mathbb{Q}}_{re}$-forms of $\mathfrak{n}^{\overline{\mathbb{Q}}}$. So we have
$$\mathcal{F}(\mathbb{C}/\mathbb{Q}) \simeq \mathcal{F}(\overline{\mathbb{Q}}/\mathbb{Q}) \text{ and } \mathcal{F}(\mathbb{C}/\mathbb{R}) \simeq \mathcal{F}(\overline{\mathbb{Q}}/\overline{\mathbb{Q}}_{re})$$

Since a Lie algebra is a vector space endowed with a tensor (its Lie bracket), it follows from [29], III-2, Proposition 1 that
$$\mathcal{F}(\overline{\mathbb{Q}}/\mathbb{Q}) = H^1(\mathrm{Gal}(\mathbb{Q}), \mathbf{G}(\overline{\mathbb{Q}})), \text{ and}$$
$$\mathcal{F}(\overline{\mathbb{Q}}/\overline{\mathbb{Q}}_{re}) = H^1(\mathrm{Gal}(\overline{\mathbb{Q}}_{re}), \mathbf{G}(\overline{\mathbb{Q}})).$$
Moreover since $\mathbf{U}$ is unipotent, we have
$$H^1(\mathrm{Gal}(\mathbb{Q}), \mathbf{G}(\overline{\mathbb{Q}})) \simeq H^1(\mathrm{Gal}(\mathbb{Q}), \overline{\mathbf{G}}(\overline{\mathbb{Q}})), \text{ and}$$
$$H^1(\mathrm{Gal}(\mathbb{Q}_{re}), \mathbf{G}(\overline{\mathbb{Q}})) \simeq H^1(\mathrm{Gal}(\mathbb{Q}_{re}), \overline{\mathbf{G}}(\overline{\mathbb{Q}})).$$
There is a commutative diagram of pointed sets

$$
\begin{array}{ccc}
\mathcal{F}(\mathbb{C}/\mathbb{Q}) & \xrightarrow{\ \theta\ } & \mathcal{F}(\mathbb{C}/\mathbb{R}) \\
\downarrow & & \downarrow \\
\mathcal{F}(\overline{\mathbb{Q}}/\mathbb{Q}) & \xrightarrow{\ \theta'\ } & \mathcal{F}(\overline{\mathbb{Q}}/\overline{\mathbb{Q}}_{re}) \\
\downarrow & & \downarrow \\
H^1(\mathrm{Gal}(\mathbb{Q}), \mathbf{G}(\overline{\mathbb{Q}})) & \xrightarrow{\ \pi\ } & H^1(\mathrm{Gal}(\overline{\mathbb{Q}}_{re}), \mathbf{G}(\overline{\mathbb{Q}})) \\
\downarrow & & \downarrow \\
H^1(\mathrm{Gal}(\mathbb{Q}), \overline{\mathbf{G}}(\overline{\mathbb{Q}})) & \xrightarrow{\ \overline{\pi}\ } & H^1(\mathrm{Gal}(\overline{\mathbb{Q}}_{re}), \overline{\mathbf{G}}(\overline{\mathbb{Q}}))
\end{array}
$$

where θ is the map $\mathbb{R} \otimes_{\mathbb{Q}} -$, θ' is the map $\overline{\mathbb{Q}}_{re} \otimes_{\mathbb{Q}} -$, π and $\overline{\pi}$ are restrictions maps. It is tautological that $\mathcal{L}(N) = \mathrm{Ker}\,\theta$. Since all vertical maps are bijective, it follows that $\mathcal{L}(N)$ is isomorphic to $\mathrm{Ker}\,\overline{\pi}$.

$\square$

10 Some Nilpotent Lie groups of infinite complexity.

This chapter is devoted to the analysis to a class of CSC nilpotent Lie groups $\mathcal{N}$, for which the classification of commensurable classes and the computation of their complexity are very explicitely connected with the arithmetic of complex quadratic fields.

For $K = \mathbb{R}$ or $\mathbb{C}$, let $O(2,K)$ be the group of linear automorphisms of $\mathbb{R}^2$ preserving the quadratic form $x^2 + y^2$. Let $\mathcal{L}$ be the class of nilpotent Lie algebras $\mathfrak{n}^R$ over $\mathbb{R}$ satisfying the following properties

(i) $\mathfrak{n}^R$ has a $\mathbb{Q}$-form

(ii) $\mathfrak{n}^R/[\mathfrak{n}^R, \mathfrak{n}^R] \simeq \mathbb{R}^2$ has dimension two

(iii) the Lie algebra $\mathfrak{n}^{\mathbb{C}} := \mathbb{C} \otimes \mathfrak{n}^{\mathbb{R}}$ has a special grading

(iv) for $K = \mathbb{R}$ or $\mathbb{C}$, the image of $\mathrm{Aut}\,\mathfrak{n}^K$ in $GL(\mathfrak{n}^K/[\mathfrak{n}^K, \mathfrak{n}^K])$ is $O(2,K)$.

Let be the class of CSC nilpotent Lie groups N whose Lie algebra $\mathfrak{n}^{\mathbb{R}}$ is in $\mathcal{L}$.

It should be noted that the class $\mathcal{N}$ is *not* empty. There is one Lie group $N_{112} \in \mathcal{N}$ of dimension 112, see [22]. Indeed [22] contains a general method to find nilpotent Lie algebras with a prescribed group of automorphisms, modulo its unipotent radical. For the group $O(2, \mathbb{R})$, N_{112} is the Lie group of minimal dimension obtained with this method. However it is difficult to provide more details, without going to very long explanations.

From now on, N will be any Lie group in class $\mathcal{N}$, ξ_0 will be one commensurable class of lattices in N and $\mathfrak{n} := \mathfrak{n}(\xi_0)$ will be the corresponding corresponding $\mathbb{Q}$ form of $\mathfrak{n}^R$. As before, set $\mathfrak{n}^K = K \otimes \mathfrak{n}$ for any field $K \subset \mathbb{C}$.

Let $\mathbf{G} = \mathbf{Aut}\,\mathfrak{n}$ the algebraic automorphism group of $\mathfrak{n}$, let $\mathbf{U}$ be its unipotent radical and set $\overline{\mathbf{G}} = \mathbf{G}/\mathbf{U}$. By hypothesis, $\overline{\mathbf{G}}$ is the algebraic group $O(2)$.

10.1 The $\mathbb{Z}$-grading of $\mathfrak{n}^{\mathbb{C}}$
Since $\overline{\mathbf{G}}(\mathbb{C}) = O(2, \mathbb{C})$, a maximal torus $\mathbf{H}$ of $\mathbf{G}(\mathbb{C})$ has dimension 1. Therefore $\mathfrak{n}^{\mathbb{C}}$ has a $\mathbb{Z}$-grading
$$\mathfrak{n}^{\mathbb{C}} = \oplus_{k \in \mathbb{Z}}\,\mathfrak{n}_k^{\mathbb{C}},$$
satisfying the following properties
    (i) the grading is essentially unique, namely any other grading is a multiple of the given grading,
    (ii) $\dim \mathfrak{n}_k^{\mathbb{C}} = \dim \mathfrak{n}_{-k}^{\mathbb{C}}$ for any k. In particular $\mathfrak{n}^{\mathbb{C}}$ does *not* admit a (nontrivial) non-negative grading, and
    (iii) the grading is *not* defined over $\mathbb{R}$.
Indeed since $\overline{\mathbf{G}}(\mathbb{C}) = O(2, \mathbb{C})$, the normalizer $\mathbf{K}(\mathbb{C})$ of $\mathbf{H}(\mathbb{C})$ has two connected components, and any $\sigma \in \mathbf{K}(\mathbb{C}) \setminus \mathbf{K}(\mathbb{C})^0$ exchanges $\mathfrak{n}_k^{\mathbb{C}}$ and $\mathfrak{n}_{-k}^{\mathbb{C}}$, what shows Assertion (ii). Since $\overline{\mathbf{G}}(\mathbb{R}) = O(2, \mathbb{R})$, no torus of $\mathbf{G}(\mathbb{R})$ is split, what implies Assertion (iii).

Moreover, the grading is not very special, so $\mathrm{fcp}(\xi) = \infty$ for any commensurable class ξ. For the forthcoming computation of $\mathrm{cp}(\xi)$, the following quantity will be involved
$$e(N) = \sum_{k>0}\,k \dim \mathfrak{n}_k^{\mathbb{C}}.$$
For example, for the Lie group N_{112} of [22], we have $e(N_{112}) = 126$.

10.2 Classification of commensurable lattices in N

Lemma 27. *Let $N \in \mathcal{N}$. Up to conjugacy, there is a bijection between*
    (i) the commensurable class of cocompact lattices in N, and
    (ii) the positive definite quadratic form on $\mathbb{Q}^2$.

Proof. Let q_0 be a given definite quadratic form on $\mathbb{Q}^2$. It determines a $\mathbb{Q}$-form of the algebraic group $O(2)$, and $H^1(\mathrm{Gal}(\mathbb{Q}), O(2, \overline{\mathbb{Q}}))$ classifies the quadratic forms on $\mathbb{Q}^2$, while the kernel of
$$H^1(\mathrm{Gal}(\mathbb{Q}), O(2, \overline{\mathbb{Q}})) \to H^1(\mathrm{Gal}(\overline{\mathbb{Q}}_{re}), O(2, \overline{\mathbb{Q}}))$$
classifies the positive definite quadratic forms on $\mathbb{Q}^2$. Thus the lemma follows from Corollary 10.

$\square$

The classification of positive definite quadratic forms q on $\mathbb{Q}^2$ is well known. Up to conjugacy, q can be written as

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$$q(x, y) = ax^2 + ady^2,$$

where a, d are positive and d is a square-free integer. Then q is determined by the following two invariants

(i) its discriminant $-d$, viewed as an element of $\mathbb{Q}^*/\mathbb{Q}^{*2}$,

(ii) the value a, viewed as an element in $\mathbb{Q}^*/N_{K/\mathbb{Q}}(K^*)$, where $K = \mathbb{Q}(\sqrt{-d})$. Equivalently, this means that $q(\mathbb{Q}^2 \setminus 0) = aN_{K/\mathbb{Q}}(K^*)$.

For any positive definite quadratic forms q on $\mathbb{Q}^2$, let $\xi(q)$ be the corresponding commensurable class (or more precisely, the conjugacy class of the commensurable class). By Theorem 9, $\mathrm{cp}\,\xi(q)$ only depends on $O(q)$, therefore it only depends on the discriminant $-d$.

10.3 The function $F(d)$

Let d be a positive square-free integer. Set $K = \mathbb{Q}(\sqrt{-d})$, let $\mathcal{O}$ be its ring of integers, let R be te set of roots of unity in K and set $K_1 = \{z \in K | z\bar{z} = 1\}$. For $z \in K^*$, recall that the integer $d(z)$ is defined by $d(z) = N_{K/\mathbb{Q}}(\pi_z) = \mathrm{Card}\,\mathcal{O}/\pi_z$, where π_z is the ideal $\pi_z = \{a \in \mathcal{O} | az \in \mathcal{O}\}$. Set

$$F(d) = \mathrm{Min}_{z \in K_1 \setminus R}\, d(z).$$

We will now show two formulas for $F(d)$. Indeed $F(d)$ is the norm of some specific ideal in $K = \mathbb{Q}(\sqrt{-d})$, and it is also the minimal solution of some diohantine equation.

Let $\mathcal{J}$ be the set of all ideals π of $\mathcal{O}$ such that π and $\bar{\pi}$ are coprime and π^2 is principal.

Lemma 28. *We have*
$$F(d) = \mathrm{Min}_{\pi \in \mathcal{J}}\, N_{k/\mathbb{Q}}(\pi).$$
In particular, we have $F(1) = 5$ and $F(3) = 7$.

Proof. The map $z \mapsto \pi_z$ induces a bijection $(K_1 \setminus R)/R \simeq \mathcal{J}$, from which the first assertion follows. Moreover, if $\mathrm{Cl}(K) = \{0\}$, then $F(d)$ is the smallest split prime number. Therefore $F(1) = 5$ and $F(3) = 7$. $\qquad\square$

Let's consider the following diophantine equation

$(\mathcal{E})$ $\qquad 4n^2 = a^2 + db^2$, with $n > 0$, $a > 0$ and $b \neq 0$.

A solution (n, a, b) of $(\mathcal{E})$ is called *primitive* if $\gcd(n, a) = 1$. Let $\mathrm{Sol}\,(\mathcal{E})$ (respectively $\mathrm{Sol}_{prim}(\mathcal{E})$) be the set of solutions (respectively of primitive solutions) of $(\mathcal{E})$.

Let $\pi \in \mathcal{J}$. Since π^2 is principal, there are integers $a(\pi) > 0$ and $b(\pi)$ such that $a(\pi) + b(\pi)\sqrt{-d}$ is a generator of $4\pi^2$. Moreover, let's assume that

$d \neq 1$ or 3. Then $R = \{\pm 1\}$ and the integers $a(\pi)$ and $b(\pi)$ are uniquely determined. Thus there is a map $\theta : \mathcal{J} \to \mathrm{Sol}\,(\mathcal{E})$ defined by
$$\theta(\pi) = (N_{K/\mathbb{Q}}(\pi), a(\pi), b(\pi)).$$

Lemma 29. *Under the hypothesis that $d \neq 1$ or 3, the map θ induces a bijection from $\mathcal{J}$ to $\mathrm{Sol}\,_{prim}(\mathcal{E})$. In particular*
$$F(d) = \mathrm{Min}_{(n,a,b)\in \mathrm{Sol}\,(\mathcal{E})}\, n.$$

Proof. Step 1: proof that $\theta(\mathcal{J}) \subset \mathrm{Sol}\,_{prim}(\mathcal{E})$. An algebraic integer $z \in \mathcal{O}$ is called primitive if there are no integer $d > 1$ such that z/d is an algebraic integer. Equivalently, there are no integer $d > 1$ such that $d \mid z + \bar{z}$ and $d^2 \mid z\bar{z}$.

Let $\pi \in \mathcal{J}$ and set $z = 1/2(a(\pi) + b(\pi)\sqrt{-d})$. Since $z + \bar{z} = a(\pi)$ and $z.\bar{z} = N_{K/\mathbb{Q}}(\pi^2)$, z is an algebraic integer which is a generator of π^2. Since π^2 and $\bar{\pi}^2$ are coprime, z is primitive. Since $z.\bar{z} = N_{K/\mathbb{Q}}(\pi)^2$, it follows that $N_{K/\mathbb{Q}}(\pi)$ and $a(\pi)$ are coprime. Hence $\theta(\pi) \in \mathrm{Sol}\,_{prim}(\mathcal{E})$ and the claim is proved.

Step 2: proof that $\theta(\mathcal{J}) = \mathrm{Sol}\,_{prim}(\mathcal{E})$. Let $(n,a,b) \in \mathrm{Sol}\,_{prim}(\mathcal{E})$ and $z = 1/2(a(\pi) + b(\pi)\sqrt{-d})$. Since $z \neq \bar{z}$, $z + \bar{z} = a$ and $z\bar{z} = n$, the number z is an algebraic integer. Set $\tau = z\mathcal{O}$ and let
$$\tau = \pi_1^{m_1} \ldots \pi_k^{m_k}$$
be the factorization of τ into a product of prime ideals of $\mathcal{O}$, where, as usual we assume that $\pi_i \neq \pi_j$ for $i \neq j$ and all m_i are positive.

For $1 \leq i \leq k$, let p_i be the characteristic of the field $\mathcal{O}/\pi_i$. Since n and a are coprime, τ and $\bar{\tau}$ are coprime. It follows that $\bar{\pi}_i$ does not divide τ. In particular $\pi_i \neq \bar{\pi}_i$ and $N_{K/\mathbb{Q}}(\pi_i) = p_i$. Since π_i and $\bar{\pi}_i$ are the only two ideals over p_i, we have $m_i = v_{p_i}(N_{K/\mathbb{Q}}(\tau)) = v_{p_i}(n^2)$. Since each m_i is even, we have $\tau = \pi^2$ for some ideal $\pi \in \mathcal{J}$. Therefore $\theta(\pi) = (n,a,b)$, and the claim is proved.

Step 3. It follows easily that θ is a bijection from $\mathcal{J}$ to $\mathrm{Sol}\,_{prim}(\mathcal{E})$. In particular $F(d) = \mathrm{Min}_{(n,a,b)\in \mathrm{Sol}\,_{prim}(\mathcal{E})}\, n$, from which the lemma follows. $\square$

10.4 Complexity computation

Theorem 11. *Let q be a positive definite quadratic form on $\mathbb{Q}^2$ of discriminant $-d$. Then we have*
$$\mathrm{cp}\,\xi(q) = F(d)^{e(\mathfrak{n}^\mathbb{C})}$$

Proof. Step 1. Let $G \subset \mathrm{End}_{\mathbb{Q}}(K)$ be the group generated by the multiplication by elements in K_1 and by the complex conjugation. We have $G \simeq O(2)$ and $SO(2) \simeq K_1$. As a $O(q)$-module, there is an isomorphism
$$V \simeq \mathbb{Q}(\sqrt{-d}),$$
where $V = \mathfrak{n}(\xi(q))/[\mathfrak{n}(\xi(q)), \mathfrak{n}(\xi(q))]$.

Step 2. Let $\overline{\mathcal{S}}(\mathfrak{n}(\xi(q))$ be the image of $\mathcal{S}(\mathfrak{n}(\xi(q))$ in $O(q)$. We claim that
$$\overline{\mathcal{S}}(\mathfrak{n}(\xi(q)) = K_1 \setminus R.$$
Indeed $O(q)$ can be identified with a Levi factor of $\mathbf{G}(\mathbb{Q})$ and let $\rho : O(q) \to \mathbf{G}(\mathbb{Q})$ a corresponding lift. Any element in $R \cup O(q) \setminus SO(q)$ has finite order, hence we have
$$\overline{\mathcal{S}}(\mathfrak{n}(\xi(q)) \subset K_1 \setminus R.$$
Let $z \in K_1 \setminus R$. It is clear that z is not an algebraic integer. Since the grading is special, we have
$$\mathfrak{z}^{\mathbb{C}} = \oplus_{k \neq 0} \mathfrak{z}_k^{\mathbb{C}}.$$
Since the eigenvalues of $\rho(z)$ on $\mathfrak{z}_k$ is z^k, it follows that z belongs to $\overline{\mathcal{S}}(\mathfrak{n}(\xi(q))$, what proves the point.

Step 3. Let $z \in K_1 \setminus R$. We have $\bar{z} = \bar{z}^{-1}$. Therefore by Lemma 15 we have
$$\mathrm{ht}\, \rho(z) = \prod_{k \geq 1} d(z^k)^{\dim \mathfrak{n}_k^{\mathbb{C}}} = d(z)^{e(N)}.$$
Therefore Theorem 4 implies Theorem 5. $\qquad\square$

Since $F(d) \geq \frac{\sqrt{1+d}}{2}$, it follows that

Corollary 12. *The group N has infinite complexity.*

Since $F(7) = F(15) = 2$ and $F(d) \geq 3$ otherwise, it follows that

Corollary 13. *If the positive definite quadratic form q has discriminant -7 or -15 we have*
$$\mathrm{cp}\, \xi(q) = 2^{e(N)},$$
and $\mathrm{cp}\, \xi(q) \geq 3^{e(N)}$ *otherwise.*

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