LASOTA-YORKE MAPS WITH HOLES: CONDITIONALLY INVARIANT PROBABILITY MEASURES AND INVARIANT PROBABILITY MEASURES ON THE SURVIVOR SET.

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Abstract. Let $T : I \longrightarrow I$ be a Lasota-Yorke map on the interval I, let Y be a non trivial sub-interval of I and $g^0 : I \longrightarrow \mathbb{R}^+$, be a strictly positive potential which belongs to BV and admits a conformal measure m. We give constructive conditions on Y ensuring the existence of absolutely continuous (w.r.t. m) conditionally invariant probability measures to non absorption in Y. These conditions imply also existence of an invariant probability measure on the set X_∞ of points which never fall into Y. Our conditions allow rather "large" holes.

Applications de type Lasota-Yorke à trou : mesure de probabilité conditionellement invariante et mesure de probabilité invariante sur l'ensemble des survivants.

Résumé. Soient $T : I \longrightarrow I$ une application de type Lasota-Yorke sur l'intervalle I, Y un sous intervalle non trivial et $g^0 : I \longrightarrow \mathbb{R}^+$ un potentiel strictement positif qui admet une mesure conforme m. Nous donnons des conditions constructives sur Y qui assurent l'existence d'une mesure de probabilité absolument continue (par rapport à m), invariante conditionellement à la non absorption dans Y. Ces conditions impliquent aussi l'existence d'une mesure de probabilité, invariante par T et supportée dans l'ensemble X_∞ des ponits qui ne tombent pas dans le trou. Nos conditions autorisent des trous relativement gros.

Introduction

The notion of conditionally invariant probability measures *c.i.p.m.* was introduced for countable state Markov chains with absorbing state in [19]. More precisely, if (U_n) is a Markov chain with law $\mathbb{P}$ and taking values in a countable set $E \cup \partial$, if $\tau_\partial = \inf\{n \geq 0 : U_n = \partial\}$ is the hitting time of ∂, then a probability measure ν concentrated on E, is called a c.i.p.m. (conditioned to stay in E) if $\mathbb{P}_\nu\{U_n \in A \mid \tau_\partial > n\} = \nu(A)$ for every $A \subset E$ and $n \geq 0$. It was proven in [10] that geometric absorption was a necessary and sufficient condition for the existence of c.i.p.m. for a wide class of Markov chains. In [17] and later in [6, 7, 8] the existence of such measures was investigated for topological Markov chains and

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Markov expanding dynamical systems with holes. More recently, these questions were also studied for Anosov systems in [4] where small holes are considered, the existence of a c.i.p.m. is obtained by a perturbative argument. General conditions ensuring existence of c.i.p.m have been given in [9] where it has been proven that Φ-mixing systems satisfying the Gibbs property for some invariant measure μ admits a c.i.p.m. which is absolutely continuous with respect to μ.

In this article, we are concerned with Lasota-Yorke maps (these systems are in general neither Φ-mixing nor Gibbs).
Let $T : I \longrightarrow I$ be a Lasota-Yorke map on the interval I, let Y be a non trivial sub-interval of I and $g^0 : I \longrightarrow \mathbb{R}^+$, $\inf g^0 > 0$, be a potential which belongs to BV and admits a conformal measure m (see definition and assumptions below).[1]

Some results have been obtained for such maps with holes, limited to the case in which the potential is given by the Jacobian of the map, in [5] and [1] for very small holes and under some additional geometrical assumption on the holes. Our goal here is on the one hand to find constructive conditions allowing not necessarily small holes and on the other to show that a smallness condition alone suffices.
The plan of the paper is as follows. In section one some general facts are recalled and the main theorems proved in the paper are stated. Section two is devoted to obtaining a special type of Lasota-Yorke like inequality that will be the basis for future arguments. Section three uses the previous results to establish that the transfer operator is a contraction in an appropriate (projective) metric. From this results the wanted statistical properties readily follows as is shown in section four. Section five investigate the Hausdorff dimension of the set of the points that never visit the hole. In section six we investigate many concrete examples and show that the theory so far developed does apply to maps with fairly large holes even in the absence of a Markov structure. Finally, section seven points out that if one is concerned only with pertubative results (i.e. rather small holes) then results of the type obtained in the previous sections follow under much more general hypothesis. It should be remarked that, although we do not investigate this explicitly, the size of the holes for which the latter result applies can be (at least in principle) explicitly computed since the perturbation theory we use is constructive.

1. Statements and Results

Let us fix some notations. Let $I \subset \mathbb{R}$ and $T : I \to I$ be a Lasota-Yorke map i.e. there exists a partition $\mathcal{Z}$ (mod. a finite number of points) of I on subintervals such that T is C^1 on each $\overline{Z}$, $Z \in \mathcal{Z}$ and monotonic. Let $\mathcal{Z}^{(n)}$ be the monotonicity partition of T^n.

Let $g^0 : I \longrightarrow \mathbb{R}^+$, be a strictly positive potential which belongs to BV and admits a conformal measure m. By $\mathcal{L}_0$ we designate the usual Perron-Frobenius operator (or transfer operator) associated to the dynamic and g^0. The operator $\mathcal{L}_0$ acts on $L^1(m)$ and BV:

$$\mathcal{L}_0 f(x) = \sum_{Ty=x} f(y) g^0(y). \tag{1.1}$$

Recall that a measure m is called g^0-conformal if it satisfies:

$$\mathcal{L}_0^* m = cm \text{ where } c := e^{P(g^0)},$$

[1] In fact, all the following can be easily extended to the case in which Y is a finite collection of sub-intervals. We choose not to do so to keep the exposition as simple as possible.

and

$$P(g^0) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{Z\in\mathcal{Z}^{(n)}} \sup_Z g_n^0.$$

Define also $\Theta(g^0)$ to be such that $\log\Theta(g^0) := \lim_{n\to\infty} \frac{1}{n} \log \sup_I g_n^0$. Our standing assumptions on g^0 will be the following:

Condition 0.
- *$\inf g^0 > 0$,*
- *the potential g^0 is* contracting *i.e., $\Theta(g^0) < e^{P(g^0)}$,*
- *the potential g^0 belongs to the space BV of functions of bounded variation.*
- *there exists a g^0-conformal probability measure m.*

Remark 1.1. *It is known (see* [12], [3], [16]*) that if g^0 belongs to BV then so does g_n^0 for all $n\in\mathbb{N}$ and that this together with the contracting condition are sufficient to ensure the existence of a g^0-conformal non atomic probability measure provided the partition is generating.*

It is known (see [12], [3], [16]) that if g^0 belongs to BV then so does g_n^0 for all $n\in\mathbb{N}$ and that this together with the contracting condition are sufficient to ensure the existence of a g^0-conformal non atomic probability measure.

Next, consider a sub-interval $Y\subset I$, the hole. To avoid trivial considerations, we assume $m(Y)m(Y^c)\neq 0$, X_0 denotes the complementary of the hole: $X_0 = I\setminus Y$. X_n will denote the set of points that have not fallen into the hole at time n: $X_n = \bigcap_{i=0}^n T^{-i}X_0$. We will also denote by $g = g^0\mathbf{1}_{X_0}$, $g_n(x) = g(x)\times\cdots\times g(T^{n-1}x)$ and $\Theta = \Theta(g)$.

Conditionally invariant probability measures (c.i.p.m. for short) are probability measures ν satisfying:

$$\forall A\in\mathcal{B}\ \forall n\in\mathbb{Z}_+\ \nu(T^{-n}A\cap X_n) = \nu(A)\nu(X_n), \qquad (*)$$

where $\mathcal{B}$ is the Borel σ-algebra. Condition $(*)$ implies that ν must be supported in X_0 and, if $\nu(T^{-1}X_0) =: \alpha\in]0,1]$, that $\nu(X_n) = \alpha^n$, i.e. with respect to ν, the entrance time into Y has exponential law.
Of course, we are not interested in all c.i.p.m., but on those that have some reasonable properties with respect to the potential g^0. We will consider only absolutely continuous with respect to m c.i.p.m. (*a.c.c.i.p.m.* for short). To this aim, an useful tool is the transfer operator $\mathcal{L}$ defined by

$$\mathcal{L}(f) = \mathcal{L}_0(f\mathbf{1}_{X_0})\,. \tag{1.2}$$

The usefulness of $\mathcal{L}$ is readily clarified.

Lemma 1.1. *The following two assertions hold true.*

1. *Let $\nu = \mathbf{1}_{X_0}h\cdot m$ be a probability measure absolutely continuous with respect to m. Then, ν is an a.c.c.i.p.m. if and only if $\mathcal{L}h = c\alpha h$ for some $\alpha\in]0,1]$.*
2. *Let $\alpha\in]0,1]$ and $h\in L^1(m)$ be such that $\mathcal{L}h = c\alpha h$, let μ be a probability measure on I such that $\mathcal{L}^*\mu = c\alpha\mu$. Then $\mu$is supported in X_∞ and $\lambda = h\mu$ is T-invariant.*

Proof. **1**. Let $\nu = (\mathbf{1}_{X_0} h)m$ and assume $\mathcal{L}h = c\alpha h$. We will make extensively use of the following two easily obtained properties on the iterates of $\mathcal{L}$:

$$\forall\ f \in L^1(m),\ \forall\ n \in \mathbb{Z}_+\ \mathcal{L}^n(f) = \mathcal{L}_0^n(f\mathbf{1}_{X_{n-1}}) \tag{1.3}$$

$$\forall\ f, \varphi \in L^1(m),\ \forall\ n \in \mathbb{Z}_+\ \int_{X_0} \varphi \mathcal{L}^n f dm = c^n \int_{X_n} \varphi \circ T^n \cdot f dm. \tag{1.4}$$

Let $A \in \mathcal{B}$, (1.3, 1.4) give:

$$\begin{aligned} \nu(T^{-n}A \cap X_n) &= \int \mathbf{1}_A \circ T^n \cdot \mathbf{1}_{X_n} \cdot h dm \\ &= \frac{1}{c^n} \int_{X_0} \mathbf{1}_A(\mathcal{L}^n h) dm = \alpha^n \nu(A). \end{aligned}$$

In particular, for $A = I$, we get $\nu(X_n) = \alpha^n$ thus, for any $A \in \mathcal{B}$, $\nu(T^{-n}A \cap X_n) = \nu(A)\nu(X_n)$.
Conversely, assume $\nu = (\mathbf{1}_{X_0} h)m$ is a a.c.c.i.p.m.. Then, by definition of c.i.p.m., there exists $\alpha \in]0,1]$ such that, for any $A \in \mathcal{B}$, $\nu(T^{-n}A \cap X_n) = \alpha^n \nu(A)$. So,

$$\forall\ A \in \mathcal{B},\ \int_{X_0} \mathbf{1}_A \cdot \frac{\mathcal{L}^n h}{c^n} dm = \alpha^n \int_{X_0} \mathbf{1}_A \cdot h dm,$$

we deduce that $\mathcal{L}^n h = (c\alpha)^n h$.
2. Let μ be a probability measure on I, assume that $\mathcal{L}^* \mu = c\alpha\mu$, $\alpha \in]0,1]$, then

$$\forall\ n \in \mathbb{Z}_+,\ \forall\ f \in L^1(m)\ (c\alpha)^n \mu(f) = \int_I \mathcal{L}^n f d\mu.$$

Assume that f is zero on X_{n-1}. Then

$$(c\alpha)^n \mu(f) = \mu(\mathcal{L}^n f) = \mu(\mathcal{L}_0^n(\mathbf{1}_{X_{n-1}} f)) = 0,$$

thus $\mu(f) = 0$. We deduce that μ has its support contained in X_∞.
The fact that for h such that $\mathcal{L}h = c\alpha h$ the measure $\lambda = h\mu$ is T-invariant is a direct computation. □

In the next section we will introduce two conditions on the holes (see Condition 1 and Condition 2) under which the following statements hold.
Our main result is the following.

Theorem A. *Assume that Conditions 0, 1 and 2 are satisfied. Then there exists a unique conditionally invariant probability measure $\nu = hm$ which is absolutely continuous with respect to m. There exists a unique probability measure μ supported in X_∞ and which satisfies $\mu(\mathcal{L}f) = \rho\mu(f)$, with $\rho \leq c$, for any bounded function f. The measure $\lambda = h\mu$ is the only T invariant measure supported on X_∞ and absolutely continuous with respect to μ. Moreover, there exists $\kappa < 1$ such that for any $f \in BV$ and any $A \in \mathcal{B}$:*

$$\left\| \frac{\mathcal{L}^n f}{\rho^n} - h\mu(f) \right\|_\infty \leq Ct\kappa^n \|f\|_{BV},$$

$$|m(T^{-n}A|X_{n-1}) - \nu(A)| \leq Ct\kappa^n$$

$$\textit{and } |\nu(A|X_{n-1}) - \lambda(A)| \leq Ct\kappa^n.$$

A sub product of our Theorem A will be the following result on the Hausdorff dimension of the set X_∞ of survivors. For any $0 \le t \le 1$, define

$$\mathcal{L}_t f(x) = \sum_{Ty=x} (g^0)^t(y)\mathbf{1}_{X_0}(y)f(y)$$

and by Θ_t, ρ_t and $P(t)$ the number corresponding to Θ, ρ, P in the case $t = 1$ (see Definition 2.1 for the definition of ρ).
We will say that g^0 has the Bounded Distortion property if there exists $C > 1$ such that for all $n \in \mathbb{N}$, $Z \in \mathcal{Z}^{(n)}$ and x, $y \in Z$,

$$\frac{g_n^0(x)}{g_n^0(y)} \le C. \tag{1.5}$$

We will say that T has large images if

$$\inf_{n\in\mathbb{N}} \inf_{Z\in\mathcal{Z}^{(n)}} m(T^n Z) > 0. \tag{1.6}$$

We will say that T has large images with respect to Y if for all $n \in \mathbb{N}$, for all $Z \in \mathcal{Z}^{(n)}$, $Z \cap X_\infty \neq \emptyset$, $T^n(Z \cap X_{n-1}) \supset X_\infty$.

Theorem B. *Let $g^0 = \frac{1}{T'}$. Assume that for all $0 \le t \le 1$, Conditions 0, 1 and 2 are satisfied. Then, there exists a unique $0 < t_0 \le 1$ such that for $0 \le t < t_0$, $\rho_t > 1$ and for $1 \ge t > t_0$, $\rho_t < 1$. If T has large images and large images with respect to Y then, $HD(X_\infty) = t_0$.*

The two theorems above will follow from Theorems 4.4 and 5.1.

As we will see in section 6, Theorems A, B apply to maps with fairly large holes, in fact this is the case in which they are of interest. If, on the contrary, one is willing to settle for small holes, then it is possible to apply a perturbative approach which yields the following stronger result.[2]

Theorem C. *Assume g^0 is satisfies Condition 0. If the Lasota-Yorke map $T : I \to I$ has a unique invariant measure μ_0 absolutely continuous with respect the conformal measure m, and the systems (I, T, μ_0) is mixing, then there exists $\varepsilon > 0$ such that, for each hole Y, $m(Y) \le \varepsilon$, the conclusions of Theorem A apply.*

Theorem C is proven in section 7. In view of Lemma 1.1, we are led to start our investigation by constructing eigenvalues and eigenfunctions for $\mathcal{L}$. As usually in these topics, a Lasota-Yorke inequality is useful.

2. Transfer operator and Lasota-Yorke inequalities with holes.

As already mentioned, our point of view is to consider the Transfer operator $\mathcal{L}$ as associated to the potential $g = g^0\mathbf{1}_{X_0}$, that is a positive, but not strictly positive, weight. Weights of such type, and more general, have been studied in quite some detail. In particular the existence of a quasi-invariant and an invariant measure is proven in [3] under very mild technical assumptions plus the hypothesis that the standard bound Θ for the essential spectral radius of $\mathcal{L}$ be strictly less than the spectral radius of $\mathcal{L}$. Yet, the arguments used there are non constructive (quasi-compactness) and both the problem of when such a condition is satisfied and the problem of the uniqueness of the above measure are not addressed. Here we will restrict ourselves to a slightly less general setting and use a different, constructive, approach patterned after some previous results for strictly positive weights (see

[2]In fact, the hypothesis that (I, T, μ_0) is mixing is superfluous and here is used only to make an easy comparison with Theorem A which conditions insure that the invariant measure is unique.

[16]). The present approach will allow us, in the following sections, to find explicit conditions for the existence and the properties of the quasi-invariant and invariant probability measures.

First of all we need to impose a condition on our system.

Condition 1. *Let $D_n := \{x \in I \mid \mathcal{L}^n 1(x) \neq 0\}$. We will consider only systems that satisfy*

C1: $D_\infty := \bigcap_{n\in\mathbb{N}} D_n \neq \emptyset$.

Notice that if $x \notin D_n$ then $\mathcal{L}^n f(x) = 0$ for each $f \in L^\infty([0,1])$ since

$$|\mathcal{L}^n f(x)| \leq \mathcal{L}^n |f|(x) \leq \|f\|_\infty \mathcal{L}^n 1(x) = 0.$$

Accordingly, for each $n \in \mathbb{N}$ holds

$$\mathcal{L}^n f = \mathbf{1}_{D_n} \mathcal{L}^n f. \tag{2.1}$$

Equation (2.1) in particular means that if $x \notin D_n$, then

$$\mathcal{L}^{n+1} 1(x) = \mathcal{L}^n(\mathcal{L}1)(x) = 0,$$

hence $x \notin D_{n+1}$, that is $D_{n+1} \subset D_n$.

We can now define the functional

$$\Lambda(f) := \lim_{n\to\infty} \inf_{x\in D_n} \frac{\mathcal{L}^n f(x)}{\mathcal{L}^n 1(x)}. \tag{2.2}$$

The above definition needs a few comments to convince the reader that it is well posed. To start with notice that Condition 1 implies that the ratio is well defined. Second the existence of the limit is assured by the fact that the sequence is increasing and bounded, indeed

$$\begin{aligned} \inf_{x\in D_{n+1}} \frac{\mathcal{L}^{n+1} f(x)}{\mathcal{L}^{n+1} 1(x)} &= \inf_{x\in D_{n+1}} \frac{\mathcal{L}\mathbf{1}_{D_n}\left[\mathcal{L}^n 1 \frac{\mathcal{L}^n f}{\mathcal{L}^n 1}\right]}{\mathcal{L}^{n+1} 1} \\ &\geq \inf_{x\in D_n} \frac{\mathcal{L}^n f(x)}{\mathcal{L}^n 1(x)} \inf_{x\in D_{n+1}} \frac{\mathcal{L}\mathbf{1}_{D_n}[\mathcal{L}^n 1]}{\mathcal{L}^{n+1} 1} \\ &= \inf_{x\in D_n} \frac{\mathcal{L}^n f(x)}{\mathcal{L}^n 1(x)}; \end{aligned} \tag{2.3}$$

and

$$-\|f\|_\infty \leq \inf_{x\in D_n} \frac{\mathcal{L}^n f(x)}{\mathcal{L}^n 1(x)} \leq \|f\|_\infty.$$

The relevant properties of the above functional are the following:[3]

- $\Lambda(1) = 1$;
- Λ is continuous in the L^∞ norm;
- $f \geq g$ implies $\Lambda(f) \geq \Lambda(g)$ (monotonicity);
- $\Lambda(\lambda f) = \lambda\Lambda(f)$ (homogeneity);
- $\Lambda(f+g) \geq \Lambda(f) + \Lambda(g)$ (super-additivity);
- $\forall b \in \mathbb{R}$, $\Lambda(f+b) = \Lambda(f) + b$;
- if for $p \subset I$ there exists $n \in \mathbb{N}$ such that $p \cap X_n = \emptyset$, then $\Lambda(\mathbf{1}_p) = 0$.[4]

All the above follows immediately from the definition.

[3]Essentially the properties of Λ are similar to the ones of an inner measure. In the following we will see that, under certain conditions, it is indeed a measure.

[4]This follows remembering equation (1.3).

Remark 2.1. *Note that, at the moment, it is not clear if the functional is linear or not, yet homogeneity and super-additivity imply at least convexity.*

Definition 2.1. *Set* $\rho = \Lambda(\mathcal{L}1)$.

Lemma 2.2. *Under condition* **C1** *we have* $\rho \le c$.

Proof. Let

$$\rho_n := \inf_{x\in D_n} \frac{\mathcal{L}^{n+1}1(x)}{\mathcal{L}^n 1(x)}, \tag{2.4}$$

then by (2.2) $\lim_{n\to\infty} \rho_n = \rho$. Accordingly,

$$\mathbf{1}_{D_n}\mathcal{L}^n 1 \rho_n \le \mathbf{1}_{D_{n+1}}\mathcal{L}^{n+1}1.$$

Integrating the above equation with respect to m, and remembering (2.1), yields

$$e^{-P}\rho_n \le \frac{m(X_n)}{m(X_{n-1})} \le 1$$

which produces the wanted result by taking the limit $n \to \infty$. □

To continue we need to impose one extra condition on the system. To do so we need some notation. Let $\mathcal{Z}^{(n)}$ be the partition of smoothness (or monotonicity) intervals of T^n. Next let $\mathcal{A}_n$ be the set of finite partitions in intervals $A = \{A_i\}$ such that $\bigvee_{A_i} g_n \le 2\|g_n\|_\infty$.[5] Given $n \in \mathbb{N}$ and $A \in \mathcal{A}_i$ let $\hat{\mathcal{Z}}^{(n)}$ be the coarsest partition in intervals among all the ones finer than both A and $\mathcal{Z}^{(n)}$ and enjoying the property that the elements of the partition are either disjoint or contained in X_{n-1}. Finally, let

$$\mathcal{Z}_*^{(n)} = \{Z \in \hat{\mathcal{Z}}^{(n)} \mid Z \subset X_{n-1}\},$$
$$\mathcal{Z}_b^{(n)} = \{Z \in \hat{\mathcal{Z}}^{(n)} \mid Z \subset X_{n-1} \text{ and } \Lambda(\mathbf{1}_Z) = 0\}$$
$$\text{and } \mathcal{Z}_g^{(n)} = \{Z \in \hat{\mathcal{Z}}^{(n)} \mid Z \subset X_{n-1} \text{ and } \Lambda(\mathbf{1}_Z) > 0\}.$$

As we will see in the proof of Lemma 2.5, the elements of $\mathcal{Z}_b^{(n)}$ are the problematic ones and those of $\mathcal{Z}_g^{(n)}$ are the good ones. We allow $\mathcal{Z}_b^{(n)}$ to be non empty provided it satisfies the following condition **C2**.

Definition 2.3. *We will call* contiguous *two elements of* $\mathcal{Z}_*^{(n)}$ *that are either contiguous, in the usual sense, or separated by a connected component of* $Y_n := \bigcup_{i=0}^{n-1} T^{-i}Y$.

Condition 2. *We will consider only systems that satisfy the following condition:*

- **C2:** *There exists constants* $K \ge 0$, *and* $\xi \ge 1$, *such that for each* $n \in \mathbb{N}$ *there exists* $A \in \mathcal{A}_n$ *such that at most* $K\xi^n$ *elements of* $\mathcal{Z}_b^{(n)}$ *are contiguous. In addition,* $\xi\Theta < \rho$.

Note that this implies, in particular $\Theta < \rho$.

Remark 2.2. *Note that condition* **C2** *implies that there exists* $\bar{n} \in \mathbb{N}$ *such that* $D_n = D_{\bar{n}}$ *for all* $n \ge \bar{n}$, *since if the latter were false it would follow* $\rho = 0$.

The following is yet another simple consequence of **C2**.

Lemma 2.4. *Condition 2 implies that for all* $n \in \mathbb{N}$, $\mathcal{Z}_g^{(n)} \ne \emptyset$.

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[5]Such partitions always exist, if in doubt see [18] Lemma 6.

Proof. Suppose that $\mathcal{Z}_g^{(n)} = \emptyset$ for some n, then it must be $\mathcal{Z}_g^{(m)} = \emptyset$ for all $m \geq n$. Assume that $\mathcal{Z}_g^{(n)} = \emptyset$, then $\mathcal{Z}_*^{(n)} = \mathcal{Z}_b^{(n)}$, thus the number of elements in $\mathcal{Z}_*^{(n)}$ is smaller than $K\xi^n$ (the elements of $\mathcal{Z}_b^{(n)}$ must be all contiguous). Then,

$$\mathcal{L}^n 1(x) \leq \sum_{Z\in\mathcal{Z}_*^{(n)}} \sup g^{(n)} \leq \sup g^{(n)} K\xi^n.$$

On the other hand, remembering (2.4), we have, for each $x \in D_\infty$,

$$|g^{(n)}|_\infty K\xi^n \geq \prod_{i=0}^{n-1} \frac{\mathcal{L}^{i+1}1(x)}{\mathcal{L}^i 1(x)} \geq \prod_{i=0}^{n-1} \rho_i.$$

Next, taking the logarithm of both sides and the limit for $n \to \infty$, we get

$$\ln\xi + \ln\Theta \geq \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \ln\rho_i = \ln\rho$$

(recall that by definition $\rho = \lim \rho_i$), contrary to condition **C2**. □

Under Condition 2 we will show that the cone

$$\mathcal{C}_a := \{h \in \mathrm{BV} \mid h \not\equiv 0;\ h \geq 0;\ \bigvee h \leq a\Lambda(h)\} \tag{2.5}$$

is strictly invariant for the Transfer operator $\mathcal{L}$.

The first step is to obtain a suitable Lasota-York type inequality.

Lemma 2.5. *For any $\theta \geq \Theta\xi$, $h \in BV$, we have*

$$\bigvee \mathcal{L}^n h \leq C_\theta \theta^n \bigvee h + K_n \Lambda(|h|),$$

where C_θ and K_n do not depend on h.

Proof. Notice that, if $Z \in \hat{\mathcal{Z}}^{(n)} \backslash \mathcal{Z}_*^{(n)}$, then $\mathcal{L}^n(h\mathbf{1}_Z) = 0$ for each $h \in \mathrm{BV}$, since $Z \cap X_{n-1} = \emptyset$.

We can then write

$$\mathcal{L}^n h = \sum_{Z\in\mathcal{Z}_*^{(n)}} \mathcal{L}^n(\mathbf{1}_Z h) = \sum_{Z\in\mathcal{Z}_*^{(n)}} (\mathbf{1}_Z g_n h)\circ T_Z^{-n}.$$

Accordingly,

$$\bigvee \mathcal{L}^n h \leq \sum_{Z\in\mathcal{Z}_*^{(n)}} \bigvee \mathbf{1}_{T^n Z}(g_n h)\circ T_Z^{-n}.$$

We will compute separately each term of the sum.

$$\begin{aligned}
\bigvee \mathbf{1}_{T^n Z}(g_n h)\circ T_Z^{-n} &\leq \bigvee_Z h g_n \ + \ 2\sup_Z |h\cdot g_n| \\
&\leq 3\bigvee_Z h g_n \ + 2\inf_Z |h\cdot g_n| \\
&\leq 3\|g_n\|_\infty \bigvee_Z h + 3\sup_Z |h| \bigvee_Z g_n + 2\inf_Z |h\cdot g_n| \\
&\leq 3\|g_n\|_\infty \bigvee_Z h + 6\|g_n\|_\infty \sup_Z |h| + 2\|g_n\|_\infty \inf_Z |h| \\
&\leq 9\|g_n\|_\infty \bigvee_Z h + 8\|g_n\|_\infty \inf_Z |h|.
\end{aligned} \tag{2.6}$$

Next, note that if $Z \in \mathcal{Z}_g^{(n)}$, then by definition, there exists $\varepsilon_n > 0$ such that $\inf_{Z \in \mathcal{Z}_g^{(n)}} \Lambda(\mathbf{1}_Z) \geq 2\varepsilon_n > 0$, it is possible to choose $N_n \in \mathbb{N}$ such that, for each $x \in D_{N_n}$,

$$\inf_{Z \in \mathcal{Z}_g^{(n)}} \frac{\mathcal{L}^{N_n}\mathbf{1}_Z(x)}{\mathcal{L}^{N_n}1(x)} \geq \varepsilon_n.$$

Accordingly, for each $x \in D_{N_n}$, $h \in \mathrm{BV}$ and $Z \in \mathcal{Z}_g^{(n)}$ holds

$$\mathcal{L}^{N_n}(|h|\mathbf{1}_Z(x)) \geq \inf_Z |h| \mathcal{L}^{N_n}\mathbf{1}_Z(x) \geq \inf_Z |h|\varepsilon_n \mathcal{L}^{N_n}1(x)$$

To deal with the $Z \in \mathcal{Z}_b^{(n)}$ we must use condition **C2**. Note that the elements of $\mathcal{Z}_g^{(n)}$ can be separated by, at most, $K\xi^n$ elements of $\mathcal{Z}_b^{(n)}$. For each $Z \in \mathcal{Z}_b^{(n)}$ let $I_\pm(Z)$ be the union of the contiguous elements of $\mathcal{Z}_b^{(n)}$ on the left and on the right of Z, respectively. Clearly, for each $Z' \subset I_-(Z)$ (or $Z' \subset I_+(Z)$), holds

$$\inf_{Z'} |h| \leq \inf_Z |h| + \bigvee_{I_-(Z)} h.$$

Accordingly,

$$\sum_{Z \in \mathcal{Z}_b^{(n)}} \inf_Z |h| \leq 2K\xi^n \left[\sum_{Z \in \mathcal{Z}_g^{(n)}} \inf_Z |h| + \bigvee h \right].$$

We can then conclude

$$\begin{aligned}
\bigvee \mathcal{L}^n h &\leq \|g_n\|_\infty (9 + 16K\xi^n) \bigvee h \\
&\quad + 8(2K\xi^n + 1)\|g_n\|_\infty \varepsilon_n^{-1} \sum_{Z \in \mathcal{Z}_*^{(n)}} \frac{\mathcal{L}^{N_n}|h|\mathbf{1}_Z(x)}{\mathcal{L}^{N_n}1(x)} \\
&\leq \|g_n\|_\infty (9 + 16K\xi^n) \bigvee h \\
&\quad + 8(2K\xi^n + 1)\|g_n\|_\infty \varepsilon_n^{-1} \frac{\mathcal{L}^{N_n}|h|(x)}{\mathcal{L}^{N_n}1(x)}.
\end{aligned}$$

Taking the inf on x in the previous expression and noticing that, by hypothesis, there must exists C_θ such that $(9 + 16K\xi^n)\|g_n\|_\infty \leq C_\theta \theta^n$ yields the result. $\square$

3. Transfer Operator and Invariant Cones

Hilbert metric. In this section, we introduce a theory developed by G. Birkhoff [2], which is highly powerful to analyzing of the so called positive operators.

We will apply it to study the Perron-Frobenius operator for our maps. This strategy has been first implemented in [11] to estimate the decay of correlations for some random dynamical systems. Then, this strategy had been used by many authors. Let us mention C. Liverani [14] and M. Viana [20] for Anosov and Axiom A diffeomorphisms. They used Birkhoff cones to obtain exponential decay of correlations. We use this technique in a way very close to [15] and [16].

Definition 3.1. *Let $\mathcal{V}$ be a vector space. We will call convex cone a subset $\mathcal{C} \subset \mathcal{V}$ which enjoys the following properties*

(i) $\mathcal{C} \cap -\mathcal{C} = \emptyset$

(ii) $\forall \lambda > 0 \quad \lambda\mathcal{C} = \mathcal{C}$

(iii) $\mathcal{C}$ *is a convex set*

(iv) $\forall f, g \in \mathcal{C}\ \forall \alpha_n \in \mathbb{R}\ \ \alpha_n \to \alpha,\ \ g - \alpha_n f \in \mathcal{C} \Rightarrow g - \alpha f \in \mathcal{C} \cup \{0\}$.

We now define the Hilbert metric on $\mathcal{C}$:

Definition 3.2. *The distance $d_{\mathcal{C}}(f,g)$ between two points f,g in $\mathcal{C}$ is given by*

$$\begin{aligned} \alpha(f,g) &= \sup\{\lambda > 0 | g - \lambda f \in \mathcal{C}\} \\ \beta(f,g) &= \inf\{\mu > 0 | \mu f - g \in \mathcal{C}\} \\ d_{\mathcal{C}}(f,g) &= \log \frac{\beta(f,g)}{\alpha(f,g)} \end{aligned}$$

where we take $\alpha = 0$ or $\beta = \infty$ when the corresponding sets are empty.

The distance $d_{\mathcal{C}}$ is a pseudo-metric, because two elements can be at an infinite distance from each others, and it is a projective metric because any two proportional elements have a null distance.

The next theorem, due to G. Birkhoff [2], will show that every positive linear operator is a contraction, provided that the diameter of the image is finite.

Theorem 3.3. *Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two vector spaces, $\mathcal{C}_1 \subset \mathcal{V}_1$ and $\mathcal{C}_2 \subset \mathcal{V}_2$ two convex cone (see definition above) and $L : \mathcal{V}_1 \to \mathcal{V}_2$ a positive linear operator (which means $L(\mathcal{C}_1) \subset \mathcal{C}_2$). Let $d_{\mathcal{C}_i}$ be the Hilbert metric associated to the cone $\mathcal{C}_i$. If we denote*

$$\Delta = \sup_{f,g \in L(\mathcal{C}_1)} d_{\mathcal{C}_2}(f,g) \quad ,$$

then

$$d_{\mathcal{C}_2}(Lf, Lg) \le \tanh\left(\frac{\Delta}{4}\right) d_{\mathcal{C}_1}(f,g) \ \ \forall f, g \in \mathcal{C}_1$$

$(\tanh(\infty) = 1)$.

Theorem 3.3 alone is not completely satisfactory: given a cone $\mathcal{C}$ and its metric $d_{\mathcal{C}}$, we do not know if $(\mathcal{C}, d_{\mathcal{C}})$ is complete. This aspect is taken care by the following lemma, which allows to link the Hilbert metric to a suitable norm defined on $\mathcal{V}$.

Lemma 3.4. [16] *Let $\|\cdot\|$ be a norm on $\mathcal{V}$ such that*

$$\forall f, g \in \mathcal{V}\ \ g - f,\ g + f \in \mathcal{C} \Rightarrow \|g\| \le \|f\|$$

and let $\ell : \mathcal{C} \to \mathbb{R}^+$ be a homogeneous and order preserving function, i.e.

$$\begin{aligned} \forall f \in \mathcal{C}, \forall \lambda \in \mathbb{R}^+ \qquad & \ell(\lambda f) = \lambda \ell(f) \\ \forall f, g \in \mathcal{C} \qquad & g - f \in \mathcal{C} \ \Rightarrow \ell(f) \le \ell(g)\ , \end{aligned}$$

then

$$\forall f, g \in \mathcal{C}\ \ \ell(f) = \ell(g) > 0 \Rightarrow \|f - g\| \le (\mathrm{e}^{d_{\mathcal{C}}(f,g)} - 1) \min(\|f\|, \|g\|)$$

Remark 3.1. *In the previous lemma, one can choose $\ell(\cdot) = \|\cdot\|$ which fulfills the hypothesis. An interesting case is also when ℓ is a linear functional positive on $\mathcal{C}$. However, we are concerned with the possibly nonlinear $\ell = \Lambda$.*

Invariant cone.

From now on, we fix $\theta \in \mathbb{R}$ such that $\Theta\xi \le \theta < \rho$.

Proposition 3.5. *There exists $n_* \in \mathbb{N}$ and $a_0 > 0$ such that, for each $n \ge n_*$, if $a \ge a_0$, then the cone $\mathcal{C}_a$ is not empty and*

$$\mathcal{L}^n \mathcal{C}_a \subset \mathcal{C}_a$$

with finite diameter.

Before proving the above proposition we need few auxiliary results.

Lemma 3.6. *For each $n \in \mathbb{N}$ holds*

$$\Lambda(\mathcal{L}^n 1) \geq \rho^n.$$

Proof. For each $g \in \mathrm{BV}$, $g \geq 0$ and $x \in D_{n+1}$, holds

$$\frac{\mathcal{L}^{n+1}g(x)}{\mathcal{L}^n 1(x)} \geq \frac{\mathcal{L}\left[\mathbf{1}_{D_n}\left(\frac{\mathcal{L}^n g}{\mathcal{L}^n 1}\right)\mathcal{L}^n 1\right](x)}{\mathcal{L}^n 1(x)} \geq \frac{\mathcal{L}^{n+1}1(x)}{\mathcal{L}^n 1(x)} \inf_{D_n} \frac{\mathcal{L}^n g}{\mathcal{L}^n 1}$$

and, taking the inf on x and the limit $n \to \infty$ we have

$$\Lambda(\mathcal{L}g) \geq \Lambda(\mathcal{L}1)\Lambda(g). \tag{3.1}$$

The lemma follows by iterating (3.1). □

Lemma 3.7. *There exists $n_0 \in \mathbb{N}$ and $a_0 \in \mathbb{R}^+$ such that for all $a \geq a_0$ holds*

$$\mathcal{L}^n \mathcal{C}_a \subset \mathcal{C}_{a/2}\ \forall n \geq n_0 \quad \textit{and} \quad \mathcal{L}^n \mathcal{C}_a \subset \mathcal{C}_{2aC_\theta}\ \forall n \geq 0.$$

Proof. First of all, it is obvious that $h \geq 0$ implies $\mathcal{L}^n h \geq 0$. Next we choose $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$, $C_\theta^2 \theta^n \rho^{-n} \leq \frac{1}{4}$. Let $h \in \mathcal{C}_a$ then for each $n \in \mathbb{N}$ we write $n = kn_0 + m$, $m < n_0$, and (recall Lemma 2.5)

$$\begin{aligned}\bigvee \mathcal{L}^n h &\leq C_\theta \theta^{n_0} \bigvee \mathcal{L}^{(k-1)n_0+m} h + K_{n_0} \Lambda(\mathcal{L}^{(k-1)n_0+m} h)\\ &\leq C_\theta^{k+1}\theta^n \bigvee h + \sum_{i=0}^{k-1}(C_\theta\theta^{n_0})^i K_{n_0}\Lambda(\mathcal{L}^{(k-i-1)n_0+m}h)\\ &\quad + C_\theta^k \theta^{kn_0} K_m \Lambda(h).\end{aligned} \tag{3.2}$$

Thus, (use (3.1))

$$\bigvee \mathcal{L}^n h \leq \left[\left(a + \frac{K_m}{C_\theta \theta^m}\right)\frac{C_\theta^{k+1}\theta^n}{\rho^n} + \sum_{i=0}^{k-1}\left(\frac{C_\theta \theta^{n_0}}{\rho^{n_0}}\right)^i \frac{K_{n_0}}{\rho^{n_0}}\right]\Lambda(\mathcal{L}^n h).$$

Clearly the worst case is when $k = 0$, then if $a_0 \geq \max_{i \leq n_0} \frac{K_i}{C_\theta \rho^i}$ we have

$$\bigvee \mathcal{L}^n h \leq 2aC_\theta \Lambda(\mathcal{L}^n h).$$

When $k > 0$ instead

$$\bigvee \mathcal{L}^n h \leq \left[\frac{1}{4}\left(a + \frac{K_m}{C_\theta \theta^m}\right) + 2K_{n_0}\rho^{-n_0}\right]\Lambda(\mathcal{L}^n h).$$

Hence, for all $n \geq n_0$ and $a \geq 8K_{n_0}\rho^{-n_0} + \max_{i \leq n_0} \frac{K_i}{C_\theta \rho^i} := a_0$,

$$\bigvee \mathcal{L}^n h \leq \frac{a}{2}\Lambda(\mathcal{L}^n h).$$

□

The above Lemma shows the invariance of the cone but has also many other implications the first of which being the following.

Lemma 3.8. *There exists a constant $B > 0$ such that, for each $h \in BV$, $h \geq 0$ and $m \in \mathbb{N}$,*

$$\Lambda(\mathcal{L}^m 1)\Lambda(h) \leq \Lambda(\mathcal{L}^m h) \leq B\Lambda(\mathcal{L}^m 1)\Lambda(h).$$

Proof. The first inequality follows trivially by iterating (3.1). For the second, consider $n, m \in \mathbb{N}$ and $x \in D_{n+m}$, then

$$\begin{aligned}\frac{\mathcal{L}^{m+n}h(x)}{\mathcal{L}^n 1(x)} &= \frac{\mathcal{L}^{m+n}h(x)}{\mathcal{L}^{n+m}1(x)}\frac{\mathcal{L}^{n+m}1(x)}{\mathcal{L}^n 1(x)} \\ &\leq \frac{\mathcal{L}^{m+n}h(x)}{\mathcal{L}^{n+m}1(x)}\|\mathcal{L}^m 1\|_\infty,\end{aligned}$$

which, by taking the inf on x and the limit $n \to \infty$ yields

$$\Lambda(\mathcal{L}^m h) \leq \|\mathcal{L}^m 1\|_\infty \Lambda(h).$$

Next, since $1 \in \mathcal{C}_a$, Lemma 3.7, implies

$$\bigvee \mathcal{L}^m 1 \leq 2aC_\theta \Lambda(\mathcal{L}^m 1).$$

Sublemma 3.9. *For each $f \in BV$ holds: for all x*

$$f(x) \leq \Lambda(f) + \bigvee f.$$

Proof. For x and y,

$$f(x) \leq f(y) + \bigvee f,$$

fix x, using the properties of Λ we get:

$$f(x) \leq \Lambda(f) + \bigvee f.$$

□

Thus

$$\|\mathcal{L}^m 1\|_\infty \leq \Lambda(\mathcal{L}^m 1) + \bigvee \mathcal{L}^m 1 \leq (2aC_\theta + 1)\Lambda(\mathcal{L}^m 1),$$

from which the result follows with $B := 2aC_\theta + 1$. □

Lemma 3.10. *For each $\varepsilon > 0$ there exists n_0 such that for each $n \geq n_0$, the partition $\mathcal{Z}^{(n)}$ has the property*

$$\sup_{Z \in \mathcal{Z}^{(n)}} \Lambda(\mathbf{1}_Z) \leq \varepsilon.$$

Proof. Choose $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $C_\theta \theta^n \rho^{-n} \leq \varepsilon$, this is possible due to condition **C2**. Then, for $Z \in \mathcal{Z}^{(n)}$

$$\mathcal{L}^n \mathbf{1}_Z(x) = \sum_{y \in T^{-n}x} g_n(y)\mathbf{1}_Z(y) \leq \|g_n\|_\infty \leq C_\theta \theta^n.$$

Accordingly, for each $x \in D_{n+m} \subset D_m$,

$$\begin{aligned}\frac{\mathcal{L}^{n+m}\mathbf{1}_Z(x)}{\mathcal{L}^{n+m}1(x)} &\leq C_\theta \theta^n \frac{1}{\frac{\mathcal{L}^m \mathcal{L}^n 1(x)}{\mathcal{L}^m 1(x)}} \\ &\leq C_\theta \theta^n \frac{1}{\inf_{z \in D_m} \frac{\mathcal{L}^m \mathcal{L}^n 1(z)}{\mathcal{L}^m 1(z)}}.\end{aligned}$$

Taking the infimum with respect to x and the limit $m \to \infty$, the above relations yields

$$\Lambda(\mathbf{1}_Z) \leq C_\theta \theta^n \frac{1}{\Lambda(\mathcal{L}^n 1)} \leq C_\theta \theta^n \rho^{-n} \leq \varepsilon, \tag{3.3}$$

where we have used Lemma 3.6. □

Lemma 3.11. *For each $a \geq a_0$ there exists $n \in \mathbb{N}$ such that, for each $h \in \mathcal{C}_a$ there exists $Z \in \mathcal{Z}_g^{(n)}$ such that*

$$\inf_{x \in Z} h(x) \geq \frac{1}{4}\Lambda(h).$$

Proof. For each $n, m \in \mathbb{N}$, $n < m$, we can write[6]

$$\mathcal{L}^m h(x) = \sum_{Z \in \hat{\mathcal{Z}}^{(n)}} \mathcal{L}^m(h\mathbf{1}_Z)(x) = \sum_{Z \in \mathcal{Z}_*^{(n)}} \mathcal{L}^m(h\mathbf{1}_Z)(x)$$

We will then prove the Lemma arguing by contradiction. Suppose that the Lemma it is not true then, since by Condition **C2** and Lemma 2.4, $\mathcal{Z}_g^{(n)} \neq \emptyset$, we have

$$\begin{aligned}
\mathcal{L}^m h(x) &= \sum_{Z \in \mathcal{Z}_g^{(n)}} \mathcal{L}^m(h\mathbf{1}_Z)(x) + \sum_{Z \in \mathcal{Z}_b^{(n)}} \mathcal{L}^m(h\mathbf{1}_Z)(x) \\
&\leq \sum_{Z \in \mathcal{Z}_g^{(n)}} \mathcal{L}^m\mathbf{1}_Z(x)\frac{\Lambda(h)}{4} + \sum_{Z \in \mathcal{Z}_g^{(n)}} \mathcal{L}^m\mathbf{1}_Z(x) \bigvee_Z h \\
&\quad + \|h\|_\infty \sum_{Z \in \mathcal{Z}_b^{(n)}} \mathcal{L}^m\mathbf{1}_Z(x) \\
&\leq \mathcal{L}^m 1(x)\frac{\Lambda(h)}{4} + \sum_{Z \in \mathcal{Z}_g^{(n)}} \left[\Lambda(\mathcal{L}^m\mathbf{1}_Z) + \bigvee \mathcal{L}^m\mathbf{1}_Z\right] \bigvee_Z h \\
&\quad + \|h\|_\infty \sum_{Z \in \mathcal{Z}_b^{(n)}} \mathcal{L}^m\mathbf{1}_Z(x),
\end{aligned}$$

Where we have used Sublemma 3.9. To proceed notice that if $Z \in \mathcal{Z}_b^{(n)}$, then Lemma 3.8 implies

$$\Lambda(\mathcal{L}^m\mathbf{1}_Z) \leq B\Lambda(\mathcal{L}^m 1)\Lambda(\mathbf{1}_Z) = 0.$$

Hence, inequality (3.2) and Sublemma 3.9 imply

$$\mathcal{L}^m\mathbf{1}_Z \leq \bigvee \mathcal{L}^m\mathbf{1}_Z \leq 2C_\theta^{m/n_0+1}\theta^m \leq 2C_\theta\left(C^{\frac{1}{n_0}}\theta\rho^{-1}\right)^m \Lambda(\mathcal{L}^m 1).$$

On the other hand, if $Z \in \mathcal{Z}_g^{(n)}$, by the same arguments we obtain

$$\begin{aligned}
\bigvee \mathcal{L}^m\mathbf{1}_Z \leq & 2C_\theta^{[m/n_0]+1}\theta^m + 2K_{n_0}\rho^{-n_0}\Lambda(\mathcal{L}^m\mathbf{1}_Z) \\
&\leq \left[2C_\theta\left(C_\theta^{\frac{1}{n_0}}\theta\rho^{-1}\right)^m + 2K_{n_0}\rho^{-n_0}B\Lambda(\mathbf{1}_Z)\right]\Lambda(\mathcal{L}^m 1),
\end{aligned}$$

where we have used Lemma 3.8.

Accordingly, setting $\sigma := C_\theta^{\frac{1}{n_0}}\theta\rho^{-1} \leq 4^{-\frac{1}{n_0}}$,

$$\Lambda(\mathcal{L}^m 1)\Lambda(h) \leq \Lambda(\mathcal{L}^m h)$$

[6]See Lemma 2.5 for the definition of $\hat{\mathcal{Z}}^{(n)}$ and $\mathcal{Z}_*^{(n)}$.

and

$$
\begin{aligned}
\Lambda(\mathcal{L}^m h) \leq \Lambda(\mathcal{L}^m 1)\frac{\Lambda(h)}{4}& \\
+ \sum_{Z\in\mathcal{Z}_g^{(n)}} &\left[B\Lambda(\mathbf{1}_Z) + 2C_\theta\sigma^m + 2K_{n_0}\rho^{-n_0}B\Lambda(\mathbf{1}_Z)\right]\bigvee_Z h\Lambda(\mathcal{L}^m 1) \\
+ \sum_{Z\in\mathcal{Z}_b^{(n)}} &2C_\theta\sigma^m\|h\|_\infty\Lambda(\mathcal{L}^m 1).
\end{aligned}
$$

Dividing by $\Lambda(\mathcal{L}^m 1)$ the above inequalities and taking the limit $m\to\infty$ yields the announced contradiction

$$
\begin{aligned}
\Lambda(h) \leq & \frac{\Lambda(h)}{4} + \sum_{Z\in\mathcal{Z}_g^{(n)}} B(2K_{n_0}\rho^{-n_0}+1)\Lambda(\mathbf{1}_Z)\bigvee_Z h \\
\leq & \frac{\Lambda(h)}{4} + B(2K_{n_0}\rho^{-n_0}+1)\bigvee h \sup_{Z\in\mathcal{Z}_g^{(n)}}\Lambda(\mathbf{1}_Z) \\
\leq & \left[\frac{1}{4} + aB(2K_{n_0}\rho^{-n_0}+1)\sup_{Z\in\mathcal{Z}_g^{(n)}}\Lambda(\mathbf{1}_Z)\right]\Lambda(h) \\
\leq & \frac{1}{2}\Lambda(h),
\end{aligned}
$$

where we have chosen n large enough and we have used Lemma 3.10.

□

We are now ready to go back to the main result of this section

Proof of Proposition 3.5. We start by observing that $h\in\mathcal{C}_a$ implies $\Lambda(h)>0$, otherwise h would be constant and such a constant would be zero.

Second note that, if $\mathcal{K}_\varepsilon := \{h\in BV \mid \|h\|_\infty<\varepsilon,\ \bigvee h<\varepsilon\}$, then $1+\mathcal{K}_\varepsilon\subset\mathcal{C}_a$, for ε sufficiently small. That is, $\mathcal{C}_a$ contains an open set and thus has non empty interior.

Invariance has been already proved in Lemma 3.7, to obtain finite diameter choose $n\in\mathbb{N}$ so that Lemma 3.11 applies. For each $h\in\mathcal{C}_a$ there exists $Z\in\mathcal{Z}_g^{(n)}$ such that, for each $x\in D_m$,

$$
\mathcal{L}^m h(x) \geq \frac{1}{4}\Lambda(h)\inf_{D_m}\frac{\mathcal{L}^m\mathbf{1}_Z}{\mathcal{L}^m 1}\inf_{D_m}\mathcal{L}^m 1.
$$

To conclude just chose m so large that, for each $Z\in\mathcal{Z}_g^{(n)}$ holds

$$
\inf_{D_m}\frac{\mathcal{L}^m\mathbf{1}_Z}{\mathcal{L}^m 1} \geq \frac{\Lambda(\mathbf{1}_Z)}{2},
$$

and notice that $\inf_{D_m}\mathcal{L}^m 1>0$ since $\inf g>0$. We get for m large enough,

$$
\inf\mathcal{L}^m h \geq \frac{\Lambda(h)}{4}\cdot\inf_{Z\in\mathcal{Z}_g^{(n)}}\frac{\Lambda(\mathbf{1}_Z)}{2}\cdot\inf\mathcal{L}^m 1
$$

and, using Sublemma 3.9 and Lemma 3.8,

$$
\sup\mathcal{L}^m h \leq \Lambda(\mathcal{L}^m h) + \bigvee\mathcal{L}^m h \leq \left[B\Lambda(\mathcal{L}^m 1)+\frac{a}{2}\right]\Lambda(h).
$$

Set $\inf_{Z\in\mathcal{Z}_g^{(n)}} \frac{\Lambda(\mathbf{1}_Z)}{2} := A$. We get (see [15] Lemma 3.5 for the details) that:

$$\operatorname{diam}_{\mathcal{C}_a}\mathcal{L}^m(\mathcal{C}_a) \le 2\log\left[\frac{\max\left(\frac{3}{2}, B\Lambda(\mathcal{L}^m 1)+\frac{a}{2}\right)}{\min\left(\frac{1}{2}, \frac{A\inf\mathcal{L}^m 1}{4}\right)}\right] < \infty.$$

□

4. Escape Rates and Invariant Measure

Lemma 4.1. *There exists a unique $h_* \in \mathcal{C}_a$ and $\lambda \ge \rho$, such that $\mathcal{L}h_* = \lambda h_*$, moreover supp $(h_*) = D_\infty$.*

Proof. By standard arguments it follows from Theorem 3.3, Lemma 3.4 and Proposition 3.5 that, for each $g \in \mathcal{C}_a$, $\frac{\mathcal{L}^n g}{\Lambda(\mathcal{L}^n g)}$ is a Cauchy sequence in L^∞. This means that for each $g \in \mathcal{C}_a$ there exists $h_g \in \mathcal{C}_a$ such that $\frac{\mathcal{L}^n g}{\Lambda(\mathcal{L}^n g)} \to h_g$. In addition, there must exist $\lambda_g > 0$ such that $\mathcal{L}h_g = \lambda_g h_g$. In fact, since $\frac{\Lambda(\mathcal{L}^{n+1}g)}{\Lambda(\mathcal{L}^n g)} \in [\rho, B\rho]$, by Lemma 3.8, there exists a convergent subsequence $\{n_j\}$, let $\lambda_g := \lim_{j\to\infty} \frac{\Lambda(\mathcal{L}^{n_j+1}g)}{\Lambda(\mathcal{L}^{n_j}g)}$. Thus

$$\mathcal{L}h_g = \lim_{j\to\infty}\frac{\mathcal{L}^{n_j+1}g}{\Lambda(\mathcal{L}^{n_j}g)} = \lim_{j\to\infty}\frac{\mathcal{L}^{n_j+1}g}{\Lambda(\mathcal{L}^{n_j+1}g)}\lim_{j\to\infty}\frac{\Lambda(\mathcal{L}^{n_j+1}g)}{\Lambda(\mathcal{L}^{n_j}g)} = \lambda_g h_g.$$

We will show now that given $f, g \in \mathcal{C}_a$ we have $h_f = h_g = h_*$

$$\begin{aligned}\|h_f - h_g\|_\infty &\le \left(e^{d_{\mathcal{C}_a}(h_f,h_g)} - 1\right)\|h_f\|_\infty \\ &\le \left(e^{d_{\mathcal{C}_a}(\mathcal{L}^n h_f,\mathcal{L}^n h_g)} - 1\right)\|h_f\|_\infty\end{aligned}$$

which goes to zero when n goes to infinity. This implies $\lambda_g = \lambda_h := \lambda$ and $\mathcal{L}(h_*) = \lambda h_*$, as well. The claimed relation from ρ and λ follows from the following chain of inequalities

$$\begin{aligned}\Lambda(\mathcal{L}h_*) &= \lim_{n\to\infty}\inf_{D_n}\frac{\mathcal{L}^{n+1}h_*}{\mathcal{L}^n 1} = \lim_{n\to\infty}\inf_{D_{n+1}}\frac{\mathcal{L}^{n+1}h_*}{\mathcal{L}^n 1} \\ &\ge \lim_{n\to\infty}\inf_{D_{n+1}}\frac{\mathcal{L}^{n+1}h_*}{\mathcal{L}^{n+1}1}\inf_{D_n}\frac{\mathcal{L}^{n+1}1}{\mathcal{L}^n 1} = \Lambda(h_*)\rho,\end{aligned}$$

where we have used twice Remark 2.2. Finally, since $\Lambda(h_*) > 0$, it follows that $\mathcal{L}^n h_*|_{D_\infty} > 0$ which implies $h_*|_{D_\infty} > 0$. □

Lemma 4.2. *The functional Λ (restricted to BV) is linear, positive, and enjoys the property $\Lambda(\mathcal{L}f) = \lambda\Lambda(f)$ for all $f \in BV$. Moreover, $\lambda = \rho$.*

Proof. Let $f \in \mathcal{C}_a$. For all integers n, k and $x \in D_\infty$

$$\begin{array}{ccccc} \frac{\mathcal{L}^{n+k}f(x)}{\mathcal{L}^n f(x)} & = & \frac{\mathcal{L}^{n+k}f(x)}{\Lambda(\mathcal{L}^{n+k}f)} & \frac{\Lambda(\mathcal{L}^{n+k}f)}{\Lambda(\mathcal{L}^n f)} & \frac{\Lambda(\mathcal{L}^n f)}{\mathcal{L}^n f(x)} \\ \downarrow & & \downarrow & \downarrow & \downarrow \\ \lim_{n\to\infty}\frac{\mathcal{L}^{n+k}f(x)}{\mathcal{L}^n f(x)} & = & h_*(x) & \lambda^k & h_*(x)^{-1} \end{array}$$

so

$$\lim_{n\to\infty}\sup_{D_\infty}\left|\frac{\mathcal{L}^{n+k}f(x)}{\mathcal{L}^n f(x)} - \lambda^k\right| = 0.$$

But

$$\begin{aligned}\sup_{D_\infty}\left|\frac{\mathcal{L}^n f}{\mathcal{L}^n 1}-\frac{\mathcal{L}^{n+k}f}{\mathcal{L}^{n+k}1}\right| &\leq \sup_{D_\infty}\frac{\mathcal{L}^{n+k}f}{\mathcal{L}^{n+k}1}\left|\frac{\mathcal{L}^n f}{\mathcal{L}^{n+k}f}\frac{\mathcal{L}^{n+k}1}{\mathcal{L}^n 1}-1\right| \\ &\leq \|f\|_\infty \sup_{D_\infty}\left|\frac{\mathcal{L}^n f}{\mathcal{L}^{n+k}f}\frac{\mathcal{L}^{n+k}1}{\mathcal{L}^n 1}-1\right|\end{aligned}$$

and since the sequences $\frac{\mathcal{L}^{n+k}f}{\mathcal{L}^n f}$ and $\frac{\mathcal{L}^{n+k}1}{\mathcal{L}^n 1}$ have the same limit λ^k, $\frac{\mathcal{L}^n f}{\mathcal{L}^n 1}|_{D_\infty}$ is a Cauchy sequence, hence converges to a function ν_f. Moreover, if we take two points $x, y \in D_\infty$, we have

$$\begin{aligned}|\nu_f(x)-\nu_f(y)| &= \lim_{n\to\infty}\left|\frac{\mathcal{L}^n f}{\mathcal{L}^n 1}(x)-\frac{\mathcal{L}^n f}{\mathcal{L}^n 1}(y)\right| \\ &= \lim_{n\to\infty}\left|\frac{\mathcal{L}^n f}{\mathcal{L}^n 1}(y)\right|\cdot\left|\frac{\mathcal{L}^n f(x)\mathcal{L}^n 1(y)}{\mathcal{L}^n 1(x)\mathcal{L}^n f(y)}-1\right| \\ &\leq \|f\|_\infty \limsup_{n\to\infty}\left(\mathrm{e}^{d_{\mathcal{C}_+}(\mathcal{L}^n f,\mathcal{L}^n 1)}-1\right) \\ &\leq \|f\|_\infty \lim_{n\to\infty}\left(\mathrm{e}^{d_{\mathcal{C}_a}(\mathcal{L}^n f,\mathcal{L}^n 1)}-1\right)=0,\end{aligned}$$

where $\mathcal{C}_+ := \{h \in BV \mid h \geq 0\}$. Therefore, $\nu_f(x) = \Lambda(f)$ for all $x \in D_\infty$. Hence, $\Lambda(f) = \lim_{n\to\infty}\frac{\mathcal{L}^n f}{\mathcal{L}^n 1}$ for all $f \in \mathcal{C}_a$. Nevertheless, if $f \in BV$, the function $(f + a^{-1}\bigvee f - \inf f) \in \mathcal{C}_a$, so $\Lambda(f) = \lim \frac{\mathcal{L}^n f}{\mathcal{L}^n 1}$ for all $f \in BV$. Clearly, Λ is linear by the linearity of the limit.

Next, as $\mathcal{L}f \in BV$, we know that

$$\Lambda(\mathcal{L}f) = \lim_{n\to\infty}\frac{\mathcal{L}^{n+1}f}{\mathcal{L}^n 1} = \lim_{n\to\infty}\frac{\mathcal{L}^{n+1}f}{\mathcal{L}^{n+1}1}\frac{\mathcal{L}^{n+1}1}{\mathcal{L}^n 1} = \Lambda(f)\Lambda(\mathcal{L}1) = \rho\Lambda(f) \quad .$$

But then $\rho = \lambda$ is obtained by taking $f = h_*$. Notice that all the convergences take place at an exponential rate. □

Lemma 4.3. *The functional Λ can be interpreted as a non-atomic measure μ, i.e.*

$$\Lambda(f) = \int f d\mu \quad \forall f \in BV(I, m).$$

In addition, $\mathrm{supp}\mu \subset X_\infty$ *and the measure $h_*\mu$ is T–invariant.*

Proof. Clearly, Λ can be extended to all continuous functions since it is continuous in the sup norm and continuous functions can be uniformly approximated by bounded variation functions. Hence by Riesz theorem there exists a measure μ such that $\Lambda(f) = \mu(f)$ on each continuous function. Lemma 3.10 implies immediately that the measure μ is non atomic. Moreover it must agree with Λ on the characteristic function of each interval. Indeed, let J be an interval, since μ is a Borel measure, for each $\varepsilon > 0$ there exists a larger open interval $\tilde{J}$ such that $\mu(\tilde{J}) - \mu(J) \leq \varepsilon$ moreover Lemma 3.10 implies that $\tilde{J}$ can be chosen so that $\Lambda(\mathbf{1}_{\tilde{J}} - \mathbf{1}_J) \leq \varepsilon$. Thus, choosing a continuous function f such that $\mathbf{1}_J \leq f \leq \mathbf{1}_{\tilde{J}}$, holds[7]

$$\begin{aligned}\Lambda(\mathbf{1}_J) - \mu(\mathbf{1}_J) \leq \Lambda(f) - \mu(f) + \Lambda(\mathbf{1}_{\tilde{J}} - \mathbf{1}_J) \leq \varepsilon \\ \mu(\mathbf{1}_J) - \Lambda(\mathbf{1}_J) \leq \mu(f) - \Lambda(f) + \mu(\mathbf{1}_{\tilde{J}} - \mathbf{1}_J) \leq \varepsilon.\end{aligned}$$

[7]The existence of such a function is insured by Urysohn's Lemma.

Since a function in BV can be uniformly approximated by a finite linear combination of characteristic functions of intervals it follows that $\mu(f) = \Lambda(f)$ for each function of bounded variation. The conclusion of the lemma follows from Lemma 1.1 $\square$

In conclusion, we have proved the following result.

Theorem 4.4. *Assume g^0 is a contracting potential which belongs to BV. Assume that Condition 1 and Condition 2 are satisfied. Then there exists a unique conditionally invariant probability measure $\nu = hm$ which is absolutely continuous with respect to m. There exists a unique probability measure μ whose support is contained in X_∞ and which satisfies $\mu(\mathcal{L}f) = \rho\mu(f)$ for any bounded function f. Moreover, there exists $\kappa < 1$ such that for any $f \in BV$ and any $A \subset I$:*

$$\left\| \frac{\mathcal{L}^n f}{\rho^n} - h\mu(f) \right\|_\infty \leq Ct\kappa^n \|f\|_{BV}$$

and $|m(T^{-n}A|X_{n-1}) - \nu(A)| \leq Ct\kappa^n.$

5. The Hausdorff dimension of X_∞

In this section we assume that T is uniformly expanding i.e. $\inf |T'| > 1$. Remark that this implies that the partition $\mathcal{Z}^{(n)}$ is generating. For $g^0 = \frac{1}{T'}$, and $t \geq 0$, let $\mathcal{L}_t$ be the transfer operator with hole associated to $(g^0)^t$ i.e.:

$$\mathcal{L}_t f(x) = \sum_{Ty=x} (g^0)^t(y)\mathbf{1}_{X_0}(y)f(y),$$

and let Θ_t, ρ_t and $P(t)$ be the numbers corresponding to Θ, ρ, P, in case $t = 1$. Recall that in the case $g^0 = \frac{1}{T'}$, $P = P(g^0) = 0$.

Theorem 5.1. *Let $g^0 = \frac{1}{T'}$. Assume that for all $0 \leq t \leq 1$, Conditions 0, 1 and 2 are satisfied. Then, there exists a unique $0 < t_0 \leq 1$ such that for $0 \leq t < t_0$, $\rho_t > 1$ and for $1 > t > t_0$, $\rho_t < 1$. If T has large images and large images with respect to Y then, $HD(X_\infty) = t_0$.*

Proof. The hypothesis of Theorem 5.1 allow us to apply Theorem A to the operators $\mathcal{L}_t$ for all $0 \leq t \leq 1$. Let us denote by μ_t the conformal measure associated to $g^t = (g^0)^t \cdot \mathbf{1}_{X_0}$ (i.e. $\mathcal{L}_t^*\mu_t = \rho_t\mu_t$).
The application $t \mapsto \rho_t$ is strictly decreasing. Indeed, remark that: for all $x \in I$,

$$\mathcal{L}_t^n 1(x) \leq \sup g_n^{t-t'} \mathcal{L}_{t'}^n 1(x)$$

taking the power $\frac{1}{n}$ and the limit gives: $\rho_t \leq \Theta^{t-t'} \cdot \rho_{t'}$ so that $\rho_t < \rho_{t'}$ provided $t > t'$ (recall that $g < 1$ and remark that Theorem A implies: $\lim \left(\mathcal{L}_t^n 1(x)\right)^{\frac{1}{n}} = \rho_t$ for all x). Moreover, $\rho_1 \leq e^{P(1)} = 1$ (see Lemma 2.2), so there exists a unique number $0 \leq t_0 \leq 1$ such that for $1 > t > t_0$, $\rho_t < 1$ and for $0 \leq t < t_0$, $\rho_t > 1$.
The following lemma is a direct consequence of the bounded distortion and large images hypothesis.

Lemma 5.2. *Assume that $g^0 = \frac{1}{T'}$. Assume that for all $0 \leq t \leq 1$, Conditions 0, 1 and 2 are satisfied. For all $0 \leq t \leq 1$, there exists $K > 0$, such that for all $n \in \mathbb{N}$ and $Z \in \mathcal{Z}^{(n)}$, if $\mu_t(Z) > 0$ then for all $x \in Z$,*

$$K^{-1} \leq \frac{(g_n^0)^t(x)}{\rho_t^n \mu_t(Z)} \text{ and } K^{-1} \leq \frac{g_n^0(x)}{m(Z)}. \tag{5.1}$$

If moreover T has large images and large images with respect to Y then

$$\frac{(g_n^0)^t(x)}{\rho_t^n \mu_t(Z)} \leq K \text{ and } \frac{g_n^0(x)}{m(Z)} \leq K \tag{5.2}$$

where m is the Lebesgue measure.

Proof. First of all, remark that the large images with respect to Y property implies that for all $0 \leq t \leq 1$, the support of μ_t is X_∞. So, $\mu_t(Z) > 0$ if and only if $Z \cap X_\infty \neq \emptyset$. In addition, $Z \in \mathcal{Z}^{(n)}$ with $Z \cap X_\infty \neq \emptyset$, $\mu_t(T^n Z) = 1$. Now, compute

$$\begin{aligned}\mu_t(Z) &= \int \mathbf{1}_Z d\mu_t = \frac{1}{\rho_t^n} \int \mathcal{L}_t^n \mathbf{1}_Z d\mu_t = \frac{1}{\rho_t^n} \int\limits_{T^n Z} \left[(g_n^0)^t \mathbf{1}_{X_{n-1}}\right] \circ T_Z^{-n} d\mu_t \\ &= \frac{1}{\rho_t^n} \int\limits_{T^n Z} (g_n^0)^t \circ T_Z^{-n} d\mu_t,\end{aligned} \tag{5.3}$$

(recall that we assume $T^n(Z \cap X_{n-1} \supset X_\infty)$). The bounded distortion property implies, for $x \in Z$,

$$K^{-1} \mu_t(T^n Z)(g_n^0)^t(x) \leq \int\limits_{T^n Z} (g_n^0)^t \circ T_Z^{-n} d\mu_t \leq K \mu_t(T^n Z)(g_n^0)^t(x).$$

This gives (5.1) for μ_t and (5.2) for μ_t using the large images property (which implies $\mu_t(T^n Z) = 1$). The computation is the same for m, recalling that m is g^0 conformal with eigenvalue 1 and using the large image property. □

Fix $\varepsilon > 0$ and $n \in \mathbb{N}$ such that for all $Z \in \mathcal{Z}^{(n)}$, the diameter of Z is less than ε. Let $\mathcal{F} = \{Z \in \mathcal{Z}^{(n)} \mid Z \cap X_\infty \neq \emptyset\}$. It is a cover of X_∞ of diameter less than ε. In what follows, x_Z denotes any element of Z.

$$\begin{aligned}\sum_{Z \in \mathcal{F}} (\mathrm{diam} Z)^t &\leq K^t \sum_{Z \in \mathcal{F}} (g_n^0)^t(x_Z) \text{ using (5.1)} \\ &\leq K^{2t} \rho_t^n \sum_{Z \in \mathcal{F}} \mu_t(Z) \text{ using (5.2)} \\ &= K^{2t} \rho_t^n \mu_t(X_\infty) = K^{2t} \rho_t^n.\end{aligned}$$

By our choice of n, it is clear that $n \to \infty$ when $\varepsilon \to 0$. If $t > t_0$ then $\rho_t < 1$ and the above expression goes to zero. Hence we conclude $HD(X_\infty) \leq t_0$.
Let us prove the converse inequality. We use the following Young's result.

Theorem 5.3. [21] *Let X be a metric space, let $Z \subset X$ assume there exists a probability measure μ such that $\mu(Z) > 0$, for any $x \in Z$, define:*

$$\underline{d_\mu}(x) = \liminf_{\varepsilon \to 0} \frac{\log \mu(B(x, \varepsilon))}{\log \varepsilon},$$

if for all $x \in Z$, $\underline{d_\mu}(x) \geq d$ then $HD(Z) \geq d$.

Take $x \in X_\infty$ and $\varepsilon > 0$, let

$$n_0 = \inf\left\{n \in \mathbb{N} \;\middle|\; \exists\, y \in B(x, \varepsilon) \;:\; g_n^0(y) \leq 2K\, \varepsilon\right\} - 1.$$

Accordingly, there exists $y_0 \in B(x,\varepsilon)$ such that,

$$g^0_{n_0}(y_0)g^0(T^{n_0}y_0) = g^0_{n_0+1}(y_0) \leq 2K\ \varepsilon \ \text{ so,}$$

$$2K\varepsilon < \ g^0_{n_0}(y_0) \ \leq \frac{2K\varepsilon}{\inf g^0}. \tag{5.4}$$

Using Lemma 5.2 and (5.4) we get:

$$2\varepsilon \leq \ \mathrm{diam} Z_{n_0}(y_0) \leq \frac{2K^2\varepsilon}{\inf g^0} := C_1\varepsilon.$$

Let $B_1 = B(x,\varepsilon) \setminus Z_{n_0}(y_0)$. If $B_1 \neq \emptyset$, then let us define:

$$n_1 = \inf\left\{n \in \mathbb{N} \ \middle|\ \exists\ y \in B_1 \ :\ g^0_n(y) \leq 2K\ \varepsilon\right\} - 1.$$

Hence, there exists $y_1 \in B_1$ such that

$$2\varepsilon \leq \ \mathrm{diam} Z_{n_1}(y_1) \leq C_1\varepsilon$$

by the same arguments as before. Remark that $n_1 \geq n_0$ by construction, so $Z_{n_0}(y_0) \cap Z_{n_1}(y_1) = \emptyset$. This implies that $Z_{n_0}(y_0) \cup Z_{n_1}(y_1) \supset B(x,\varepsilon)$. Indeed if it were not the case, we could find $y_2 \in B_1 \setminus Z_{n_1}(y_1)$ and $n_2 \geq n_1$ such that:

$$2\varepsilon \leq \ \mathrm{diam} Z_{n_2}(y_2) \leq C_1\varepsilon.$$

By construction, we would obtain three disjoint intervals, with diameter larger than 2ε, all intersecting $B(x,\varepsilon)$, but this is clearly impossible.
Therefore, we have shown that $B(x,\varepsilon) \supset Z_{n_0} \cup Z_{n_1}$, where the second set may be empty. We have

$$\mu_t(Z_{n_i}) \quad \leq \quad \frac{K^t}{\rho_t^{n_i}} \left(\mathrm{diam} Z_{n_i}\right)^t,$$

by (5.3) and (5.2). So,

$$\begin{aligned}
\frac{\log \mu_t(B(x,\varepsilon))}{\log \varepsilon} \quad &\geq \quad \frac{\log\left(\mu_t(Z_{n_0}) + \mu_t(Z_{n_1})\right)}{\log \varepsilon} \\
&\geq \quad t\ \frac{\log K}{\log \varepsilon} + \frac{\log\left(\rho_t^{-n_0}\mathrm{diam}(Z_{n_0})^t + \rho_t^{-n_1}\mathrm{diam}(Z_{n_1})^t\right)}{\log \varepsilon} \\
&\geq \quad t\ \frac{\log KC_1}{\log \varepsilon} + \frac{\log(\rho_t^{-n_0} + \rho_t^{-n_1})}{\log \varepsilon} + t
\end{aligned}$$

Since, for ε small enough, n_0 and n_1 are arbitrarily large and for $t < t_0$, $\rho_t > 1$, we can assume $\rho_t^{-n_0} + \rho_t^{-n_1} < 1$ so, $\frac{\log(\rho_t^{-n_0}+\rho_t^{-n_1})}{\log \varepsilon} > 0$. Therefore, taking the $\liminf$, $\underline{d_\mu}(x) \geq t$ for all $t < t_0$. We conclude that $HD(X_\infty) \geq t_0$. □

6. Examples

In this section we give verifiable criteria to insure conditions **C1, C2** in concrete situations and we discuss some explicit examples.

Condition **C1** is rather mild and in most cases can be checked easily (for example the presence of a full branch outside the hole suffices).

Next notice that, setting

$$\rho_n := \inf_{x \in D_n} \frac{\mathcal{L}^{n+1}1(x)}{\mathcal{L}^n 1(x)},$$

then $\rho \geq \rho_n$ (see (2.3)), hence one can verify condition **C2** by using some ρ_n (which is explicitly computable) rather than ρ. The main problem is then to control the number of contiguous elements in $\mathcal{Z}_b^{(n)}$. This of course is a case by case matter, yet it is possible to make some rather general statements. Let us exemplify the situation by looking at few relevant examples.

Markov maps with non Markov hole. Let us give examples of Markov maps with a non Markov hole. Recall that T is said to be *Markov* with respect to the partition $\mathcal{Z}$ if for all $Z \in \mathcal{Z}$, TZ is exactly a union of some elements of $\mathcal{Z}$. We call Y a *Markov hole* if T is Markov and $Y \in \mathcal{Z}^{(n)}$ for some n;[8] up to replacing $\mathcal{Z}$ by $\mathcal{Z}^{(n)}$, we may always assume that a Markov hole is an element of $\mathcal{Z}$. Let Y be such a Markov hole, let $\hat{\mathcal{Z}}$ be the set of elements of $\mathcal{Z}$ that are not Y. We call Y an *aperiodic Markov hole* if there exists $N \in \mathbb{N}$ such that for all $n \geq N$, for all Z, $Z' \in \mathcal{Z}$, there are $Z_1, \ldots, Z_n \in \hat{\mathcal{Z}}$ such that the $(n+1)$-cylinder $Z \cap T^{-1}Z_1 \cap \cdots \cap T^{-n}Z_n \cap T^{-n-1}Z'$ is non empty. For expanding Markov maps with an aperiodic Markov hole, Theorem A has been proved in [7].
We are now in position to give examples of Markov maps with non Markov hole such that $\mathcal{Z}_b^{(n)} = \emptyset$.

Lemma 6.1. *Let T be a Markov map with Lipschitz derivative, let $\tilde{Y}$ be an aperiodic Markov hole. Let $Y \subset \tilde{Y}$ be a hole such that there exists $p \in \mathbb{N}$ and $C \in \mathcal{Z}^{(p)}$ such that $C \subset \tilde{Y} \setminus Y$ and $C \subset X_{p-1}$. Then for the map T with hole Y one can choose $\xi = 1$ in condition 2 (indeed, for all n, $\mathcal{Z}_b^{(n)} = \emptyset$, hence one can choose $K = 0$ as well).*

Proof. First of all, remark that since T' is Lipschitz, there exists a constant $K(T)$ such that for all $Z \in \mathcal{Z}^{(n)}$, $\bigvee_Z g_n^0 \leq K(T)\|g_n^0\|_\infty$ so that we may avoid the use of partitions $\mathcal{A}$ in the definition of $\mathcal{Z}_*^{(n)}$ (see section 2). Take $n \in \mathbb{N}$, we are going to prove that $\Lambda(\mathbf{1}_Z) > 0$ for all $Z \in \mathcal{Z}_*^{(n)}$, to this aim, it suffices to prove that for some k, $\mathcal{L}^k\mathbf{1}_Z > 0$. In other words, it suffices to prove that for some k, every $x \in I$ has a k-preimage in $Z \cap X_{k-1}$.
Take $Z \in \mathcal{Z}_*^{(n)}$, according to the definitions,

$$Z \supset \bigcap_{i=0}^{n-1} T^{-i}C_i := \tilde{Z}$$

where C_i is either an element of $\hat{\mathcal{Z}}$ or is equal to C, $\tilde{Z}$ is a p'-cylinder with $n \leq p' \leq pn$. Then using the aperiodicity of $\tilde{Y}$, we have that for all $q \geq N$, any $x \in I$ has a $(p'+q)$-preimage in $Z \cap X_{p'+q-1}$. □

We conclude with a concrete Markov example with non Markov hole. Consider a partition of I into two subintervals Z_0 and Z_1. Take T uniformly expanding and increasing on each Z_i, with Lipschitz derivative and such that $TZ_i = I$, $i = 0, 1$. Take $\tilde{Y} = Z_0 \cap T^{-1}Z_0$, it is clear that $\tilde{Y}$ is an aperiodic Markov hole (see Figure 1).
First consider $Y = [0, \alpha] \subset \tilde{Y}$, there exists $p \in \mathbb{N}$ such that $C = Z_0 \cap T^{-1}Z_0 \cap \bigcap_{i=2}^{p-1} T^{-i}Z_1 \subset \tilde{Y} \setminus Y$, this C satisfies the hypothesis of Lemma 6.1 so the map T with hole Y satisfies condition **C2** provided it satisfies condition $\Theta < \rho$.

[8] In fact, one could work with $Y = \cup Y_i$ and $Y_i \in \mathcal{Z}^{(n)}$, $i = 1, \ldots, k$.

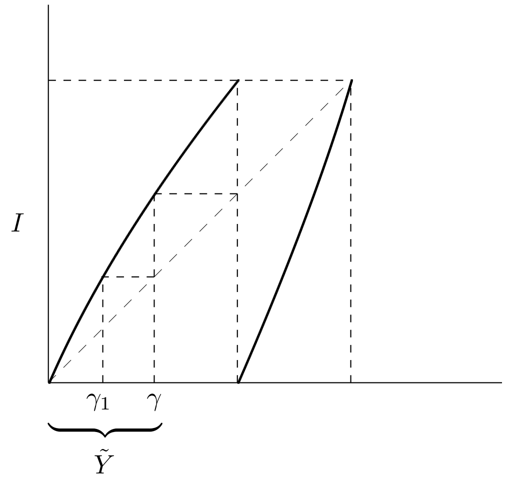

Figure 1: Aperiodic Markov hole

Second, consider $Y = [\varepsilon, \gamma]$ with γ such that $\tilde{Y} = [0, \gamma]$, let γ_1 be such that $Z_0 \cap T^{-1}Z_0 \cap T^{-2}Z_1 = [\gamma_1, \gamma]$. If $\varepsilon \geq \gamma_1$ then Y satisfies the hypothesis of Lemma 6.1: for p large enough, the cylinder $C := \bigcap_{i=0}^{p-2} T^{-i}Z_0 \cap T^{-p+1}Z_1$ is a p-cylinder included in $X_{p-1} \cap \tilde{Y} \setminus Y$.
If $\varepsilon < \gamma_1$ then it is easy to see that for all n, $\mathcal{Z}_b^{(n)}$ satisfies condition **C2** with $K = 1$ and $\xi = 1$ (the elements of $\mathcal{Z}_b^{(n)}$ are those made up with the interval $[0, \varepsilon]$ and they are never contiguous).

In this last example we have seen that some special cases can be easily handled even if $Z_b^{(n)} \neq \emptyset$. The next examples go further in this direction.

Non Markov maps.
Let $I = [0, 1]$, for $\beta > 1$ and consider the β-map $T(x) = \beta x (\text{mod } 1)$ and the potential $g^0 := DT^{-1} = \beta^{-1}$. If $\beta \notin \mathbb{N}$, then the map it is not Markov. We will consider only such cases and we will designate by $[\beta]$ the integer part of β. Let $\gamma = \frac{[\beta]}{\beta}$ and $Y = [\gamma_1, 1]$ with $\gamma < \gamma_1 < 1$. Denote the element of $\mathcal{Z}$ by $Z_1, \ldots, Z_{[\beta+1]}$, it is clear that for p large enough, $C := Z_{[\beta]} \cap \bigcap_{i=1}^{p-1} T^{-i}Z_1$ is included in X_{p-1}, this leads to the conclusion that there are no contiguous elements of $\mathcal{Z}_b^{(n)}$. So, Condition **C2** is satisfied (with $K = \xi = 1$) provided Condition **C1** is. Note that since the behavior of the map inside the hole it is completely irrelevant we could modify the map inside the hole to be Markov, accordingly this case bears no difference with the ones discussed in the previous subsection.
On the contrary, if we consider the case $Y = [\gamma, \gamma_1]$ we have a Non-Markov map with an hole. In this case then the number of contiguous elements of $\mathcal{Z}_b^{(n)}$ is bounded by 2^n (since the worst case scenario is when the preimages of a contiguous group join the preimages of another group across the hole[9]–see Lemma 6.3 for a similar discussion in a more general context) so that Condition **C2** is satisfied provided

[9]Note that this is a general bound, better bounds may be available for specific values of β

$\frac{2}{\beta} < \rho$. We remark that $\rho \geq \frac{[\beta]}{\beta}$, so that Condition **C2** will be satisfied for $\beta \geq 3$. In particular, the map in figure 2, with $\beta = \frac{7}{2}$ does satisfy our conditions.

In addition notice that one can consider also bigger holes that encompass more than one element of the dynamical partition. For example if $\beta = \frac{9}{2}$ and the hole is of the form $Y = [\frac{4}{9}, 1-\varepsilon]$, $\varepsilon \in (0, \frac{1}{9})$, then the map has three full branches outside the hole hence $\rho \geq \frac{2}{3}$ while the maximal number of contiguous elements in $\mathcal{Z}_b^{(n)}$ is still 2^n, thus **C2** is satisfied. Note that in this case we can have holes with size almost $\frac{1}{3}$ which is rather large. In fact, even more dramatic examples can be easily produced.

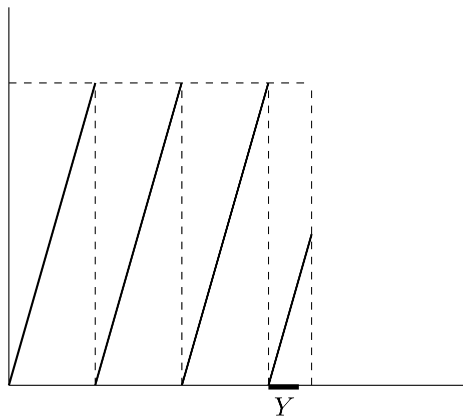

Figure 2: Non Markov β-map with a hole ($\beta = \frac{7}{2}$)

We have seen that it is possible to insure condition **C2** by using the combinatorial properties of a Markov map or the special behavior of β-maps. Some of the above discussion can be generalized by requiring the existence of well behaved elements in the partition: let $\mathcal{Z}_f^{(n)}$ be the collection of elements in $\mathcal{Z}_*^{(n)}$ such that $T^n Z = [0,1]$. Call $\mathcal{Z}_u^{(n)}$ the collection of the others.

Definition 6.2. *For $\xi > 0$, we call a map ξ* full branched *(ξ-f.b. for short) if there exists $K > 0$ such that the number of contiguous elements in $\mathcal{Z}_u^{(n)}$ does not exceed $K\xi^n$.*

Obviously a ξ-f.b. map satisfies condition **C2**, provided $\Theta\xi < \rho$, since if $Z \in \mathcal{Z}_f^{(n)}$ then $\Lambda(\mathbf{1}_Z) > 0$. The point is that it may be easy to verify that a map is ξ-f.b. as the next lemma shows.

Lemma 6.3. *Calling C_n the maximal number of contiguous elements in $\mathcal{Z}_u^{(n)}$, holds*

$$C_n \leq 2\sum_{i=0}^{n-1}(C_1+2)^i C_1.$$

Proof. The proof is by induction on n. Clearly it is true for $n = 1$. Let us suppose it true for n. The elements of the partition $\mathcal{Z}_*^{(n+1)}$ are formed by $\{T^{-1}Z \cap Z_1\}$ where $Z \in \mathcal{Z}_*^{(n)}$ and $Z_1 \in \mathcal{Z}_*^{(1)}$. Now if $Z_1 \in \mathcal{Z}_f^{(1)}$, the elements maintain the same nature (i.e. if $Z \in \mathcal{Z}_u^{(n)}$ then $T^{-1}Z \cap Z_1 \in \mathcal{Z}_u^{(n+1)}$ and if $Z \in \mathcal{Z}_f^{(n)}$ then $T^{-1}Z \cap Z_1 \in \mathcal{Z}_f^{(n+1)}$). So we have in Z_1 at most C_n contiguous elements of $\mathcal{Z}_u^{(n+1)}$. The only problem can arise when a block of contiguous elements ends at the boundary of Z_1 since in such a case it can still be contiguous to other elements of $\mathcal{Z}_u^{(n+1)}$. Yet, if the contiguous elements of Z_1 are in $\mathcal{Z}_f^{(1)}$, then there can be at most a block of length $2C_n$. One must then analyze what can happen if $Z_1 \in \mathcal{Z}_u^{(1)}$. In this case a set of contiguous elements can either have only partial preimage in Z_1, hence we get a shorter groups of contiguous elements or all the group can have preimage. In this last case the worst case scenario is when the elements contiguous to the groups (that must belong to $\mathcal{Z}_f^{(n)}$) are cut while taking preimages. This means that at most two new contiguous elements can be generated, but in this case the group must end at the boundary of Z_1. Since there are at most C_1 contiguous elements in $\mathcal{Z}_u^{(1)}$ in this way we can generate, at most, $C_1(C_n + 2)$ contiguous elements that, again in the worst case scenario, can be contiguous to two blocks belonging to the neighboring elements in $\mathcal{Z}_f^{(1)}$. Accordingly

$$C_{n+1} \leq C_1(C_n + 2) + 2C_n = (C_1 + 2)C_n + 2C_1 \leq 2\sum_{i=0}^{n}(C_1 + 2)^i C_1,$$

where we have used the induction hypothesis. □

The Lemma says that if $\rho\Theta^{-1} > C_1 + 2$, then the hypothesis **C2** is verified. The interest of this condition is that it applies to general non Markov maps provided the potential is sufficiently contracting and there are enough full branches outside the hole.[10]

7. Small Holes

In this section we will see that, if one is interested only in very small holes then results stronger than the one in the previous sections can be readily obtained by regarding the system with holes as a small perturbation of the system without holes.

The basic idea is to consider the transfer operator $\mathcal{L}$ as a small perturbation of the operator $\mathcal{L}_0$. Of course, the norm of the difference of the above operators equal 2 both in the L^1 and BV norm, hence standard perturbation theory does not apply directly (but see [5] for an indirect application), yet they are close as operators from BV to L^1.

Definition 7.1. *For each operator $\mathcal{L} : BV(I, m) \to L^1(I, m)$ let*

$$|||\mathcal{L}||| := \sup_{\|f\|_{BV} \leq 1} |\mathcal{L}f|_1.$$

Then the exact statement of the closeness of the two operators is given by the following lemma. Let $\mathcal{L}_Y$ be the transfer operator associated to the hole Y.

Lemma 7.2. *If $\mathcal{L}_0$ and $\mathcal{L}_Y$ are the two operators defined in (1.1) and (1.2), respectively, then*

$$|||\mathcal{L}_0 - \mathcal{L}_Y||| \leq e^{P(g^0)} m(Y).$$

[10]Note that if a map does not satisfy immediately such a criteria, some of its powers may.

Proof. For each $f \in BV$ holds

$$\begin{aligned}|\mathcal{L}_0(f) - \mathcal{L}_Y(f)|_1 =& |\mathcal{L}_0(\mathbf{1}_Y f)|_1 \leq e^{P(g^0)}|\mathbf{1}_Y f|_1\\ \leq& e^{P(g^0)}|f|_\infty m(Y) \leq e^{P(g^0)}\|f\|_{BV} m(Y)\end{aligned}$$

from which the lemma follows. □

The above notion of closeness is the one employed in [13], it is then natural to try to verify the conditions of the abstract perturbation result contained in such a paper.

For the reader convenience let us summarize the above mention result specialized to the simple case under consideration.

Theorem 7.3 ([13])**.** *If there exists constants $A, B > 0$, independent of Y, and $\theta \in (\Theta(g^0), e^{P(g^0)})$ such that, for each $f \in BV$,*

$$\begin{aligned}\|\mathcal{L}_0^n f\|_{BV} \leq A\theta^n \|f\|_{BV} + B|f|_1\\ \|\mathcal{L}_Y^n f\|_{BV} \leq A^n\theta^n \|f\|_{BV} + B|f|_1\end{aligned}$$

then for each $\theta_1 \in (\theta, 1)$ and $\delta \in (0, 1-\theta_1)$, there exists $\varepsilon_0 > 0$ such that if $|||\mathcal{L}_0 - \mathcal{L}_Y||| < \varepsilon_0$ then the spectrum of $\mathcal{L}_Y$ outside the disk $\{z \in \mathbb{C} \mid |z| \leq \theta_1\}$ is δ-close, with multiplicity, to the one of $\mathcal{L}_0$.

Clearly, from the Theorem 7.3 Theorem C readily follows. In fact, if the map T has a unique invariant measure μ_0 absolutely continuous with respect to m, this means that $\mathcal{L}_0$ has $e^{P(g^0)}$ as an isolated eigenvalues and, if the systems (I, T, μ_0) is mixing, this means that there are no other eigenvalues of modulus one, which in turn implies the existence of a spectral gap. Let λ_1, $|\lambda_1| < 1$ be the second largest eigenvalue then, in the above theorem, choose $\theta_1 \geq \max\{\theta, \lambda_1\}$ and $\delta = \frac{e^{P(g^0)} - \theta_1}{2}$. Theorem 7.3 implies that, for sufficiently small holes, the spectrum of $\mathcal{L}$ outside the disk $\{z \in \mathbb{C} \mid |z| \leq \theta_1\}$ consists of only one eigenvalue λ_0 (that moves continuously from $e^{P(g^0)}$ as the hole gets larger) of multiplicity one and of modulus larger than $1 - \delta$.[11] The projector Π ($\mathcal{L}_Y \Pi = \Pi \mathcal{L}_Y = \lambda_0 \Pi$) associated to such an eigenvalue is of the form $\Pi(f) = h\mu(f)$ where $\mathcal{L}_Y h = \lambda_0 h$, gives the quasi invariant measure and $h\mu$ is the invariant measure.[12]

Hence, to conclude we need only verify the hypotheses of Theorem 7.3.

Lemma 7.4. *For each $\theta \in (\Theta(g^0),\, e^{P(g^0)})$ there exists $A, B > 0$, independent of Y, such that, for each $f \in BV$,*

$$\begin{aligned}\|\mathcal{L}_0^n f\|_{BV} \leq A\theta^n \|f\|_{BV} + B|f|_1\\ \|\mathcal{L}_Y^n f\|_{BV} \leq A\theta^n \|f\|_{BV} + B|f|_1\end{aligned}$$

Proof. The first inequality is nothing else that the usual Lasota-Yorke inequality, the second is proved by a simplified version of Lemma 2.5.

Remember that $\mathcal{A}_n$ is the set of finite partitions in intervals $A = \{A_i\}$ such that $\bigvee_{A_i} g_n \leq 2\|g_n\|_\infty$. Given $n \in \mathbb{N}$ and $A \in \mathcal{A}_i$ let $\widetilde{\mathcal{Z}}^{(n)}$ be the coarsest partition in intervals among all the ones finer than both A and $\mathcal{Z}^{(n)}$. For each $Z \in \mathcal{Z}_*^{(n)}$ let

[11] In fact, the results in [13] imply that there exist constants $C > 0$ such that $e^{P(g^0)} - \lambda_0 \leq Cm(Y)$, provided δ is chosen small enough.

[12] See [12] for the proof that μ is a measure.

$\tilde{Z} \in \widetilde{\mathcal{Z}}^{(n)}$ be such that $Z \subset \tilde{Z}$. We have then the following analogous of equation (2.6):

$$\begin{aligned}\bigvee \mathbf{1}_{T^n Z}(g_n h) \circ T_Z^{-n} &\leq \bigvee_Z h g_n \ + \ 2 \sup_Z |h \cdot g_n| \\ &\leq \bigvee_Z h g_n \ + 2 \inf_{\tilde{Z}} |h \cdot g_n| + 2 \bigvee_{\tilde{Z}} h g_n \\ &\leq 9 \|g_n\|_\infty \bigvee_{\tilde{Z}} h + 8 \|g_n\|_\infty \inf_{\tilde{Z}} |h|.\end{aligned} \tag{7.1}$$

Sublemma 7.5. *For each $\tilde{Z} \in \hat{\mathcal{Z}}^{(n)}$, $\#\{Z \in \mathcal{Z}_*^{(n)} \mid Z \subset \tilde{Z}\} \leq n+1$.*

Proof. Since, by definition, $T^i|_{\tilde{Z}}$, $i \leq n$, is invertible, then $T^{-i}Y$ can have at most one preimage in $\tilde{Z}$. Accordingly, $Y_n \cap \tilde{Z}$ can consist of, at most, n subintervals, hence X_n can have, at most, $n+1$ connected components which are exactly $\{Z \in \mathcal{Z}_*^{(n)} \mid Z \subset \tilde{Z}\}$. □

By Sublemma 7.5 it follows that we can sum over $Z \in \mathcal{Z}_*^{(n)}$ and obtain

$$\bigvee \mathcal{L}^n h \leq 9(n+1)\|g_n\|_\infty \bigvee h + 8(n+1)\|g_n\|_\infty \sup_{\tilde{Z} \in \hat{\mathcal{Z}}^{(n)}} m(Z)^{-1} \int |h| dm.$$

Since there exists $\bar{n} \in \mathbb{N}$: $\theta^{\bar{n}} > 9(\bar{n}+1)\|g_{\bar{n}}\|_\infty$, the result follows by choosing

$$\begin{aligned}&A := \sup_{n \leq \bar{n}} 9(n+1)\|g\|_\infty \\ &B := 2(1-\theta^{\bar{n}})^{-1} \sup_{n \leq \bar{n}} 8(n+1)\|g\|_\infty \sup_{\tilde{Z} \in \hat{\mathcal{Z}}^{(n)}} m(Z)^{-1},\end{aligned}$$

and using the same iteration scheme employed in the proof of Lemma 3.7. Notice that, as announced, A and B do not depend on the hole Y. □

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