Quantile oriented sensitivity analysis for insurance companies' solvency.

Véronique Maume-Deschamps
Joint work with Kevin Elie-Dit-Cosaque.

ICIAM 2019, July 16th

Lyon 1
Math Institut Camille Jordan
SCOR The Art & Science of Risk

Plan

1. Context
2. Global sensitivity analysis
3. Simulation studies

Solvency Capital Requirement

SCR (Solvency Capital Requirement): regulatory capital calculated using a risk measure that the re-insurer must have to absorb potential losses. In Europe, the **SCR** can be calculated into two ways:

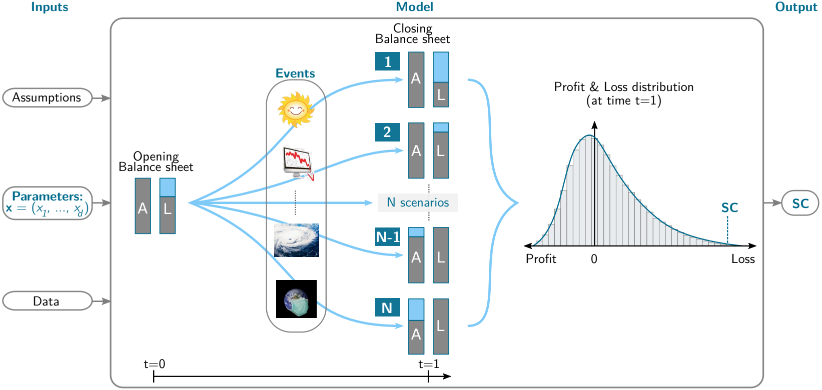
The Standard Formula

- Detailed in Solvency II rules
- The default option
- Formula factor based calculation
- Not firm specific
- A means of calculating SCR only - no wider significance

The internal model

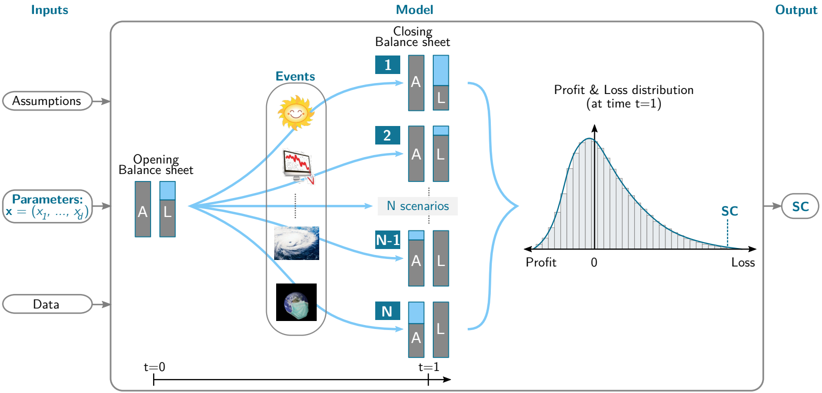
- Standards set out in Solvency II rules
- Regulator pre-approval required
- Specific to individual firm and risk profile
- Must be widely used and play an important part in firm's system of governance

Modelling insurance companies' solvency



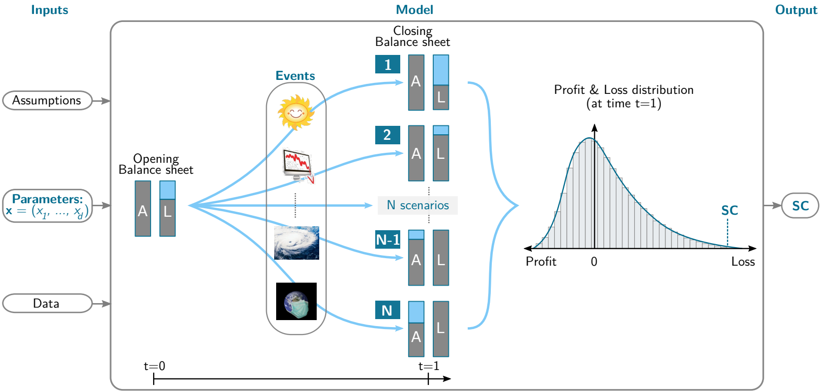
In the Solvency II directive of the European Union, the SCR it is the $VaR(99.5\%)$ of the P&L distribution.

Modelling insurance companies' solvency



- P&L distribution depends on input parameters $\mathbf{x} = (x_1, ..., x_d)$.
- The model input variables $\mathbf{x} = (x_1, \ldots, x_d)$ are not perfectly known.

Modelling insurance companies' solvency



Interest of lower quantile levels for a better understanding of the internal model.

General framework

Model

$$f : \left| \begin{array}{ccc} \mathbb{R}^d & \rightarrow & \mathbb{R} \\ \mathbf{x} = (x_1, \ldots, x_d) & \mapsto & y = f(\mathbf{x}) \end{array} \right.$$

with

- f: mathematical or numerical model,
- $\mathbf{x}$: uncertain input parameters,
- y: model's output.

The uncertainty on the input parameters is modelled by a probability distribution $\mathbb{P}$ on $\mathbb{R}^d$ and we get

$$Y = f(X_1, \ldots, X_d)$$

with the vector $\mathbf{X} = (X_1, \ldots, X_d)$ distributed as $\mathbb{P}$.

General framework

Model

$$f : \left| \begin{array}{ccc} \mathbb{R}^d & \rightarrow & \mathbb{R} \\ \mathbf{x} = (x_1, \ldots, x_d) & \mapsto & y = f(\mathbf{x}) \end{array} \right.$$

with

- f: mathematical or numerical model,
- $\mathbf{x}$: uncertain input parameters,
- y: model's output.

Sensitivity Analysis (SA)

The study of how uncertainty in the output of a model (numerical or otherwise) can be apportioned to different sources of uncertainty in the model's inputs (Saltelli et al. *(2004) e.g.).*

Sobol indices

Independent X_i's. Defined by Sobol (1993) .

$$S_i = \frac{\text{var}\left(\mathbb{E}[Y|X_i]\right)}{\text{var}(Y)}$$

$$S_i = \frac{\text{var}(Y) - \mathbb{E}\left(\text{var}[Y|X_i]\right)}{\text{var}(Y)}$$

$$S_i = \frac{\mathbb{E}\left[(Y - \mathbb{E}[Y])^2\right] - \mathbb{E}\left(\mathbb{E}\left[(Y - \mathbb{E}[Y|X_i])^2 |X_i\right]\right)}{\mathbb{E}\left[(Y - \mathbb{E}[Y])^2\right]}$$

$$S_i = \frac{\min_\theta \mathbb{E}\left[(Y - \theta)^2\right] - \mathbb{E}\left(\min_\theta \mathbb{E}\left[(Y - \theta)^2 |X_i\right]\right)}{\min_\theta \mathbb{E}\left[(Y - \theta)^2\right]}$$

Quantile oriented sensitivity analysis

QOSA: Quantile Oriented Sensitivity Analysis) index: (Fort *et al.* 2016)

$$S_i^\alpha = \frac{\min_{\theta\in\mathbb{R}} \mathbb{E}\left[\psi(Y,\theta)\right] - \mathbb{E}\left[\min_{\theta\in\mathbb{R}} \mathbb{E}\left[\psi\left(Y,\theta\right)|X_i\right]\right]}{\min_{\theta\in\mathbb{R}} \mathbb{E}\left[\psi(Y,\theta)\right]}$$

$$S_i^\alpha = \frac{\mathbb{E}\left[\psi\left(Y,q_\alpha(Y)\right)\right] - \mathbb{E}\left[\psi\left(Y,q_\alpha(Y|X_i)\right)\right]}{\mathbb{E}\left[\psi(Y,q_\alpha(Y))\right]}$$

with the contrast function $\psi : (y,\theta) \mapsto (y-\theta)(\alpha - \mathbf{1}_{y\leq\theta})$, $\alpha \in [0,1]$

Quantile oriented sensitivity analysis

QOSA: Quantile Oriented Sensitivity Analysis) index: (Fort *et al.* 2016)

$$S_i^\alpha = \frac{\mathbb{E}\left[\psi\left(Y, q_\alpha(Y)\right)\right] - \mathbb{E}\left[\psi\left(Y, q_\alpha(Y|X_i)\right)\right]}{\mathbb{E}\left[\psi(Y, q_\alpha(Y))\right]}$$

Properties:

- $0 \leq S_i^\alpha \leq 1$
- $S_i^\alpha = 0 \iff Y$ and X_i are independent
- $S_i^\alpha = 1 \iff Y$ is X_i measurable

Estimating QOSA

Estimating Sobol' index may avoid the estimation of the conditional distribution by using $\text{var}\left(\mathbb{E}[Y|X_i]\right) = \text{Cov}(Y, Y')$ with

$$Y' = f(\mathbf{X}'),\ \mathbf{X}' = (X'_1, \ldots, X'_{i-1}, X_i, X'_{i+1}, \ldots, X'_n)$$

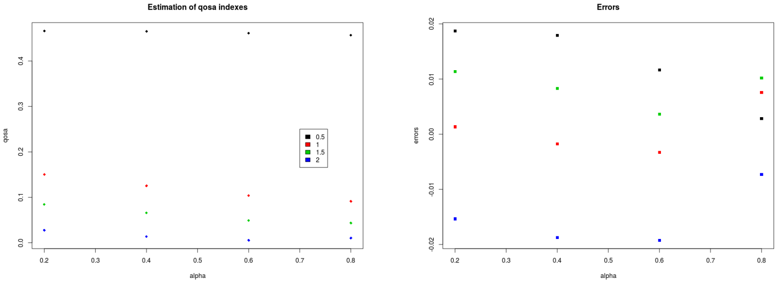
X'_j independent copy of X_j.
The estimation of QOSA' index requires to estimate the conditional distribution $Y|X_i$.

- Kernel methods (Maume-Deschamps & Niang 2018; Browne *et al.* 2017) optimal window width difficult to calibrate, requires a large number of calls to the costly function f.
- Random Forest method Less calls to f, time consuming nevertheless.

Sum of exponential laws

In case $X_i \rightsquigarrow \mathcal{E}(\lambda_i)$, $\lambda_i \in \mathbb{R}^+$ distinct; $Y = \sum_{i=1}^{n} X_i$ a semi-closed form formula may be obtained by using calculations from Marceau (2014) .

Learning sample of size 15000, 500 trees, estimation of the conditional distributions with sample sizes 1500

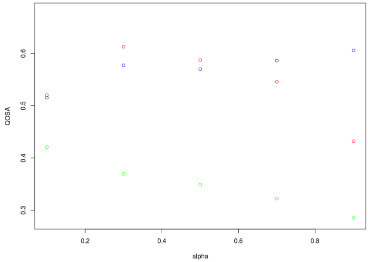


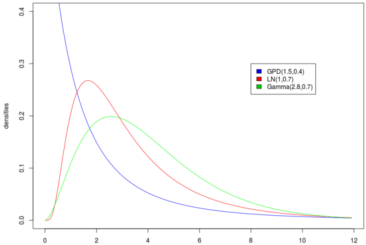
Toys insurance model

- $X_1 \rightsquigarrow \text{GDP}(1.5, 0.4)$
- $X_2 \rightsquigarrow \mathcal{LN}(1, 0.7)$
- $X_3 \rightsquigarrow \Gamma(2.8, 0.7)$

$$Y = X_1 + X_2 + X_3$$

Learning sample of size 15000, 500 trees, estimation of the conditional distributions with sample sizes 1500





Conclusion

- Qantile Oriented Sensitivity Analysis allow to quantify the inputs' impact on a quantile of the model's output, and not only its volatility. Hence it is a significant extension of the sensitivity analysis.
- Estimation is time consuming because of the conditional law.
- Efficient estimating methods are still required.
- Interest of SCOR for application to their Operational Risk Model.
- Dependent inputs, recent works by Mara & Tarantola (2012), Owen & Prieur (2016), Iooss & Prieur (2017), Benoumechiara & Elie-dit-Cosaque (2018) for variance based indices.

References I

Benoumechiara, N. and Elie-Dit-Cosaque, K. (2018).
Shapley effects for sensitivity analysis with dependent inputs: bootstrap and kriging-based algorithms.
arXiv preprint arXiv:1801.03300.

Browne, T., Fort, J.-C., Iooss, B., and Le Gratiet, L. (2017).
Estimate of quantile-oriented sensitivity indices.

Fort, J.-C., Klein, T., and Rachdi, N. (2016).
New sensitivity analysis subordinated to a contrast.
Communications in Statistics-Theory and Methods, 45(15):4349–4364.

Iooss, B. and Prieur, C. (2017).
Shapley effects for sensitivity analysis with dependent inputs: comparisons with sobol'indices, numerical estimation and applications.
arXiv preprint arXiv:1707.01334.

References II

Mara, T. A. and Tarantola, S. (2012).
Variance-based sensitivity indices for models with dependent inputs.
Reliability Engineering & System Safety, 107:115–121.

Marceau, E. (2013).
Modélisation et évaluation quantitative des risques en actuariat: Modèles sur une période.
Springer.

Maume-Deschamps, V. and Niang, I. (2018).
Estimation of quantile oriented sensitivity indices.
Statistics & Probability Letters, 134:122–127.

Owen, A. B. and Prieur, C. (2016).
On shapley value for measuring importance of dependent inputs.
arXiv preprint arXiv:1610.02080.

References III

Saltelli, A., Tarantola, S., Campolongo, F., and Ratto, M. (2004).
Sensitivity analysis in practice: a guide to assessing scientific models.
John Wiley & Sons.

Sobol, I. M. (1993).
Sensitivity estimates for nonlinear mathematical models.
Mathematical Modelling and Computational Experiments, 1(4):407–414.

Thank you

Gracias.

Thank you.