Spatial risk measures

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Plan

Plan

The notion of risk

Lisboa's earthquake in 1755 marks (at least in Europe) the beginning of the concept of risk and random behavior of natural phenomenon.

Nowadays, the notion of risk is strongly related with probabilistic models and occurs in various domains such as:

- environnement (calibration of buildings such as dams, extreme events forecast),
- insurance (claim amonts estimation),
- finance (portefolio's evaluation)
- ...

Risk assessment

Risk is related with a random outcome, Risk can be measured - theory of risk measures, Risk can be managed - theory of decision making under risk.

- Regulatory rules in insurance or finance impose norms on risk assessing.
- Environmental risk assessment in order to minimize the effects of human activities on the environment.
- Decision makers of ecological policy require measures on the environmental risk.

Univariate risk measures

Consider a random variable X on Ω, it may be the wind speed, the temperature, a claim amount.... F_X is its distribution function.
A Risk measure is a function of X, valued in $\mathbb{R}$. The choice of a risk measure depends on the purpose.
First examples:

- Expected value: $\mathbb{E}(X)$ gives information of the mean behavior.
- Variance: $Var = E((X - E(X))^2)$ measures the average deviation of X with respect to its mean.
- Median: $Me = \inf\{t,\ F_X(t) \geq 0.5\}$, this is the value that X should not exceed with probability $\frac{1}{2}$.
- Quantiles: let $\alpha \in]0,1[$, the α-quantile is $q_\alpha = \inf\{t,\ F_X(t) \geq \alpha\}$, this is the value that X should not exceed with probability α.

Univariate risk measures

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- Quantiles: let $\alpha \in]0,1[$, the α-quantile is $q_\alpha = \inf\{t,\ F_X(t) \geq \alpha\}$, this is the value that X should not exceed with probability α.

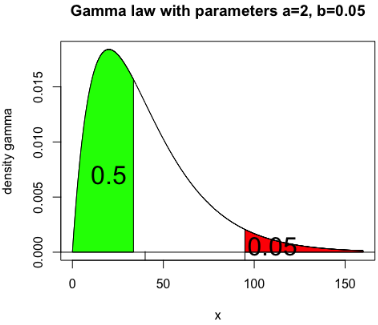
Remark: if $X \rightsquigarrow \mathcal{N}(\mu, \sigma^2)$, the alpha quantile is given by $q_\alpha = \mu + \sigma\phi^{-1}(\alpha)$ with ϕ the distribution function of a $\mathcal{N}(0,1)$ law.

An example

Consider $X \rightsquigarrow \Gamma(a,b)$, i.e. it has density:

$$\frac{b^a}{\Gamma(a)} \mathrm{e}^{-bx} x^{a-1} \mathbb{1}_{\mathbb{R}^+}(x), \quad \mathbb{E}(X) = \frac{a}{b}, \; \mathrm{Var}(X) = \frac{a}{b^2}$$

.



In case $a = 2$, $b = 0.05$, we have

- $\mathbb{E}(X) = 40$,
- $\mathrm{Me} = 33.57$,
- $\mathrm{Var} = 800$,
- $q_{0.95} = 94.88$, compare with $\mu + \sigma\phi^{-1}(0.95) = 86.53$.

Properties of univariate risk measures

A risk measure ρ is:

- invariant by translation if for any $a \in \mathbb{R}$, $\rho(X + a) = \rho(X) + a$. It means that adding a constant risk increases the risk with that constant amount.
- positive homogeneous if for any $a > 0$, $\rho(aX) = a\rho(X)$. The measure is not affected by a change of unity.
- sub-additivite, if for any random variables X and Y, $\rho(X + Y) \leq \rho(X) + \rho(Y)$. Diversification effect.
- a.s. monotone, if $X \leq Y$ a.s. then $\rho(X) \leq \rho(Y)$.

Following Artzen et al (1999), a risk measure is coherent if it satisfies the four above axioms.

Coherent risk measures

$X \rightsquigarrow \mathbb{E}(X)$ and $X \rightsquigarrow \text{Var}(X)$ are coherent.
$X \rightsquigarrow q_\alpha(X)$ is not coherent (it is not sub-additive).

Nevertheless, the quantile function is much used (imposed by regulatory rules in finance / insurance, it is called Value at Risk = VaR), related to return time in environnement.

Remark: $\alpha \rightsquigarrow q_\alpha(X)$ is increasing.

Relationship with return time

Consider an event E whose occurence probability is p, e.g., E is: *the river level is greater than 5m.* If the occurences of E in time are independent, the law of the appearing time of E is

Relationship with return time

Consider an event E whose occurence probability is p, e.g., E is: *the river level is greater than 5m.* If the occurences of E in time are independent, the law of the appearing time of E is geometric with parameter p :

the probability that E appears for the first time after n repetitions is $p(1-p)^{n-1}$ (probability that the river level is above 5m for the first time after n years).
The expectation of this law is

$$\tau = \frac{1}{p},$$

this is the mean time required to observe E, it is called return time of E. In mean, one has to wait $\frac{1}{p}$ years to see the river level above 5m.

Relationship with return time

Consider a 99%-quantile $q_{0.99}$ of a random variable X, E the event *X is above $q_{0.99}$*, $\mathbb{P}(E) = 1\%$ and the associated return time is 100. If the scale time is the year, X exceeds $q_{0.99}$ in mean once each 100 years (centennial flood).

Vectorial / spatial context

Multivariate problematics: $X = (X_1, \ldots, X_d)$ a random vector e.g.

- different lines of business in insurance or finance,
- rainfall amount and duration, flow in case of flood,
- height of waves H, duration of storm D, direction of waves A to study see storms.

The different coordinates may be aggregated $\Rightarrow$ univariate random variable if meaningful (aggregate portfolio, magnitude of storm proportional to $H \times D$...).
In any case a multivariate modelisation is required to take the dependencies into account.
Many ways to define multi-variate risk measure, depend on the purpose.

Vectorial / spatial context

Spatial contexts $\mathcal{S}$ a region of interest. $X(s)$, $s \in \mathcal{S}$ random variable at each location $s \Rightarrow$ spatial process $(X(s))_{s\in\mathcal{S}}$, e.g.

- precipitation at each location,
- temperature...

Stationary spatial processes:

$$(X(s_1), \dots, X(s_k)) \stackrel{\mathcal{L}}{=} (X(s_1 + h), \dots, X(s_k + h))$$

for any $s_i \in \mathcal{S}$, $i = 1, \dots, k$ and h with $s_i + h \in \mathcal{S}$.

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Models for the dependence structure, with

- Covariogram: $\mathrm{Cov}(X(s), X(s+h)) = \gamma(h)$, depends only on $\|h\|$ in the isotropic case.
- Bivariate exponent measure V_h: $\mathbb{P}(X(s) \leq x, X(s+h) \leq y) = e^{-V_h(x,y)}$.

Our purpose

Considering a spatial process $(X(s))_{s \in \mathcal{S}}$ and an area $\mathcal{A} \subset \mathcal{S}$,

- define a risk measure $\mathcal{R}(\mathcal{A}, X)$,
- study its axiomatic properties.

For different spatial processes Gaussian processes, max-stable processes, max-mixture processes,

- compute $\mathcal{R}(\mathcal{A}, X)$,
- study the behavior of $\mathcal{R}(\mathcal{A}, X)$,
- evaluate the impact of the dependence structure on the risk measure.

Plan

1. Introduction
2. Spatial risk measures
   - Definitions
   - Axiomatic properties
   - General form
3. Gaussian processes
4. Max-stable and max-mixture processes
5. Conclusion

Normalized loss function

Previous work from Keef *et al.* (2009) or Koch (2015) where the risk is evaluated by the expectation of an integrated loss function. Let $\mathcal{D}_X$ be a positive function of X called a damage function e.g. $\mathcal{D}_{X,u} = (X-u)^+$ or $\mathcal{D}_X^\nu = X^\nu$.

Definition (Normalized loss function)

Let $\mathcal{A} \subset \mathcal{S}$,

$$L(\mathcal{A}, \mathcal{D}_X) = \frac{1}{|\mathcal{A}|} \int_{\mathcal{A}} \mathcal{D}_X(s) \quad \mathrm{d}s.$$

Spatial risk measure

Spatial risk measure composed from two components: the expectation and variance of the normalized loss,

$$\begin{aligned} \mathcal{R}(\mathcal{A}, \mathcal{D}_X) &= \{\mathbb{E}[L(\mathcal{A}, \mathcal{D}_X)], \mathrm{Var}\big(L(\mathcal{A}, \mathcal{D}_X)\big)\}, \\ &=: \{\mathcal{R}_0(\mathcal{A}, \mathcal{D}_X), \mathcal{R}_1(\mathcal{A}, \mathcal{D}_X)\} \end{aligned}$$

For stationary processes, $\mathbb{E}[L(\mathcal{A}, \mathcal{D}_X)]$ gives informations on the severity of the phenomenon.

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$\mathbb{E}[L(\mathcal{A},\mathcal{D}_X)] = \frac{1}{|\mathcal{A}|}\int_{\mathcal{A}} \mathbb{E}(\mathcal{D}_X(s))ds = \mathbb{E}(\mathcal{D}_X(s))$ does not depend on $\mathcal{A}$.

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For stationary processes, $\mathbb{E}[L(\mathcal{A}, \mathcal{D}_X)]$ gives informations on the severity of the phenomenon.
$\mathrm{Var}\big(L(\mathcal{A}, \mathcal{D}_X)\big)$ is impacted by the dependence structure.
Remark:

$$\mathrm{Var}\big(L(\mathcal{A}, \mathcal{D}_X)\big) = \frac{1}{|\mathcal{A}|^2} \int_{\mathcal{A}\times\mathcal{A}} \mathrm{Cov}\big(\mathcal{D}_X(s), \mathcal{D}_X(t)\big) \mathrm{d}s\mathrm{d}t.$$

Axioms for risk measures

Natural axioms for risk measures (extension of coherence axioms by Artzner *et al.* (1999) and the work by Koch (2015)).
$\mathcal{A}$, $\mathcal{A}_1$, $\mathcal{A}_2$ subsets of $\mathcal{S}$.

1. Invariance by translation. Let $v \in \mathcal{S}$, $\mathcal{R}(\mathcal{A} + v, \mathcal{D}) = \mathcal{R}(\mathcal{A}, \mathcal{D})$.
2. Anti-monotonicity If $|\mathcal{A}_1| \leq |\mathcal{A}_2|$, then $\mathcal{R}(\mathcal{A}_2, \mathcal{D}) \leq \mathcal{R}(\mathcal{A}_1, \mathcal{D})$.
3. Sub-additivity If $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$, then $\mathcal{R}(\mathcal{A}_1 \cup \mathcal{A}_2, \mathcal{D}) \leq \mathcal{R}(\mathcal{A}_1, \mathcal{D}) + \mathcal{R}(\mathcal{A}_2, \mathcal{D})$.
4. Super sub-additivity If $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$, then $\mathcal{R}(\mathcal{A}_1 \cup \mathcal{A}_2, \mathcal{D}) \leq \min_{i=1,2} [\mathcal{R}(\mathcal{A}_i, \mathcal{D})]$.

Axiomatic properties of $\mathcal{R}_1(\mathcal{A}, \mathcal{D})$

Invariance by translation directly follows from the stationarity:

$$\begin{aligned}
\mathcal{R}_1(\mathcal{A}, \mathcal{D}) &= \frac{1}{|\mathcal{A}|^2} \int_{\mathcal{A}\times\mathcal{A}} \mathrm{Cov}\big(\mathcal{D}_X(s), \mathcal{D}_X(t)\big)\mathrm{d}s\mathrm{d}t \\
&= \frac{1}{|\mathcal{A}|^2} \int_{(\mathcal{A}+v)\times(\mathcal{A}+v)} \mathrm{Cov}\big(\mathcal{D}_X(s+v), \mathcal{D}_X(t+v)\big)\mathrm{d}s\mathrm{d}t
\end{aligned}$$

(change of variable)

$$= \frac{1}{|\mathcal{A}+v|^2} \int_{(\mathcal{A}+v)\times(\mathcal{A}+v)} \mathrm{Cov}\big(\mathcal{D}_X(s), \mathcal{D}_X(t)\big)\mathrm{d}s\mathrm{d}t$$

by stationarity $(X(s), X(t)) \overset{\mathcal{L}}{=} (X(s+v), X(t+v))$

$$= \mathcal{R}_1(\mathcal{A}+v, \mathcal{D}).$$

Axiomatic properties of $\mathcal{R}_1(\mathcal{A}, \mathcal{D})$

Sub-additivity follows from Cauchy-Schwarz inequality:

$$\mathcal{R}_1(\mathcal{A}_1 \cup \mathcal{A}_2, \mathcal{D}_X) = \mathrm{Var}\big(L(\mathcal{A}_1 \cup \mathcal{A}_2, \mathcal{D}_X)\big)$$

$$= \frac{1}{(|\mathcal{A}_1| + |\mathcal{A}_2|)^2} \Bigg[|\mathcal{A}_1|^2 \mathcal{R}_1(\mathcal{A}_1, \mathcal{D}_X) + |\mathcal{A}_2|^2 \mathcal{R}_1(\mathcal{A}_2, \mathcal{D}_X) + 2\mathrm{Cov}\left(\int_{\mathcal{A}_1} \mathcal{D}_X(s)ds, \int_{\mathcal{A}_2} \mathcal{D}_X(s)ds \right) \Bigg].$$

$$\leq \frac{1}{(|\mathcal{A}_1| + |\mathcal{A}_2|)^2} \Big[|\mathcal{A}_1|^2 \mathcal{R}_1(\mathcal{A}_1, \mathcal{D}_X) + |\mathcal{A}_2|^2 \mathcal{R}_1(\mathcal{A}_2, \mathcal{D}_X) + 2|\mathcal{A}_1||\mathcal{A}_2| \sqrt{\mathcal{R}_1(\mathcal{A}_1, \mathcal{D}_X)} \sqrt{\mathcal{R}_1(\mathcal{A}_2, \mathcal{D}_X)} \Big]$$

by using the Cauchy-Schwarz inequality,

$$\leq \mathcal{R}_1(\mathcal{A}_1, \mathcal{D}_X) + \mathcal{R}_1(\mathcal{A}_2, \mathcal{D}_X).$$

Axiomatic properties of $\mathcal{R}_1(\mathcal{A}, \mathcal{D})$

Anti-monotonicity (equivalent to super sub-additivity) is more difficult to get.
May be obtained for specific models and for $\mathcal{A}$ either a disk or a square.

From now one, we consider isotropic processes.

If $\mathcal{A}$ is either a disk or a square

If $\mathcal{A}$ is either a disk or a square, $\mathcal{R}_1(\mathcal{A}, \mathcal{D}_X)$ rewrites:

When $\mathcal{A}$ is a disk of radius R

$$\mathcal{R}_1(\mathcal{A}, \mathcal{D}_X) = \int_{h=0}^{2R} f_{disk}(h, R)\mathrm{Cov}\left(\mathcal{D}_X(s), \mathcal{D}_X(s+h)\right) \quad \mathrm{d}h.$$

Where $f_{disk}(h, R)$ is the density of distance between two points uniformly drawn on a disk (see Moltchanov (2012)), that is

$$f_{disk}(h, R) = \frac{2h}{R^2}\left(\frac{2}{\pi}\mathrm{acos}(\frac{h}{2R}) - \frac{h}{\pi R}\sqrt{1 - \frac{h^2}{4R^2}}\right).$$

If $\mathcal{A}$ is either a disk or a square

If $\mathcal{A}$ is either a disk or a square, $\mathcal{R}_1(\mathcal{A}, \mathcal{D}_X)$ rewrites:
When $\mathcal{A}$ is a square with side R

$$\mathcal{R}_1(\mathcal{A}, \mathcal{D}_X) = \int_{h=0}^{\sqrt{2}R} f_{square}(h, R) \mathrm{Cov}\big(\mathcal{D}_X(s), \mathcal{D}_X(s+h)\big) \quad \mathrm{d}h.$$

Where $f_{square}(h, R)$ is given by: for $h \in [0, R]$,

$f_{square}(h, R) = \frac{2\pi h}{R^2} - \frac{8h^2}{R^3} + \frac{2h^3}{R^4}$, and for $h \in [R, \sqrt{2}R]$, let $b = \frac{h^2}{R^2}$

$$\begin{aligned} f_{square}(h, R) &= \frac{2h}{R^2} \Bigg\{ -2 - b + 3\sqrt{b-1} + \frac{b+1}{\sqrt{b-1}} \\ &\quad + 2\arcsin\Big(\frac{2-b}{b}\Big) - \frac{4}{b\sqrt{1 - \frac{(2-b)^2}{b^2}}} \Bigg\}, \end{aligned}$$

If $\mathcal{A}$ is either a disk or a square

These two formulas show that if you can compute $\mathrm{Cov}\left(\mathcal{D}_X(s), \mathcal{D}_X(s+h)\right)$, then the risk measure reduces to a one dimensional integration.
This can be achieved in some cases.

Plan

1. Introduction
2. Spatial risk measures
3. Gaussian processes
   - Explicit forms
   - Application to environmental data
4. Max-stable and max-mixture processes
5. Conclusion

Gaussian processes

The process $(X(s))_{s\in\mathcal{S}}$ is a Gaussian process if and only if, for any any $d \in \mathbb{N}$, $(s_1, \dots, s_d) \in \mathcal{S}^d$, the random vector $(X(s_1), \dots, X(s_d))$ is a Gaussian vector.

$\Rightarrow$ a stationary Gaussian process is determined by:

- $\mu = \mathbb{E}(X(s))$, $\sigma^2 = \mathrm{Var}(X(s))$,
- the covariance structure: $\gamma(h) = \mathrm{Cov}(X(s), X(s+h))$ or equivalently the correlation function $\rho(h) = \mathrm{Corr}(X(s), X(s+h))$.

In what follows, φ is the density of a standard normal law and Φ its distribution function, $\overline{\Phi} = 1 - \Phi$ its survival distribution function.

The excess risk measure

Consider a fixed threshold u, $\mathcal{D}_{X,u} = (X - u)^+$.
This means that $\mathcal{R}_1(\mathcal{A}, \mathcal{D}^+_{X,u})$ is the variance of the average of X over the threshold u on the area A.
We consider a standard Gaussian process (i.e. $\mu = 0$ and $\sigma = 1$), with positive auto-correlation function ρ, a simple calculation gives:

$$\mathcal{R}_0(\mathcal{A}, \mathcal{D}^+_{X,u}) = \mathbb{E}(L(\mathcal{A}, \mathcal{D}^+_{X,u})) = \varphi(u) - u(\overline{\Phi}(u).$$

The excess risk measure

The variance of $L(\mathcal{A}, \mathcal{D}^+_{X,u})$ may be obtained by using results from Rosenbaum (1961) on moments of truncated bivariate normal distributions. If $\mathcal{A}$ is a disk,

$$\mathcal{R}_1(\mathcal{A}, \mathcal{D}^+_{X,u}) = \mathrm{Var}(L(\mathcal{A}, \mathcal{D}^+_{X,u})) = \int_{h=0}^{2R} f_{disk}(h, R)\mathcal{G}(h, u)\mathrm{d}h.,$$

with

$$\mathcal{G}(h, u) = \ell\left(u, u, \rho(h)\right)\left(\rho(h) + u^2\right) - 2u\varphi(u)\overline{\Phi}\left(\frac{u(1-\rho(h))}{(1-\rho^2(h))^{1/2}}\right) + \left(1-\rho^2(h)\right)^{1/2}\varphi\left(\frac{u}{(1+\rho(h))^{1/2}}\right)^2 - \left(\varphi(u) - u\overline{\Phi}(u)\right)^2;$$

and $\ell\left(u, v, \rho(h)\right)$ is the total probability of a truncated bivariate standard normal distribution with correlation function ρ.

$$\ell\left(u, v, \rho(h)\right) = \frac{1}{2\pi(1-\rho^2(h))^{1/2}} \int_u^\infty \int_v^\infty e^{\left\{\frac{-1}{2(1-\rho(h))^2}[x^2 - 2\rho(h)xy + y^2]\right\}} \mathrm{d}x\mathrm{d}y.$$

The excess risk measure

Change of variables to get the risk measure for general isotropic Gaussian processes. Let Y be an isotropic Gaussian process with mean μ and variance σ^2. $X = \frac{Y-\mu}{\sigma}$ is an isotropic and standard Gaussian process.

Corollary

The spatial risk measure $\mathbb{R}(\mathcal{A}, \mathcal{D}^+_{Y,u})$ *statisfies*

$$\mathcal{R}(\mathcal{A}, \mathcal{D}^+_{Y,u}) = \left\{ \sigma \mathbb{E}[L(\mathcal{A}, \mathcal{D}^+_{X,u_0})], \sigma^2 \mathrm{Var}\left(L(\mathcal{A}, \mathcal{D}^+_{X,u_0})\right) \right\},$$

with $u_0 = (u - \mu)/\sigma$.

Behaviour of $\mathcal{R}_1(\mathcal{A}, \mathcal{D}^+_{X,u})$

The previous formula provides the behavior of $\lambda \rightsquigarrow \mathcal{R}_1(\lambda\mathcal{A}, \mathcal{D}^+_{X,u})$ and it implies anti-monotonicity for disk or square.

Corollary

Let X be an isotropic Gaussian process with auto-correlation function ρ. Let $\mathcal{A} \subset \mathcal{S}$ be either a disk or a square.
The mapping $\lambda \mapsto \mathcal{R}_1(\lambda\mathcal{A}, \mathcal{D}^+_{X,u})$ is non-increasing if and only if $h \mapsto \rho(h)$, $h > 0$ is non-increasing and non-negative.
If $h \mapsto \rho(h)$ is decreasing to 0 as h goes to infinity,

$$\lim_{\lambda \to \infty} \mathcal{R}_1(\lambda\mathcal{A}, \mathcal{D}^+_{X,u}) = 0.$$

Let $\mathcal{A}_1$, $\mathcal{A}_2$ be either squares or disks with $|\mathcal{A}_1| \leq |\mathcal{A}_2|$ then

$$\mathcal{R}_1(\mathcal{A}_2, \mathcal{D}^+_{X,u}) \leq \mathcal{R}_1(\mathcal{A}_1, \mathcal{D}^+_{X,u}).$$

Behavior for different correlation functions

Gaussian processes behave differently, according to their correlation function $h \rightsquigarrow \rho(h)$. Five Gaussian models

1. Spherical correlation function:

$$\rho_\theta^{sph}(h) = \left[1 - 1.5\left(\frac{h}{\theta}\right) + \frac{1}{2}\left(\frac{h}{\theta}\right)^3\right]\mathbf{1}_{\{h>\theta\}}.$$

2. Cubic correlation function :

$$\rho_\theta^{cub}(h) = \left[1-7\left(\frac{h}{\theta}\right)+\frac{35}{2}\left(\frac{h}{\theta}\right)^2-\frac{7}{2}\left(\frac{h}{\theta}\right)^5+\frac{3}{5}\left(\frac{h}{\theta}\right)^7\right]\mathbf{1}_{\{h>\theta\}}.$$

3. Exponential correlation functions:

$$\rho_\theta^{exp}(h) = \exp\left[-\frac{h}{\theta}\right],$$

4. Gaussian correlation functions:
5. Matérn correlation function:

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1. Spherical correlation function:
2. Cubic correlation function :
3. Exponential correlation functions:
4. Gaussian correlation functions:

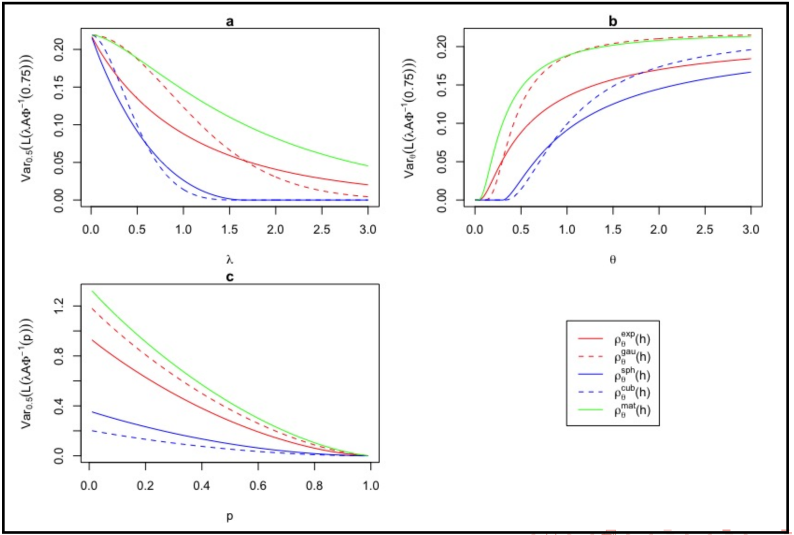
$$\rho_{\theta}^{gau}(h) = \exp\left[-\left(\frac{h}{\theta}\right)^2\right];$$

5. Matérn correlation function:

$$\rho^{mat}(h) = \frac{1}{\Gamma(\kappa)2^{\kappa-1}}(h/\theta)^{\kappa}K_{\kappa}(h/\theta),$$

where Γ is the gamma function, K_{κ} is the modified Bessel function of second kind and order $\kappa > 0$, κ is a smoothness parameter and θ is a scaling parameter.

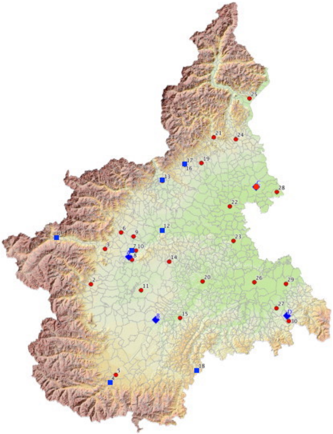
Behavior for different correlation functions



Environmental data

Data on pollution in Piemonte data, measured by the concentration in PM_{10}. The observed values of PM_{10} are frequently larger than the legal level fixed by the European directive $2008/50/EC$.

The log of PM_{10} has been fitted on an isotropic Gaussian process with Matérn auto-correlation function (previous work from Bande *et al.* 2006), with parameters $\kappa = 1$, $\theta = 100$, $\mu = 3.69$ and $\sigma^2 = 1.2762$.



Risk measure for the PM concentration

$$\left(\mathcal{R}_0(\mathcal{A},\mathcal{D}^+_{Y,\log u}),\mathcal{R}_1(\mathcal{A},\mathcal{D}^+_{Y,\log u})\right),$$

with $Y = \log PM_{10}$, $\mathcal{A}$ a square of side 10km and u the legal level, i.e. $u = 50$.

$$\mathcal{R}_0(\mathcal{A},\mathcal{D}^+_{Y,\log u}) = \mathbb{E}(L(\mathcal{A},\mathcal{D}^+_{Y,\log u})) = 0.3483621,$$

$$\mathcal{R}_1(\mathcal{A},\mathcal{D}^+_{Y,\log u}) = \mathrm{Var}(L(\mathcal{A},\mathcal{D}^+_{Y,\log u})) = 0.4119461.$$

$L(\mathcal{A},\mathcal{D}^+_{Y,\log u})$ is the average over the square $\mathcal{A}$ of the values of Y that exceed the legal threshold $\log u$.
The standard deviation of $L(\mathcal{A},\mathcal{D}^+_{Y,\log u})$ (~ 0.6) is large with respect to its expectation $\Rightarrow$ the dependence structure of the underlying process highly impacts the random variable $L(\mathcal{A},\mathcal{D}^+_{Y,\log u})$.

Plan

Max-stable spatial processes

Gaussian processes not well suited for e.g. rainfall, wind... $\Rightarrow$ max-stable processes, unit Fréchet margins, dependence structure given by the exponent measure function V, that is:

$$\mathbb{P}(X(s) \leq x) = e^{-\frac{1}{x}},\ \mathbb{P}\big(X(s) \leq x_1, X(t) \leq x_2\big) = \exp(-V_{s,t}(x_1, x_2)).$$

The process is isotopic if $V_{s,t}(x_1, x_2)$ depends only on $h = \|t - s\|$.

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Max-stable processes are Asymptotically Dependent in the sense that either $X(s)$ and $X(s+h)$ are independent or

$$\chi(h) = \lim_{u \to 1} \mathbb{P}\big(F(X(s)) > u | F(X(s+h)) > u\big) > 0.$$

See the course by Jean-Noël Bacro for more details.

Examples of max-stable processes

Smith Model (Gaussian extreme value model)

$$V_h(x_1, x_2) = \frac{1}{x_1}\Phi\left(\frac{\tau(h)}{2} + \frac{1}{\tau(h)}\log\frac{x_2}{x_1}\right) + \frac{1}{x_2}\Phi\left(\frac{\tau(h)}{2} + \frac{1}{\tau(h)}\log\frac{x_1}{x_2}\right);$$

$\tau(h) = \sqrt{h^T \Sigma^{-1} h}$ and $\Phi(\cdot)$ the standard normal cumulative distribution function.

Schlather Models (Extremal Gaussian Model)

$$V_h(x_1, x_2) = \frac{1}{2}\left(\frac{1}{x_1} + \frac{1}{x_2}\right)\left[1 + \sqrt{1 - 2(\rho(h) + 1)\frac{x_1 x_2}{(x_1 + x_2)^2}}\right].$$

Inverse max-stable processes

Let X' be a max-stable process as above margin, consider

$$X(s) = g(X'(s)) = -\frac{1}{\log\{1 - e^{-1/X'(s)}\}} \quad s \in \mathcal{S}.$$

Then X has unit Fréchet margin and bivariate survivor function

$$\mathbb{P}(X(s_1) > x_1, X(s+h) > x_2) = \exp\big(-V_h(g(x_1), g(x_2))\big).$$

Defined by Ledford and Tawn.
Inverse max-stable processes are Asymptotically Independent in the sense that $\chi(h) = 0$ for any h.

Max-mixture processes

Wadsworth and Tawn proposed to mix max-stable and inverse max-stable processes, studied also by Bacro *et al.*: Let X be a max-stable process, with exponent measure function V_h^X. Let Y be an inverse max-stable process with and exponent measure function V_h^Y. Let $a \in [0,1]$ and define

$$Z(s) = \max\{aX(s), (1-a)Y(s)\}, \quad s \in \mathcal{S}.$$

Z has unit Fréchet marginals. Its bivariate distribution function is given by $\mathbb{P}\left(Z(s) \leq z_1, Z(s+h) \leq z_2\right) =$

$$e^{-aV_h^X(z_1,z_2)}\left[e^{\frac{-(1-a)}{z_1}} + e^{\frac{-(1-a)}{z_2}} - 1 + e^{-V_h^Y(g_a(z_1),g_a(z_2))}\right],$$

where $g_a(z) = g(\frac{z}{1-a})$.

Damage function

We shall consider the damage function

$$\mathcal{D}_X^{\nu}(s) = |X(s)|^{\nu},$$

for $0 < \nu < \frac{1}{2}$ (so that the order two moment exists).
Used e.g. in analyzing the negative effects due to the wind speed.
Koch 2015, computed the risk measure associated to $\mathcal{D}_X^{\nu}$ for Smith processes.

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Properties of moments of Fréchet distributions give that is X as unit Fréchet marginal distributions,

$$\mathbb{E}(L(\mathcal{A}, \mathcal{D}_X^{\nu}) = \Gamma(1 - \nu).$$

Computation of the risk measure

Computation using the formula when $\mathcal{A}$ is a square:

$$\mathcal{R}_1(\mathcal{A}, \mathcal{D}_X^{\nu}) = \int_{h=0}^{\sqrt{2}R} \mathcal{Q}(h,\nu) f_{square}(h,R)\mathrm{d}h,$$

with

$$\mathcal{Q}(h,\nu) = \nu^2 \int_0^{\infty}\int_0^{\infty} x_1^{\nu-1}x_2^{\nu-1}\left[G_h^X(x_1,x_2) - F(x_1)F(x_2)\right]\mathrm{d}x_1\mathrm{d}x_2$$

and $G_h^X = \mathbb{P}(X(s) \leq x_1, X(s+h) \leq x_2)$.

Properties of the risk measure

The asymptotic dependence properties are reflected in the risk measure.

Property

Let Z be an isotropic and stationary max-mixture spatial process. Assume that the mappings $h \mapsto V_h^X(x_1, x_2)$ and $V_h^Y(x_1, x_2)$ are non decreasing for any $(x_1, x_2) \in \mathbb{R}_+^2$. Moreover, we assume that

$$V_h^X(x_1; x_2) \longrightarrow \frac{1}{x_1} + \frac{1}{x_2} \text{ and } V_h^Y(x_1, x_2) \longrightarrow \frac{1}{x_1} + \frac{1}{x_2} \quad \text{as} \quad h \to \infty$$

$\forall x_1, x_2 \in \mathbb{R}_+$. Let $\mathcal{A} \subset \mathcal{S}$ be either a disk or a square,

$$\lim_{\lambda \to \infty} \mathcal{R}_1(\lambda \mathcal{A}, \mathcal{D}_Z^{\nu}) = 0.$$

Properties of the risk measure

The asymptotic dependence properties are reflected in the risk measure.

Property

Let Z be an isotropic and stationary max-mixture spatial process. Assume that the mappings $h \mapsto V_h^X(x_1, x_2)$ and $V_h^Y(x_1, x_2)$ are non decreasing for any $(x_1, x_2) \in \mathbb{R}_+^2$. If there exists V_0 (resp. V_1) an exponent measure function of a non independent max-stable (resp. inverse max-stable) bivariate random vector, such that $V_h^X \longrightarrow V_0$ and $V_h^Y \longrightarrow V_1$ as $h \to \infty$, then

$$\lim_{\lambda \to \infty} \mathcal{R}_1(\lambda \mathcal{A}, \mathcal{D}_Z^{\nu}) > 0.$$

Plan

Conclusion

- We have propose several risk measures for different spatial processes. These spatial risk measures take into account the spatial dependence structure of the processes.
- We have provided computational tools to calculate these measures.
- Risk measure might be useful for detection / attribution purposes.
- Renormalization of $L(\lambda\mathcal{A}, \mathcal{D}_Z^{\nu})$ so that it converges to a non trivial distribution ?

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Thank you

Gracias por vuestra atención.

Thank you for your attention.

Merci pour votre attention.