On the estimation of aggregated quantiles with marginal and dependence informations.

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Plan

General problematic

$(X_1, \ldots, X_d)$ random vector of risks. Write

$$S = \sum_{i=1}^{d} X_i, \text{ the aggregated risk.}$$

Regulatory rules, Risk management purposes, Environmental risks ... $\Longrightarrow$ need to estimate / approximate high level quantiles of S:

$$F_S^{-1}(\alpha) = \mathrm{VaR}_\alpha(S), \text{ for } \alpha \text{ near } 1,$$

where F_S is the distribution function of S.

Why we consider quantiles?

If X is a random variable, its distribution function is

$$F_X(t) = \mathbb{P}(X \leq t).$$

F_X^{-1} is the generalized inverse of F_X or the quantile function: for $\alpha \in]0,1[$,

$$F_X^{-1}(\alpha) = \inf\{t \in \mathbb{R},\ F_X(t) \geq \alpha\}.$$

If X is a continuous random variable, then $\mathbb{P}(X \leq u) = \alpha$ if $u = F_X^{-1}(\alpha)$.
So, quantiles give a thresholds which X may exceed with probability $1 - \alpha$.

Examples

- Insurance: X describes the distribution of the claim amonts, regulatory rules impose to insurance companies to estimate $F_X(\alpha)$ for α near to 1. The quantiles $F_X^{-1}(\alpha)$ are called Value at Risk and denoted $\mathrm{VaR}_\alpha(X)$.
- Hydrology: X may describe a flood level. Computing $F_X^{-1}(\alpha)$ is required to calibrate a barage e.g. (or a dam).
- Many other field: finance, wind electricity...

Our purpose

$(X_1, \ldots, X_d)$ random vector of risks.
The X_i may be different lines of business in insurance contexts.

$$S = \sum_{i=1}^{d} X_i.$$

$\Longrightarrow$ Estimation of $\mathrm{VaR}_\alpha(S)$.
The law of S (and thus $\mathrm{VaR}_\alpha(S)$) depends on the law of $(X_1, \ldots, X_d)$ (marginal laws and dependence structure).

Toy example

X_1 and X_2 are normally distributed ($\mathcal{N}(0,1)$) with different dependence structures:

- X_1 and X_2 are independent, $S_1 = X_1 + X_2 \rightsquigarrow \mathcal{N}(0,2)$.
- $X_1 = X_2$ (perfect dependence), $S_2 = X_1 + X_2 \rightsquigarrow \mathcal{N}(0,4)$.
- $X = (X_1, X_2)$ is a gaussian vecteur with correlation 0.5 (moderate dependence), $S_2 = X_1 + X_2 \rightsquigarrow \mathcal{N}(0,3)$.

Toy example

Quantiles at different levels for the three models.

α	0.7	0.9	0.95	0.99	0.995
Quantiles for S_1	0.74	1.81	2.33	3.29	3.64
Quantiles for S_2	1.05	2.56	3.29	4.65	5.15
Quantiles for S_3	0.91	2.22	2.85	4.03	4.46

High quantiles of aggregated risks

- High dimensional problem (d may be large),
- Marginal laws (laws of the X_i's) are usually known (or well estimated), some information on the dependence is available,
- Even if the law of $(X_1, \ldots, X_d)$ is known, the effective computation of

$$\mathrm{VaR}_\alpha(S), \text{ for } \alpha \text{ near } 1,$$

may be difficult to do,

High quantiles of aggregated risks

- Even if the law of $(X_1, \ldots, X_d)$ is known, the effective computation of

$$\mathrm{VaR}_\alpha(S), \quad \text{for } \alpha \text{ near } 1,$$

may be difficult to do,
the distribution function of S is given by:

$$F_S(t) = \int_{\mathbb{R}^d} \mathbf{1}_{\{x_1+\cdots+x_d \leq t\}} f_X(x_1, \ldots, x_d) dx_1 \ldots dx_d.$$

$\Longrightarrow$ Efficient methods are still welcome.

One proposition

Assume that the X_i's laws are known.
Information on the dependence is available through

- a (quite small) $(X_1, \ldots, X_d)$ sample and
- some expert opinion (e.g the dependence structure between X_1 and X_2 is completely known) and / or
- some knowledge of the join tail ($\mathbb{P}(X_1 \geq u_1, \ldots, X_d \geq u_d)$ is known for some $(u_1, \ldots, u_d)$).

We use checkerboard copulas to estimate $\mathrm{VaR}_\alpha(S)$.

Copulas

Recall that if F is the distribution function of $X = (X_1, \ldots, X_d)$, Sklar's Theorem implies that there exists a distribution function C in $[0,1]^d$ whose marginal laws are uniformely distributed on $[0,1]$, such that

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)),$$

where F_i is the distribution function of X_i and F is the distribution function of the vector X.

If the marginals of X are absolutely continuous then C is unique. It is the copula associated to X.

Modeling dependence with copulas

Below are some simple examples of copulas.

- Independent copula: $C(u_1, \ldots, u_d) = u_1 \times \cdots \times u_d$, $u_i \in [0, 1]$. If $X_1, \ldots, X_d$ are independent then $F(x_1, \ldots, x_d) = F_1(x_1) \times \cdots \times F_d(x_d)$.

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- Comonotonic copula: $C(u_1, \ldots, u_d) = \min(u_1, \ldots, u_d)$, $u_i \in [0,1]$. The X_i are comonotonic if there exists increasing functions f_i such that $X_i = f_i(U)$ with $U \rightsquigarrow [0,1]$, in that case, $F(x_1, \ldots, x_d) = \min(f_i^{-1}(x_i))$.

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- Clayton copula: for $\theta > 0$,
  $C_\theta(u_1, \ldots, u_d) = \left(u_1^{-\frac{1}{\theta}} + \cdots + u_d^{-\frac{1}{\theta}} - (d-1)\right)^{-\theta}$. Useful for strong dependence for $u_i \sim 0$.

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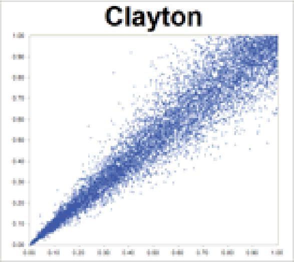

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- Survival Clayton copula (dual of the Clayton copula): for $\theta > 0$, $C_\theta^*(u_1, \ldots, u_d) = \mathbb{P}(U_1 > 1 - u_1, \ldots, U_d > 1 - u_d)$ with $(U_1, \ldots, U_d)$ having C_θ as distribution function. Useful for strong dependence for $u_i \sim 1$.

Useful property on copulas

Lemma

If C is a distribution function on $[0,1]^d$, then C is a copula if and only if $C(x) = x_k$ for all $x \in [0,1]^d$ with $x_i = 1$, $i \neq k$.

The condition above is necessary and sufficient to have uniform marginal laws.

Plan

The checkerboard copula: definition

The cherckerboard copula, introduced by Mikusinski (2010) is an approximation of a copula C.

μ is the probability measure associated to C on $[0,1]^d$:

$$\mu([0,x]) = C(x), x = (x_1, \dots, x_d) \in [0,1]^d,\ [0,x] = \prod_{i=1}^{d} [0, x_i].$$

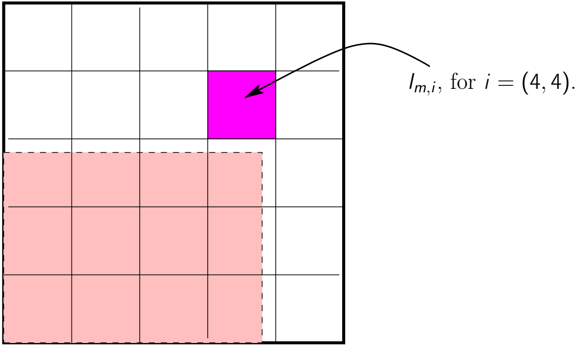
Consider $(I_{i,m})_{i \in \{1,\dots m\}^d}$ the partition (modulo a 0 measure set) of $[0,1]^d$ given by the m^d squares:

$$I_{i,m} = \prod_{j=1}^{d} \left[\frac{i_j - 1}{m}, \frac{i_j}{m} \right],\ i = (i_1, \dots, i_d).$$

The checkerboard copula: definition

The cherckerboard copula of order m is defined on $[0,1]^d$ by: (λ is the Lebesgue measure)

$$C^*_m(x) = \sum_i m^d \mu(I_{i,m}) \lambda([0,x] \cap I_{i,m}).$$



The checkerboard copula: definition

The cherckerboard copula of order m is defined on $[0,1]^d$ by: (λ is the Lebesgue measure)

$$C_m^*(x) = \sum_i m^d \mu(I_{i,m}) \lambda([0,x] \cap I_{i,m}).$$

From a probabilistic point of view,

$$C_m^*(x) = \sum_i \mu(I_{i,m}) \mathbb{P}(U \leq x | U \in I_{i,m}).$$

with U a random vector of $\mathbb{R}^d$ of i.i.d. uniform laws on $[0,1]$.

Approximation by the checkerboard copula

Proposition

C_m^* is a copula which approximate C:

$$\sup_{x \in [0,1]^d} |C_m^*(x) - C(x)| \leq \frac{d}{2m}.$$

Proof:

To prove that C_m^* is a copula, it suffices to notice that $C_m^*(x) = x_k$ if $x_j = 1$ for $j \neq k$, this is an simple computation.

Approximation by the checkerboard copula

Proposition

C_m^* is a copula which approximate C:

$$\sup_{x\in[0,1]^d} |C_m^*(x) - C(x)| \leq \frac{d}{2m}.$$

Proof:

For any $x \in [0,1]^d$ with $x = \frac{i}{m}$, $i \in \{1,\ldots,m\}^d$, $C_m^*(x) = C(x)$.
For $a \in \{1,\ldots,m\}$ and $k \in \{1,\ldots,d\}$,

$$B_a^{k+} = \left\{x \in [0,1]^d,\ \frac{a}{m} - \frac{1}{2m} < x_k \leq \frac{a}{m}\right\} \text{ and}$$

$$B_a^{k-} = \left\{x \in [0,1]^d,\ \frac{a-1}{m} < x_k \leq \frac{a}{m} - \frac{1}{2m}\right\}.$$

Approximation by the checkerboard copula

Proposition

C^*_m is a copula which approximate C:

$$\sup_{x\in[0,1]^d} |C^*_m(x) - C(x)| \leq \frac{d}{2m}.$$

Proof:

Denote by μ^*_m the probability measure on $[0,1]^d$, associated to C^*_m.
If $x \in I_{i,m}$, $i = (1_1, \ldots, i_d)$ then,

$$|C^*_m(x) - C(x)| \leq \sum_{k=1}^{d} |\mu^*_m(B^{k-}_{i_k}) - \lambda(B^{k-}_{i_k})| \mathbf{1}_{B^{k-}_{i_k}}(x) + \sum_{k=1}^{d} |\mu^*_m(B^{k+}_{i_k}) - \lambda(B^{k+}_{i_k})| \mathbf{1}_{B^{k+}_{i_k}}(x)$$

Approximation by the checkerboard copula

Proposition

C_m^* is a copula which approximate C:

$$\sup_{x\in[0,1]^d} |C_m^*(x) - C(x)| \leq \frac{d}{2m}.$$

Proof:

$$\mu_m^*(B_{i_k}^{k-}) = \lambda(B_{i_k}^{k-}) = \mu_m^*(B_{i_k}^{k+}) = \lambda(B_{i_k}^{k+}) = \frac{1}{2m}.$$

Approximation by the checkerboard copula

Proposition

C_m^* is a copula which approximate C:

$$\sup_{x\in[0,1]^d} |C_m^*(x) - C(x)| \leq \frac{d}{2m}.$$

Proof:

$$\begin{aligned}|C_m^*(x) - C(x)| &\leq \sum_{k=1}^{d} \min(\mu_m^*(B_{i_k}^{k-}), \lambda(B_{i_k}^{k-}))\mathbf{1}_{B_{i_k}^{k-}}(x) + \\ &\sum_{k=1}^{d} \min(\mu_m^*(B_{i_k}^{k+}), \lambda(B_{i_k}^{k+}))\mathbf{1}_{B_{i_k}^{k+}}(x) = \frac{d}{2m}.\end{aligned}$$

The checkerboard copula with additional information

We may include some kind of information in the checkerboard copula, mainly:

The copula of a subvector $\mathbf{X}^J$, $J \subset \{1, \ldots, d\}$, C^J is known, $|J| = k < d$.

The checkerboard copula with additional information

We may include some kind of information in the checkerboard copula, mainly:

The copula of a subvector $\mathbf{X}^J$, $J \subset \{1, \dots, d\}$, C^J is known, $|J| = k < d$.

Let μ^J be the probability measure on $[0,1]^k$ associated to C^J.
For $i = (i_1, \dots, i_d)$, let $x = (x_1, \dots, x_d) \in [0,1]^d$, $x^J = (x_j)_{j \in J}$, $x^{-J} = (x_j)_{j \notin J}$ and

$$I^J_{i,m} = \left\{ x \in [0,1]^k \; / \; x_j \in \left[\frac{i_j - 1}{m}, \frac{i_j}{m} \right], \; j \in J \right\},$$

$$I^{-J}_{i,m} = \left\{ x \in [0,1]^{d-k} \; / \; x_j \in \left[\frac{i_j - 1}{m}, \frac{i_j}{m} \right], \; j \notin J \right\}.$$

Checkerboard with information on a sub-vector

Define

$$\mu_m^J([0,x]) = \sum_{i\subset\{1,\dots,m\}^d} \frac{m^{d-k}}{\mu^J(I_{i,m}^J)} \mu(I_{i,m})\lambda([0,x^{-J}]\cap I_{i,m}^{-J})\mu^J([0,x^J]\cap I_{i,m}^J).$$

Let $C_m^J(x) = \mu_m^J([0,x])$.
From a probabilistic point of view,

$$C_m^J(x) = \sum_i \mu(I_{i,m})\mathbb{P}(U^{-J} \leq x^{-J},\ U^J \leq x^J | U \in I_{i,m}).$$

with U a random vector of $\mathbb{R}^d$, with U^{-J} and U^J independent, U^{-J} a random vector of $\mathbb{R}^{d-k}$ of i.i.d. uniform laws on $[0,1]$ and U^J distributed as C^J.

Checkerboard with information on a sub-vector

Define

$$\mu_m^J([0,x]) = \sum_{i\subset\{1,\ldots,m\}^d} \frac{m^{d-k}}{\mu^J(I_{i,m}^J)}\mu(I_{i,m})\lambda([0,x^{-J}]\cap I_{i,m}^{-J})\mu^J([0,x^J]\cap I_{i,m}^J).$$

Let $C_m^J(x) = \mu_m^J([0,x])$.

Proposition

C_m^J is a copula, it approximates C: $\sup_{x\in[0,1]^d} |C_m^J(x) - C(x)| \leq \frac{d}{2m}$.

If X^J and X^{-J} are independent then,

$$\sup_{x\in[0,1]^d} |C_m^J(x) - C(x)| \leq \frac{d-k}{2m}.$$

Information on the tail

We may also add information on the tail.

Definition

Let $t \in]0,1[$ and $E = \left(\prod_{i=1}^{d}[0,t]^d\right)^c$, assume that $\mu_C(E)$ is known (information on the tail).
The checkerboard copula with extra information on the tail is defined by:

$$C_m^{\mathcal{E}}(x) = \mu_C(E^c)C_m^*(x/t)\mathbf{1}_{E^c}(x) + \frac{\mu_C(E)}{\lambda(E)}\lambda([0,x]\cap E),$$

where C_m^* is the checkerboard copula with partition: $J_{i,m} = t \cdot I_{i,m}$.

$C_m^{\mathcal{E}}$ is a copula, it approximates C.

Plan

An estimation procedure

Assume the marginal laws are known, a (quite small sample) of **X** is available.

1. Estimate μ by $\widehat{\mu}$ using the empirical copula. Empirical copula.
2. Construct the empirical checkerboard copula:

$$\widehat{C}_m^*(x) = \sum_i m^d \widehat{\mu}(I_{i,m}) \lambda([0,x] \cap I_{i,m})$$

or if subvector information is available:

$$\widehat{C}_m^J(x) = \sum_{i \subset \{1,\dots,d\}} \frac{m^{d-k}}{\mu^J(I_{i,m}^J)} \widehat{\mu}(I_{i,m}) \lambda([0,x^{-J}] \cap I_{i,m}^{-J}) \mu^J([0,x^J] \cap I_{i,m}^J).$$

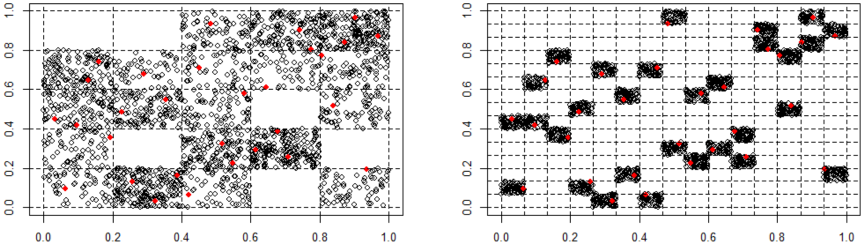
3.
4.
5.

An estimation procedure

Assume the marginal laws are known, a (quite small sample) of $\mathbf{X}$ is available.

1.
2.
3. Simulate a sample of size N from the copula $\widehat{C}_m$, (or $\widehat{C}_m^J$) for N large:
$$(u_1^{(1)}, \dots, u_d^{(1)}), \dots, (u_1^{(N)}, \dots, u_d^{(N)})$$
4. Get a sample of S using the marginals transform:
$$\sum_{i=1}^{d} F_i^{-1}(u_i^{(1)}), \dots, \sum_{i=1}^{d} F_i^{-1}(u_i^{(N)}).$$
5. Estimate the distribution function F_S of S empirically using the sample above $\Rightarrow \widehat{F}_S$.

An estimation procedure



An estimation procedure

Let X_n^* be a random vector with the same marginal laws as X and whose dependence structure is given by the empirical checkerboard copula. Let F_S^* be the distribution function of S.

Proposition

Let $A\sqrt{n} \leq m \leq n$, assume that S is absolutely continuous and C has continuous partial derivatives (Fermanian et al (2004)),

$$\|F_S - F_S^*\|_\infty = O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right).$$

Proposition

If m divides n, then $\widehat{C}_m^*$ is a copula.

An estimation procedure

Proposition

If m divides n, then $\widehat{C}^*_m$ is a copula.

Sketch of proof: prove that $\widehat{C}^*_m(x) = x_1$ for any $x \in [0,1]^d$ with $x_j = 1$ for $j \neq 1$.
For $\ell \in \{1, \dots, m\}$, consider:

$$B^1_\ell = \left\{ x \in [0,1]^d,\ \frac{\ell-1}{m} < x_1 \leq \frac{\ell}{m} \right\} = \left] \frac{\ell-1}{m}, \frac{\ell}{m} \right] \times [0,1]^{d-1}.$$

C_n is concentrated on n points of $[0,1]^d$ whose coordinates are of the form $\frac{j}{n}$, $j = 1, \dots, n$.If $k = n/m$, the number of masses of C_n on each strip B^1_ℓ, $\ell = 1, \dots, m$ is exactly k, $\Longrightarrow \widehat{\mu}(B^1_\ell) = \frac{k}{n} = \frac{1}{m}$.
The result follows by a simple computation.

The Pareto - Clayon model

A model for which Δ may be calculated will serve as a benchmark.

$$\mathbb{P}(X_1 > x_1, \ldots, X_d > x_d \,|\, \Lambda = \lambda) = \prod_{i=1}^{d} e^{-\lambda x_i},$$

that is, conditionally on the value of Λ the marginals of $\mathbf{X}$ are independent and exponentially distributed.

Λ Gamma distributed $\Rightarrow$ X_i are Pareto distributed with dependence given by a survival Clayton copula.

These models have been initially studied by Oakes (1989) and Yeh (2007) .

Exact formula for $\mathrm{VaR}_\alpha(S)$ using the so-called Beta prime distribution (see Dubey (1970)).

The Pareto - Clayton model: exact formula

$\Lambda \rightsquigarrow \Gamma(\alpha, \beta)$, so that the X_i are Pareto (α, β) distributed with the dependence structure is described by a survival Clayton copula with parameter $1/\alpha$.

$\Rightarrow$ S is the so-called Beta prime distribution (see Dubey (1970)):

$$F_S(x) = F_\beta\left(\frac{x}{1+x}\right).$$

where F_β is the c.d.f. of the Beta$(d\beta, \alpha)$ distribution.

The inverse of F_S (or VaR function of S) can also be expressed in function of the inverse of the Beta distribution

$$F_S^{-1}(p) = \frac{F_\beta^{-1}(p)}{1 - F_\beta^{-1}(p)}.$$

Simulations

Pareto-Clayton model:

- in dimension 2, with parameter $\alpha = 1$. The size of the multivariate sample is 30,
- in dimension 3, with information on the sub-vector (X_1, X_2), the size of the multivariate sample is 30,
- in dimension 10, with parameter $\alpha = 2$. The size of the multivariate sample is 75 and 150.

Comparaison with the direct estimation.

Dimension 2

Mean and relative mean squared error for different quantile levels, $N = 1000$, several value of $m|n$ tested.

	Quantile 80%	Quantile 90%	Quantile 95%	Quantile 99%	Quantile 99.5%	Quantile 99.9%
Exact value	2.5	4.1	6.4	16.0	23.2	53.4
Empirical	2.5 (26%)	4.0 (31%)	6.1 (39%)	12.2 (72%)	13.2 (70%)	14.0 (78%)
ECBC (m=6)	2.6 (9%)	4.4 (8%)	6.6 (6%)	14.8 (8%)	20.8 (11%)	45.7 (15%)
ECBC (m=15)	2.5 (12%)	4.2 (13%)	6.8 (11%)	15.5 (9%)	21.5 (10%)	46.4 (14%)
ECBC (m=30)	2.5 (13%)	4.2 (15%)	6.6 (17%)	15.8 (13%)	22.0 (12%)	47.0 (14%)

Dimension 3

$X = (X_1, \ldots, X_3)$ with

- $X_1 = X_2 = Y/2$,
- X_3 distributed as Y, a Pareto r.v. with parameter $\alpha = 2$.
- The copula of (Y, X_3) is assumed to be a survival Clayton of parameter 1/2.

So that $S = X_1 + X_2 + X_3 \stackrel{\mathcal{L}}{=} Y_1 + Y_2$ with $Y = (Y_1, Y_2)$ a Pareto-Clayton vector defined above.

Simulations without and with the additional information on (X_1, X_2) (comonotonic copula).

Dimension 3

	Quantile 80%	Quantile 90%	Quantile 95%	Quantile 99%	Quantile 99.5%	Quantile 99.9%
Exact	2.5	4.1	6.4	16.0	23.2	53.4
ECBC (m=6)						
No information	2.7 (13%)	4.6 (13%)	6.6 (7%)	14.0 (13%)	19.1 (18%)	40.7 (24%)
Information on (X_1, X_2)	2.6 (9%)	4.4 (8%)	6.6 (6%)	14.8 (8%)	20.8 (11%)	45.7 (15%)
ECBC (m=10)						
No information	2.5 (12%)	4.6 (13%)	7.0 (12%)	14.5 (11%)	19.8 (15%)	41.3 (23%)
Information on (X_1, X_2)	2.5 (11%)	4.3 (9%)	6.7 (9%)	15.2 (8%)	21.2 (10%)	46.1 (15%)
ECBC (m=30)						
No information	2.5 (14%)	4.2 (16%)	6.8 (19%)	15.9 (14%)	21.4 (14%)	43.3 (21%)
Information on (X_1, X_2)	2.5 (13%)	4.2 (16%)	6.6 (17%)	15.8 (13%)	21.9 (13%)	47.1 (14%)

Dimension 10

Mean and relative standard deviation for different quantile levels, $N = 1000$.

	VaR 80%	VaR 90%	VaR 95%	VaR 99%	VaR 99.5%	VaR 99.9%
Exact value	12.2	19.2	29	70.1	100.8	230.5
Empirical, $n = 75$	12.6 (12%)	20 (15%)	29.9 (19%)	62.2 (39%)	75.8 (58%)	86.7 (71%)
Checkerboard, $n = 75$	12.5 (10%)	20.1 (13%)	31.2 (14%)	74.8 (20%)	92.4 (20%)	152.6 (16%)
Empirical, $n = 150$	12.4 (8%)	19.6 (11%)	30.3 (14%)	67.3 (27%)	89.9 (38%)	121 (59%)
Checkerboard, $n = 150$	12.4 (7%)	19.6 (9%)	29.8 (12%)	75.4 (16%)	107.6 (21%)	173.9 (19%)

Using information on the tail (dimension 2)

Same dimension 2 model as above. $m = 6$, information on the tail added.

	Quantile 80%	Quantile 90%	Quantile 95%	Quantile 99%	Quantile 99.5%	Quantile 99.9%
Exact value	2.5	4.1	6.4	16.0	23.2	53.4
Empirical	2.5	4.0	6.1	12.2	13.2	14.0
	(26%)	(31%)	(39%)	(72%)	(70%)	(78%)
ECBC (m=6)						
t=1	2.6	4.4	6.6	14.8	20.8	45.7
	(9%)	(8%)	(6%)	(8%)	(11%)	(15%)
t=0.99	2.6	4.4	6.4	14.2	22.7	49.5
	(9%)	(8%)	(5%)	(11%)	(3%)	(8%)
t=0.95	2.7	4.1	6.1	15.6	21.8	46.8
	(10%)	(5%)	(4%)	(3%)	(6%)	(13%)

Plan

Conclusion

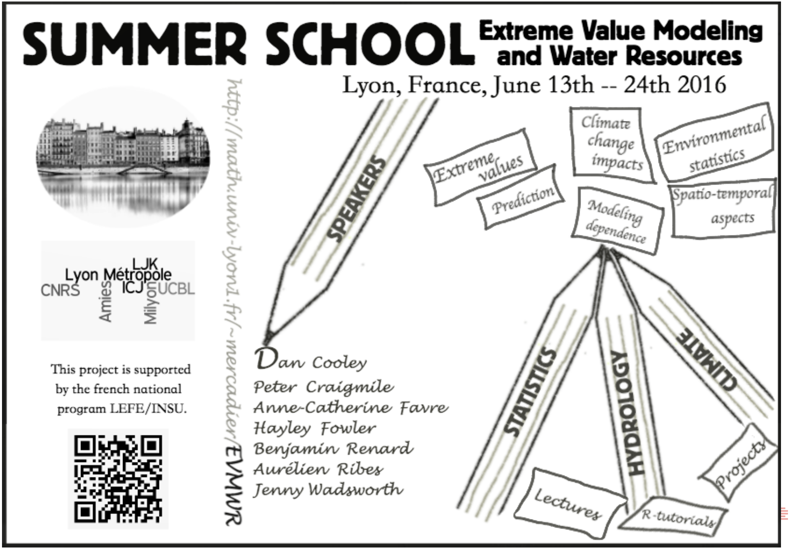
- Efficient methods to estimate the aggregated VaR.
- Efficient even in (relatively) high dimension with (relatively) small samples.
- Additional information / expert opinion may be taken into account: dependence structure on a sub-vector or on the tail.

ToDo Determine optimally m.

ToDo Quantify the information gain.

ToDo Develop efficient procedures to simulate a sample from the checkerboard copula with partial information (tailor copula of a sub-vector).

Thank you for your attention



Empirical Copula

Deheuvels (1979) defined the empirical copula.

Definition

Let $X^{(1)}, \dots X^{(n)}$ be n independent copies of $\mathbf{X}$ and $R_i^{(1)}, \dots, R_i^{(n)}$, $i = 1, \dots, d$ their marginals ranks, i.e.,

$$R_i^{(j)} = \sum_{k=1}^{n} 1\{X_i^{(j)} \geq X_i^{(k)}\},\ i = 1, \dots, d,\ j = 1, \dots, n.$$

The empirical copula C_n of $X^{(1)}, \dots X^{(n)}$ is defined as

$$C_n(u) = \frac{1}{n} \sum_{k=1}^{n} 1 \left\{ \frac{1}{n} R_1^{(k)} \leq u_1, \dots, \frac{1}{n} R_d^{(k)} \leq u_d \right\}.$$