Global sensitivity analysis and quantification of uncertainty

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Plan

General problematic

Inputs variables - parameters - $X_1, \ldots, X_k$.

Ouput $Y = f(X_1, \ldots, X_k)$.

How does the uncertainty on the X_i's impact the uncertainty on Y?

Some examples

- Y is be the water high or the first time that the water level is above some threshold in hydrology,
- Y is the flood level,
- Y is the price of an option or the default probability in credit risk.

$X_1, \ldots, X_k$ are the parameters of the model (wind strength, nature of the soil, precipitation, volatility, mean return, ...). Y could be obtained by solving an EDS or a PDE or by optimization procedures ...

Notations

Let $Y = f(X_1, \ldots, X_k)$ be the output with $X_1, \ldots, X_k$ independent random variables.
Denote

$$X_\alpha = (X_i, i \in \alpha) \text{ for } \alpha \subset \{1, \ldots, k\}.$$

Sobol's decomposition of the output

$Y = f(X)$ can be decomposed into (see Sobol (1995 or 2001) e.g.)

$$f(X_1, \ldots, X_k) = \sum_{\alpha \subset \{1,\ldots,k\}} f_\alpha(X_\alpha),$$

with

1. $f_\varnothing = \mathbb{E}(f(X))$,
2. $\displaystyle\int f_\alpha d\mu_{X_i} = 0$ if $i \in \alpha$,
3. $\displaystyle\int f_\alpha \cdot f_\beta d\mu_X = 0$ if $\alpha \neq \beta$.

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$$f_\varnothing = \mathbb{E}(f(X)),$$

for $i \in \{1, \ldots, k\}$

$$f_i(X_i) = \mathbb{E}(f(X) \mid X_i) - f_\varnothing.$$

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$$f_i(X_i) = \mathbb{E}(f(X) \mid X_i) - f_\varnothing.$$

For $\alpha \subset \{1, \ldots, k\}$,

$$f_\alpha(X_\alpha) = \mathbb{E}(f(X) \mid X_\alpha) - \sum_{\beta \subsetneq \alpha} f_\beta(X_\beta).$$

Decomposition of the variance

A direct application of the above definitions leads to the decomposition:

$$\mathrm{var}(Y) = \mathrm{var}(f(X)) = \sum_{\alpha \subset \{1,\ldots,k\}} \mathrm{var}(f_\alpha(X_\alpha)) = \sum_{\alpha \subset \{1,\ldots,k\}} \mathbb{E}(f_\alpha(X_\alpha)^2).$$

Simple indices

The impact of the variation of X_i on the variation of $Y = f(X)$ may be measured by the Sobol index:

$$S_i = \frac{\mathrm{var}(\mathbb{E}(f(X) \mid X_i))}{\mathrm{var}(Y)} = \frac{\mathbb{E}(f_i(X_i)^2)}{\mathrm{var}(Y)}.$$

It is the relative impact of X_i on the variation of $Y = f(X)$.

We have:

$$\sum_{i\in\{1,\ldots,k\}} S_i \leq 1.$$

The equality is achieved when there is no interactions.

Total indices

Interactions between the variables $X_1, \dots, X_k$, they are identified by the f_α, with $|\alpha| \geq 2$.
Total Sobol indices take into account the impact of the interactions:

$$S_{T_i} = \frac{\sum_{\alpha \ni i} \mathrm{var}(f_\alpha(X_\alpha))}{\mathrm{var}(Y)} = \frac{\sum_{\alpha \ni i} \mathbb{E}((f_\alpha(X_\alpha)^2))}{\mathrm{var}(Y)}.$$

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Our aim is to study the impact of a replacement $X_i \to X_i^*$ on the Sobol indices S_i and S_{T_i}.
The more X_i is uncertain, the greater S_i and S_{T_i}?

The stochastic order, the convex order

Stochastic orders: different ways to - partially - order random variables.

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Stochastic orders: different ways to - partially - order random variables.

X_1 and X_1^* two random variables.

- X_1^* is smaller than X_1 for the standard stochastic order ($X_1^* \leq_{\mathrm{st}} X_1$) if and only if, for any bounded non decreasing function f,

$$\mathbb{E}(f(X_1^*)) \leq \mathbb{E}(f(X_1)).$$

- X_1^* is smaller than X_1 for the convex order ($X_1^* \leq_{\mathrm{cx}} X_1$) if and only if, for any bounded convex function f,

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These are not *location free* orders. Remark that

$$\begin{aligned} X_1^* \leq_{\text{st}} X_1 &\Rightarrow \mathbb{E}(X_1^*) \leq \mathbb{E}(X_1). \\ X_1^* \leq_{\text{cx}} X_1 &\Rightarrow \mathbb{E}(X_1^*) = \mathbb{E}(X_1). \end{aligned}$$

Some variability orders

We shall consider orders designed to take into account the variability and are location free.

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X_1^* and X_1 two random variables.

- F_* and F their distribution functions,
- F_*^{-1} and F^{-1} their generalized inverse (or the quantile function),
- $\overline{F}_* = 1 - F_*$, $\overline{F} = 1 - F$ their survival functions.

Some variability orders

We shall consider orders designed to take into account the variability and are location free.

- X_1^* is smaller than X_1 for the dilatation order ($X_1^* \leq_{\text{dil}} X_1$) if and only if $(X_1^* - \mathbb{E}(X_1^*)) \leq_{\text{cx}} (X_1 - \mathbb{E}(X_1))$,
- X_1^* is smaller than X_1 for the dispersive order ($X_1^* \leq_{\text{disp}} X_1$) if and only if $F^{-1} - F_*^{-1}$ is non decreasing,
- If X_1^* and X_1 have finite means, then X_1^* is smaller than X_1 for the excess wealth order ($X_1^* \leq_{\text{ew}} X_1$) if and only if, for all $p \in]0, 1[$,

$$\int_{[F_*^{-1}(p),\infty[} \overline{F}_*(x)dx \leq \int_{[F^{-1}(p),\infty[} \overline{F}(x)dx.$$

Scale invariant orders

- X_1^* is smaller than X_1 for the star order ($X_1^* \leq_* X_1$) if and only if
$$\frac{F^{-1}}{F_*^{-1}} \text{ is non decreasing,}$$
- X_1^* is smaller than X_1 for the Lorenz ($X_1^* \leq_{\text{Lorenz}} X_1$) if and only if
$$\frac{X_1^*}{\mathbb{E}(X_1^*)} \leq_{\text{cx}} \frac{X_1}{\mathbb{E}(X_1)}.$$

Properties and relationships I.

Property (see eg the book *Stochastic orders* by Shaked-Shanthikumar 2007)

1. $\leq_{disp} \Longrightarrow \leq_{ew} \Longrightarrow \leq_{dil}$.
2. $\leq_{*} \Longrightarrow \leq_{Lorenz}$.
3. $X_1^* \leq_* X_1 \Longleftrightarrow \log X_1^* \leq_{disp} \log X_1$.
4. *If* X_1^* *and* X_1 *are random variables with* $X_1^* \leq_{disp} X_1$ *and* $X_1^* \leq_{st} X_1$ *then for all non decreasing and convex or non increasing concave function* φ, $\varphi(X_1^*) \leq_{disp} \varphi(X_1)$.

Properties and relationships II.

As a corollary, we have that

$$X_1^* \leq_{\text{disp}} X_1 \text{ and } X_1^* \leq_{\text{st}} X_1 \Rightarrow \text{var}(\varphi(X_1^*)) \leq \text{var}(\varphi(X_1))$$

for any non decreasing and convex or non increasing concave function φ.

More properties on stochastic orders.

Sketch of results

For which order and under which conditions on f,

$$X_i^* \leq X_i \Longrightarrow S_i^* \leq S_i$$

or

$$X_i^* \leq X_i \Longrightarrow S_{T_i}^* \leq S_{T_i}?$$

Where S_i^* and $S_{T_i}^*$ are Sobol indices for
$Y^* = f(X_1, \ldots, X_{i-1}, X_i^*, X_{i+1}, \ldots, X_k)$.
Write $X^* = (X_1, \ldots, X_{i-1}, X_i^*, X_{i+1}, \ldots, X_k)$.

Result when there is no interactions

No interactions, Sobol's decomposition writes:

$$f(X) = \sum_{i=1}^{k} f_i(X_i) + f_\varnothing .$$

Theorem

Assume

- *f is convex and componentwise non decreasing (or concave and componentwise non increasing).*
- *X_i^* is independent of $(X_1, \ldots, X_k)$.*
- *$X_i^* \leq_{ew} X_i$ and $-\infty < \ell_* \leq \ell$, where ℓ and ℓ_* are the left end points of the support of X_i^* and X_i.*

Then $S_i^* \leq S_i$.

Idea of the proof

Write $\varphi_j(X_j) = \mathbb{E}(f(X)|X_j)$, so that $f_j = \varphi_j - f_\varnothing$, $\varphi_j(X_j)$ is non decreasing and convex. $f(X^*)$ writes:

$$f(X^*) = \sum_{j \neq i} f_j(X_j) + f_i(X_i^*) + f_\varnothing.$$

$$\mathrm{var}(Y^*) = \sum_{j \neq i} \mathbb{E}(f_j(X_j)^2) + \mathrm{var}(f_i(X_i^*)) = \sum_{j \neq i} \mathrm{var}(\varphi_j(X_j)) + \mathrm{var}(\varphi_i(X_i^*)).$$

Finally,

$$S_i^* = \frac{\mathrm{var}(\varphi_i(X_i^*))}{\sum_{j \neq i} \mathrm{var}(\varphi_j(X_j)) + \mathrm{var}(\varphi_i(X_i^*))}$$

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$$f(X^*) = \sum_{j \neq i} f_j(X_j) + f_i(X_i^*) + f_\varnothing.$$

$$\mathrm{var}(Y^*) = \sum_{j \neq i} \mathbb{E}(f_j(X_j)^2) + \mathrm{var}(f_i(X_i^*)) = \sum_{j \neq i} \mathrm{var}(\varphi_j(X_j)) + \mathrm{var}(\varphi_i(X_i^*)).$$

Also, we have

$$S_i = \left[1 + \frac{\sum_{j \neq i} \mathrm{var}(\varphi_j(X_j))}{\mathrm{var}(\varphi_i(X_i))} \right]^{-1} \quad S_i^* = \left[1 + \frac{\sum_{j \neq i} \mathrm{var}(\varphi_j(X_j))}{\mathrm{var}(\varphi_i(X_i^*))} \right]^{-1}.$$

$$\mathrm{var}(\varphi_i(X_i^*)) \leq \mathrm{var}(\varphi_i(X_i)), \implies S_i^* \leq S_i.$$

Products of convex functions

Theorem

If f writes:

$$f(X_1, \ldots, X_k) = g_1(X_1) \times \cdots \times g_k(X_k) + K$$

with $K \in \mathbb{R}$ and the $\log g_i$'s convex and non decreasing functions.
Let X_i^* be independent of X and $X_i^* \leq_{disp} X_i$ and $X_i^* \leq_{st} X_i$.
Then $S_{T_i}^* \leq S_{T_i}$.

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Remark: If X_i^* and X_i have ℓ_* and ℓ as finite left end points of their support then $X_i^* \leq_{\text{disp}} X_i$ and $\ell_* = \ell \implies X_i^* \leq_{\text{st}} X_i$.

Idea of the proof.

Extensions

The previous result holds in some extended cases described below.

1. Let $\{I_a\}_{a\in A}$ be a partition of $\{1,\ldots,k\}$ and assume that

$$f(X) = \sum_{a\in A}\prod_{j\in I_a} g_j(X_j)$$

with $\log g_j$ non decreasing and convex. If X_i^* is independent of X and $X_i^* \leq_{\text{disp}} X_i$ and $X_i^* \leq_{\text{st}} X_i$. Then $S^*_{T_i} \leq S_{T_i}$.

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2. Let $f(X)=\varphi_1(X_i)\prod_{j\neq i} g_j(X_j)+\varphi_2(X_i)$ with $\log g_j$, $\log\varphi_1$ and $\log\varphi_2$ non decreasing and convex. If
   - X_i^* is independent of X and $X_i^* \leq_{\text{disp}} X_i$ and $X_i^* \leq_{\text{st}} X_i$.
   - $\frac{\text{var}(\varphi_2(X_i^*))}{\mathbb{E}(\varphi_1(X_i^*))^2} \leq \frac{\text{var}(\varphi_2(X_i))}{\mathbb{E}(\varphi_1(X_i))^2}$ and $\frac{\text{cov}(\varphi_1(X_i^*),\varphi_2(X_i^*))}{\mathbb{E}(\varphi_1(X_i^*))^2} \leq \frac{\text{cov}(\varphi_1(X_i),\varphi_2(X_i))}{\mathbb{E}(\varphi_1(X_i))^2}$.

   Then $S_{T_i}^* \leq S_{T_i}$.

Examples

- Flood event (river stage in Shopshire, UK).
- Value at Risk in the classical Black and Sholes model.
- Price of zero coupon in the Vasicek model.

Flood event

©H.L. Cloke, F. Pappenberger, P.-P Renaud, *Multi-method global sensitivity analysis for modelling floodplain hydrological processes.* Hydrological processes, **22**, (2008).

Table I. Specified ranges and distributions of factors. Factors 4 and 5 are exchangeable between the two soil moisture algorithms (Brooks–Corey and van Genuchten)

Factor	Description	Symbol	Unit	Distribution	Mean	Min (0·001 quantile)	Max (0·999 quantile)
1	Saturated moisture content	θ_S	—	Normal ($\sigma = 0{\cdot}09$)	0·41	0·132	0·688
2	Residual moisture content	θ_R	—	Normal ($\sigma = 0{\cdot}01$)	0·0954	0·065	0·125
3	Saturated hydraulic conductivity	K_S	ms^{-1}	Log normal (A = −14·82, B = 1·24)	$9{\cdot}93 \times 10^{-7}$	$1{\cdot}51 \times 10^{-10}$	$1{\cdot}01 \times 10^{-4}$
4a	Brooks–Corey, pore size distribution index	λ	—	Normal ($\sigma = 0{\cdot}1$)	0·318	0·017	0·619
5a	Brooks–Corey, air entry pressure	h_S	m	Log normal (A = −0·382, B = 0·710)	0·880	0·074	6·275
4b	van Genuchten alpha	α	m^{-1}	Log normal (A = −4·22, B = 0·719)	1·9	0·16	13·56
5b	van Genuchten, n	n	—	Normal ($\sigma = 0{\cdot}1$)	1·32	1·02	1·62
6	Storage parameter	S	—	Uniform	$0{\cdot}1 \times 10^{-3}$	$0{\cdot}1 \times 10^{-4}$	$0{\cdot}1 \times 10^{-2}$
7	Upslope pressure	UP	m	Uniform	Measured value	−0·5	0·5
8	River stage	R_S	m	Uniform	Measured value	−0·5	0·5
9	Rainfall (precipitation)	PPT	m	Uniform	Measured value	90%	100%

Flood event

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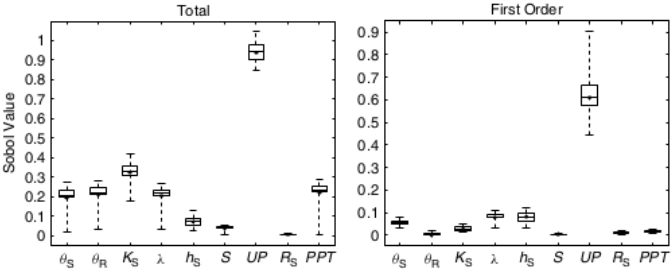

Sensibility of the VaR

Simplest model (Black-Sholes). L is a loss of a portfolio of the form $L = S_T - K$ where K is positive and where S_T is the value at time T of a geometric brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \ \ t \in [0, T].$$

The Value at Risk is given by

$$VaR_\alpha(L) = S_0 \exp\left(\mu T + \sigma\sqrt{T}\mathcal{N}^{-1}(\alpha)\right) - K.$$

The parameters are μ and σ. This is a case of a product of *log* non decreasing and convex functions.
We have chosen for σ and μ several uniform, truncated normal and truncated exponential laws (ordered with respect to the dispersive and stochastic orders).

Sensibility of the VaR

Results for $\alpha = 0.9$.
$\mathcal{N}_{\mathrm{T}}$ stands for a truncated, on $[0, 2]$ normal law.
$\mathcal{E}_T$ stands for a truncated, on $[0, 1]$ exponential law.

μ^*	μ	σ^*	σ	$S^*_{T_\mu}$	S_{T_μ}	$S^*_{T_\sigma}$	S_{T_σ}
$\mathcal{U}[0,1]$	-	$\mathcal{U}[0,1]$	$\mathcal{U}[0,2]$	0.41	0.2	0.64	0.87
$\mathcal{U}[0,2]$	-	$\mathcal{U}[0,1]$	$\mathcal{N}_{\mathrm{T}}(0.5,2)$	0.73	0.48	0.36	0.69
$\mathcal{U}[0,1]$	-	$\mathcal{E}_{\mathrm{T}}(5)$	$\mathcal{E}_{\mathrm{T}}(1)$	0.53	0.4	0.52	0.66
$\mathcal{U}[0,1]$	$\mathcal{N}_{\mathrm{T}}(0.5,2)$	$\mathcal{U}[0,1]$	-	0.4	0.73	0.65	0.35

Vasicek model

Vasicek model: model for short interest rate (or for default intensity) given by the solution of an Ornstein Ulenbeck type stochastic differential equation i.e:

$$dr_t = a(b - r_t)dt + \sigma dW_t$$

where a, b and σ positive parameters and W_t is a standard brownian motion.

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The price at time t of a zero coupon bond with maturity T (or the survival probability in a credit risk model) is given by :

$$P(t,T) = A(t,T)e^{-r(t)B(t,T)}$$

with

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a}$$

$$A(t,T) = \exp\left((b - \frac{\sigma^2}{2a^2})(B(t,T) - T + t) - \frac{\sigma^2}{4a}B^2(t,T) \right)$$

Vasicek model

Results for the initial rate $r_0 = 0.1$.

param.	law	S_T	param.	law	S_T	param.	law	S_T
a	$\mathcal{U}[0,1]$	0.41	a	$\mathcal{U}[0,1]$	0.48	a	$\mathcal{U}([0,1])$	0.25
b	$\mathcal{U}[0,1]$	0.52	b^*	$\mathcal{U}[0,2]$	0.57	b	$\mathcal{U}([0,1])$	0.13
σ	$\mathcal{U}[0,1]$	0.18	σ	$\mathcal{U}[0,1]$	0.06	σ^*	$\mathcal{N}_T(0.5,2)$	0.7

Conclusion

+ Some compatibility between risk theory (via stochastic orders) and Sobol indices.
- The order of Sobol indices may change when changing the law of the parameters.

ToDo Hydrological applications.

ToDo Find the class of functions f for which the ordering on Sobol indices may be done.

ToDo Use the results presented to find bounds on Sobol indices (use of smallest elements for the dispersive or ew orders).

Thanks for your attention.

Other properties of stochastic orders

Property (E Fagiuoli, F Pellerey, and M Shaked 1999.)

X_1^* and X_1 two finite means random variables with supports bounded from below by ℓ_* and ℓ. If $X_1^* \leq_{ew} X_1$ and $-\infty < \ell_* \leq \ell$ then for all non decreasing and convex functions h_1, h_2 for which $h_i(X_1^*)$ and $h_i(X_1)$ $i = 1, 2$ have order two moments,

$$\mathrm{cov}(h_1(X_1^*), h_2(X_1^*)) \leq \mathrm{cov}(h_1(X_1), h_2(X_1)).$$

Other properties of stochastic orders

Property (Shaked-Shanthikumar 2007)

- $X_1^* \leq_{ew} X_1$ *if and only if*

$$\frac{1}{1-p}\int_p^1 (F^{-1}(u) - F_*^{-1}(u))du$$

is non decreasing in $p \in]0,1[$.

- $X_1^* \leq_{disp} X_1$ *if and only if for all* $c \in \mathbb{R}$, *the curve of* $F_*(\cdot - c)$ *crosses that of* F *at most once. When they cross, the sign is* $-,+$.

Back.

Idea of the proof I.

$$f_i(X_i) = (g_i(X_i) - \mathbb{E}(g_i(X_i)) \prod_{j \neq i} \mathbb{E}(g_j(X_j)),$$

The form of f gives:

$$\begin{aligned} f_\alpha(X_\alpha) &= \sum_{\beta \subset \alpha} (-1)^{|\alpha| - |\beta|} \prod_{j \in \beta} g_j(X_j) \prod_{j \notin \beta} \mathbb{E}(g_j(X_j)) \\ &= \prod_{j \notin \alpha} \mathbb{E}(g_j(X_j)) \prod_{j \in \alpha} (g_j(X_j) - \mathbb{E}(g_j(X_j))) \,. \end{aligned}$$

Idea of the proof II.

We write

$$f_{T_i} = \sum_{i \in \alpha} f_\alpha$$

Then, one gets

$$f_{T_i}(X) = (g_i(X_i) - \mathbb{E}(g_i(X_i)) \prod_{j \neq i} g_j(X_j).$$

Moreover,

$$f_\alpha(X_\alpha) = \prod_{j \notin \alpha} \mathbb{E}(g_j(X_j)) \prod_{j \in \alpha} (g_j(X_j) - \mathbb{E}(g_j(X_j)))\,.$$

Idea of the proof III.

Compute the variances:

$$\operatorname{var} f_{T_i} = \operatorname{var}(g_i(X_i)) \prod_{j \neq i} \mathbb{E}(g_j(X_j)^2),$$

if $i \notin \alpha$,

$$\operatorname{var} f_\alpha(X_\alpha) = \mathbb{E}(g_i(X_i))^2 \operatorname{var}\left(\prod_{\substack{j \neq i \\ j \notin \alpha}} \mathbb{E}(g_j(X_j)) \prod_{j \in \alpha} (g_j(X_j) - \mathbb{E}(g_j(X_j))) \right).$$

Idea of the proof IV.

The total Sobol indices rewrite

$$S_{T_i} = \left[1 + \frac{\sum_{\alpha \not\ni i} \mathrm{var}(f_\alpha(X_\alpha))}{\mathrm{var}(f_{T_i}(X))} \right]^{-1} \text{ and } S^*_{T_i} = \left[1 + \frac{\sum_{\alpha \not\ni i} \mathrm{var}(f_\alpha(X_\alpha))}{\mathrm{var}(f^*_{T_i}(X^*))} \right]^{-1}.$$

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The result follows if

$$\frac{\mathrm{var}\, g_i(X_i^*)}{\mathbb{E}(g_i(X_i^*))^2} \le \frac{\mathrm{var}\, g_i(X_i)}{\mathbb{E}(g_i(X_i))^2}.$$

We have

$$\log g_i(X_i^*) \le_{\mathrm{disp}} \log g_i(X_i) \iff g_i(X_i^*) \le_* g_i(X_i)$$

$$\implies \quad g_i(X_i^*) \le_{\mathrm{Lorenz}} g_i(X_i) \implies \frac{\mathrm{var}\, g_i(X_i^*)}{\mathbb{E}(g_i(X_i^*))^2} \le \frac{\mathrm{var}\, g_i(X_i)}{\mathbb{E}(g_i(X_i))^2}.$$