Some Copula's approximations.

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Plan

General problematic

$(X_1, \ldots, X_d)$ random vector of risks. Write

$$S = \sum_{i=1}^{d} X_i, \text{ the aggregated risk.}$$

Regulatory rules, Risk management purposes, Environmental risks ... $\Longrightarrow$ need to estimate / approximate (relatively) high level quantiles of S:

$$F_S^{-1}(\alpha) = \mathrm{VaR}_\alpha(S),$$

where F_S is the distribution function of S.

Examples

- Insurance: X describes the distribution of the claim amonts, regulatory rules impose to insurance companies to estimate $F_X(\alpha)$ for $\alpha = 0.995$.
- Hydrology: X may describe a flood level. Computing $F_X^{-1}(\alpha)$ is required to calibrate a barrage e.g. (or a dam).
- Many other field: finance, wind electricity...

Our purpose

$(X_1, \ldots, X_d)$ random vector of risks.
The X_i may be different lines of business in insurance contexts.

$$S = \sum_{i=1}^{d} X_i.$$

$\Longrightarrow$ Estimation of $\mathrm{VaR}_\alpha(S)$.
The law of S (and thus $\mathrm{VaR}_\alpha(S)$) depends on the law of $(X_1, \ldots, X_d)$ (marginal laws and dependence structure).

Quantiles of aggregated risks

- High dimensional problem (d may be large),
- Marginal laws (laws of the X_i's) are usually known (or well estimated), some information on the dependence is available,
- Even if the law of $(X_1, \ldots, X_d)$ is known, the effective computation of

$$\mathrm{VaR}_\alpha(S),$$

may be difficult to do,

Quantiles of aggregated risks

- Even if the law of $(X_1, \ldots, X_d)$ is known, the effective computation of

$$\mathrm{VaR}_\alpha(S),$$

may be difficult to do,
the distribution function of S is given by:

$$F_S(t) = \int_{\mathbb{R}^d} \mathbf{1}_{\{x_1+\cdots+x_d \leq t\}} f_X(x_1, \ldots, x_d) dx_1 \ldots dx_d.$$

$\Longrightarrow$ Efficient methods are still welcome.

One proposition

Assume that the X_i's laws are known.
Information on the dependence is available through

- a (quite small) $(X_1, \dots, X_d)$ sample and
- some expert opinion (e.g the dependence structure between X_1 and X_2 is completely known) and / or
- some knowledge of the join tail ($\mathbb{P}(X_1 \geq u_1, \dots, X_d \geq u_d)$ is known for some $(u_1, \dots, u_d)$).

We use check-erboard-min copulas to estimate $\text{VaR}_\alpha(S)$.

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We assume that X has continuous marginals and we shall denote by C the copula associated to X.

Plan

The checkerboard copula: definition

The cherckerboard copula, introduced in dimension 2 by Li *et al.* (1998) and Mikusinski and Taylor (2010) is an approximation of a copula C. Durante *et al.* (2015) also consider related approximations known as patchwork copulas.

μ is the probability measure associated to C on $[0,1]^d$:

$$\mu([0,x]) = C(x), x = (x_1, \dots, x_d) \in [0,1]^d, \ [0,x] = \prod_{i=1}^{d} [0, x_i].$$

Consider $(I_{i,m})_{i \in \{1,\dots m\}^d}$ the partition (modulo a 0 measure set) of $[0,1]^d$ given by the m^d squares:

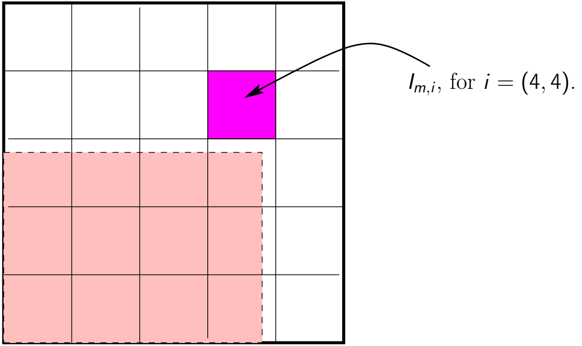
$$I_{i,m} = \prod_{j=1}^{d} \left[\frac{i_j - 1}{m}, \frac{i_j}{m} \right], \ i = (i_1, \dots, i_d).$$

The checkerboard copula: definition

λ denotes the Lebesgue measure.

The checkerboard copula of order m is defined on $[0,1]^d$ by:

$$C^*_m(x) = \sum_i m^d \mu(I_{i,m}) \lambda([0,x] \cap I_{i,m}).$$



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From a probabilistic point of view,

$$C^*_m(x) = \sum_i \mu(I_{i,m}) \mathbb{P}(U \leq x | U \in I_{i,m}).$$

with U a random vector of $\mathbb{R}^d$ of i.i.d. uniform laws on $[0,1]$.

The checkmin copula

In the previous construction, replace the independent copula by the comonotonic copula.
In other words, replace U on $I_{i,m}$ by $U^*_{i,m}$ with

$$(U^*_{i,m})_1 \rightsquigarrow \mathcal{U}([\frac{i_1 - 1}{m}, \frac{i_1}{m}]) \text{ and } (U^*_{i,m})_j = (U^*_{i,m})_1 - \frac{i_1}{m} + \frac{i_j}{m}.$$

$$C^\dagger_m(x) = \sum_i m\mu(I_{i,m}) \min(x_j - \frac{i_j - 1}{m}, \frac{1}{m}).$$

Approximation by the check-erboard-min copula

In what follows, C_m^o is either C_m^* or $C_m^\dagger$.

Proposition

C_m^o is a copula which approximates C:

$$\sup_{x \in [0,1]^d} |C_m^o(x) - C(x)| \leq \frac{d}{2m}.$$

Gives a more precise bound on the approximation of C by C_m^o by a factor 2, than the one presented in dimension 2 in Li *et al.* (1998).

Plan

An estimation procedure

Assume the marginal laws are known, a (quite small sample) of $\mathbf{X}$ is available.

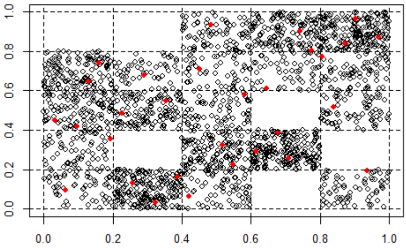
1. Estimate μ by $\widehat{\mu}$ using the empirical copula. Empirical copula.
2. Simulate a sample of size N from the copula $\widehat{C}^*_m$

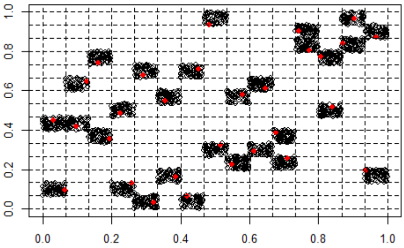
$$\widehat{C}^*_m(x) = \sum_i m^d \widehat{\mu}(I_{i,m}) \lambda([0,x] \cap I_{i,m}).$$

$$(u_1^{(1)}, \dots, u_d^{(1)}), \dots, (u_1^{(N)}, \dots, u_d^{(N)})$$

3.

An estimation procedure





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$$\widehat{C}_m^*(x) = \sum_i m^d \widehat{\mu}(I_{i,m}) \lambda([0,x] \cap I_{i,m}).$$

$$(u_1^{(1)}, \ldots, u_d^{(1)}), \ldots, (u_1^{(N)}, \ldots, u_d^{(N)})$$

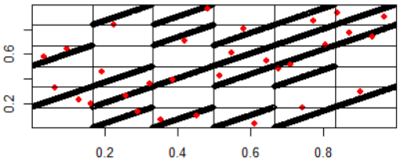
3. Get a sample of S using the marginals transform:

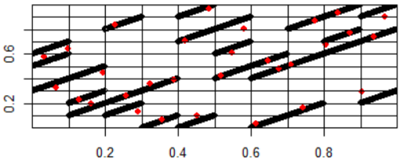
$$\sum_{i=1}^{d} F_i^{-1}(u_i^{(1)}), \ldots, \sum_{i=1}^{d} F_i^{-1}(u_i^{(N)}).$$

4. Estimate the distribution function F_S of S empirically using the sample above $\Rightarrow \widehat{F}_S$.

An estimation procedure

Similar construction for the checkmin copula $\Longrightarrow \widehat{C}^o_m$.





Convergence results for $\widehat{C}^o_m$.

Proposition

Let m divide n, we have:

$$\sup_{t\in[0,1]} |\widehat{C}^o_m(t) - C(t)| \leq O_{\mathbb{P}}\left(\frac{1}{\sqrt{n}}\right) + \frac{d}{2\,m}.$$

Convergence results to F_S.

Estimate $F_S(t)$ by

$$\mathbb{P}\left(\sum_{i=1}^{n}(T^{-}(U_m^o))_i \leq t\right) = F_m^o(t)$$

where $U_m^o \rightsquigarrow \widehat{C}_m^o$ and $T^{-}(u_1, \ldots, u_d) = (F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d))$.
With a regularity condition due to Mainik, we obtain the convergence of F_m^o to F_S.

Proposition

Under the regularity assumption, if m divides n,

$$\sup_{t \in \mathbb{R}} |F_S(t) - F_m^o(t)| = O_{\mathbb{P}}(\frac{1}{\sqrt{n}}) + O(\frac{1}{m}).$$

The Pareto - Clayon model

- Pareto marginal distributions (parameters a, b).
- Survival Clayton copula (parameter $\frac{1}{a}$).

Exact formula for $\mathrm{VaR}_\alpha(S)$ using the so-called Beta prime distribution (see Dubey (1970)).

Gaussian example

- Lognormal marginal distributions.
- Gaussian copula.

Pareto-Clayton model

RMSE in % of the exact value for the Pareto-Clayton model of parameters 3 and 1, in dimension 25, for a sample size $n = 80$, 100 runs.

	90%	95%	99%	99.5%	99.9%
Exact value	23.08	31.28	59.10	76.41	135.89
ECBC, $m = 5$	4%	14%	40%	48%	63%
ECBC, $m = 20$	9%	9%	21%	31%	52%
ECBC, $m = 40$	9%	11%	18%	26%	48%
ECBC, $m = 80$	9%	12%	23%	25%	44%
ECBC, median	5%	8%	31%	41%	59%
ECMC, $m = 5$	3%	4%	6%	7%	13%
ECMC, $m = 20$	5%	6%	14%	17%	23%
ECMC, $m = 40$	6%	7%	15%	19%	27%
ECMC, $m = 80$	7%	10%	16%	21%	32%
ECMC, median	3%	4%	9%	11%	15%
Gaussian cop.	3%	10%	27%	34%	48%
Surv. Clayt.	2%	3%	5%	6%	12%
Clayton copula	10%	23%	46%	54%	66%
Empirical cop.	9%	12%	23%	31%	56%

Gaussian lognormal example

RMSE in % of the exact value for the Gaussian lognormal model with $\rho = 0.1$, dimension 25, for a sample size $n = 80$, 100 runs.

	90%	95%	99%	99.5%	99.9%
Near exact value	111.65	129.81	176.99	200.82	270.14
ECBC, $m = 5$	4%	6%	10%	11%	13%
ECBC, $m = 20$	3%	4%	8%	9%	11%
ECBC, $m = 40$	4%	4%	9%	9%	11%
ECBC, $m = 80$	4%	5%	10%	11%	12%
ECBC, median	3%	5%	9%	10%	11%
ECMC, $m = 5$	3%	11%	33%	44%	72%
ECMC, $m = 20$	3%	3%	7%	10%	22%
ECMC, $m = 40$	3%	4%	7%	8%	15%
ECMC, $m = 80$	4%	5%	8%	10%	13%
ECMC, median	2%	4%	17%	24%	41%
Gaussian copula	2%	2%	3%	4%	6%
Survival Clayton	2%	3%	9%	12%	20%
Clayton copula	7%	9%	13%	14%	14%
Empirical cop.	6%	9%	16%	22%	35%

Gaussian lognormal example

More simulations. RMSE in % of the exact value for the Gaussian lognormal model with $\rho = 0.1$, dimension 25, for a sample size $n = 80$, 100 runs.

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Plan

Conclusion

- Efficient methods to estimate the aggregated VaR.
- Efficient even in (relatively) high dimension with (relatively) small samples.
- Additional information / expert opinion may be taken into account: dependence structure on a sub-vector or on the tail.

ToDo Determine optimally m.

ToDo Quantify the information gain.

ToDo Develop efficient procedures to simulate a sample from the checkerboard copula with partial information (tail or copula of a sub-vector).

ToDo Estimation of the Kendall distribution and application to multivariate return time.

References I

Piotr Mikusinski and Michael D Taylor.
Some approximations of n-copulas.
Metrika, 72(3):385–414, 2010.

Satya D Dubey.
Compound gamma, beta and F distributions.
Metrika, 16(1):27–31, 1970.

Fabrizio Durante, Juan Fernández-Sánchez, José Juan Quesada-Molina, and Ùbeda-Flores Manuel.
Convergence results for patchwork copulas.
European Journal of Operational Research, 247:525–531, 2015.

Xin Li, P Mikusiński, and Michael D Taylor.
Strong approximation of copulas.
Journal of Mathematical Analysis and Applications, 225(2):608–623, 1998.

References II

Georg Mainik.
Risk aggregation with empirical margins: Latin hypercubes, empirical copulas, and convergence of sum distributions.
Journal of Multivariate Analysis, 141:197–216, 2015.

Thank you for your attention

Plan

Empirical Copula

Deheuvels (1979) defined the empirical copula.

Definition

Let $X^{(1)}, \ldots X^{(n)}$ be n independent copies of $\mathbf{X}$ and $R_i^{(1)}, \ldots, R_i^{(n)}$, $i = 1, \ldots, d$ their marginals ranks, i.e.,

$$R_i^{(j)} = \sum_{k=1}^{n} 1\{X_i^{(j)} \geq X_i^{(k)}\},\ i = 1, \ldots, d,\ j = 1, \ldots, n.$$

The empirical copula C_n of $X^{(1)}, \ldots X^{(n)}$ is defined as

$$C_n(u) = \frac{1}{n} \sum_{k=1}^{n} 1\left\{\frac{1}{n} R_1^{(k)} \leq u_1, \ldots, \frac{1}{n} R_d^{(k)} \leq u_d\right\}.$$

Back.

The checkerboard copula with additional information

We may include some kind of information in the checkerboard copula, mainly:

The copula of a subvector $\mathbf{X}^J$, $J \subset \{1, \ldots, d\}$, C^J is known, $|J| = k < d$.

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We may include some kind of information in the checkerboard copula, mainly:

The copula of a subvector $\mathbf{X}^J$, $J \subset \{1, \dots, d\}$, C^J is known, $|J| = k < d$.

Let μ^J be the probability measure on $[0,1]^k$ associated to C^J.
For $i = (i_1, \dots, i_d)$, let $x = (x_1, \dots, x_d) \in [0,1]^d$, $x^J = (x_j)_{j \in J}$, $x^{-J} = (x_j)_{j \notin J}$ and

$$I_{i,m}^J = \left\{ x \in [0,1]^k \ / \ x_j \in \left[\frac{i_j - 1}{m}, \frac{i_j}{m} \right], \ j \in J \right\},$$

$$I_{i,m}^{-J} = \left\{ x \in [0,1]^{d-k} \ / \ x_j \in \left[\frac{i_j - 1}{m}, \frac{i_j}{m} \right], \ j \notin J \right\}.$$

Check-erboard-min with information on a sub-vector

Define

$$\mu_m^J([0,x]) = \sum_{i\subset\{1,\dots,m\}^d} \frac{1}{\mu^J(I_{i,m}^J)}\mu(I_{i,m})\frac{\mu^o([0,x^{-J}]\cap I_{i,m}^{-J})}{\mu^o(I_{i,m}^{-J})}\mu^J([0,x^J]\cap I_{i,m}^J)$$

Let $C_m^J(x) = \mu_m^J([0,x])$.
Where μ^o is either the Lebesgue or the comonotonic measure on $I_{i,m}^{-J}$. From a probabilistic point of view,

$$C_m^J(x) = \sum_i \mu(I_{i,m})\mathbb{P}(U^{-J} \leq x^{-J},\ U^J \leq x^J | U \in I_{i,m}).$$

with U a random vector of $\mathbb{R}^d$, with U^{-J} and U^J independent, U^{-J} is a random vector of $\mathbb{R}^{d-k}$ either of i.i.d. uniform laws on $[0,1]$ or of comonotonic margins conditionnally to $I_{i,m}$ and U^J distributed as C^J.

Check-erboard-min with information on a sub-vector

Define

$$\mu_m^J([0,x]) = \sum_{i\subset\{1,\ldots,m\}^d} \frac{1}{\mu^J(I_{i,m}^J)}\mu(I_{i,m})\frac{\mu^o([0,x^{-J}]\cap I_{i,m}^{-J})}{\mu^o(I_{i,m}^{-J})}\mu^J([0,x^J]\cap I_{i,m}^J)$$

Let $C_m^J(x) = \mu_m^J([0,x])$.

Proposition

C_m^J is a copula, it approximates C: $\sup_{x\in[0,1]^d} |C_m^J(x) - C(x)| \leq \frac{d}{2m}$.

If X^J and X^{-J} are independent then,

$$\sup_{x\in[0,1]^d} |C_m^J(x) - C(x)| \leq \frac{d-k}{2m}.$$

Information on the tail

We may also add information on the tail.

Definition

Let $t \in]0,1[$ and $E = \left(\prod_{i=1}^{d}[0,t]^d\right)^c$, assume that $\mu_C(E)$ is known (information on the tail).
The checkerboard copula with extra information on the tail is defined by:

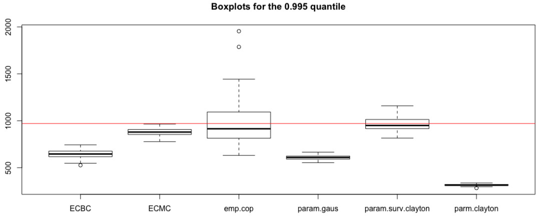
$$C_m^{\mathcal{E}}(x) = \mu_C(E^c)C_m^o(x/t)\mathbf{1}_{E^c}(x) + \frac{\mu_C(E)}{\lambda(E)}\lambda([0,x] \cap E),$$

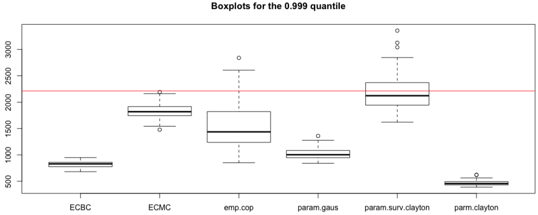
where C_m^o is the check-erboard-min copula with partition: $J_{i,m} = t \cdot I_{i,m}$.

$C_m^{\mathcal{E}}$ is a copula, it approximates C.

More simulations

Pareto-Clayton model with parameters 2 and 1, in dimension 100, $n = 400$.





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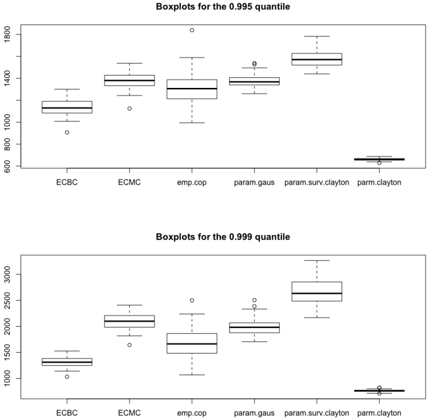
More simulations

Gaussian-lognormal model, correlations 0.25, 0.5, 0.75, dimension 100, $n = 400$.



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More simulations

Pareto-Clayton model in dimension 2, with $\beta = 1$ and $\alpha = 2$, $n = 30$
The information on the tail is introduced on $\mathcal{E}_p$, for $p = 0.95, 0.99$.

	90%	95%	99%	99.5%	99.9%
Empirical	31%	39%	72%	70%	78%
ECBC (m=6)					
No tail information	8%	6%	8%	11%	15%
Information on $\mathcal{E}_p$ p=0.99	8%	5%	11%	3%	8%
Information on $\mathcal{E}_p$ p=0.95	5%	4%	3%	6%	13%
ECBC (m=15)					
No tail information	13%	11%	9%	10%	14%
Information on $\mathcal{E}_p$p=0.99	12%	12%	11%	3%	8%
Information on $\mathcal{E}_p$ p=0.95	10%	4%	3%	6%	13%
ECBC (m=30)					
No tail information	15%	17%	13%	12%	14%
Information on $\mathcal{E}_p$ p=0.99	16%	16%	11%	3%	8%
Information on $\mathcal{E}_p$ p=0.95	11%	4%	3%	6%	13%

Back.

More simulations

$\mathbf{X} = (X_1, X_2, X_3)$ whith $X_1 = X_2 = Y/2$, $X_3 \sim Y$ where Y is Pareto distributed with $\alpha = 2$, and (Y, X_3) is a Pareto-Clayton model $\Longrightarrow X_1$ and X_2 are comonotonic (or fully dependent) and the dependence between X_1 and X_3 is given by a survival Clayton of parameter $1/2$.

	90%	95%	99%	99.5%	99.9%
ECBC (m=6)					
No information	13%	7%	13%	18%	24%
Information on (X_1, X_2)	8%	6%	8%	11%	15%
ECBC (m=10)					
No information	13%	12%	11%	15%	23%
Information on (X_1, X_2)	9%	9%	8%	10%	15%
ECBC (m=30)					
No information	16%	19%	14%	14%	21%
Information on (X_1, X_2)	16%	17%	13%	13%	14%

Back.