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Stabilizers of closed sets in the Urysohn space

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Abstract

Building on earlier work of Katětov, Uspenskij proved in [9] that the group of isometries of Urysohn's universal metric space $\mathbb{U}$, endowed with the pointwise convergence topology, is a universal Polish group (i.e it contains an isomorphic copy of any Polish group). Answering a question of Gao and Kechris, we prove here the following, more precise result: for any Polish group G, there exists a closed subset F of $\mathbb{U}$ such that G is topologically isomorphic to the group of isometries of $\mathbb{U}$ which map F onto itself.

1 Introduction

In a posthumously published article ([7]), P.S Urysohn constructed a complete separable metric space $\mathbb{U}$ that is *universal* (meaning that it contains an isometric copy of every complete separable metric space), and ω-*homogeneous* (i.e such that its isometry group acts transitively on isometric r-tuples contained in it).

In recent years, interest in the properties of $\mathbb{U}$ has greatly increased, especially since V.V Uspenskij, building on earlier work of Katětov, proved in [8] that the isometry group of $\mathbb{U}$ (endowed with the product topology) is a universal Polish group, that is to say any Polish group is isomorphic to a (necessarily closed) subgroup of it.

In [2], S. Gao and A.S Kechris used properties of $\mathbb{U}$ to study the complexity of the equivalence relation of isometry between certain classes of Polish metric spaces; as a side-product of their construction, they proved the beautiful fact that any Polish group is (topologically) isomorphic to the isometry

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group of some Polish space. A consequence of their construction is that, for any Polish group G, there exists a sequence (X_n) of closed subsets of $\mathbb{U}$ such that G is isomorphic to $Iso(\mathbb{U},(X_n)) = \{\varphi \in Iso(\mathbb{U})\colon \forall n\,(\varphi(X_n) = X_n)\}$. This led them to ask the following question (cf [2]):
Can every Polish group be represented, up to isomorphism, by a group of the form $Iso(\mathbb{U}, F)$ for a single subset $F \subseteq \mathbb{U}$?

The purpose of this article is to provide a positive answer to this question by proving the following theorem:

Theorem 1.1. *Let G be a Polish group. There exists a closed set $F \subseteq \mathbb{U}$ such that G is (topologically) isomorphic to $Iso(F)$, and every isometry of F is the restriction of a unique isometry of $\mathbb{U}$; in particular, G is isomorphic to $Iso(\mathbb{U}, F)$.*

This gives a somewhat concrete realization of any Polish group as a subgroup of $Iso(\mathbb{U})$.
The construction, which will be detailed in section 3, starts with a bounded Polish metric space X such that G is isomorphic to $Iso(X)$ (the isometry group of X, endowed with the product topology) (Gao and Kechris proved that such an X always exist see [2]). Identifying G with $Iso(X)$, we construct an embedding of X in $\mathbb{U}$ and a discrete, unbounded sequence $(x_n) \subseteq \mathbb{U}$ such that $F = X \cup \{x_n\}$ has the desired properties (here we identify X with its image via the embedding provided by our construction).

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2 Notations and definitions

If (X, d) is a complete separable metric space, we say that it is a *Polish metric space*, and often write it simply X.
To avoid confusions, we say, if (X, d) and (X', d') are two metric spaces, that f is an *isometric map* if $d(x, y) = d'(f(x), f(y))$ for all $x, y \in X$; if f is onto, then we say that f is an *isometry*.
A *Polish group* is a topological group whose topology is Polish. If X is a separable metric space, then we denote its isometry group by $Iso(X)$, and

endow it with the product topology, which turns it into a second countable topological group, and into a Polish group if X is Polish (see [1] or [5] for a thorough introduction to the theory of Polish groups).
We say that a metric space X is *finitely injective* iff for any finite subsets $K \subseteq L$ and any isometric map $\varphi\colon K \to X$ there exists an isometric map $\tilde{\varphi}\colon L \to X$ such that $\tilde{\varphi}_{|K} = \varphi$. Up to isometry, $\mathbb{U}$ is the only finitely injective Polish metric space (see [7]).
Let (X,d) be a metric space; we say that $f : X \to \mathbb{R}$ is a *Katětov map* iff

$$\forall x, y \in X \ |f(x) - f(y)| \leq d(x,y) \leq f(x) + f(y) \ .$$

These maps correspond to one-point metric extensions of X. We denote by $E(X)$ the set of all Katětov maps on X and endow it with the sup-metric, which turns it into a complete metric space.
That definition was introduced by Katětov in [4], and it turns out to be pertinent to the study of finitely injective spaces, since one can easily see by induction that a non-empty metric space X is finitely injective if, and only if,

$$\forall x_1, \ldots, x_n \in X \ \forall f \in E(\{x_1, \ldots, x_n\}) \ \exists z \in X \ \forall i = 1 \ldots n \ d(z, x_i) = f(x_i) \ .$$

If $Y \subseteq X$ and $f \in E(Y)$, define $\hat{f}\colon X \to \mathbb{R}$ (the Katětov extension of f) by

$$\hat{f}(x) = \inf\{f(y) + d(x,y)\colon y \in Y\}.$$

Then $\hat{f}$ is the greatest 1-Lipschitz map on X which is equal to f on Y; one checks easily (see for instance [4]) that $\hat{f} \in E(X)$ and $f \mapsto \hat{f}$ is an isometric embedding of $E(Y)$ into $E(X)$.
To simplify future definitions, we will say that if $f\colon X \to \mathbb{R}$ and $S \subseteq X$ are such that
$\forall x \in X \ f(x) = \inf\{f(s) + d(x,s)\colon s \in S\}$, then S is a *support of* f, or that S *controls* f.
Notice that if S controls $f \in E(X)$ and $S \subseteq T$, then T controls f.
Also, X isometrically embeds in $E(X)$ via the Kuratowski map $x \mapsto f_x$, where
$f_x(y) = d(x,y)$. A crucial fact for our purposes is that

$$\forall f \in E(X) \ \forall x \in X \ d(f, f_x) = f(x).$$

Thus, if one identifies X with its image in $E(X)$ via the Kuratowski map, $E(X)$ is a metric space containing X and such that all one-point metric extensions of X embed isometrically in $E(X)$.

We now go on to sketching Katětov's construction of $\mathbb{U}$; we refer the reader to [2], [3], [7] and [9] for a more detailed presentation and proofs of the results we will use below.
Most important for the construction is the following

Theorem 2.1. *(Urysohn) If X is a finitely injective metric space, then the completion of X is also finitely injective.*

Since $\mathbb{U}$ is, up to isometry, the unique finitely injective Polish metric space, this proves that the completion of any separable finitely injective metric space is isometric to $\mathbb{U}$.
The basic idea of Katětov's construction works like this: if one lets $X_0 = X$, $X_{i+1} = E(X_i)$ then, identifying each X_i to a subset of X_{i+1} via the Kuratowski map, let Y be the inductive limit of the sequence X_i.
The definition of Y makes it clear that Y is finitely injective, since any $\{x_1, \ldots, x_n\} \subseteq Y$ must be contained in some X_m, so that for any $f \in E(\{x_1, \ldots, x_n\})$ there exists $z \in X_{m+1}$ such that $d(z, x_i) = f(x_i)$ for all i.
Thus, if Y were separable, its completion would be isometric to $\mathbb{U}$, and one would have obtained an isometric embedding of X into $\mathbb{U}$.
The problem is that $E(X)$ is in general not separable: at each step, we have added too many functions.
Define then $E(X, \omega) = \{f \in E(X) \colon f \text{ is controlled by some finite } S \subseteq X\}$.
$E(X, \omega)$ is easily seen to be separable if X is, and the Kuratowski map actually maps X into $E(X, \omega)$, since each f_x is controlled by $\{x\}$.
Notice also that, if $\{x_1, \ldots, x_n\} \subseteq X$ and $f \in E(\{x_1, \ldots, x_n\})$, then its Katětov extension $\hat{f}$ is in $E(X, \omega)$, and $d(\hat{f}, f_{x_i}) = f(x_i)$ for all i.
Thus, if one defines this time $X_0 = X$, $X_{i+1} = E(X_i, \omega)$, then the inductive limit Y of $\cup X_i$ is separable and finitely injective, hence its completion Z is isometric to $\mathbb{U}$, and $X \subseteq Z$.
The most interesting property of this construction is that it enables one to keep track of the isometries of X: indeed, any $\varphi \in Iso(X)$ is the restriction of a unique isometry $\tilde{\varphi}$ of $E(X, \omega)$, and the mapping $\varphi \mapsto \tilde{\varphi}$ from $Iso(X)$ into $Iso(E(X, \omega))$ is an isomorphic embedding of topological groups.
That way, we obtain for all i continuous embeddings $\Psi^i \colon Iso(X) \to Iso(X_i)$, such that $\Psi^{i+1}(\varphi)_{|X_i} = \Psi^i(\varphi)$ for all i and all $\varphi \in Iso(X)$.
This in turns defines a continuous embedding from $Iso(X)$ into $Iso(Y)$, and since extension of isometries defines a continuous embedding from the

group of isometry of any metric space into that of its completion (see [8]), we actually have a continous embedding of $Iso(X)$ into the isometry group of Z, that is to say $Iso(\mathbb{U})$ (and the image of any $\varphi \in Iso(X)$ is actually an extension of φ to Z).

3 Proof of the main theorem

To prove theorem 1.1 ,we will use ideas very similar to those used in [2] ; all the notations are the same as in section 2.
We will need an additional definition, which was introduced in [2]:
If X is a metric space and $i \geq 1$, let

$$E(X,i) = \{f \in E(X) : f \text{ has a support of cardinality } \leq i\}$$

We endow $E(X,i)$ with the sup-metric.
Gao and Kechris proved the following result, of which we will give a new, slightly simpler proof:

Theorem 3.1. *(Gao-Kechris)*
If X is a Polish metric space and $i \geq 1$ then $E(X,i)$ is a Polish metric space.

Proof:
Notice first that the separability of $E(X,i)$ is easy to prove; we will prove its completeness by induction on i.
The proof for $i = 1$ is the same as in [2]; we include it for completeness.
First, let (f_n) be a Cauchy sequence in $E(X,1)$.
It has to converge uniformly to some Katětov map f, and it is enough to prove that $f \in E(X,1)$.
By definition of $E(X,1)$, there exists a sequence (y_n) such that

$$\forall x \in X \;\; f_n(x) = f_n(y_n) + d(y_n, x) \qquad (*)$$

But then let $\varepsilon > 0$, and let M be big enough that $m, n \geq M \Rightarrow d(f_n, f_m) \leq \varepsilon$.
Then, for $m, n \geq M$, one has

$$2d(y_n, y_m) = (f_n(y_m) - f_m(y_m)) + (f_m(y_n) - f_n(y_n)) \leq 2\varepsilon \ .$$

This proves that (y_n) is Cauchy, hence has a limit y.
One easily checks that $f(y) = \lim f_n(y_n)$, so that $(*)$ gives us, letting $n \to \infty$

$$\forall x \in X \ f(x) = f(y) + d(y, x) \qquad .$$

That does the trick for $i = 1$; suppose now we have proved the result for $1 \dots i-1$, and let (f_n) be a Cauchy sequence in $E(X, i)$.
By definition, there are $y_1^n, \dots y_i^n$ such that:
$\forall x \in X \ f_n(x) = \min_{1 \le j \le i} \{f_n(y_j^n) + d(y_j^n, x)\} \qquad (**)$.

Once again, (f_n) converges uniformly to some Katětov map f, and we want to prove that $f \in E(X, i)$.
Thanks to the induction hypothesis, we can assume that there is $\delta > 0$ such that for all n and all $k \neq j \le i$ one has $d(y_j^n, y_k^n) \ge 2\delta$ (if not, a subsequence of (f_n) can be approximated by a Cauchy sequence in $E(X, i-1)$, and the induction hypothesis applies).
Let $d_n = \min\{f_n(x) \colon x \in X\}$.
Then (d_n) is Cauchy, so it has a limit $d \ge 0$; up to some extraction, and if necessary changing the enumeration of the sequence, we can assume that there is $p \ge 1$ and $\delta' > 0$ such that:
- $\forall j \le p \ f_n(y_j^n) \to d$
- $\forall j > p \ \forall n \ f_n(y_j^n) > d + \delta'$.
Let $\varepsilon > 0$, $\alpha = \min(\delta, \delta', \varepsilon)$ and choose M big enough that $n, m \ge M \Rightarrow d(f_n, f_m) < \frac{\alpha}{4}$ and $|f_n(y_j^n) - d| < \frac{\alpha}{4}$ for all $j \le p$.
Then, for $n, m \ge M$ and $j \le p$ one has:
$f_n(y_j^m) < d + \frac{\alpha}{2}$, so there exists $k \le p$ such that $f_n(y_j^m) = f_n(y_k^n) + d(y_j^m, y_k^n)$.
Such a y_k^n has to be at a distance strictly smaller than δ from y_j^m: there is at most one y_k^n that can work, and there is necessarily one. Thus, one obtains, as in the case $i = 1$, that $d(y_k^n, y_j^m) \le \varepsilon$.
This means that one can assume, choosing an appropriate enumeration, that for $k \le p$ each sequence $(y_i^n)_n$ is Cauchy, hence has a limit y_k .
Define then $\tilde{f}_n \colon x \mapsto \min_{1 \le k \le p} \{f_n(y_i^n) + d(x, y_k^n)\}$.

$\tilde{f}_n \in E(X, p)$, and one checks easily, since $y_k^n \to y_k$ for all $k \le p$, that $(\tilde{f}_n)$ converges uniformly to $\tilde{f}$, where $\tilde{f}(x) = \min_{1 \le k \le p} \{f(y_k) + d(x, y_k)\}$.

If $p = i$ then we are finished; otherwise, notice that, using again the induction hypothesis, we may assume that there is $\eta > 0$ such that

$$\forall n\,\forall j > p \qquad f_n(y_j^n) < \tilde{f}_n(y_j^n) - \eta \qquad (***).$$

Now define $\tilde{g}_n$ by $\tilde{g}_n(x) = \min_{j>p}\{f_n(y_j^n) + d(x, y_j^n)\}$.
Choose M such that $n, m \geq M \Rightarrow d(f_n, f_m) < \dfrac{\eta}{4}$ and $d(\tilde{f}_n, \tilde{f}_m) < \dfrac{\eta}{4}$.

Then (***) shows that for, all $n, m \geq M$ and all $j > p$,

$$f_m(y_j^n) \leq f_n(y_j^n) + \frac{\eta}{4} \leq \tilde{f}_n(y_j^n) - \frac{3\eta}{4} \leq \tilde{f}_m(y_j^n) - \frac{\eta}{2}\ ,$$

so that $f_m(y_j^n) = f_m(y_k^m) + d(y_j^n, y_k^m)$ for some $k > p$.
Consequently, for $m, n \geq M$ and $j > p$, $f_m(y_j^n) = \tilde{g}_m(y_j^n)$; by definition, $f_m(y_j^m) = \tilde{g}_m(y_j^m)$.
This proves that for all $n, m \geq M$ one has $d(\tilde{g}_n, \tilde{g}_m) \leq d(f_n, f_m)$, so that $(\tilde{g}_n)$ is Cauchy in $E(X, i-p)$, hence has a limit $\tilde{g} \in E(X, i-p)$ by the induction hypothesis.
But then, (**) shows that, for all $x \in X$, $f(x) = \min(\tilde{f}(x), \tilde{g}(x))$, and this concludes the proof. $\Diamond$

If Y is a nonempty, closed and bounded subset of a metric space X, define

$$E(X,Y) = \{f \in E(X)\colon \exists d \in \mathbb{R}^+\ \forall x \in X\ f(x) = d + d(x,Y)\}$$

$E(X,Y)$ is closed in $E(X)$, and is isometric to $\mathbb{R}^+$.

Now we can go on to the
Proof of theorem 1.1.
Essential to our proof is the fact that for every Polish group G there exists a Polish space (X,d) such that G is isomorphic to the group of isometries of X (This result was proved by Gao and Kechris, see [2]).
So, let G be a Polish group, and X be a metric space such that G is isomorphic to $Iso(X)$.
One can assume that X contains more than two points, and (X,d) is bounded, of diameter $d_0 \leq 1$.(If not, define $d'(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$. Then (X,d') is now a bounded Polish metric space with the same topology as X, and the isometries of (X,d') are exactly the isometries of (X,d)).

Let $X_0 = X$, and define inductively bounded Polish metric spaces X_i, of diameter d_i, by:

$$X_{i+1} = \left\{ f \in E(X_i, i) \cup \bigcup_{j<i} E(X_i, X_j) \colon \forall x \in X_i \ f(x) \leq 2d_i \right\}$$

(We endow X_{i+1} with the sup-metric; since X_i canonically embeds isometrically in X_{i+1} via the Kuratowski map, we assume that $X_i \subseteq X_{i+1}$).
Remark that $d_i \to +\infty$ with i, and that each X_i is a Polish metric space.
Let then Y be the completion of $\bigcup_{i \geq 0} X_i$.
The definition of $\cup X_i$ makes it easy to see that it is finitely injective, so that Y is isometric to $\mathbb{U}$.
Also, any isometry $g \in G$ extends to an isometry of X_i, and for any i and $g \in G$ there is a unique isometry g^i of X_i such that $g^i(X_j) = X_j$ for all $j \leq i$ and ${g^i}_{|X_0} = g$ (same proof as in [4]).
Remark also that the mappings $g \mapsto g^i$, from G to $Iso(X_i)$, are continuous (see [8]).
All this enables us to assign to each g an isometry g^* of Y, given by ${g^*}_{|X_i} = g^i$, and this defines a continuous embedding of G into $Iso(Y)$ (see again [8] for details).
It is important to remark here that, if $f \in X_{i+1}$ is defined by
$f(x) = d + d(x, X_j)$ for some $d \geq 0$ and some $j < i$, then $g^*(f) = f$ for all $g \in G$ (This was the aim of the definition of X_i: adding "many" points that are fixed by the action of G).

Notice that an isometry φ of Y is equal to g^* for some $g \in G$ if, and only if, $\varphi(X_n) = X_n$ for all n.
The idea of the construction is then simply to construct a closed set F such that $\varphi(F) = F$ if, and only if, $\varphi(X_n) = X_n$ for all n. To achieve this, we will build F as a set of carefully chosen "witnesses".

The construction proceeds as follows:
First, let $(k_i)_{i \geq 1}$ be an enumeration of the non-negative integers where every number appears infinitely many times.
Using the definition of the sets X_i, we choose recursively for all $i \geq 1$ points $a_i \in \cup_{n \geq 1} X_n$(the witnesses), non-negative reals e_i, and a nondecreasing sequence of integers (j_i) such that :

- $e_1 \geq 4$; $\forall i \geq 1$ $e_{i+1} > 4e_i$.
- $\forall i \geq 1$ $j_i \geq k_i$, $a_i \in X_{j_i+1}$ and $\forall x \in X_{j_i}$ $d(a_i, x) = e_i + d(x, X_{k_i-1})$
- $\forall i \geq 1$ $\forall g \in G$ $g^*(a_i) = a_i$.

(This is possible, since at step i it is enough to fix $e_i > \max(4e_{i-1}, \mathrm{diam}(X_{k_i}))$, then find $j_i \geq \max(1 + j_{i-1}, k_i)$ such that $\mathrm{diam}(X_{j_i}) \geq e_i$, and define $a_i \in X_{j_i+1}$ by the equation above; then, by definition of g^* and of a_i, one has $g^*(a_i) = a_i$ for all $g \in G$)

Let now $F = X_0 \cup \{a_i\}_{i \geq 1}$; since X_0 is complete, and $d(a_i, X_0) = e_i \to +\infty$, F is closed.
We claim that for all $\varphi \in Iso(Y)$, one has

$$(\varphi(F) = F) \iff (\varphi \in G^*).$$

The definition of F makes one implication obvious.
To prove the converse, we need a lemma:

Lemma 3.2. *If $\varphi \in Iso(F)$, then $\varphi(X_0) = X_0$, so that $\varphi(a_i) = a_i$ for all i. Moreover, there exists $g \in G$ such that $\varphi = {g^*}_{|F}$.*

Admitting this lemma for a moment, it is now easy to conclude:
Notice that lemma 3.2 implies that G is isomorphic to the isometry group of F, and that any isometry of F extends to Y.
Thus, to conclude the proof of theorem 1.1, we only need to show that the extension of a given isometry of F to Y is unique. As explained before, it is enough to show that, if $\varphi \in Iso(Y)$ is such that $\varphi(F) = F$, then $\varphi(X_n) = X_n$ for all $n \geq 0$.
So, let $\varphi \in Iso(Y)$ be such that $\varphi(F) = F$.
It is enough to prove that $\varphi(X_n) \supseteq X_n$ for all $n \in \mathbb{N}$ (since this will also be true for φ^{-1}), so assume that this is not true, i.e there is some $n \in \mathbb{N}$ and $x \notin X_n$ such that $\varphi(x) \in X_n$.
Let $\delta = d(x, X_n) > 0$ (since X_n is complete), and pick $y \in \cup X_m$ such that $d(x, y) \leq \frac{\delta}{4}$.
Then $y \in X_m \setminus X_n$ for some $m > n$; now choose i such that $k_i = n + 1$ and $j_i \geq m$.
Then we know that
$d(\varphi(y), \varphi(a_i)) = d(y, a_i) = e_i + d(y, X_n) \geq e_i + \dfrac{3\delta}{4}$, and

$d(a_i, \varphi(y)) \leq d(a_i, \varphi(x)) + d(x, y) \leq e_i + \dfrac{\delta}{4}$, so that $d(\varphi(a_i), a_i) \geq \frac{\delta}{2}$, and this contradicts lemma 3.2. $\diamondsuit$

It only remains to give the
Proof of lemma 3.2:
Since we assumed that X_0 has more than two points and $\mathrm{diam}(X_0) \leq 1$, the definition of F makes it clear that

$$\forall x \in F \ (x \in X_0) \Leftrightarrow (\exists y \in F\colon \ 0 < d(x, y) \leq 1)$$

The right part of the equivalence is invariant by isometries of F, so this proves that $\varphi(X_0) = X_0$ for any $\varphi \in Iso(F)$. In turn, this easily implies that $\varphi(a_i) = a_i$ for all $i \geq 1$.
Thus, if one lets $g \in G$ be such that $g_{|X_0} = \varphi_{|X_0}$, we have shown that $\varphi = {g^*}_{|F}$. $\diamondsuit$

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