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Groupes d'automorphismes des structures homogènes

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Groupes d'automorphismes des structures homogènes

Résumé

Une structure dénombrable du premier ordre est dite homogène si tout isomorphisme entre deux sous-structures finiment engendrées s'étend en un automorphisme de la structure globale. C'est équivalent à une propriété d'amalgamation des sous-structures finiment engendrées, et les structures homogènes dénombrables sont aussi appelées limites de Fraïssé, en lien avec les travaux de Roland Fraïssé sur l'ordre des rationnels. Cette thèse concerne les groupes d'automorphismes des structures homogènes, avec la question centrale suivante: est-ce que le groupe automorphismes d'une structure homogène est universel pour la classe des groupes d'automorphismes de ses sous-structures ? Nous répondons positivement à cette question pour les structures homogènes dans un langage relationnel et avec la propriété d'amalgamation libre, à l'aide d'une construction par tour assez similaire à une construction de Katetov et Uspenskij dans le cas de l'espace d'Urysohn. Avec des techniques similaires, nous obtenons toute sous-structure dénombrable comme points fixes d'un automorphisme d'ordre fini prédéterminé. Cela nous permet par ailleurs d'étudier la complexité de la relation d'isomorphisme entre sous-structures dénombrables, et de montrer qu'elle se réduit boreliennement à la relation de conjugaison dans le groupe d'automorphismes. Nous continuons avec les éléments d'ordre fini, en supposant de plus que les sous-structures finies satisfont une version forte de la propriété d'extension de Hrushovski-Lascar-Herwig, et des arguments topologiques nous permettent alors de montrer que dans le groupe d'automorphismes tout élément est produit de quatre conjugués de certains éléments d'ordre fini. Nous montrons aussi des résultats similaires pour le groupe d'isométries de l'espace d'Urysohn, ou sa version bornée, la sphère d'Urysohn, en utilisant le fait que ces derniers sont très bien approximés par des espaces métriques rationnels. Enfin, revenant à la question de l'universalité du groupe automorphismes de la limite de Fraïssé, nous considérons la question plus fine de savoir si toute sous-structure dénombrable s'injecte de manière rigide, c'est-à-dire de sorte chacun de ses automorphismes s'étende en un unique automorphisme de la limite de Fraïssé. D'abord, nous introduisons une construction de telle injections rigides dans le cas des graphes homogènes. Ensuite, nous modifions cette construction dans diverses classes de graphes orientés et de structures relationnelles homogènes, pour enfin la faire fonctionner dans un contexte très general de structures dans un langage relationnel fini et avec la propriété d'amalgamation libre.

Automorphisms groups of homogeneous structures

Summary

A countable first-order structure is called homogneous when each isomorphism between two finitely generated substructures extends to an automorphism of the whole structure. This is equivalent to an amalgamation property of finitely generated substructures, and countable homogeneous structures are also called Fraïssé limits, in connection to the work of Roland Fraïssé on the order of rational numbers. The present thesis concerns automorphism groups of homogeneous structures, with the following central question: is it the case that the automorphism group of a homogeneous structure is universal for the class of automorphism groups of its substructures? We answer positively this question for homogeneous structures in a relational langage and with the free amalgamation property, by using a construction rather similar to a construction of Katetov and Uspenskij in the case of the Urysohn space.With similar techniques, we obtain any countable substructure as the set of fixed points of an automorphism of a given finite order. Besides, this allows us to study the complexity of the isomorphism relation between countable substructures, and to show that it Borel reduces to the conjugacy relation in the automorphism group. We continue with elements of finite order, assuming further that finite substructures satisfy a strong version of the Hrushovski-Lascar-Herwig extension property, and topological arguments then allow us to show that in the automorphism group any element is the product of four conjugates of certain elements of finite order. We also show similar results for the isometry group of the Urysohn space, or its bounded version, the Urysohn sphere, by using the fact that they are well approximated by rational metric spaces. Finally, concerning the question of the universality of the automorphism group of a Fraïssé limit, we consider the finer question to know whether any countable substructure embeds in a rigid way, that is, in such a way that each of its automorphisms extends in a unique automorphism of the Fraïssé limit. First, we introduce a construction of such rigid embeddings in the case of homogeneous graphs. Then, we modify this construction in various classes of oriented graphs and of homogeneous relational structures, ultimately to make it work in a very general context of structures in a finite relational langage and with the free amalgamation property.

Dedicated to Zeynep Bilge & Ali Eren

Contents

1 Introduction en Français

Une structure dénombrable du premier ordre est dite *homogène* si tout automorphisme entre deux sous-structures finiment engendrées s'étend en un automorphisme de la structure globale. Roland Fraïssé a découvert dans les années cinquante que l'homogénéité est équivalente à une propriété d'amalgamation des sous-structures finiment engendrées. Même si son travail original concernait essentiellement l'ordre des rationnels, il s'applique pratiquement dans tout contexte, et les structures homogènes sont aussi appelées limites de Fraïssé. Des propriétés similaires furent découvertes bien plus tôt par Urysohn dans les années vingt dans le cas spécifique des espaces métriques. Le premier exemple d'une limite de Fraïssé est bien sûr l'ensemble dénombrable infini sans structure (sauf l'égalité). Son groupe d'automorphismes est particulièrement riche puisqu'il s'agit de tout le groupe symétrique. Ce phénomène est en fait très géneral et la présente thèse concerne l'étude des groupes d'automorphismes des structures homogènes.

Les limites de Fraïssé ont tendance à être universelles relativement à certaines classes de structures finiment engendrées. C'est le cas par exemple de la limite de Fraïssé des graphes finis, qui est en fait universelle pour tous les graphes dénombrables. Alors la question qui apparait, et qui est formulée en termes plus généraux dans [18], est la suivante. *Quand est-ce que le groupe d'automorphismes d'une limite de Fraïssé est universel pour la classe des groupes d'automorphismes de ses sous-structures dénombrables?* Cette question va être le fil directeur de la présente thèse.

Une façon d'approcher cette question est d'essayer de considérer la limite de Fraïssé comme un objet de Cayley, une approche tentée dans [2] et [19] dans des cas combinatoires. Cela tend à produire des groupes qui agissent librement sur la limite de Fraïssé, et en particulier des groupes qui s'injectent comme sous-groupes transitifs dans le groupe de tous les automorphismes. Nous n'allons pas suivre cette approche ici mais plutôt une approche opposée. *Peut-on trouver des injections d'une structure donnée X dans la limite de Fraïssé de telle sorte que le stabilisateur de X dans le groupe de tous les automorphismes soit isomorphe au groupe des automorphismes de X?*

Une manière d'y parvenir est d'injecter X dans la limite de Fraïssé d'une façon telle que chaque automorphisme de X s'étende en un unique automorphisme de la limite de Fraïssé. Il est alors clair que le groupe d'automorphismes de X s'injecte dans le groupe de tous les automorphismes de la limite de Fraïssé; il apparait comme le stabilisateur de telles injections de X. De telles injections de X seront appelées *rigides* et nous considèrerons en détail l'existence de telles injections rigides dans des cas combinatoires dans la Section 8.

Même si X n'est pas injecté de manière rigide dans la limite de Fraïssé, il se peut qu'un automorphisme de X s'étende canoniquement en un unique automorphisme de la limite de Fraïssé. Des constructions plus simples de ce type ont été produites (par Katětov dans [20] et Uspenskij dans [33]) dans le cas métrique. Dans la section 4 nous allons adapter des constructions de ce type à des cas plus combinatoires. En fait, nous allons essayer de le faire dans le contexte des structures relationnelles qui satisfont la propriété de l'amalgamation libre, que nous allons revoir dans la Section 4. Les limites de Fraïssé ne sont pas restreintes aux classes qui satisfont la propriété d'amalgamation libre mais celles avec cette propriété forment une partie importante de la théorie.

Dans le cas de classes avec l'amalgamation libre, étant donné X, nous parvenons à construire des éléments d'ordre fini dans le groupe d'automorphismes de la limite dont l'ensemble des points fixes est isomorphe à X. Il apparait que cette construction a des conséquences sur la complexité borlienne de la relation de conjugaison dans divers groupes d'automorphismes.

Aussi, cette construction d'éléments d'ordre fini avec des propriétés prédestinées nous a conduit à l'étude des éléments d'ordre fini dans les groupes d'automorphismes; dans la Section 6 nous prouvons, sous de fortes hypothèses sur $\mathcal{K}$, qu'il existe des éléments génériques (dans le sens de la catégorie de Baire) d'ordre fini fixé dans $\mathrm{Aut}(\mathbf{K})$, où $\mathbf{K}$ est la limite de Fraïssé de $\mathcal{K}$, et nous prouvons que tout élément dans $\mathrm{Aut}(\mathbf{K})$ est un produit de quatre conjugués de cet élément générique. Dans la Section 7 nous appliquons nos résultats à l'espace d'Urysohn, en utilisant le fait que ce dernier est très bien approximé par des structures relationnelles dénombrables qui satisfont les hypothèse fortes de la Section 6.

Nous donnons maintenant une description plus précise des sections. Dans la Section 3 nous allons revoir la théorie des limites de Fraïssé, principalement dans des langages relationnels, et considérer leurs groupes d'automorphismes, que l'on peut voir comme groupes topologiques. Nous allons aussi revoir le cas spécifique de l'espace d'Urysohn pour les structures métriques.

Dans le Section 4 nous allons nous concentrer sur le cas spécifique des limites de Fraïssé dans un langage relationnel et avec la propriété d'amalgamation libre. En lien avec notre question principale, nous allons prouver le théorème suivant:

Théorème (4.9)**.** *Supposons que $\mathcal{K}$ soit une classe de Fraïssé avec la propriété d'amalgamation libre, et dénotons par $\mathbf{K}$ la limite de Fraïssé de $\mathcal{K}$. Alors, pour tout $\mathbf{X} \in \mathcal{K}_\omega$ infini il existe une injection $i\colon \mathbf{X} \to \mathbf{K}$ tel que tout automorphisme ϕ de $i(\mathbf{X})$ s'étende en un automorphisme $E(\phi)$ de $\mathbf{K}$.*

L'application d'extension $\phi \mapsto E(\phi)$ peut être prise de telle sorte que ce soit une injection continue de groupe de $\mathrm{Aut}(\mathbf{X})$ *dans* $\mathrm{Aut}(\mathbf{K})$*.*

Théorème (4.11)**.** *Soit $n \geq 2$ un entier, supposons que $\mathcal{K}$ soit une classe de Fraïssé avec la propriété d'amalgamation libre, et soit $\mathbf{K}$ la limite de Fraïssé de $\mathcal{K}$. Alors, pour tout $\mathbf{X} \in \mathcal{K}_\omega$, il existe un automorphisme ϕ de $\mathbf{K}$ tel que $\phi^n = 1$ et l'ensemble des points fixes de ϕ, vu comme sous-structure de $\mathbf{K}$, soit isomorphe à $\mathbf{X}$.*

En utilisant une construction similaire par tour, J. Melleray a prouvé dans [29] que pour tout espace polonais X, il existe une isométrie ϕ de l'espace d'Urysohn telle que l'ensemble des points fixes de ϕ soit isométrique à X. Sa preuve fonctionne aussi pour les espaces métriques rationnels (et l'espace d'Urysohn rationnel); la construction présentée ici pourrait aussi fonctionner dans le contexte de l'espace rationnel métrique, et fournir un résultat plus fort (avec ϕ étant additionnellement d'un ordre fini prédéterminé). Cependant, les idées décrites ici ne sont pas suffisantes pour obtenir un résultat similaire pour l'espace d'Urysohn, car nous n'avons aucun contrôle en prenant la complétion de l'espace construit durant la construction par tour.

Nous allons aussi considérer des questions similaires dans le cas du groupe localement fini de Philip Hall, qui est la limite de Fraïssé des groupes finis. Cela sera une de nos quelques excursions en dehors des langages relationnels.

Le Théorème 4.11 nous permet de montrer que, lorsque $\mathcal{K}$ a la propriété d'amalgamation libre, la relation d'isomorphisme sur la classe de structures dont l'age est contenu dans $\mathcal{K}$ est se rd́uit boréliennement à la relation de conjugaison dans $\mathrm{Aut}(\mathbf{K})$ (en fait, elle se réduit même à la relation de conjugaison sur les éléments d'ordre 2 dans $\mathrm{Aut}(\mathbf{K})$); voir la Section 5.1 pour une discussion rapide de la notion de Borel réductibilité. Comme corollaire, nous obtenons le résultat suivant.

Théorème (5.5)**.** *Soit $n \geq 3$ un entier. Alors:*

i. La relation de conjugaison sur $\{g \in \mathrm{Aut}(\mathcal{R})\colon g^n = 1\}$, où $\mathcal{R}$ dénote le graphe aléatoire, est S_∞-universel.

ii. Soit $m \geq 3$ un entier et soit G_m la limite de Fraïssé de la classe de graphes K_m-libres. Alors la relation de conjugaison sur $\{g \in \mathrm{Aut}(G_m)\colon g^n = 1\}$ est S_∞-universelle.

La partie (i) ci-dessus a d'abord été prouvée par Coskey, Ellis et Schneider dans un article récent [5].

Dans la Section 6, motivé par nos résultats préliminaires sur les éléments d'ordre fini, nous étudions les propriétés des éléments d'ordre fini dans des classes qui satisfont une version forte de la propriété d'extension de Hrushovski–Herwig–Lascar(voir [13], [31]), que nous appelons *propriété d'extension isomorphe* (IEP), et une propriété d'amalgamation, appelée *propriété d'amalgamation isomorphe* (IAP), qui se comporte bien envers l'extension de morphismes. Comme les définitions sont un peu techniques nous ne les détaillons pas ici; notons simplement pour l'instant que les exemples de classes $\mathcal{K}$ avec à la fois (IAP) et (IEP) incluent:

- la classe des espaces métriques dénombrables où la distance prend ses valeurs dans un sous-groupe dénombrable de $(\mathbf{R}, +)$.
- La classe des graphes K_n-libres pour un n.

Alors nous obtenons le résultat suivant:

Théorème (6.15)**.** *Soit $\mathcal{K}$ une classe de Fraïssé avec (IAP) et (IEP), et g_i un élément générique d'ordre i. Alors, pour tout quadruple $\bar{i}$ d'entiers permettant les extensions, et tout $g \in G$, il existe $h_1, \dots, h_4$ tel que chaque h_j soit conjugué à g_{i_j} et $g = h_1 \dots h_4$.*

En particulier quand i est un seul entier permettant les extensions, cela montre que chaque élément de G est un produit de quatre conjugués de g_i.

Dans la Section 7, nous nous tournons vers des applications de ces résultats pour l'étude du groupe d'isométries de l'espace d'Urysohn $\mathbf{U}$ et sa version bornée, la sphère d'Urysohn $\mathbf{U}_1$. La raison pour laquelle nos résultats précédents peuvent s'appliquer est que pour tout uple fini $\bar{g}$ d'éléments de $\mathrm{Iso}(\mathbf{U})$, il existe un sous-espace métrique dense dénombrable X de $\mathbf{U}$ qui est isométrique à la limite d'une classe de Fraïssé d'espaces métriques comme ceux étudiés dans la Section 6, et tel que $\bar{g}$ stabilise X. Alors nous pouvons appliquer des résultats sur le groupe d'automorphismes de X pour en déduire des résultats sur $\mathrm{Iso}(\mathbf{U})$. Ce raisonnement nous conduit au résultat suivant.

Théorème (7.2)**.** *Tout élément de $\mathrm{Iso}(\mathbf{U})$ est un commutateur et un produit d'au plus quatre éléments d'ordre n pour tout $n \geq 2$. Le même résultat est valide pour $\mathrm{Iso}(\mathbf{U}_1)$.*

Cela nous amène à étudier les éléments d'ordre fini de $\mathrm{Iso}(\mathbf{U})$ et $\mathrm{Iso}(\mathbf{U}_1)$; nous prouvons qu'il y a des éléments génériques d'ordre n pour tout n (ce qui peut surprendre, puisque les classes de conjugaison sont maigres dans chaque groupe) et obtenons l'énoncé suivant:

Théorème (7.7)**.** *Pour tout entier n il existe un élément g_n dont la classe de conjugaison est comaigre dans $\{g \in \mathrm{Iso}(\mathbf{U})\colon g^n = 1\}$. Chaque $g \in \mathrm{Iso}(\mathbf{U})$ est un produit d'au plus quatre conjugués de g_n.*

Un résultat similaire est aussi valide pour $\mathrm{Iso}(\mathbf{U}_1)$. Finalement, nous parvenons à injecter l'élément générique d'ordre 2 dans un *flot* continu de $(\mathbf{R}, +)$ dans $\mathrm{Iso}(\mathbf{U})$ ou $\mathrm{Iso}(\mathbf{U}_1)$.

Théorème (7.10)**.** *Soit n un entier. Alors un élément générique de $\Omega_n(\mathrm{Iso}(\mathbf{U}))$ s'injecte dans un flot. Dans le cas de $\mathrm{Iso}(\mathbf{U}_1)$, un élément générique s'injecte dans un flot qui est n-Lipschitzien de $(\mathbf{R}, +)$ dans $(\mathrm{Iso}(\mathbf{U}_1), d_u)$.*

(ici d_u dénote la métrique *uniforme* sur $\text{Iso}(\mathbf{U}_1)$). Un corollaire immédiat de cela et de nos résultats précédents est que $\text{Iso}(\mathbf{U}_1, d_u)$ est connexe par arcs.

Le contenu des sections 4, 5, 6 et 7 est l'objet d'un article en commun avec Melleray soumis pour publication et intitulé *"Elements of finite order in automorphism groups of homogeneous structures"*.

Dans la Section 8 nous allons considérer la réponse la plus avancée pour l'universalité du groupe d'automorphismes, c'est-à-dire l'existence d'injections rigides dans la limite de Fraïssé. Là l'histoire débute dans le cas des graphes. Les graphes dénombrables homogènes ont été classifiés par Lachlan et Woodrow dans [25] et Cherlin a donné une autre preuve dans [3]. Nous allons d'abord revoir l'existence d'injections rigides dans le graphe aléatoire, vues d'abord par Henson dans [10]. Ensuite nous allons prouver un résultat similaire dans le cas de ses graphes K_n-libres de [10]. Il y avait apparemment une lacune sérieuse dans la preuve de Henson de cet énoncé dans ce cas, et la méthode que nous allons introduire pour combler cette lacune sera la base de tous les résultats suivants. Alors nous allons considérer d'autres classes de graphes ou de structures relationnelles et finalement passer à un cadre de travail général où nous montrons l'existence d'injections rigides dans une classe d'amalgamation libre (presque générale). Le théorème le plus fort que nous obtiendrons sera le suivant:

Theorem (8.30)**.** *Soit $\mathcal{K}$ une classe d'amalgamation libre et pas totalement déconnectée dans un langage relationnel fini $\mathcal{L}$ et telle que tous les singletons de $\mathcal{K}$ soient isomorphes. Soit T une structure infinie dans $\mathcal{K}_\omega$. Alors T s'injecte comme une moitié rigide de $\mathbf{K}$. De plus, il y a 2^ω classes de conjugaison de telles injections sous l'action de $Aut(\mathbf{K})$.*

Pour des références à propos des structures homogènes nous renvoyons à [15, Chapter 7] ou à l'exposition récente par Macpherson [27].

2 Introduction in English

A countable first-order structure is *homogeneous* if any isomorphism between two finitely generated substructures extends to automorphism of the full structure. It was discovered by Roland Fraïssé in the fifties that homogeneity is equivalent to an amalgamation property of finitely generated substructures. Even though his original work was mainly concerned with the order of the rationals, it applies basically in any context, and countable homogeneous structures are also called Fraïssé limits. Similar properties were discovered much earlier by Urysohn in the twenties in the specific case of metric spaces. The first example of a Fraïssé limit is of course the countably infinite set with no structure (except equality). Its automorphism group is a particularly rich group since it is the full symmetric group. This phenomenon is indeed quite general and the present thesis is about the study automorphism groups of homogeneous structures.

Fraïssé limits tend to be universal relatively to a certain class of finitely generated structures. For example this property holds for the Fraïssé limit of finite graphs, which is indeed universal for all countable graphs. Now the question arising, and formulated in more general terms in [18], is the following. *When is it the case that the automorphism group of a Fraïssé limit is universal for the class of automorphism groups of its countable substructures?* This general question is going to be the main theme of the present thesis.

One way to approach this question is to try to consider the Fraïssé limit as a Cayley object, an approach taken in [2] and [19] in specific combinatorial cases. This tends to give groups that act freely on the Fraïssé limit, and in particular groups that embed as transitive subgroups of the full automorphism group. We will not consider this approach here but rather an opposite one. *Can one find embeddings of a given structure X into the Fraïssé limit such that the setwise stabilizer of X in the full automorphism group is isomorphic to the automorphism group of X?*

One way to achieve this is to embed X into the Fraïssé limit in such a way that each automorphism of X extends to a unique automorphism of the Fraïssé limit. Then it is clear that the automorphism group of X embeds into the automorphism group of the Fraïssé limit; it will appear exactly as the setwise stabilizers of such embeddings of X. Such embeddings of X will be called *rigid* and we will consider the existence of such rigid embeddings at length in combinatorial cases in Section 8.

Even if X is not embedded rigidly into the Fraïssé limit, it might happen that an automorphism of X extends canonically to a unique automorphism of the Fraïssé limit. Simpler constructions of this type were done (by Katětov in [20] and Uspenskij in [33]) in the metric case. In Section 4 we will adapt constructions of this type to more combinatorial cases. Namely, we will try to do this in the context of relational structures where the class of finite structures satisfy the free amalgamation property, which will be reviewed in Section 4 . Fraïssé limits are not restricted to classes satisfying the free amalgamation property but those with this property form an important part of the theory.

In the case of classes with free amalgamation, given X, we manage to build elements of finite order in the automorphism group of the limit whose set of fixed points is isomorphic to X. This construction turns out to have consequences on the Borel complexity of the relation of conjugacy in various automorphism groups.

Also, this construction of elements of finite order with preordained properties led us to the study of elements of finite order in automorphism groups; in Section 6 we prove, under strong assumptions on $\mathcal{K}$, that there exist generic (in the sense of Baire category) elements of some fixed finite order in $\mathrm{Aut}(\mathbf{K})$, where $\mathbf{K}$ is the Fraïssé limit of $\mathcal{K}$, and show that any element in $\mathrm{Aut}(\mathbf{K})$ is a product of four conjugates of this generic element. In Section 7 we apply our results to the

Urysohn space, using the fact that it is very well approximated by countable relational structures satisfying the strong assumptions of Section 6.

We now give a more detailed description of the sections. In Section 3 we will review the theory of Fraïssé limits, mostly in relational languages, and consider their automorphism groups, which can be seen as topological groups. We will also review the specific case of the Urysohn space for metric structures.

In Section 4 we will focus on the specific case of Fraïssé limits of classes in a relational language and with the free amalgamation property. In connection to our main question, we will prove the following:

Theorem (4.9)**.** *Assume that $\mathcal{K}$ is a Fraïssé class with the free amalgamation property, and denote by $\mathbf{K}$ the Fraïssé limit of $\mathcal{K}$. Then, for any infinite $\mathbf{X} \in \mathcal{K}_\omega$ there exists an embedding $i\colon \mathbf{X} \to \mathbf{K}$ such that any automorphism ϕ of $i(\mathbf{X})$ extends to an automorphism $E(\phi)$ of $\mathbf{K}$.*

This extension map $\phi \mapsto E(\phi)$ may be taken to be a continuous group embedding from $\mathrm{Aut}(\mathbf{X})$ to $\mathrm{Aut}(\mathbf{K})$.

Theorem (4.11)**.** *Let $n \geq 2$ be an integer, assume that $\mathcal{K}$ is a Fraïssé class with the free amalgamation property, and let $\mathbf{K}$ denote the Fraïssé limit of $\mathcal{K}$. Then, for any $\mathbf{X} \in \mathcal{K}_\omega$, there exists an automorphism ϕ of $\mathbf{K}$ such that $\phi^n = 1$ and the set of fixed points of ϕ, seen as a substructure of $\mathbf{K}$, is isomorphic to $\mathbf{X}$.*

Using a similar tower construction, J. Melleray proved in [29] that for any Polish metric space X, there exists an isometry ϕ of the Urysohn space such that the set of fixed points of ϕ is isometric to X. His proof also works for rational metric spaces (and the rational Urysohn space); the construction presented here would also apply in the rational metric space setting, and provide a stronger result (with ϕ being additionally of some preordained finite order). However, the ideas described here are not sufficient to obtain a similar result for the Urysohn space, because we have no control of what happens when taking the metric completion of the space built during the tower construction.

We will also consider similar questions in the case of Philip Hall's locally finite group, which is the Fraïssé limit of finite groups. This will be one of the few excursions outside of relational languages.

Theorem 4.11 enables one to show that, whenever $\mathcal{K}$ has the free amalgamation property, the relation of isomorphism on the class of structures whose age is contained in $\mathcal{K}$ Borel reduces to the relation of conjugacy in $\mathrm{Aut}(\mathbf{K})$ (actually, it even reduces to the relation of conjugacy on elements of order 2 in $\mathrm{Aut}(\mathbf{K})$); see Section 5.1 for a quick discussion of the notion of Borel reducibility. As a corollary, we obtain the following result.

Theorem (5.5)**.** *Let $n \geq 2$ be an integer. Then:*

i. The conjugacy relation on $\{g \in \mathrm{Aut}(\mathcal{R})\colon g^n = 1\}$, where $\mathcal{R}$ denotes the random graph, is S_∞-universal.

ii. Let $m \geq 3$ be an integer and G_m denote the Fraïssé limit of the class of K_m-free graphs. Then the conjugacy relation on $\{g \in \mathrm{Aut}(G_m)\colon g^n = 1\}$ is S_∞-universal.

Part (i) above was first proved by Coskey, Ellis and Schneider in a recent paper [5].

In Section 6, motivated by our earlier results about elements of finite order, we study properties of elements of finite order in classes which satisfy a strong version of the Hrushovski–Lascar–Herwig extension property (see [13], [31]), which we call the *isomorphic extension property* (IEP) and an amalgamation property, called the *isomorphic amalgamation property* (IAP) that behaves well with respect to extension of morphisms. Since the definitions are a bit technical we do not detail them here; let us just note for now that examples of classes $\mathcal{K}$ with both the (IAP) and the (IEP) include:

- the class of countable metric spaces whose distance takes values in a countable subgroup of $(\mathbf{R}, +)$.
- The class of K_n-free graphs for some n.

Then we obtain the following result:

Theorem (6.15)**.** *Let $\mathcal{K}$ be a Fraïssé class with the (IAP) and the (IEP), $\mathbf{K}$ be the Fraïssé limit and g_i be a generic element of order $i \geqslant 2$. Then, for any quadruple $i_1, \ldots, i_4$ of integers allowing extensions, and any $g \in Aut(\mathbf{K})$, there exist $h_1, \ldots, h_4$ such that each h_j is conjugate to g_{i_j} and $g = h_1 \ldots h_4$.*

In the particular when i is a single integer allowing extensions, this shows that every element of G is a product of four conjugates of g_i.

In Section 7, we turn to applications of our earlier results to the study of the isometry group of the Urysohn space $\mathbf{U}$ and its bounded counterpart, the Urysohn sphere $\mathbf{U}_1$. The reason why our earlier results can be applied is that for any finite tuple $\bar{g}$ of elements of $\mathrm{Iso}(\mathbf{U})$, there exists a dense countable metric subspace X of $\mathbf{U}$ which is isometric to the limit of a Fraïssé class of metric spaces as studied in Section 6, and such that $\bar{g}$ stabilizes X. Then one can apply results about the automorphism group of X to deduce results about $\mathrm{Iso}(\mathbf{U})$. This reasoning yields the following result.

Theorem (7.2)**.** *Every element of* $\mathrm{Iso}(\mathbf{U})$ *is a commutator and a product of at most four elements of order n for all $n \geq 2$. The same result is true for* $\mathrm{Iso}(\mathbf{U}_1)$*.*

This leads us to studying elements of finite order in $\mathrm{Iso}(\mathbf{U})$ and $\mathrm{Iso}(\mathbf{U}_1)$; we show that there are generic elements of order n for all n (perhaps surprisingly, since conjugation classes are meager in both groups) and obtain the following:

Theorem (7.7)**.** *For any integer n there exists an element g_n whose conjugacy class is comeager in $\{g \in \mathrm{Iso}(\mathbf{U}) \colon g^n = 1\}$. Any $g \in \mathrm{Iso}(\mathbf{U})$ is a product of at most four conjugates of g_n.*

A similar result holds for $\mathrm{Iso}(\mathbf{U}_1)$. Finally, we manage to embed the generic element of order 2 in a continuous *flow* from $(\mathbf{R}, +)$ into $\mathrm{Iso}(\mathbf{U})$ or $\mathrm{Iso}(\mathbf{U}_1)$.

Theorem (7.10)**.** *Let n be an integer. Then a generic element of $\Omega_n(\mathrm{Iso}(\mathbf{U}))$ embeds in a flow. In the case of* $\mathrm{Iso}(\mathbf{U}_1)$*, a generic element embeds in a flow which is n-Lipschitz from $(\mathbf{R}, +)$ to $(\mathrm{Iso}(\mathbf{U}_1), d_u)$.*

(here d_u denotes the *uniform* metric on $\mathrm{Iso}(\mathbf{U}_1)$). An immediate corollary of this is and our earlier results is that $\mathrm{Iso}(\mathbf{U}_1, d_u)$ is path-connected.

The material included in sections 4, 5, 6 and 7 is joint work with Melleray and is submitted for publication under the title of *"Elements of finite order in automorphism groups of homogeneous structures"*.

In Section 8 we will consider the most advanced answer for the universality of the automorphism group, namely the existence of rigid embeddings into the Fraïssé limit. Here the story starts in the case of graphs. Countable homogeneous graphs were classified by Lachlan and Woodrow in [25] and Cherlin gave another proof in [3]. We will first review the existence of rigid embeddings into the random graph, first done by Henson in [10]. Then we will prove a similar statement in the case of his K_n-free graphs in [10]. There was apparently a serious gap in Henson's proof of this statement in this case, and the method we will introduce to fill this gap will be the basis of all subsequent results. Then we will consider other classes of graphs or of relational structures and finally pass to a general framework where we show the existence of rigid embeddings in a (almost general) free amalgamation class. The strongest theorem we will obtain is the following:

Theorem (8.30)**.** *Let $\mathcal{K}$ be a not totally disconnected free amalgamation class in a finite relational language $\mathcal{L}$ and assume that all the one-point structures in $\mathcal{K}$ are isomorphic. Let T be an infinite structure in $\mathcal{K}_\omega$. Then T embeds as a rigid moiety into $\mathbf{K}$. Moreover, there are 2^ω many such embeddings which are not conjugate in $Aut(\mathbf{K})$.*

For general references about homogeneous structures we refer to [15, Chapter 7] or the recent survey by Macpherson [27].

3 Preliminaries

3.1 Fraïssé classes and limits

We begin by recalling the basic vocabulary of model theory; we refer to [15] for a more detailed treatment.

Definition 3.1. A (relational) *language* is a family $\mathcal{L} = (R_i, n_i)_{i \in I}$ where R_i is a relation symbol for all $i \in I$ and n_i is a nonnegative integer indicating the arity of R_i. We say that $\mathcal{L}$ is countable if I is (note: in this thesis, "countable" means finite or equipotent to ω).

Convention. In this thesis, all the languages are assumed to be relational and countable, except in some cases we will indicate specifically.

Definition 3.2. Given a language $\mathcal{L} = (R_i, n_i)_{i \in I}$, an $\mathcal{L}$*-structure* $\mathbf{M}$ is :

- A set M (the *universe* of $\mathbf{M}$).
- A family of subsets $R_i^{\mathbf{M}}$ of M^{n_i}.

For any $i \in I$, any $\bar{x} \in M^{n_i}$, we write

$$\mathbf{M} \models R(\bar{x}) \Leftrightarrow \bar{x} \in R_i^{\mathbf{M}} .$$

We say that $\mathbf{M}$ is *countable* if its universe is. We always assume that $\mathcal{L}$ contains a distinguished relational symbol $=$ which is interpreted by the equality relation in any $\mathcal{L}$-structure.

Definition 3.3. Let $\mathcal{L} = (R_i, n_i)_{i \in I}$ be a language and $\mathbf{M}$ be a $\mathcal{L}$-structure. A *substructure* of $\mathbf{M}$ is an $\mathcal{L}$-structure $\mathbf{N}$ such that:

- The universe N of $\mathbf{N}$ is a subset of the universe M of $\mathbf{M}$.
- For any $i \in I$, $R_i^{\mathbf{N}} = R_i^{\mathbf{N}} \cap M^{n_i}$, in other words

$$\forall \bar{x} \in N^{n_i} \ (\mathbf{N} \models R_i(\bar{x})) \Leftrightarrow (\mathbf{M} \models R_i(\bar{x}))$$

An *embedding* from a $\mathcal{L}$-structure $\mathbf{N}$ to another $\mathcal{L}$-structure $\mathbf{M}$ is a map $f \colon N \to M$ such that, for any $i \in I$, and any $(x_1, \ldots, x_{n_i}) \in N^{n_i}$, one has

$$\mathbf{N} \models R(x_1, \ldots, x_{n_i}) \Leftrightarrow \mathbf{M} \models R(f(x_1), \ldots, f(x_{n_i})) .$$

Note that, since our language contains a special symbol for the equality relation, an embedding is necessarily injective. Also, as we are working over a relational language, the empty structure is a substructure of any $\mathcal{L}$-structure.

An *isomorphism* is a surjective embedding (thus, an embedding from $\mathbf{N}$ into $\mathbf{M}$ is the same thing as an isomorphism of $\mathbf{N}$ with a substructure of $\mathbf{M}$).

Definition 3.4. If T is a graph, and A is a subset of vertices of T, $T|A$ denotes the induced subgraph of T on A. Note that this corresponds to the notion of substructure defined above.

It is easy to get bogged down in notation in our setting, so, to simplify things a little bit, we will often use the same symbol for a structure and its universe, and write things like "$M \models R(\bar{x})$" or "let A be a substructure of M", etc. Also, if $f \colon M \to N$ is a function and $\bar{x} = (x_1, \dots, x_k) \in M^k$, we will denote by $f(\bar{x})$ the tuple $(f(x_1), \dots, f(x_k))$.

We now recall the terminology of Fraïssé classes.

Definition 3.5. Let $\mathcal{L}$ be a language, and $\mathbf{M}$ be a $\mathcal{L}$-structure. The *age* of $\mathbf{M}$ is the collection of all finite $\mathcal{L}$-structures which are isomorphic to a substructure of $\mathbf{M}$.

Definition 3.6. Let $\mathcal{L}$ be a language and $\mathcal{K}$ be a class of finite $\mathcal{L}$-structures. One says that:

i. $\mathcal{K}$ is *countable* if $\mathcal{K}$ has countably many members up to isomorphism.

ii. $\mathcal{K}$ is *hereditary* if $\mathcal{K}$ is closed under embeddings, i.e if $A \in \mathcal{K}$ and B is an $\mathcal{L}$-structure embedding in A then $B \in \mathcal{K}$.

iii. $\mathcal{K}$ has the *joint embedding property* if for any $A, B \in \mathcal{K}$ there exists $C \in \mathcal{K}$ such that both A, B embed in C.

iv. $\mathcal{K}$ has the *amalgamation property* if for any $AB, C \in \mathcal{K}$, and any embeddings $i \colon A \to B$, $j \colon A \to C$, there exists $D \in \mathcal{K}$ and embeddings $\beta \colon B \to D$, $\gamma \colon C \to D$ such that $\beta \circ i = \gamma \circ j$.

A class satisying the four properties above is called a *Fraïssé class*.

It is clear that, whenever $\mathbf{M}$ is a countable $\mathcal{L}$-structure, its age satisfies the first three properties above. The last one, however, is not satisfied in general; note that, since all our languages are relational, in our context the amalgamation property implies the joint embedding property, since the empty structure must belong to $\mathcal{K}$ if $\mathcal{K}$ is nonempty and joint embedding is the same thing as amalgamating over the empty set.

Definition 3.7. Let $\mathcal{L}$ be a language, and $\mathbf{M}$ be a countable $\mathcal{L}$-structure. We say that $\mathbf{M}$ is *homogeneous* if any isomorphism between finite substructures of $\mathbf{M}$ exends to an automorphism of $\mathbf{M}$.

It is easy to see that whenever $\mathbf{M}$ is homogeneous, its age satisfies the amalgamation property. The converse is true, in the following sense.

Theorem 3.8 (Fraïssé [6])**.** *Let $\mathcal{L}$ be a language and $\mathcal{K}$ a Fraïssé class of $\mathcal{L}$-structures. Then there exists a unique (up to isomorphism) $\mathcal{L}$-structure $\mathbf{K}$ which is homogeneous and such that age$(\mathbf{K}) = \mathcal{K}$. This structure is called the* Fraïssé limit *of $\mathcal{K}$.*

Note that the Fraïssé limit $\mathbf{M}$ of a class $\mathcal{K}$ of $\mathcal{L}$-structures may be characterized, up to isomorphism, by the following universal property (sometimes called Alice's Restaurant axiom):

For any finite $A \subseteq M$, any $B \in \mathcal{K}$ and any embedding $j \colon A \to B$, there exists $\tilde{B} \subseteq M$ such that $A \subseteq \tilde{B}$ and an isomorphism $\phi \colon B \to \tilde{B}$ such that $\phi \circ j = i$, where i denotes the inclusion map from A to $\tilde{B}$.

Less formally, this is saying that any extension of A to a finite $\mathcal{L}$-structure belonging to $\mathcal{K}$ is realized by a subsructure of $\mathbf{M}$ containing A.

3.2 Automorphism groups of countable structures

Definition 3.9. Let $\mathcal{L}$ be a language and $\mathbf{M}$ be a countable $\mathcal{L}$-structure. We denote its automorphism group by $\mathrm{Aut}(\mathbf{M})$, and endow it with the *permutation group topology*, whose basic open sets are of the form

$$\{g \in \mathrm{Aut}(\mathbf{M}) \colon g(m_1) = n_1, \ldots, g(m_k) = n_k\}$$

where k is an integer and m_i, n_i are elements of M for all $i \in \{1, \ldots, k\}$.

The topology defined above is a group topology (i.e, the group operations are continuous), and it is Polish, which means that there exists a complete separable metric inducing the permutation group topology on $\mathrm{Aut}(\mathbf{M})$.

A particularly important example is the case when $\mathcal{L}$ is reduced to the equality symbol, and M is a countable infinite set. Then its automorphism group is the group of all permutations of a countable infinite set; we denote this group by S_∞. To describe the topology a bit more explicitly in this case, assume that $M = \mathbf{N}$ is the set of integers. For any $\sigma, \tau \in S_\infty$, define

$$n(\sigma, \tau) = \inf\{i \colon \sigma(i) \neq \tau(i)\} \in \mathbf{N} \cup \{+\infty\}$$

and let

$$d(\sigma, \tau) = 2^{-n(\sigma,\tau)} + 2^{-n(\sigma^{-1},\tau^{-1})} \ .$$

Then d is a complete metric on S_∞ inducing the permutation group topology. Closed subgroups of S_∞ are interesting from the decriptive set theoretic point of view for many reasons and are intimately linked with automorphism groups of countable structures because of the following folklore result.

Theorem 3.10 (see e.g. [1])**.** *Let $\mathcal{L}$ be a language and $\mathbf{M}$ a countable $\mathcal{L}$-structure. Then $\mathrm{Aut}(\mathbf{M})$, endowed with its permutation group topology, is isomorphic (as a topological group) to a closed subgroup of S_∞.*

Conversely, for any closed subgroup G of S_∞, there exists a language $\mathcal{L}$ and a homogeneous *$\mathcal{L}$-structure $\mathbf{M}$ such that G is isomorphic (as a topological group) to $\mathrm{Aut}(\mathbf{M})$ endowed with its permutation group topology.*

In the remainder of this thesis, whenever we mention a topological property of $\mathrm{Aut}(\mathbf{M})$, it is implicitly assumed that we are talking about the permutation group topology. The fact that $\mathrm{Aut}(\mathbf{M})$ is a Polish group, so we can use Baire category techniques, will be essential to us. We refer the reader to [21], [8] and references therein for information on Polish groups and the Baire-category vocabulary and techniques.

3.3 The Urysohn space

In this subsection we define the Urysohn space and its variations. First, consider the class of finite metric spaces equipped with rational distances; we can see this as a class of first-order structures in a countable language (add a binary relational symbol R_q for each rational number q, and set $R_q(x, y)$ iff $d(x, y) = q$). It can easily be checked that it has the amalgamation property, and hence it is a Fraïssé class. The Fraïssé limit of this class is called the rational Urysohn space. It is in particular a countable rational homogeneous metric space. A similar space can be defined for uniformly bounded finite rational metric spaces and it gives rise to the bounded rational Urysohn space (of a given diameter, which can be assumed to be 1).

By taking the completion of these two metric spaces, we get respectively the Urysohn space $\mathbf{U}$ and the bounded Urysohn space $\mathbf{U}_1$ (by restricting ourselves to spaces of diameter $\leqslant 1$). The metric space $\mathbf{U}$ is characterized up to isometry, among all separable complete metric spaces, by the following property:

> For any finite subset $A \subseteq \mathbf{U}$, for any abstract one-point metric extension $A \cup \{z\}$ of A, there exists $\tilde{z} \in \mathbf{U}$ such that $d(\tilde{z}, a) = d(z, a)$ for all $a \in A$.

Of course, $\mathbf{U}_1$ is characterized by a similar condition among complete separable metric spaces of diameter at most 1.

The isometry group of $\mathbf{U}$ may be endowed with the pointwise convergence topology; a basis of open neighborhoods for $g \in \mathrm{Iso}(\mathbf{U})$ is given by

$$\{h \in \mathrm{Iso}(\mathbf{U}) \colon \forall a \in A \; d(g(a), h(a)) < \epsilon\},$$

where A is a finite subset of $\mathbf{U}$ and $\epsilon > 0$.

A similar definition makes sense for $\mathrm{Iso}(\mathbf{U}_1)$. Endowed with this topology, $\mathrm{Iso}(\mathbf{U}_1)$ and $\mathrm{Iso}(\mathbf{U})$ are Polish groups; given the groups we discussed in the previous section, we should mention that they are not isomorphic (as topological, or even abstract groups) to subgroups of S_∞. A theorem of Uspenskij [33] states that any other Polish group embeds, as a topological group, into either of those two groups, i.e they are *universal* Polish groups.

4 Free amalgamation and tower constructions

A particularly convenient way to study automorphism groups of Fraïssé limits is given by tower constructions. These work as follows: Inductively build an increasing chain of structures while making sure that the union of this chain will have the universality property characterizing Fraïssé limits. The point is that during the construction one can enforce some properties that will be then hold true in the limit. For instance one can use this to show that the automorphism group of the random graph is universal for the automorphism groups of countable graphs. This actually extends to classes with the free amalgamation property which we discuss below. The key idea in our constructions is that, given a Fraïssé class $\mathcal{K}$ with the free amalgamation property, and a structure X whose age is contained in $\mathcal{K}$, one can build a space $E(X)$ of "1-point extensions" of X in such a way that $E(X)$ is again a countable structure whose age is contained in $\mathcal{K}$ and automorphisms of X uniquely extend to automorphisms of $E(X)$. This idea was previously used in the context of metric spaces by Katětov [20].

4.1 Free amalgams

We quickly recall the definition of a free amalgam; given our setup below, we adopt a presentation that differs slightly from the classical approach.

Let $\mathcal{L}$ be a language, $\mathcal{A}$ an $\mathcal{L}$-structure, $(\mathcal{X}_j)_{i\in J}$ a family of $\mathcal{L}$-structures and $f_j\colon \mathcal{A} \to \mathcal{X}_j$ be an embedding from A to X_i for all j. We define an $\mathcal{L}$-structure $\mathcal{Y}$, which we will call the free amalgam of the family $(\mathcal{X}_j)$ over the embeddings f_j, or less formally *the free amalgam of the* X_j*'s over* A, as follows:

- First, we consider the disjoint union $Z = \sqcup_j X_j$, and define an equivalence relation $\sim$ on Z by saying that $f_j(a) \sim f_k(a)$ for all j,k and there are no other nontrivial relations. Then we set $Y = \sqcup_j X_{j_{/\sim}}$, and let Y be the universe of $\mathcal{Y}$.
- Next, we need to turn Y into a $\mathcal{L}$-structure; modulo the obvious identifications, we view X_j as a subset of Y, so that $Y = \cup X_j$, $X_j \cap X_k = A$ for all j,k. Then, if n is an integer and R is a n-ary relation symbol of $\mathcal{L}$, for any $\bar{y} \in Y^n$ we set

$$(Y \models R(\bar{y})) \Leftrightarrow (\exists j \in J\ \forall k \in \{1,\dots,n\}\ y_k \in X_j \text{ and } X_j \models R(\bar{y}))\ .$$

Informally, the free amalgam of the X_j's over A is an $\mathcal{L}$-structure Y with universe $\cup X_j$, with $X_j \cap X_k = A$ for all $j \neq k$, such that each X_j is a substructure of Y and no tuple which meets $X_j \setminus A$ and $X_k \setminus A$ for some $j \neq k$ satisfies any relation in $\mathcal{L}$.

Definition 4.1. Let $\mathcal{K}$ be a language. We say that a class $\mathcal{K}$ of $\mathcal{L}$-structures has the *free amalgamation property* if whenever $A, B, C \in \mathcal{K}$ and $i\colon A \to B$, $j\colon A \to C$ are embeddings, the free amalgam of B, C over i, j belongs to $\mathcal{K}$.

Example 1. The class of all finite graphs (seen as a class of structures in the language $\mathcal{L}$ whose only relation symbol besides the equality is binary) has the free amalgamation property, while the class of tournaments does not (it does have the amalgamation property). If K_m denotes the complete graph on m vertices, a graph which does not contain a substructure isomorphic to K_m is called a *K_m-free graph*. The class of all K_m-free graphs also has the free amalgamation property and its Fraïssé limit for $n \geqslant 3$, denoted $\mathbf{K_m}$, is characterized by the property that for any finite disjoint subsets A and B such that A is K_{m-1}-free, there exists a vertex $v \in \mathbf{K_m}$ such that v is adjacent to every vertex in A and not adjacent to any vertex in B. ([15, Exercise 7.4-7])

The other conditions defining a Fraïssé class are easy to check in all the examples above; the Fraïssé limit of the class of all finite graphs is often called the *Radó graph*, or the *random graph*, while the various Fraïssé limits of K_m-free graphs (as m varies) are sometimes called the Henson graphs; this is because C. Ward Henson was the first to consider those graphs in [10].

4.2 The class $\mathcal{K}_\omega$

In this subsection, we fix a language $\mathcal{L}$ and a class $\mathcal{K}$ of finite $\mathcal{L}$-structures, and we assume that $\mathcal{K}$ is a Fraïssé class with the free amalgamation property.

Definition 4.2. We let $\mathcal{K}_\omega$ denote the class of all countable $\mathcal{L}$-structures whose age is contained in $\mathcal{K}$.

Observe that, if $\mathbf{M}$ is an $\mathcal{L}$-structure admitting an increasing chain of substructures $\mathbf{M}_i$ such that each $\mathbf{M}_i$ belongs to $\mathcal{K}_\omega$ and $\mathbf{M} = \cup \mathbf{M}_i$, then $\mathbf{M}$ also belongs to $\mathcal{K}_\omega$.

Proposition 4.3. *Assume that $\mathbf{A} \in \mathcal{K}_\omega$, $(\mathbf{X}_i)_{i\in I}$ is a countable family of element of $\mathcal{K}_\omega$ and that $f_i \colon \mathbf{A} \to \mathbf{X}_i$ is an embedding for all i. Then the free amalgam of the $\mathbf{X}_i$'s over $\mathbf{A}$ belongs to $\mathcal{K}_\omega$.*

Proof. Since the union of an increasing chain of elements of $\mathcal{K}_\omega$ still belongs ot $\mathcal{K}_\omega$, it is clearly enough to prove that whenever $A, B, C \in \mathcal{K}_\omega$ and A embeds in B, C, then the free amalgam D of B and C over A belongs to $\mathcal{K}_\omega$. As in the definition of the free amalgam, we view D as being equal to $B \cup C$ with $B \cap C = A$. Pick a nonempty finite substructure Z of D, and let $Z_B = Z \cap B$, $Z_C = Z \cap C$. If either Z_B, Z_C is empty then $Z \in \mathcal{K}$ because both B and C belong to $\mathcal{K}_\omega$, so we assume that Z_B, Z_C are nonempty. Then $Z_A = Z_B \cap Z_C$ belongs to $\mathcal{K}$, and Z is the free amalgam of Z_B and Z_C over Z_A, so $Z \in \mathcal{K}$. □

Definition 4.4. For $X \in \mathcal{K}_\omega$, denote by $\mathcal{L}_X$ the language obtained by adding to $\mathcal{L}$ a constant symbol c_x for all $x \in X$.[1]

If $Y = X \cup \{y\}$ is an element of $\mathcal{K}_\omega$, $y \notin X$, seen as an $\mathcal{L}_X$-structure by interpreting each c_x by x, the *quantifier-free type of y over X* is the family of all quantifier-free $\mathcal{L}_X$-formulas ϕ with one free variable z such that $Y \models \phi(y)$.

We say that a set p of quantifier-free $\mathcal{L}_X$ formulas with one free variable is a *quantifier-free type over X* if there exists $Y = X \cup \{y\}$ such that p is the quantifier-free type of y. We call Y the *structure associated to p*; up to obvious identifications it is uniquely determined by p.

In less formal terms: if Y is a structure of the form $X \cup \{y\}$, $y \notin X$, then the quantifier-free type of y over X is the complete description of the relations between elements of X, and relations between elements of X and y. To shorten notation a bit, below we write q.f type instead of quantifier-free type.

Definition 4.5. Let $X \in \mathcal{K}_\omega$ and p be a q.f type over X. Then, for any automorphism f of X, we define $f(p)$ as the q.f type with one free variable z defined by:

$$\phi(z, c_{x_1}, \ldots, c_{x_n}) \in f(p) \Leftrightarrow \phi(z, c_{f^{-1}(x_1)}, \ldots, c_{f^{-1}(x_n)}) \in p \; .$$

[1] Here we do not respect our convention that languages are relational.

Intuitively, $f(p)$ describes an element such that, for any $x_1, \dots, x_n$ in X, $f(p)$ satisfies the same relations with $f(x_1), \dots, f(x_n)$ as p does with $x_1, \dots, x_n$.

Definition 4.6. Let $X \in \mathcal{K}_\omega$. We say that a type p over X is *finitely induced* if there exists a finite substructure A of X, and an element $B = A \cup \{b\}$ of $\mathcal{K}$, such that the structure associated to p is isomorphic to the free amalgam of B and X over A.

Remark. This is the same thing as saying that in the structure $X \cup \{y\}$ associated to p, there are only finitely many elements $x_1, \dots, x_n$ which are contained in a tuple containing y and satisfying one of the relations in $\mathcal{L}$.

Note that, if $\mathcal{K}$ is not reduced to a singleton, then the free amalgamation property implies that $\mathcal{K}$ is infinite, and for any $X \in \mathcal{K}_\omega$ there always exists at least one finitely-induced q.f type over X (there may exist only one, a situation that presents itself for instance when $\mathcal{K}$ is the class of pure sets). In the remainder of the thesis, we always assume that the classes we consider are infinite.

4.3 Tower constructions

Now we establish basic properties of objects introduced above which we then use to prove the main results of this section.

Definition 4.7. Le $\mathbf{X}$ be an element of $\mathcal{K}_\omega$, $\{p_i\}_{i \in I}$ be an enumeration of all the finitely induced q.f types over $\mathbf{X}$, and for each i let $\mathbf{Y}_i$ denote the structure associated to p_i. Let $E(\mathbf{X})$ denote the free amalgam of the $\mathbf{Y}_i$'s over $\mathbf{X}$.

Since there are countably many finitely induced q.f types over $\mathbf{X}$, and $\mathcal{K}_\omega$ is stable under free amalgamation, $E(\mathbf{X})$ belongs to $\mathcal{K}_\omega$. Also, $\mathbf{X}$ naturally embeds in $E(\mathbf{X})$, and $E(\mathbf{X}) \setminus \mathbf{X}$ is always nonempty.

The definition of $E(\mathbf{X})$ is motivated by Katětov's construction of the Urysohn space [20]; in that context the next proposition was first pointed out by Uspenskij [33].

Proposition 4.8. *Assume $\mathbf{X} \in \mathcal{K}_\omega$. Then each automorphism of $\mathbf{X}$ extends uniquely to an automorphism of $E(\mathbf{X})$, and the extension morphism is an embedding of topological groups from* $\mathrm{Aut}(\mathbf{X})$ *to* $\mathrm{Aut}(E(\mathbf{X}))$.

Proof. Let ϕ be an automorphism of X. For each $y \in E(X) \setminus X$, with type denoted by p, there exists a unique $z \in E(X) \setminus X$ such that the type of z is equal to $\phi(p)$. To extend ϕ to an automorphism $E(\phi)$, one has no choice but to set $E(\phi)(y) = z$; this proves the uniqueness. The fact that this extension is indeed an automorphism of $E(X)$ is also obvious, since by definition two different elements of $E(X) \setminus X$ do not belong to any tuple satisfying a relation of $\mathcal{L}$.

The uniqueness of the extension ensures that $\phi \mapsto E(\phi)$ is a homomorphism, and it is obviously injective. The last remaining task is to show that it is continuous; to see that, pick $x_1, \dots, x_n \in X$ and $y_1, \dots, y_m \in E(X) \setminus X$. For $i \in \{1, \dots, m\}$, let A_i be a finite substructure of X witnessing the fact that the q.f type of y_i is finitely induced, and let $A = \{x_1, \dots, x_n\} \cup \cup_i A_i$. Then A is a finite substructure of X, and for any ϕ which coincides with the identity on A one must have $\phi(x_i) = x_i$ for all $i \in \{1, \dots, n\}$ and $\phi(y_i) = y_i$ for all $i \in \{1, \dots, m\}$. We have just proved that, for any neighborhood V of the identity in $\mathrm{Aut}(E(X))$, there exists a neighborhood U of the identity in $\mathrm{Aut}(X)$ such that $E(U) \subseteq V$, hence E is continuous. $\square$

Theorem 4.9. *Assume that $\mathcal{K}$ is a Fraïssé class with the free amalgamation property, and denote by $\mathbf{K}$ the Fraïssé limit of $\mathcal{K}$. Then, for any infinite $\mathbf{X} \in \mathcal{K}_\omega$ there exists an embedding $i\colon \mathbf{X} \to \mathbf{K}$ such that any automorphism ϕ of $i(\mathbf{X})$ extends to an automorphism $E(\phi)$ of $\mathbf{K}$.*

The extension map $\phi \mapsto E(\phi)$ may be taken to be a continuous group embedding from $\mathrm{Aut}(\mathbf{X})$ *to* $\mathrm{Aut}(\mathbf{K})$.

In particular, this result shows that $\mathrm{Aut}(\mathbf{K})$ is universal for all permutation groups of the form $\mathrm{Aut}(\mathbf{X})$, $\mathbf{X} \in \mathcal{K}_\omega$, answering partially the general question at the outset of [18]. In the Urysohn space context, the above result was proved by Uspenskij [33]; the proof below is essentially Uspenskij's proof, translated to our setting.

Proof. Starting from $X_0 = X$, we build an increasing chain of structures in $\mathcal{K}_\omega$ by setting $X_{i+1} = E(X_i)$ for all $i < \omega$, viewing X_i as a substructure of X_{i+1} via the natural embedding from X_i to $E(X_i)$. Then $X_\infty = \cup X_i$ belongs to $\mathcal{K}_\omega$. Using Proposition 4.8, we see that any automorphism φ of X_i uniquely extends to an automorphism $E(\phi)$ such that $E(\phi)(X_i) = X_i$ for all i, and that $\phi \mapsto E(\phi)$ is a topological group embedding from $\mathrm{Aut}(X)$ to $\mathrm{Aut}(X_\infty)$.

Fix a finite substructure A of X_∞, and an embedding $j\colon A \to B$ for some $B \in \mathcal{K}$. Let $B = j(A) \cup \{b_1, \dots, b_n\}$. There must exist some $i < \omega$ such that $A \subset X_i$, and an easy induction argument shows that there exists $\tilde{b}_1, \dots, \tilde{b}_n \in X_{i+n}$ such that $\tilde{B} = A \cup \{\tilde{b}_1, \dots, \tilde{b}_n\}$ is isomorphic to B via an isomorphism $\phi\colon B \to \tilde{B}$ such that $\phi \circ j = i$ (where i stands for the inclusion map from A to $\tilde{B}$). This shows that X_∞ is isomorphic to the Fraïssé limit of $\mathcal{K}$, and we are done. $\square$

We emphasize that Theorem 4.9 has the following corollary, answering partially the general question about the universality of the automorphism groups of Fraïssé limits.

Corollary 4.10. *Under the assumptions and notation of Theorem 4.9,* $\mathrm{Aut}(\mathbf{X})$ *appears as a closed subgroup of* $\mathrm{Aut}(\mathbf{K})$.

An elaboration on the proof of Theorem 4.9 yields the following.

Theorem 4.11. *Let $n \geq 2$ be an integer, assume that $\mathcal{K}$ is a Fraïssé class with the free amalgamation property, and let $\mathbf{K}$ denote the Fraïssé limit of $\mathcal{K}$. Then, for any $\mathbf{X} \in \mathcal{K}_\omega$, there exists an automorphism ϕ of $\mathbf{K}$ such that $\phi^n = 1$ and the set of fixed points of ϕ, seen as a substructure of $\mathbf{K}$, is isomorphic to $\mathbf{X}$.*

Proof. We build inductively an increasing sequence (X_i, ϕ_i) such that each X_i belongs to $\mathcal{K}_\omega$, ϕ_i is an automorphism of X_i, and:

i. $X_0 = X$, $\phi_0 = id_X$.

ii. For all $i < \omega$, $X_i \subseteq E(X_i) \subseteq X_{i+1}$ and ϕ_{i+1} exends ϕ_i.

iii. For all $i < \omega$, one has $\phi_i^n = 1$.

iv. For all $i < \omega$, the set of fixed points of ϕ_i is equal to X_0.

In point (ii), the inclusion $X_i \subseteq E(X_i)$ is to be understood as the natural embedding of X_i in $E(X_i)$.

Then, set $X_\infty = \cup X_i$, $\phi_\infty = \cup \phi_i$. As in the proof of Theorem 4.9, (ii) ensures that X_∞ is the Fraïssé limit of $\mathcal{K}$; points (iii) and (iv) ensure that $\phi_\infty^n = 1$ and that the set of fixed points of ϕ_∞ is equal to X_0, hence, as a substructure of X_∞, is isomorphic to X.

Thus, we only need to explain how to carry out the construction. The first step is imposed, so we set $X_0 = X$, $\phi_0 = id_X$. Assume that (X_i, ϕ_i) has been constructed. We let X_{i+1} be the free amalgam of n copies of $E(X_i)$ over X_i. To define ϕ_{i+1}, let $Y_0 \dots, Y_{n-1}$ be n copies of $E(X_i)$ such that $X_{i+1} = \cup Y_k$, $Y_j \cap Y_k = X_i$ for all $j \neq k \in \{0, \dots, n\}$, and pick $y \in Y \setminus X_i$. There exists a unique j such that $y \in Y_j$; let p denote the q.f type of y over X_i. There exists a unique element z of Y_{j+1} (addition here being modulo n) such that the q.f type of z is equal to $\phi(p)$; we set $\phi_{i+1}(y) = z$.

It is clear that ϕ_{i+1} is then an automorphism of X_{i+1}, whose set of fixed points is the same as the set of fixed points of ϕ_i, hence is equal to X_0. To check that $\phi_{i+1}^n = 1$, it suffices to notice that, for any $y \in Y_j \setminus X_i$, with q.f type denoted by p, $\phi_{i+1}^n(y)$ is by definition the unique element of $Y_{j+n} = Y_j$ whose type is equal to $\phi_i^n(p) = p$. Hence $\phi_{i+1}^n(y) = y$, and the proof is complete. □

4.4 An example in the functional case

In this section we are going to study an example of a universal object when the language contains functions. More specifically we are going to study Philip Hall's countable universal locally finite group. The results in this section are mostly due to Philip Hall and can be found in [24]. The class considered is the class $\mathcal{C}$ of all finite groups. Notice that $\mathcal{C}$ does not have free amalgamation property, so this case does not fit in the present chapter about free amalgamation classes. However we will see in this case a tower construction much similar to that in Theorem 4.9.

Definition 4.12. The locally finite group U is universal if

a. every finite group can be embedded into U

b. any two isomorphic finite subgroups of U are conjugate in U

Theorem 4.13. *If U is a universal group, then*

a. For any two isomorphic finite subgroups A and B of U, every isomorphism of A onto B is induced by an inner automorphism of U;

b. If A is a subgroup of the finite group B, then every embedding of A into U can be extended to an embedding of B into U;

c. The group U contains an isomorphic copy of every countable, locally finite group;

d. If C_m denotes the set of all elements of order $m > 1$ of U, then C_m is a single class of conjugate elements and $U = C_m C_m$; in particular U is a simple group.

Proof. a. Let α be any isomorphism of A onto B. By assumption, U contains finite subgroups C and G where C is isomorphic to A, G normalizes C, and G acts on C as its full group of automorphisms. Since the group U is universal there exists elements a and b in U such that $A^a = B^b = C$. Clearly, the mapping $x \to x^{a^{-1}\alpha b}$ determines an automorphism of the finite group C. Thus there exists an element g in G with $x^{a^{-1}\alpha b} = x^g$ for every element $x \in C$. Hence, if $y \in A$, we have

$$y^\alpha = (y^a)^{a^{-1}\alpha} = y^{agb^{-1}}$$

and so transformation by the element agb^{-1} induces the isomorphism α of A onto B.

b. There exists embeddings $\varphi : A \to U$ and $\psi : B \to U$. Clearly, $\psi^{-1}\varphi$ induces an isomorphism of A^ψ onto A^φ. By a. there exists an element $g \in U$ inducing this isomorphism, that is, $a^{\psi g} = a\varphi$ for every $a \in A$. But then the map $b \to b^{\psi\varphi}$ is an embedding of B into U, and its restriction to A is just φ.

c. Let G be any countable, locally finite group. Then G contains a local system of finite subgroups G_i, linearly ordered with respect to inclusion. Let n be any natural number such that for all natural numbers $i \leqslant n$, embeddings $\varphi_i : G \to U$ have been determined such that, if $i+1 \leqslant n$, the embedding φ_i is the restriction to G_i of the embedding φ_{i+1}. By b. there is an embedding φ_{n+1} of G_{n+1} into U extending φ_n. So, inductively, one may choose a sequence $\{\varphi_i\}_{i \in \mathbb{N}}$, and this sequence determines an embedding of G into U.

d. Clearly, the group U is simple if it is generated by every non-trivial conjugacy class of elements. If u and v are any two elements of order m of U, then a. applied to the subgroups $\langle u \rangle$ and $\langle v \rangle$ shows that u and v are conjugate in U. Now let x be any element of U of order n. Suppose that there exists a finite 2-generator group $\langle a, b \rangle$ where a and b both have order m, and the element ab has order n. Then there exists an embedding φ of $\langle a, b \rangle$ into U. Since $a^{\varphi} b^{\varphi}$ and x both have order n, there exists an element g in the universal group U such that $(a^{\varphi} b^{\varphi})^{g} = a^{\varphi g} b^{\varphi g} = x$. But this exhibits the arbitrary element x of U as a product of two elements of order m. Thus the following lemma completes the proof of 4.13. □

Lemma 4.14. *For any integers $m > 1$ and $n \geqslant 1$ there exists a finite 2-generator group $\langle a, b \rangle$ such that a and b both have order m; and the product ab has order n.*

Proof. Let $\langle a \rangle$ be a cyclic group of order m, $\langle c \rangle$ a cyclic group of order n, and denote by G the standard wreath product $\langle c \rangle \wr \langle a \rangle$ of $\langle c \rangle$ by $\langle a \rangle$. The base group B of G is the set of all mappings β of $\langle a \rangle$ into $\langle c \rangle$ with point wise multiplication and regarded as an $\langle a \rangle$-module via

$$\beta^{a} : x \mapsto (ax)^{\beta}, \qquad x \in \langle a \rangle, \beta \in B.$$

Let φ denote the mapping of $\langle a \rangle$ into $\langle c \rangle$ given by

$$(a)\varphi = c$$

$$(a^2)\varphi = c^{-1}$$

$$(a^i)\varphi = 1, \text{ for } 3 \leqslant i \leqslant m.$$

(Note that $m \geqslant 2$). Then for any element $x \in \langle a \rangle$, one has

$$(x)\varphi^{a} \cdot \varphi^{a^2} \cdot \ldots \cdot \varphi^{a^m} = (ax)\varphi \cdot (a^2 x)\varphi \cdot \ldots \cdot (a^m x)\varphi = \prod_{i=1}^{m} (a^i)\varphi = 1,$$

and thus

$$\varphi^{a} \cdot \varphi^{a^2} \cdot \ldots \cdot \varphi^{a^m} = 1.$$

Put $b = a^{-1}\varphi$. Then $ab = \varphi$, which has order n since $(a)\varphi^{i} = c^{i}$. Also

$$b^{i} = \varphi^{a} \cdot \varphi^{a^2} \cdot \ldots \cdot \varphi^{a^i} \cdot a^{-i} \neq 1 \text{ for } 1 \leqslant i < m,$$

since $a^{-i} \notin B$, and

$$b^{m} = \varphi^{a} \cdot \varphi^{a^2} \cdot \ldots \cdot \varphi^{a^m} = 1.$$

Thus the element b has order m. Therefore, the subgroup $\langle a, b \rangle$ of G is a finite group such that a and b have order m and the product ab has order n.

□

Let G be a locally finite group and $\bar{S}$ the full symmetric group on the set G. If ρ denotes the regular representation of G in $\bar{S}$ notice that $(x\langle y\rangle)^{y^\rho} = x\langle y\rangle y = x\langle y\rangle$ for all x and y in G. Let $S = \{\sigma \in \bar{S};\ there\ exists\ a\ finite\ subgroup\ F_\sigma\ of\ G\ satisfying\ (xF_\sigma)^\sigma = xF_\sigma for\ all\ x \in G\}$. This set S is in fact a locally finite group for if $T = \langle \sigma_1, \sigma_2, ..., \sigma_r\rangle$ where the $\sigma_i \in S$, then clearly for $F = \langle F_{\sigma_1}, F_{\sigma_2}, ..., F_{\sigma_r}\rangle$ we have $(xF)^\sigma = xF$ for every $\sigma \in T$ and $x \in G$. This embeds T into a cartesian product of copies of the symmetric group on the finite set F and therefore T is finite. We call S the *constricted symmetric group on* G. Notice that if G is finite then $S = \bar{S}$. Observe also that S depends upon the group structure of G and so is not a canonical subgroup of $\bar{S}$.

Lemma 4.15. *Let G be a locally finite group and denote by ρ the regular representation of G in the constricted symmetric group S on G. Then any two finite isomorphic subgroups of G^ρ are conjugate in S.*

Proof. Let K and K^* be finite isomorphic subgroups of G and put $H = \langle K, K^*\rangle$. Denote by $x \to x^*$ an isomorphism of K onto K^*. Let $\{x_i \in i \in I\}$ be a complete set of left coset representatives of H in G, $\{y_1, ..., y_r\}$ a complete set of left coset representatives of K in H and $\{y_1^*, ..., y_r^*\}$ a complete set of left coset representatives of K^* in H. Define the element σ of S by

$$(x_i y_j x)^\sigma = x_i y_j^* x^*$$

for $i \in I, 1 \leqslant i \leqslant r$ and $x \in K$.

Clearly $\sigma \in S$, we may take $H = F_\sigma$. For any $k \in K$ we have that

$$(x_i y_j^* x^*)^{\sigma^{-1} k^\rho \sigma} = (x_i y_j x)^{k^\rho \sigma} = (x_i y_j x k)^\sigma = x_j y_j^* (xk)^* = (x_j y_j^* x^*) k^{*\rho}$$

using that $*$ is a homomorphism. Hence $\sigma^{-1} k^\rho \sigma = k^{*\rho}$ for every $k \in K$, so that K^ρ and $K^{*\rho}$ are conjugate in S. ☐

Theorem 4.16. *There exists countable universal groups and any two such groups are isomorphic.*

Proof. Define inductively a direct system of finite groups and embeddings as follows. Let U_1 be any finite group of order at least 3. If $n \geqslant 1$ and the group U_n is already chosen, let U_{n+1} be the symmetric group on the set U_n and embed U_n into U_{n+1} via its right regular representation. This family of groups and embeddings clearly forms a direct system. Put $U = \varinjlim U_n$. Obviously U is countable, focally finite group. For convenience of notation we shall identify the group U_n with its image in U.

The order $|U_n|$ tends to infinity with n. Hence, if G is any finite group, then there exists an integer n such that $|G| \leqslant |U_n|$. But in this case, the group G is isomorphic to a subgroup of U_{n+1} and hence of U. If G and H are any two isomorphic finite subgroups of U, then there exists an integer j such that $\langle G, H\rangle \subseteq U_j$. By Lemma 4.15 the subgroups G and H are conjugate in the subgroup U_{j+1} of U. Therefore, the group U is universal.

Let V be any other countable universal group. Then there is a local system $\{V_n\}_{n\in\mathbb{N}}$ of finite subgroups of V linearly ordered by inclusion. Let φ be any embedding of U_r into V, such an embedding exists since U_r is finite and V is universal. Then one has $U_r^\varphi \subsetneqq V_s$, for some integer s. By Theorem 4.13 there exists an embedding ψ of V_s into U such that the composite map $\varphi\psi$ induces the identity on U. There is an integer r' with $V_s^\psi \subsetneqq U_{r'}$. Again by 4.13 there exists an embedding φ' of $U_{r'}$ into V such that $\psi\varphi'$ induces the identity map on V_s.

By choosing an arbitrary embedding φ_1 of U_1 into V one may in this way choose inductively two strictly ascending sequences of integers $1 = r_1 < r_2 < \dots$ and $0 < s_1 < s_2 < \dots$ and two sequences of proper embeddings

$$\varphi_i : U_{r_i} \to V_{s_i} \text{ and } \psi_i : V_{s_i} \to U_{r_i+1} \qquad i \in \mathbb{N}$$

such that $\varphi_i\psi_i$ is the identity on U_{r_i} and $\psi_i\varphi_{i+1}$ is the identity on V_{s_i}. It follows that for each index i the embeddings φ_{i+1} and ψ_{i+1} are, respectively, extensions of φ_i and ψ_i. Thus, the sequences $\{\varphi_i\}_{i\in\mathbb{N}}$ and $\{\psi_i\}_{i\in\mathbb{N}}$ determine monomorphisms $\varphi : U \to V$ and $\psi : V \to U$ such that $\varphi\psi$ is the identity on U and $\varphi\psi$ is the identity on V. Therefore each of these maps is an isomorphism. $\square$

In the case of locally finite groups, we now get the following analogue of Theorem 4.9.

Theorem 4.17. *Let G be a countably infinite locally finite group. Then G embeds into the Philip Hall Group U as a subgroup of infinite index such that every automorphism of G extends to an automorphism of U.*

Proof. Let $G = G_0$. Take the constricted symmetric group on G_0 and call it G_1. Assume that G_n is constructed, then let G_{n+1} be the constricted symmetric group on G_n. Let $G_\omega = \bigcup G_n$. We claim that G_ω is isomorphic to U. First we need to show that G_ω is a universal group. But since G_ω embeds symmetric groups of arbitrary size, it embeds every finite group.

Now let A and A' be two isomorphic finite subgroups of G_ω. Then there exists $k \in \mathbb{N}$ such that $A, A' \subseteq G_k \subseteq G_\omega$. Then by Lemma 4.15, A and A' are conjugate in G_{k+1} and thus in G_ω.

Now if we take an automorphism f of G, it canonically extends to an automorphism f^1 of G_1. Consequently f^1 extends to an automorphism of G_2. Inductively they extend to an automorphism of G_ω. $\square$

In Section 8 we will try to make similar constructions as in Theorems 4.9 and 4.17, requiring in addition that each automorphism extends *uniquely* to an automorphism of the Fraïssé limit. More specifically we will consider this question in the case of classes of relational structures with the free amalgamation property. Requiring the uniqueness of extensions in the case of Philip Hall's universal group seems to be a difficult problem about locally finite groups.

5 An application to the complexity of some conjugation problems

In this section, we want to apply Theorem 4.11 to show that, if $\mathcal{K}$ is a Fraïssé class with the free amalgamation property, then the isomorphism relation on $\mathcal{K}_\omega$ is always reducible to the conjugacy relation in $\mathrm{Aut}(\mathbf{K})$. This will in particular give the exact complexity of the conjugacy relation in the automorphism group of the Henson graphs and the random graph; for the random graph, this result was first proved in [5].

We begin by recalling what the complexity of a classification problem is and the classical setup used to study this problem.

5.1 Definable equivalence relations and Borel reducibility

A very common type of problem in mathematics is trying to classify objects up to some natural notion of equivalence: for instance, classifying groups in a certain class up to isomorphism, metric spaces up to isometry... As it turns out, it is often possible to view the class of objects that one is trying to classify as a standard Borel space, and the equivalence relation one is trying to understand as a definable equivalence relation. Then, one is led to try and assign complete invariants for the equivalence relation in a computable way. It turns out not to be possible in most cases, but one can still try to compare the relative levels of complexity of various classification problems; one way to do this is via the notion of Borel reducibility of definable equivalence relations, which was introduced by Friedman and Stanley [7] and which we discuss now.

Definition 5.1. Let X be a standard Borel space, and E be an equivalence relation on X. We say that an equivalence relation E on X is Borel (resp. analytic) if it is a Borel (resp. analytic) subset of $X \times X$.

For instance, if X is a Polish space, G is a Polish group and G acts continuously on X, then the equivalence relation induced by the action of G is analytic (but not Borel in general). The simplest conceivable relations with continuum many classes are those which one can classify using real numbers as complete invariants, in a definable way.

Definition 5.2. Let X be a standard Borel space, and E an equivalence relation on X. Say that E is *smooth* if there exists a Borel map $f\colon X \to \mathbf{R}$ such that

$$\forall x, x' \in X \; (xEx') \Leftrightarrow (f(x) = f(x')) \, .$$

A smooth equivalence relation is necessarily Borel, since it is the inverse image of the diagonal of $\mathbf{R}$ via $f \times f$. There are many examples of Borel equivalence relations which are not smooth, the most classical one being perhaps the Vitali equivalence relation on $\mathbf{R}$, defined by

$$xEy \Leftrightarrow x - y \in \mathbf{Q} \, .$$

In general, given an apparently complicated classification problem, there is little hope that it will turn out to be smooth. But one can still compare complexities.

Definition 5.3. Let X, Y be standard Borel spaces and E, F be equivalence relations on X, Y respectively. We say that $f\colon X \to Y$ is a *Borel reduction* of E to Y if one has

$$\forall x, x' \in X \; (xEx') \Leftrightarrow (f(x)Ff(x')) \, .$$

If there exist a Borel reduction from E to F, we say that E is (Borel) reducible to F; if E reduces to F and F reduces to E, we say that E and F are bireducible.

Thus, an equivalence relation is smooth if and only if it Borel reduces to the equality relation on $\mathbf{R}$; intuitively, if there exists a Borel reduction f from E to F, then, via composition by f, the problem of classifying elements of X for the equivalence relation E has been reduced to the problem of classifying elements of Y for the equivalence relation F. Thus, one may then think of F as being more complicated than E. Note that the fact that one asks f to be definable (in our case, Borel) is extremely important: without any condition on f, we would simply be comparing the cardinalities of the quotient sets X/E and Y/F.

Borel reducibility induces a quasiorder on equivalence relations. This quasiorder has been studied a lot since the seminal work of Friedman and Stanley, and we now know that it is very complicated. A subclass of particular interest to us is the class of relations which are given by Borel actions of Polish groups. For any Polish group G Becker and Kechris proved that there exists a *G-universal* Borel G-action (see [1]), that is to say, there exists a relation E_G which is induced by a Borel action of G (hence, E_G is analytic) and is such that for any other relation F induced by an action of G, F Borel reduces to E. In this thesis, we will describe a few new examples of S_∞-universal equivalence relations. Actions of S_∞ are of particular interest to logicians, and our next task is to discuss why.

5.2 The space of $\mathcal{L}$-structures

In this subsection we fix a countable language $\mathcal{L} = (R_i, n_i)_{i \in I}$. For any $i \in I$, we let $\mathcal{X}_i = 2^{\mathbf{N}^{n_i}}$, endowed with the product topology, and set

$$\mathcal{X}_\mathcal{L} = \prod_{i \in I} \mathcal{X}_i \ .$$

Endowed with the product topology, this is a compact topological space (homeomorphic to the Cantor space); for each i we denote by $\pi_i \colon \mathcal{X}_\mathcal{L} \to \mathcal{X}_i$ the coordinate projection.

To each element X of $\mathcal{X}_\mathcal{L}$ we may associate an infinite, countable $\mathcal{L}$-structure $\tilde{X}$ by setting, for all $\bar{n} \in \mathbf{N}^{n_i}$:

$$\tilde{X} \models R_i(\bar{n}) \Leftrightarrow \pi_i(X)(\bar{n}) = 1 \ .$$

Conversely, any $\mathcal{L}$-structure with universe $\mathbf{N}$ defines an element of $\mathcal{X}_\mathcal{L}$; thus we may see $\mathcal{X}_\mathcal{L}$ as the *space of all infinite countable $\mathcal{L}$-structures (with universe* $\mathbf{N}$*)*.

For $X \in \mathcal{X}_\mathcal{L}$ and $A \subseteq \mathbf{N}$ we denote by $\tilde{X}_{|A}$ the structure on A induced by $\tilde{X}$. We note for future reference that, for B a finite $\mathcal{L}$-structure and $A \subseteq \mathbf{N}$, the set $\{X \in \mathcal{X}_\mathcal{L} \colon \tilde{X}_{|A}$ is isomorphic to $B\}$ is closed in $\mathcal{X}_\mathcal{L}$ (it is clopen if $\mathcal{L}$ is a finite language).

We say that $X, Y \in \mathcal{X}_\mathcal{L}$ are *isomorphic* if $\tilde{X}, \tilde{Y}$ are isomorphic as $\mathcal{L}$-structures. The permutation action of S_∞ on $\mathbf{N}$ naturally extends to a continuous action of S_∞ on $\mathcal{X}_\mathcal{L}$, defined by setting, for all $X \in \mathcal{X}_\mathcal{L}$, $\sigma \in S_\infty$, $i \in I$ and $\bar{n} \in \mathbf{N}^{n_i}$,

$$\pi_i(\sigma \cdot X)(\bar{n}) = \pi_i(X)(\sigma^{-1}(\bar{n})) \ .$$

Then, X and Y are isomorphic if, and only if, there exists $\sigma \in S_\infty$ such that $\sigma \cdot X = Y$; thus, thinking of $\mathcal{X}_\mathcal{L}$ as the space of all infinite countable $\mathcal{L}$-structures, the relation of isomorphism of infinite countable $\mathcal{L}$-structures is given by a continuous action of S_∞ on the compact space $\mathcal{X}_\mathcal{L}$, called the *logic action* of S_∞ on $\mathcal{X}_\mathcal{L}$. Hence, understanding the relation of isomorphism among all $\mathcal{L}$-structures turns out to be the same as trying to classify elements of a Polish space up to equivalence under a continuous action of S_∞.

Often, one is not interested in all $\mathcal{L}$-structures; let us simply give an example. Assume that $\mathcal{L}$ is the language whose only nonequality symbol is a binary relation, and consider the space $\mathcal{G} \subseteq \mathcal{X}_\mathcal{L}$ whose elements encode a *graph* with universe $\mathbf{N}$, where by a graph we mean an irreflexive, symmetric binary relation. Then $\mathcal{G}$ is compact, invariant under the logic action, and we can think of the relation induced by that action as the *relation of isomorphism between infinite countable graphs*. Thus, the relation of isomorphism of infinite countable graphs may be seen as being induced by a continuous action of S_∞ on a compact space. Understanding the complexity of this relation is then of interest. It was proved in [7], at the very beginning of the theory of Borel complexity of equivalence relations, that this relation is S_∞-universal, as are many (but not all!) natural classification problems for first-order structures.

As a side remark, we note that it might well be that the way we encoded the isomorphism relation, say, on graphs, has an influence on the level the isomorphism relation occupies in the Borel complexity hierarchy - i.e. , maybe there is another standard Borel space that one can reasonably think of as being the space of countable graphs, and on that space the isomorphism relation is not S_∞-universal. Let us simply note that, whenever a given classification problem has been encoded in different ways, this problem has not occurred. For instance, the classification of Polish spaces up to isometry has been encoded in two different ways by Gao–Kechris [9] and by Clemens [4] and in both cases the classification problem was found to be universal for relations induced by a Borel action of a Polish group. It is not clear whether one can turn this heuristic into a mathematical statement - as a starting point, this would certainly require a good notion of what a "reasonable" coding of a given classification problem is, and we do not know of any such notion.

For more details on the rich theory of definable equivalence relations (in particular those that are induced by definable actions of Polish groups) we refer to [1], [8], [22] and [14].

5.3 Reduction of isomorphism to conjugacy

We now turn to the statement and proof of our main result in this section. We use the notations of section 5.2; define

$$\mathcal{X}_\mathcal{K} = \{X \in \mathcal{X}_\mathcal{L} \colon \tilde{X} \in \mathcal{K}_\omega\}$$

Then $\mathcal{X}_\mathcal{K}$ is a Borel subset of $\mathcal{X}_\mathcal{L}$ (it is even G_δ if $\mathcal{L}$ is finite), and $\mathcal{X}_\mathcal{L}$ is invariant under the logic action of S_∞. As in section 5.2, we call the relation induced by the logic action the isomorphism relation on infinite elements of $\mathcal{K}_\omega$. Our aim is to prove the following result.

Theorem 5.4. *Let $n \geq 2$ be an integer, $\mathcal{L}$ be a countable relational language, and $\mathcal{K}$ be a Fraïssé class of $\mathcal{L}$-structures with the free amalgamation property. Denote by $\mathbf{K}$ the Fraïssé limit of $\mathcal{K}$. Then, for any integer n, the isomorphism relation on infinite elements of $\mathcal{K}_\omega$ (as encoded above) is Borel reducible to the conjugacy relation in $\{g \in \mathrm{Aut}(\mathbf{K}) \colon g^n = 1\}$.*

Proof. We claim that it is possible to define a Borel map $X \mapsto \phi(X)$ from $\mathcal{X}_\mathcal{K}$ to $\mathrm{Aut}(\mathbf{K})$ such that $\phi(X)$ is conjugate to the automorphism of $\mathbf{K}$ built in the proof of Theorem 4.11. Assuming that this is possible, let us see why this map is a Borel reduction of the isomorphism relation on infinite elements of $\mathcal{K}_\omega$ to the conjugacy relation in $\{g \in \mathrm{Aut}(\mathbf{K}) \colon g^n = 1\}$.

First, letting $(\tilde{X}_i, \phi_i(X))$ and $(\tilde{Y}_i, \phi_i(Y))$ denotes the sequences constructed by applying the construction of 4.11 to $\tilde{X}, \tilde{Y}$, we see that if $\tilde{X}, \tilde{Y}$ are isomorphic we may inductively build a sequence of isomorphisms $g_n\colon \tilde{X}_i \to \tilde{Y}_i$ such that $g_i\phi_i(X)g_i^{-1} = \phi_i(Y)$. This yields an isomorphism $g\colon \tilde{X}_\infty \to \tilde{Y}_\infty$ such that $g\phi(X)g^{-1} = \phi(Y)$; identifying $\tilde{X}_\infty, \tilde{Y}_\infty$ with $\mathbf{K}$ we see that $\phi(X)$, $\phi(Y)$ are conjugate. Conversely, if $\phi(Y) = g\phi(X)g^{-1}$ for some $g \in \mathrm{Aut}(\mathbf{K})$, then g must map the set of fixed points of $\phi(X)$ onto the set of fixed points of $\phi(Y)$. Hence the sets of fixed points of $\phi(X), \phi(Y)$ are isomorphic substructures of M, that is, X and Y are isomorphic.

Now, we sketch why it is possible to define a map $X \mapsto \phi(X)$ as above; we leave to the reader the easy (but technical) verification that all the choices made in the construction can be made in a Borel way.

Our first aim is to build a map $\Psi\colon \mathcal{X}_\mathcal{K} \to \mathcal{X}_\mathcal{K}$ and a map $\phi\colon \mathcal{X}_\mathcal{K} \to S_\infty$ such that $\Psi(X)$ is isomorphic to $\mathbf{K}$ and $\phi(X)$ is an automorphism of $\Psi(X)$ conjugate to the isomorphism built in Theorem 4.11. To do this, we use an inductive construction, building (in a Borel way) a sequence of nonempty pairwise disjoint sets $A_j(X) \subseteq \mathbf{N}$ whose union is equal to $\mathbf{N}$ and, letting $B_j(X) = \cup_{k\le j} A_k(x)$, defining inductively $\Psi(X)_{|B_j(X)}$ and $\phi(X)_{|B_j(X)}$. These sets correspond to the steps of the construction in the proof of Theorem 4.11: $\Psi(X)_{|A_0(X)}$ will be isomorphic to X and $\phi(X)_{|A_0(X)}$ will be the identity; $\Psi(X)_{|A_1(X)}$ will be isomorphic to the free amalgam of n copies of $E(X)$ over X, etc.

Before the construction, we fix an enumeration $\{p_r\}_{r<\omega}$ of all the q.f types over finite structures in $\mathcal{K}$. Then the first step of the construction is as follows: fix an infinite, coinfinite set $A_0 \subset \mathbf{N}$, and a bijection $f_0\colon \mathbf{N} \to A_0$. Then set $\phi(X)_{|A_0} = id_{|A_0}$, and define $\Psi(X)_{|A_0}$ by setting, for all $i \in I$ and any n_i-tuple $\bar{m}$ contained in A_0,

$$\pi_i(\Psi(X))(\bar{m}) = 1 \Leftrightarrow \pi_i(X)(f_0^{-1}(\bar{m})) = 1\,.$$

Then we set $A_0(X) = A_0$ for all $X \in \mathcal{X}_\mathcal{K}$.

This takes care of the first step. Now, assume that $B_j(X), \Psi(X)_{|B_j(X)}, \phi(X)_{|B_j(X)}$ have been built, and that $B_j(X)$ is coinfinite. We let $k_j(X)$ be the smallest integer not belonging to $B_j(X)$. Then, one can find (in a Borel way):

- An infinite, coinfinite subset $C(X)$ which contains $k_j(X)$ and does not intersect $B_j(X)$.
- A partition of $C(X)$ in n infinite subsets $C_0(X), ..., C_{n-1}(X)$ satisfying $k_j(X) \in C_0(X)$.
- A bijection $G(X)$ of $C(X)$ onto itself such that $G(X)^n = 1$ and $G(X)(C_i(X)) = C_{i+1}(X)$ (here addition is to be understood modulo n).
- A bijection $H(X)$ from $\mathbf{N}$ to $C_0(X)$ with $H(X)(0) = k_j(X)$.
- A surjection $J(X)$ from $\mathbf{N}$ to the set of finite subsets of $B_j(X)$, which takes each value infinitely many times.

Next, we define inductively elements $H(X)(m_k) \in C_0(X)$ and define all relations $R_i(\bar{n})$ (in $\Psi(X)$) for all tuples $\bar{n}$ in $B_j(X) \cap C_0(X)$ whose intersection with $C_0(X)$ is equal to $\{H(x)(m_k)\}$, in such a way that the q.f type of each $H(X)(m_k)$ over $B_j(X)$ is finitely induced. Assume that these elements have been defined up to rank $k-1$. Then, we check whether there exists $l \ge k$ and a q.f type p over $\Psi_{|B_j(X)}$ which is induced by $J(X)(l)$ is and such that p is not realized by $H(X)(m_0), \dots, H(X)(m_{k-1})$ (notice that if $k = 0$ then such a type must exist because $E(X) \setminus X$ is nonempty for any $X \in \mathcal{K}_\omega$). If such a type does not exist, then $E(B_j(X)) \setminus B_j(X)$ was finite and we have finished encoding it; we set

$$A_{j+1}(X) = \bigcup_{q=0}^{n-1} G(X)^q(\{H(X)(m_0), \dots, H(X)(m_{k-1})\}$$

and

$$\phi_{j+1}(X) = G(X)_{|A_{j+1}(X)} .$$

If there exists $l \geq k$ as above, then we define m_k to be the minimum such l, and choose the minimum r such that p_r is a type over $J(X)(m_k)$ which is not realized by $H(X)(m_0), \dots, H(X)(m_{k-1})$. Then, we set the value of $\pi_i((\Psi(X))(\bar{n})$ for tuples $\bar{n}$ of $B_j(X) \cup C_O(X)$ whose intersection with $C_0(X)$ is equal to $\{H(X)(m_k)\}$ so that the q.f type of $H(x)(m_k)$ over $B_j(X)$ in $\Psi(X)$ is the q.f type induced by p_r. Then we move on to the next step of the inductive construction.

If this construction stops at some point, then we have defined $A_{j+1}(X)$ and $\phi_{j+1}(X)$; otherwise, we simply set

$$A_{j+1}(X) = \bigcup_{q=0}^{n-1} G(X)^q(\{H(m_k) \colon k < \omega\})$$

and

$$\phi_{j+1}(X) = G(X)_{|A_{j+1}(X)}$$

We still have to finish defining $\Psi(X)_{|B_{j+1}(X)}$. If $i \in I$, and $\bar{n}$ is a n_i-tuple in $B_{j+1}(X)$ which meets $A_{j+1}(X)$ in two different points, then we set $\pi_i(\Psi(X))(\bar{n}) = 0$. If $\bar{n}$ is a n_i-tuple in $B_{j+1}(X)$ which meets $A_{j+1}(X)$ in a single point belonging to $C_q(X)$ for some $q \in \{1, \dots, n-1\}$, then $\pi_i(\Psi(X))(\phi_{j+1}(X)^{-q}(\bar{n}))$ has already been defined, and we set

$$\pi_i(\Psi(X))(\bar{n}) = \pi_i(\Psi(X))(\phi_{j+1}^{-q}(\bar{n}))) .$$

This completely describes the construction of $\Psi(X)$, $\phi(X)$.

The next step is simpler: fix an element Y of $\mathcal{X}_{\mathcal{K}}$ such that $\tilde{Y}$ is isomorphic to $\mathbf{K}$; for all X we know that $\Psi(X)$ is isomorphic to Y and, using the universal property of a Fraïssé limit and a back-and-forth construction, we may build in a Borel way (choosing a minimal witness at each step of the back-and-forth) an element $\psi(X)$ of S_∞ such that $\psi(X)$ is an isomorphism from $\Psi(X)$ to Y for all $X \in \mathcal{X}_{\mathcal{K}}$. Then

$$\tilde{\phi}(X) = \psi(X)\phi(X)\psi^{-1}(X)$$

is the desired automorphism of $\tilde{Y}$ - that is to say, it is an automorphism of the Fraïssé limit of $\mathcal{K}$ which is conjugate to the automorphism built in the proof of Theorem 4.11, and the map $X \mapsto \tilde{\phi}(X)$ is Borel. □

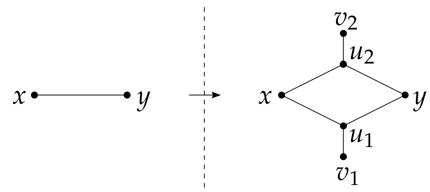
Corollary 5.5. *Let $n \geq 2$ be an integer. Then:*

i. The conjugacy relation on $\{g \in \mathrm{Aut}(\mathcal{R}) \colon g^n = 1\}$, where $\mathcal{R}$ denotes the random graph, is S_∞-universal.

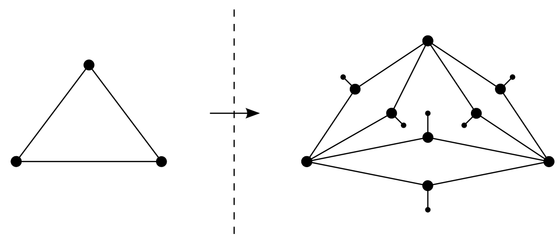
ii. Let $m \geq 3$ be an integer and G_m denote the Fraïssé limit of the class of K_m-free graphs. Then the conjugacy relation on $\{g \in \mathrm{Aut}(G_m) \colon g^n = 1\}$ is S_∞-universal.

Proof. Theorem 5.4 shows that the relation of isomorphism of graphs Borel reduces to the conjugacy relation on $\{g \in \mathrm{Aut}(\mathcal{R})\colon g^n = 1\}$ for all $n \geq 2$, so the first result is an immediate corollary of Theorem 5.4 and the fact that isomorphism of countable graphs is S_∞-universal.

For the second one, we claim that, for any m, the isomorphism relation for graphs Borel reduces to the isomorphism relation for K_m-free graphs, from which the result follows. One way to build a reduction is as follows: let G denote a countable graph; for each edge of G, add four new vertices, remove the edge, and add new edges as in the picture below.



Let $\tilde{G}$ denote the graph produced by applying this transformation to G; for example, the following picture shows what $\tilde{G}$ is if G is a triangle.



Then $\tilde{G}$ is K_m-freee for all $m \geq 3$, and the map $G \mapsto \tilde{G}$ can be coded by a Borel map (in the space of codes of graphs), thus we only need to show that, for two countable graphs G, H, one has

$$(G \cong H) \Leftrightarrow (\tilde{G} \cong \tilde{H}) .$$

The implication from left to right is obvious. To prove the converse, assume that $\phi\colon \tilde{G} \to \tilde{H}$ is an isomorphism. The v_i's constructed above are the only elements of $\tilde{G}$ with exactly one neighbour, and the u_i's are the only elements of $\tilde{G}$ with neighbour a v_i. The same thing happens in $\tilde{H}$, so ϕ must map the u_i's, v_i's that were added to G to the u_i's, v_i's that were added to H, hence ϕ maps the set $V(G)$ of vertices of G to the set $V(H)$ of vertices of H. Finally, two elements of $V(G)$ have an edge between them in G if and only if they have a common u_i as a neighbour in $\tilde{G}$, and this is preserved by ϕ, hence ϕ induces an isomorphism from G to H, and we are done. $\square$

In particular, if G is the random graph or a Henson graph, the conjugacy relation on $\mathrm{Aut}(G)$ is S_∞-universal; in the case of the random graph, this result was first proved by Coskey, Ellis and Schneider [5] . Before moving on, let us note that we do not know any example of a free amalgamation class $\mathcal{K}$ such that the isomorphism relation for infinite elements of $\mathcal{K}_\omega$ is neither smooth nor Borel complete.

6 Generic elements of finite order

In this section, we introduce a strong amalgamation property for a Fraïssé class, as well as a stregthening of Hrushovski's extension property [16]. These properties hold in Fraïssé classes of metric spaces as well as classes in a finite relational language with free amalgamation and which are defined by finitely many forbidden structures, e.g. graphs.

Then we study properties of elements of finite order in the corresponding automorphism groups, proving in particular that for the above classes there exists a generic element g_n of order n for all n and any other element is a product of 4 conjugates of g_n.

Note that the results of Macpherson–Tent–Ziegler (see [26] and [32]) show that, in most classes with the free amalgamation property, for *any* element $g \neq 1 \in \mathrm{Aut}(\mathbf{K})$ any other element may be written as a product of 16 conjugates of g and g^{-1}; Tent–Ziegler proved that the same thing holds in the isometry group of the rational Urysohn space whenever g is an unbounded isometry. So, in most cases, the results of Macpherson–Tent–Ziegler are much stronger than ours; however, the proof is different and (we think) interesting in its own right, and it works in some cases where the Macpherson–Tent–Ziegler machinery is not known to work at the moment, especially classes of Q-metric spaces where Q is bounded.

We begin by introducing the extension property mentioned above. The idea behind it comes from work of Herwig-Lascar [13] and Solecki [31].

6.1 Coherent extensions

In this subsection, we fix a *finite* language $\mathcal{L}$. We discuss an improvement by Solecki of a theorem of Herwig–Lascar.

Definition 6.1 (see [13], p. 1994)**.** Let A, B be two $\mathcal{L}$-structures. A map $f \colon A \to B$ is a *weak homomorphism* if for any integer n, any n-ary relational symbol R of $\mathcal{L}$ and any $a_1, \dots, a_n \in A$ one has

$$A \models R(a_1, \dots, a_n) \Rightarrow B \models R(f(a_1), \dots, f(a_n)) .$$

If A is a $\mathcal{L}$-structure and T is a set of $\mathcal{L}$-structures, say that A is *T-free* if there is no weak homomorphism from a structure in T to A.

This coincides with the usual terminology in the case of graphs.

Definition 6.2 (Solecki [31])**.** Let A be a set, and p_1, p_2, p_3 three partial bijections of A. Say that (p_1, p_2, p_3) is *coherent* if

$$\mathrm{dom}(p_1) = \mathrm{rng}(p_2),\ \mathrm{dom}(p_2) = \mathrm{dom}(p_3), \mathrm{rng}(p_1) = \mathrm{rng}(p_3) \text{ and } p_3 = p_1 \circ p_2 .$$

If A, B are sets and ϕ is a function from the sets of partial bijections of A to the set of partial bijections of B, say that ϕ is *coherent* if $(\phi(p_1), \phi(p_2), \phi(p_3))$ is coherent whenever (p_1, p_2, p_3) is coherent.

The following is Solecki's aforementioned improvement of the Herwig–Lascar theorem.

Theorem 6.3 (Herwig–Lascar; Solecki)**.** *Let T be a finite family of structures, A be a finite T-free structure, and P be a set of partial isomorphisms of A. If there exists a T-free structure M containing A such that each element of P extends to an automorphism of M, then there exists a finite T-free structure B containing A and $E \colon P \to \mathrm{Aut}(B)$ such that* *i.* $E(p)$ *is an extension of* p *for all* $p \in P$.

ii. E *is coherent.*

Using the notations of the above theorem, note that if A' is a substructure of A, and P contains $\mathrm{Aut}(A')$, then the coherence of the map E implies that $E\colon \mathrm{Aut}(A') \to \mathrm{Aut}(B)$ is a homomorphism (which is necessarily injective).

The assumption of Theorem 6.3 is satisfied if T is a finite set of $\mathcal{L}$-structures such that the set of all T-free structures is a Fraïssé class. Solecki also used this result to show a similar theorem for classes of metric spaces, proving the following:

Theorem 6.4 (Solecki [31])**.** *Let* A *be a finite metric space. There exists a finite metric space* B *such that* $A \subseteq B$ *as metric spaces, each partial isometry* p *of* A *extends to an isometry* $E(p)$ *of* B *and the function* E *is coherent. Moreover, the distances between points in* B *belong to the additive semi- group generated by the distances between points in* A*.*

6.2 Metric spaces as relational structures

To prove Theorem 6.4, one of the steps is to view finite metric spaces as relational structures. We quickly explain how one may do this; first, to stay within the realm of countable structures, one has to impose a condition on the set of possible values for the metric, and we introduce some ad hoc terminology.

Definition 6.5. Let Q be a countable subset of $[0, +\infty)$ containing 0. We say that Q is a *metric value set* if one of the following conditions is satisfied:

1. Q is a subsemigroup of $(\mathbf{R}; +)$.
2. Q is the intersection of an additive subsemigroup of $(\mathbf{R}, +)$ and a bounded interval, and $M_Q = \sup(Q) \in Q$.

Definition 6.6. If Q is a metric value set, a Q-metric space is a metric space (X, d) such that $d(x, x') \in Q$ for all $(x, x') \in X^2$.

Q-metric spaces may easily be turned into relational structures in a countable language $\mathcal{L}_Q$, containing a binary predicate R_q for all $q \in Q$: simply put $R_q(x, y)$ if and only if $d(x, y) = q$. Thus one may think of Q-metric spaces as relational structures in a countable language; it is well-known that, if Q is a metric value set, the class of finite Q-metric spaces is a Fraïssé class, with limit denoted by $\mathbf{U}_Q$; the countable metric space $\mathbf{U}_Q$ is characterized up to isometry, within the class of countable Q-metric spaces, by the fact that it contains a copy of any finite Q-metric space and any isometry between finite subsets extends to an isometry of the whole space.

We now discuss the amalgamation procedure we use for Q-metric spaces: let (A, d_A), (B, d_B), (C, d_C) be three finite Q-metric spaces, $i\colon A \to B$, $j\colon A \to C$ two isometric embeddings and assume that A is nonempty. We let X denote the disjoint union of B and C and define a pseudometric ρ on X as follows:

i. If Q is unbounded, $\rho(b, c) = \min\{d_B(b, i(a)) + d_C(j(a), c) \colon a \in A\}$.

ii. If Q is bounded and $M_Q = \sup(Q)$,

$$\rho(b, c) = \min\{M_Q, \min\{d_B(b, i(a)) + d_C(j(a), c) \colon a \in A\}\} .$$

In both cases it is not hard to see that ρ is indeed a pseudometric; if $x, y \in X$, say $x \sim y$ if $\rho(x, y) = 0$ and let $[x]$ denote the $\sim$-equivalence class of x. Then ρ induces a metric d_D on $D = X_{/\sim}$, and the maps $\alpha \colon b \mapsto [b]$, $\beta \colon c \mapsto [c]$ are isometric embeddings such that $\alpha \circ i = \beta \circ j$. In less formal terms, we identified the two copies of A in B, C and for $b \in B \setminus A, c \in C \setminus A$, set $d_D(b, c)$ to be equal to length of the shortest path between b and c going through A (cut off at M_Q if Q is bounded).

Note that we don't really need $\sup(Q)$ to belong to Q to make amalgamation work, but this assumption simplifies slightly the exposition and is more than enough for the applications to the Urysohn space that we have in mind.

Since we allow metric spaces to be empty, we should also explain how we amalgamate over the empty set. Assume that Q is a metric value set and B, C are two finite Q-metric spaces. We extend the metric on B, C to a metric on $B \sqcup C$ by setting $d(b, c) = \max(\operatorname{diam}(B), \operatorname{diam}(C))$ for all $b \in B, c \in C$.

6.3 The isomorphic extension property

We now present the main definition of this section.

Definition 6.7. Let $\mathcal{K}$ be a class of finite structures in a relational language $\mathcal{L}$. We say that $\mathcal{K}$ has the *isomorphic extension property* (IEP) if for any $A \in \mathcal{K}$ there exists $B \in \mathcal{K}$ such that $A \leq B$, and a map E from the set of *partial* isomorphisms of A to the set of *global* automorphisms of B such that

i. For any partial isomorphism g of A, $E(g)$ extends g.

ii. For any $A' \leq A$, E induces a homomorphism from $\operatorname{Aut}(A')$ to $\operatorname{Aut}(B)$.

If there exists E as above such that only condition (i) is satisfied, then we say that $\mathcal{K}$ has the *extension property*.

As mentioned above, whenever $\mathcal{L}$ is a finite relational language, and T is a finite set of $\mathcal{L}$-structures such that the class $\mathcal{K}$ of all finite T-free structures is a Fraïssé class, it follows from the Herwig–Lascar–Solecki theorem that $\mathcal{K}$ has the isomorphic extension property. Similarly, it follows from Theorem 6.4 that, whenever Q is a metric value set, the class $\mathcal{M}_Q$ of all finite Q-metric spaces has the isomorphic extension property.

It is well-known that, if $\mathcal{K}$ is a Fraïssé class with limit $\mathbf{K}$, the extension property of $\mathcal{K}$ translates to a topological property of $\operatorname{Aut}(\mathbf{K})$, namely that in $\operatorname{Aut}(\mathbf{K})$ there exists an increasing sequence of compact subgroups (G_n) with dense union or, equivalently, that for a generic element $\bar{g} \in \operatorname{Aut}(\mathbf{K})^n$ each $x \in \mathbf{K}$ has a finite orbit under $\langle \bar{g} \rangle$ (see [12] or [23]). We do not know whether the isomorphic extension property may be translated similarly to a property of $\operatorname{Aut}(\mathbf{K})$; it is not hard to see that it implies that $\operatorname{Aut}(\mathbf{K})$ has a dense locally finite subgroup. In this section, we are actually interested in tuples of elements of fixed finite order in $\operatorname{Aut}(\mathbf{K})$.

We also need a strong amalgamation property.

Definition 6.8. Let $\mathcal{L}$ be a relational language and $\mathcal{K}$ a class of $\mathcal{L}$-structures. We say that $\mathcal{K}$ has the *isomorphic amalgamation property* (IAP) if, for any $A, B, C \in \mathcal{K}$ and any embeddings $i \colon A \to B$, $j \colon A \to C$, there exists $D \in \mathcal{K}$ and embeddings $\alpha \colon B \to D$, $\beta \colon C \to D$ such that

i. $\alpha \circ i = \beta \circ j$.

ii. Whenever $\phi \in \mathrm{Aut}(B)$ fixes $i(A)$, $\psi \in \mathrm{Aut}(C)$ fixes $j(A)$, and $i^{-1}\phi i = j^{-1}\psi j$, there exist a partial automorphism σ of D fixing $\alpha(B)$, $\beta(C)$ respectively and such that $\alpha^{-1}\sigma\alpha = \phi$, $\beta^{-1}\sigma\beta = \psi$.

In plain words: one may amalgamate B and C over A in a sufficiently independent way such that any $\phi \in \mathrm{Aut}(B)$, $\psi \in \mathrm{Aut}(C)$ coinciding on A extend to a single partial automorphism of the amalgam. Note that any Fraïssé class with the free amalgamation property satisfies the isomorphic amalgamation property, as does the class of Q-metric spaces for any metric value set Q. Also, if $\mathcal{K}$ has both the (IAP) and the extension property (which will be the case in all our examples), then σ can be taken to be an automorphism of D.

Notation. Let $\mathcal{K}$ be a Fraïssé class with limit $\mathbf{K}$, and denote by G the automorphism group of $\mathbf{K}$. For any integer k and any $\bar{n} = (n_1, \ldots, n_k) \in \mathbf{N}^k$, we set

$$\Omega_{\bar{n}}(G) = \{\bar{g} \in G^k \colon \forall i \in \{1, \ldots, k\}\ g_i^{n_i} = 1\} .$$

We endow it with the topology induced by the topology of G^k, which turns it into a Polish space on which G acts by diagonal conjugacy:

$$g \cdot (g_1, \ldots, g_k) = (gg_1g^{-1}, \ldots, gg_ng^{-1})$$

We say that $\bar{g} \in \Omega_{\bar{n}}(G)$ is *generic* if its diagonal conjugacy class is comeager (equivalently, dense G_δ).

Proposition 6.9. *Let $\mathcal{K}$ be a Fraïssé class with the isomorphic extension property, and let G denote the automorphism group of its limit $\mathbf{K}$. Then for any integer k, and any k-tuple $\bar{n}$ of non-negative integers, there exists a generic element $\bar{g}$ in $\Omega_{\bar{n}}(G)$. The orbit of any $x \in \mathbf{K}$ under $\langle \bar{g} \rangle$ is finite.*

Proof. First, let us show that for a comeager set of $\bar{g} \in \Omega_{\bar{n}}(G)$, the orbit of any $x \in \mathbf{K}$ under $\langle \bar{g} \rangle$ is finite. Using the Baire category theorem, and the fact that for a fixed x the set $A_x = \{\bar{g} \in \Omega_{\bar{n}}(G) \colon \langle \bar{g} \rangle \cdot x \text{ is finite}\}$ is open, it is enough to show that for fixed $x \in \mathbf{K}$ the set A_x is dense. To see this, fix a nonempty open subset V of $\Omega_{\bar{n}}(G)$; without loss of generality, we may assume that there exist finite substructures $B_1, \ldots, B_k \subset \mathbf{K}$ and automorphisms $g_1, \ldots, g_k$ such that $g_i^{n_i} = 1$, $g_i(B_i) = B_i$ and

$$\forall \bar{h} \in \Omega_{\bar{n}}(G)\ (\forall i\ h_{i|B_i} = g_{i|B_i}) \Rightarrow \bar{h} \in V .$$

Set $B = \{x\} \cup \cup_{i=1}^k B_i$. Applying the isomorphic extension property to B, we may find a finite $C \subseteq \mathbf{K}$ containing B and automorphisms $h_1, \ldots, h_n$ of C such that h_i coincides with g_i on B_i, and $h_i^{n_i} = 1$. Using the usual back-and-forth construction and the isomorphic etxension property, we may extend each h_i to an automorphism of $\mathbf{K}$ such that $h_i^{n_i} = 1$. By construction, $\bar{h} = (h_1, \ldots, h_k) \in V$ and $\langle \bar{h} \rangle \cdot x \subseteq C$ is finite.

Now, we show that, for a generic $\bar{g} \in \Omega_{\bar{n}}(G)$, the following holds:

> For any finite $\langle \bar{g} \rangle$-invariant $B \subset \mathbf{K}$, any finite $C \in \mathcal{K}$ such that $B \leq C$ and any $\bar{h} \in Aut(C)$ with order $\bar{n}$ fixing B setwise and coinciding with $\bar{g}$ on B , there exists $\tilde{C} \subseteq \mathbf{K}$ such that $(\tilde{C}, \bar{g}_{|\tilde{C}}) \cong (C, \bar{h}_{|C})$.

To see this, let Σ denote the set of $\bar{g} \in \Omega_{\bar{n}}(G)$ satisfying the above condition. It is clear that Σ is G_δ; to see that it is dense, fix $B \leq C$ and $\bar{p} \in \Omega_{\bar{n}}(B)$. Pick some nonempty open set O in $\Omega_{\bar{n}}(G)$. We may assume without loss of generality that there exists a finite $D \subset \mathbf{K}$ containing B and a tuple of partial automorphisms $\bar{q} \in \Omega_{\bar{n}}(D)$ such that O consists of all tuples extending $\bar{q}$. We need to find $\bar{g}$ extending $\bar{q}$ and such that either some g_i does not coincide with p_i on B, or there exists $\tilde{C} \subseteq \mathbf{K}$ such that $(\tilde{C}, \bar{g}_{|\tilde{C}}) \cong (C, \bar{h}_{|C})$. If some q_i does not coincide with p_i on $E = C \cap D$, we have nothing to prove. In the other case, E must be fixed by $q_1, \ldots, q_n$, hence by $p_1, \ldots, p_n$. Then, we may use (IAP) and (IEP) to amalgamate D and C over E, obtaining a structure $F \in \mathcal{K}$ and automorphisms $g_1, \ldots, g_n$ of F with $g_i^{n_i} = 1$ for all i, g_i coinciding with q_i on D and with p_i on B. We may assume that $D \subseteq F \subseteq \mathbf{K}$. Then we may extend each f_i to an element of $\mathrm{Aut}(\mathbf{K})$ still denoted by g_i and such that $g_i^{n_i} = 1$. This is the tuple we were looking for.

Thus, the set of elements $\bar{g}$ satisfying the following two conditions is dense G_δ in $\Omega_{\bar{n}}(G)$:

i. The orbit of any $x \in \mathbf{K}$ under $\langle \bar{g} \rangle$ is finite.

ii. For any finite $\langle \bar{g} \rangle$-invariant $B \subseteq K$, any finite $C \in \mathcal{K}$ such that $B \leq C$ and any $\bar{h} \in \Omega_{\bar{n}}(C)$ fixing B setwise and coinciding with $\bar{g}$ on B , there exists $\tilde{C} \subseteq \mathbf{K}$ such that $(\tilde{C}, \bar{g}_{|\tilde{C}}) \cong (C, \bar{h}_{|C})$.

Since any two tuples satisfying both conditions above are easily seen to be conjugate, we are done. □

Note that essentially the same proof as above would enable one to show that if $\mathcal{K}$ is a Fraïssé class with (IAP) and (EP), then the automorphism group G of its limit has *ample generics*, i.e. there is a comeager diagonal conjugacy class in G^n for each integer n. This would also follow easily from the results of [17] or [23]. Below we will only use that, under the above conditions on $\mathcal{K}$, G has a dense conjugacy class.

Remark 1. In the case $\mathcal{K}$ is the class of Q-metric spaces for some metric values set Q, the conclusion of Proposition 6.9 follows from [30, Theorem 5], since a tuple of elements of finite order $n_1, \ldots, n_k$ is the same thing as a representation of $\mathbf{Z}_{n_1} * \ldots * \mathbf{Z}_{n_k}$, and this group has property (RZ). Rosendal's proof also uses a version of the Herwig–Lascar–Solecki theorem.

From now on, we assume that, whenever $\mathcal{K}$ is a Fraïssé class with the (IAP), $A, B, C \in \mathcal{K}$ and $i \colon A$, $j \colon A \to B$ are embeddings, then the isomorphism type of the triple (D, α, β) produced by the (IAP) only depends on the isomorphism type of (A, B, C, i, j). For a class with the free amalgamation property, we choose D to be the free amalgam of B and C over A; for a class of metric spaces, we choose for D the metric amalgam as presented in Section 6.2. Slightly abusing notation, we will often call D the I-amalgam of B and C over A.

Definition 6.10. Let $\mathcal{K}$ be a Fraïssé class with the (IAP), and n an integer. We say that *n allows extensions* if whenever $B \in \mathcal{K}$, ϕ a partial automorphism of B and $A \leq B$ are such that $\phi(A) = A$, $\phi^n_{|A} = 1$, and C is the I-amalgam of $\mathrm{dom}(\phi)$ and $\phi(\mathrm{dom}(\phi))$ over A, there exists $D \in \mathcal{K}$ containing C and ϕ_D an automorphism of D extending ϕ and such that $\phi_D^n = 1$.

This is a technical asumption on n that is very often, but not always, satisfied, as witnessed by the folowing lemma.

Lemma 6.11. *If $\mathcal{K}$ is a Fraïssé class with the free amalgamation property, or the class of finite Q-metric spaces for some metric value set Q, then any integer $n \geq 2$ allows extensions. If $\mathcal{K}$ is the class of all finite tournaments, then* 3 *allows extensions but* 2 *does not.*

Proof. We begin with the case when $\mathcal{K}$ has the free amalgamation property or is the class of finite $\mathbf{Q}$-metric spaces. Pick A, B, ϕ as in the definition of numbers allowing extensions. We may assume that B is the I-amalgam of $\mathrm{dom}(\phi)$ and $\phi(\mathrm{dom}(\phi))$.

Enumerate B as $\{b_1, \ldots, b_m\}$ with $A = \{b_1, \ldots, b_p\}$. Fix $j \in \{0, \ldots, n-1\}$. We define a structure $B^{(j)}$ with universe a m-element set $\{b_1^{(j)}, \ldots, b_m^{(j)}\}$ as follows: if R is a q-ary relational symbol of the language of $\mathcal{K}$, and $(b_{k_1}^{(j)}, \ldots, b_{k_q}^{(j)})$ is a q-tuple of elements of $B^{(j)}$, we first define $(x_1, \ldots, x_q) \in B^q$ by setting

$$\forall i \leq q \ x_i = \begin{cases} \phi^{-j}(b_{k_i}) & \text{if } k_i \leq p \\ b_{k_i} & \text{else} \end{cases} .$$

Then we set

$$B^{(j)} \models R(b_{k_1}^{(j)}, \ldots, b_{k_q}^{(j)}) \Leftrightarrow B \models R(x_1, \ldots, x_q) .$$

Then the map $\phi^{(j)} \colon B \mapsto B^{(j)}$ defined by

$$\phi^{(j)}(b_i) = \begin{cases} b_k^{(j)} & \text{if } i \leq p \text{ and } \phi^j(b_i) = b_k \\ b_i^{(j)} & \text{else} \end{cases}$$

is an isomorphism from B onto $B^{(j)}$, so each $B^{(j)}$ belongs to $\mathcal{K}$. Each $\phi^{(j)}$ naturally induces an embedding i_j of A into $B^{(j)}$. Using these identifications of A with a substructure of $B^{(j)}$, we may form the I-amalgam D of the $B^{(j)}$'s over A. It is straightforward to check that ϕ extends to an isomorphism of D of order n.

For tournaments, we should first explain our amalgamation procedure: if B, C are tournaments with a common subtournament A, we amalgamate them by saying that any element of $B \setminus A$ loses to any element of $C \setminus A$. This amalgamation procedure satisfies the (IAP) and, using the same reasoning as above, one sees that 3 allows extensions. The fact that 2 does not allow extensions is obvious, as the automorphism group of any tournament cannot contain an element of order 2. $\square$

We are almost ready to state, and prove, the main result of this section. Before that, we need to recall a definition and a well-known lemma.

Definition 6.12. Let X, Y be Polish spaces, $f \colon X \to Y$ a continuous map and $x \in X$. Say that x is *locally dense for* f if, whenever U is a neighborhood of x, $\overline{f(U)}$ is a neighborhood of $f(x)$.

Lemma 6.13 ("Dougherty's lemma")**.** *Assume that X, Y are Polish spaces, $f \colon X \to Y$ is continuous and the set of points which are locally dense for f is dense. Then $f(X)$ is not meager.*

Proof. Assume $f(X)$ is meager, and let F_n be a countable family of closed subsets of X with empty interior such that $f(X) \subseteq \cup_n F_n$. Then $X = \cup_n f^{-1}(F_n)$, so some $f^{-1}(F_n)$ must have nonempty interior, hence contain a point of local density for f. Then $\overline{f(f^{-1}(F_n))} \subseteq F_n$ has nonempty interior, a contradiction. $\square$

Proposition 6.14. *Let $\mathcal{K}$ be a Fraïssé class with the (IAP) and the (IEP), and n, m be two integers allowing extensions. Define $\pi \colon \Omega_{(n,m)}(G) \to G$ by $\pi(\sigma, \tau) \mapsto \sigma\tau$. Then any generic element of $\Omega_{(n,m)}(G)$ is locally dense for π.*

Proof. Let (σ, τ) be a generic element of $\Omega_{(n,m)}(G)$, and U be a neighborhood of (σ, τ). There exists a finite $A \subseteq \mathbf{K}$ which is $\langle \sigma, \tau \rangle$-invariant and such that

$$\forall (\phi, \psi) \in \Omega_{(n,m)}(G) \ (\phi_{|A} = \sigma_{|A},\ \psi_{|A} = \tau_{|A}) \Rightarrow (\phi, \psi) \in U \ .$$

We claim that

$$\overline{\pi(U)} \supseteq \{g \in G \colon g_{|A} = \sigma\tau_{|A}\} = O \ .$$

Proving this will yield the desired result. To see that it is true, let V be a nonempty open subset of O; we may assume without loss of generality that there exists a finite $B \subseteq \mathbf{K}$ containing A, and an automorphism g of $\mathbf{K}$ such that $g(B) = B$ and

$$\forall h \in G \ h_{|B} = g_{|B} \Rightarrow h \in V \ .$$

Let $B = \{b_1, \ldots, b_P\}$ be enumerated in such a way that $A = \{b_1, \ldots, b_p\}$ (with $p < P$ otherwise there is nothing to prove). As in the proof of Lemma 6.11, we pick an abstract P-point set $C = \{c_1, \ldots, c_P\}$ and turn it into an $\mathcal{L}$-structure as follows. Given R, a q-ary relational symbol of $\mathcal{L}$, and $(c_{k_1}, \ldots, c_{k_q})$ a q-tuple of elements of C, we first define $(x_1, \ldots, x_q) \in B$ by setting

$$\forall i \leq q \ x_i = \begin{cases} \tau^{-1}(b_{k_i}) & \text{if } k_i \leq p \\ b_{k_i} & \text{else} \end{cases} .$$

Then we set

$$C \models R(c_{k_1}, \ldots, c_{k_q}) \Leftrightarrow B \models R(x_1, \ldots, x_q) \ .$$

By construction, τ induces an isomorphism $\tilde{\tau} \colon B \to C$, defined by

$$\tilde{\tau}(b_i) = \begin{cases} c_j & \text{if } i \leq p \text{ and } \tau(b_i) = b_j \\ c_i & \text{else} \end{cases} .$$

We may also use σ to define $\tilde{\sigma} \colon C \to B$ by setting

$$\tilde{\sigma}(c_i) = \begin{cases} \sigma(b_i) & \text{if } i \leq p \\ h(b_i) & \text{else} \end{cases}$$

It is straightforward to check from the definition (and the fact that $\sigma\tau = g$ on A) that $\tilde{\sigma}$ is an isomorphism from C to B, and $\tilde{\sigma}\tilde{\tau} = g$.

We may form the I-amalgam D of B, C over A, using the map $\tilde{\tau}_{|A}$ for our embedding from A to C. Then, $(D, \tilde{\tau})$ fits our setup for numbers allowing extensions, so by assumption on m we can find E containing D such that $\tilde{\tau}$ extends to an isomorphism of $\Omega_m(E)$, still denoted by $\tilde{\tau}$. Applying the same reasoning to $(D, \tilde{\sigma}^{-1})$ and n, we find F containing D and such that $\tilde{\sigma}$ extends to an isomorphism $\tilde{\sigma}$ in $\Omega_n(F)$.

Amalgamating E, F over D, and applying the (IEP) one last time, we finally find $H \in \mathcal{K}$ containing D and $(\tilde{\sigma}, \tilde{\tau})$ extending the original $(\tilde{\sigma}, \tilde{\tau})$ and such that $\tilde{\sigma}^n = 1$, $\tilde{\tau}^m = 1$. We may assume that $H \leq \mathbf{K}$; by construction $(\tilde{\sigma}, \tilde{\tau}) \in \Omega_{n,m}(G)$. The construction ensures that $(\tilde{\sigma}, \tilde{\tau}) \in U$ and $\tilde{\sigma}\tilde{\tau}_{|B} = g_{|B}$, showing that $\tilde{\sigma}\tilde{\tau} \in V$, and we are done. $\square$

Theorem 6.15. *Let $\mathcal{K}$ be a Fraïssé class with the (IAP) and the (IEP), $\mathbf{K}$ be the Fraïssé limit and g_i be a generic element of order $i \geqslant 2$. Then, for any quadruple $i_1, \ldots, i_4$ of integers allowing extensions, and any $g \in Aut(\mathbf{K})$, there exist $h_1, \ldots, h_4$ such that each h_j is conjugate to g_{i_j} and $g = h_1 \ldots h_4$.*

In particular when i is a single integer allowing extensions, this shows that every element of G is a product of four conjugates of g_i.

Proof. Applying Proposition 6.14 (whose notations we reuse here), we know that $\pi(\Omega_{(n_1,n_2)}(G))$ is not meager. Since this set is analytic and conjugacy-invariant, and there is a dense conjugacy class in G, the $0-1$-topological law implies that $\pi(\Omega_{(n_1,n_2)}(G))$ is comeager, and an easy Baire category argument using the fact that points of local density are dense, shows that for any comeager subset A of $\Omega_{(n_1,n_2)}(G)$, $\pi(A)$ is comeager. The same argument works for (n_4, n_3).

Now, let $B_{(n_1,n_2)}$ be the set of elements which are a product of a conjugate of g_1 and a conjugate of g_2, and $B_{(n_4,n_3)}$ be the set of elements which are a product of a conjugate of g_4 and a conjugate of g_3. Since each of those sets is comeager, we may apply Pettis' lemma and obtain that $G = B_{(n_1,n_2)} \cdot B_{(n_3,n_4)}$. □

We recall that the hypothesis of this theorem (hence, its conclusion!) holds for any tuple of integers ≥ 2 in the case of Fraïssé classes of T-free structures in a finite language with the free amalgamation property, and for classes of Q-metric spaces for any metric value set Q.

Similar arguments would show, for instance, that in the automorphism group of the random tournament, there exists an element g of order 3 such that every other element is a conjugate of four conjugates of g.

7 Applications to the isometry group of the Urysohn space.

In this section we apply our results to the isometry group of Urysohn spaces. These are not countable but can be very well approximated by countable substructures (see Lemma 7.1). We denote by $\mathbf{U}$ the (unbounded) Urysohn space and by $\mathbf{U}_R$ the Urysohn space of diameter R. Our earlier results immediately imply that every element in $\mathrm{Iso}(\mathbf{U})$ (or $\mathrm{Iso}(\mathbf{U}_1)$) is a commutator and a product of 4 conjugates of elements of order n for any $n \geqslant 2$. Even in this continuous setting there is a generic element g_n of order n and we show that every other element is a product of 4 conjugates of g_n. Our results have consequences on the structure of $\mathrm{Iso}(\mathbf{U}_1)$ endowed with the uniform metric d_u: After giving a description of g_n we show that there is a continuous path from 1 to g_n in $(\mathrm{Iso}(\mathbf{U}_1), d_u)$. This along with our earlier results enables us to prove that $(\mathrm{Iso}(\mathbf{U}_1), d_u)$ is path-connected.

Notation. We say that a metric value set Q is *divisible* if $\frac{q}{n} \in Q$ for all $q \in Q$ and all $n \in \mathbf{N}^*$. We recall that metric value sets are by definition countable; also, whenever Q is divisible, it must be dense in $[0, \sup(Q)]$.

Note that if Q is divisible and unbounded, then the completion of $\mathbf{U}_Q$ is isometric to $\mathbf{U}$, while if Q is divisible and bounded with $\sup(Q) = R$ then the completion of $\mathbf{U}_Q$ is isometric to $\mathbf{U}_R$. This can be proved using the same method with which Urysohn proved that the Urysohn space is the completion of the rational Urysohn space.

Also, we recall that $\mathbf{U}$ is characterized up to isometry, among all separable complete metric spaces, by the following property:

> For any finite subset $A \subseteq \mathbf{U}$, for any abstract one-point metric extension $A \cup \{z\}$ of A, there exists $\tilde{z} \in \mathbf{U}$ such that $d(\tilde{z}, a) = d(z, a)$ for all $a \in A$.

Of course, $\mathbf{U}_1$ is characterized by a similar condition among complete separable metric spaces of diameter at most 1 and, whenever Q is a countable metric value set, $\mathbf{U}_Q$ is characterized up to isometry, among all countable Q-metric spaces, by an analogous condition, where one restricts the extension $A \cup \{z\}$ to be a Q-metric space.

Lemma 7.1. *Let $g_1, \dots, g_n$ be a finite family of isometries of $\mathbf{U}$. Then there exists a divisible countable metric value set Q and an isometric copy X of $\mathbf{U}_Q$ which is dense in $\mathbf{U}$ and such that $g_i(X) = X$ for all $i \in \{1, \dots, n\}$.*

The same lemma, with obvious modifications, would also be true for $\mathbf{U}_1$ instead of $\mathbf{U}$; this lemma was proved, earlier and independently, by Tent and Ziegler [32]. Their proof used a model-theoretic argument based on the Löwenheim–Skolem theorem, which is essentially the same idea as the proof below.

Proof. We proceed by induction: we build an increasing sequence of countable dense subspaces (Y_i) of $\mathbf{U}$ and divisible metric value sets Q_i such that:

1. For all i, Y_i is a Q_i-metric space and Y_i is $\langle \bar{g} \rangle$-invariant.
2. For any finite subset A of Y_i, and any one-point Q_i-metric extension $A \cup \{z\}$ of A, there exists $\tilde{z} \in Y_{i+1}$ such that $d(\tilde{z}, a) = d(z, a)$ for all $a \in A$.

To begin the construction, let X_0 be any countable dense subspace of $\mathbf{U}$, Y_0 be the countable set $\langle \bar{g} \rangle \cdot X_0$, and Q_0 the smallest divisible metric value set containing the values of the distance on Y_0 (this is indeed a countable set);

Now, assume that (Y_i, Q_i) has been built. Then, there are only countably many one-point Q_i-extensions of finite subsets of Y_i, and each of them is realized in $\mathbf{U}$, so we may add a countable set $\{x_j\}$ to Y_i so that all one-point Q_i-metric extensions of Y_i are realized in $X_{i+1} = Y_i \cup \{x_j\}$. Let $Y_{i+1} = \langle \bar{g} \rangle \cdot X_{i+1}$, and Q_{i+1} be the smallest divisible metric value set containing all the values of the distance on Y_{i+1}.

At the end of this construction, $Q = \bigcup Q_i$ is a countable, divisible metric value set, and $\bigcup Y_i$ is a Q-metric space which is $\langle \bar{g} \rangle$-invariant, isometric to $\mathbf{U}_Q$ and dense in $\mathbf{U}$. □

Theorem 7.2. *Every element of* $\mathrm{Iso}(\mathbf{U})$ *is a commutator and a product of at most four elements of order n for all $n \geq 2$. The same result is true for* $\mathrm{Iso}(\mathbf{U}_1)$.

Proof. Pick $g \in \mathrm{Iso}(\mathbf{U})$, and apply lemma 7.1 to find a dense, countable metric value set Q and a dense copy X of $\mathbf{U}_Q$ such that $g(X) = X$. Since $\mathrm{Iso}(\mathbf{U}_Q)$, endowed with its permutation group topology, has ample generics, every element of $\mathrm{Iso}(X)$ is a commutator, thus there exists $a, b \in \mathrm{Iso}(X)$ such that $g_{|X} = aba^{-1}b^{-1}$. These elements a, b uniquely extend to isometries of $\mathbf{U}$, still denoted by a, b, and we obtain $g = aba^{-1}b^{-1}$.

The proof that every element of $\mathrm{Iso}(\mathbf{U})$ is a product of at most four elements of order n for all $n \geq 2$ follows similarly from Lemma 7.1 and Theorem 6.15.

It is clear that exactly the same argument works for $\mathbf{U}_1$. □

Below, we explain how to obtain a stronger result: for any n there exists an element g of order n in $\mathrm{Iso}(\mathbf{U})$ such that every element of $\mathrm{Iso}(\mathbf{U})$ is a product of at most four conjugates of g (and the same fact holds for $\mathrm{Iso}(\mathbf{U}_1)$; to avoid unecessary repetitions, we focus on the case $\mathrm{Iso}(\mathbf{U})$ below). For this, we first need to show that, as in the discrete case, there exists a generic element of order n in $\mathrm{Iso}(\mathbf{U})$. This might be a bit surprising, since it is well-known that, as opposed to the discrete case, conjugacy classes are meager in $\mathrm{Iso}(\mathbf{U})$.

Notation. In what follows we let G denote $\mathrm{Iso}(\mathbf{U})$ and, for any integer n, set

$$\Omega_n(G) = \{g \in G \colon g^n = 1\} .$$

Definition 7.3. Let $n \geq 2$ and $\sigma \in \Omega_n(G)$. Say that σ has the *n-approximate extension property* if for any $\epsilon > 0$, any finite σ-invariant subset $A = \{a_1, \ldots, a_m\}$ of $\mathbf{U}$ and any (B, d_B, τ) such that $B = \{a_1, \ldots, a_m, b, \tau(b), \ldots, \tau^{n-1}(b)\}$ is a metric space containing A, τ coincides with σ on A, and $\tau^n = 1$, there exists $\tilde{b} \in \mathbf{U}$ such that :

i. $\forall i \in \{1, \ldots, m\}\ \forall j \in \{0, \ldots, n-1\}\ |d_B(a_i, \tau^j(b)) - d(a_i, \sigma^j(\tilde{b}))| \leq \epsilon$.

ii. $\forall i, j \in \{0, \ldots, n-1\}\ |d(\sigma^i(\tilde{b}), \sigma^j(\tilde{b})) - d(\tau^i(b), \tau^j(b))| \leq \epsilon$.

If the condition above is satisfied for $\epsilon = 0$, we say that σ has the *n-extension property*.

Note that, in the definition of the n-approximate extension property, we allow the possibility that $\tau^q(b) = b$ for some strict divisor q of n - in that case $(\{\tau(b), \ldots, \tau^{n-1}(b)\}, d_B)$ is a pseudometric space rather than a metric space, which is why we will have to manipulate pseudometrics below.

Intuitively, the n-approximate extension property is saying that for any finite σ-invariant set A, any abstract extension (B, τ) of $(A, \sigma_{|A})$ such that B is finite and $\tau^n = 1$ is approximated arbitrarily closely by some $(\tilde{B}, \sigma_{|\tilde{B}})$. Using a back-and-forth argument, it is easy to check that any two elements of $\Omega_n(G)$ with the n-extension property must be conjugate. From a descriptive set theoretic point of view, however, the n-approximate extension property is nicer, since it turns out to be a G_δ condition, and it is certainly easier to check than the extension property. This is why it will be useful to us to show that the two properties are actually equivalent - an unsurprising fact, in view of analogous results in Katětov's construction of the Urysohn space [20], but it turned out to be a bit trickier to write down than we expected.

To prove this equivalence, we need the following lemma; in the case A below is empty, this lemma is due to Uspenskij [34, Proposition 7.1]; the proof we give is essentially the same as Uspenskij's. That lemma was also known for along time to C. Ward Henson, who never published it, and Melleray heard it from him.

Lemma 7.4. *Let $A = \{a_1, \dots, a_n\}$ be an enumerated finite metric space. Let $B = A \cup \{b_1 \dots, b_p\}$ and $C = A \cup \{c_1, \dots, c_p\}$ be two enumerated finite pseudometric spaces containing A, and $\epsilon > 0$ be such that*

i. $\forall i, j \in \{1, \dots, p\}$ $\left|d_B(b_i, b_j) - d_C(c_i, c_j)\right| \leq 2\epsilon$.

ii. $\forall i \in \{1, \dots, n\}$ $\forall j \in \{1, \dots, p\}$ $\left|d_B(a_i, b_j) - d_C(a_i, c_j)\right| \leq \epsilon$.

Then there exists a pseudometric ρ on $X = B \cup C$ such that:

1. *ρ extends d_B, d_C.*
2. $\forall i \in \{1, \dots, p\}$ $\rho(b_i, c_i) \leq \epsilon$.
3. *If $f \colon A \to A$ is an isometry, and π is a permutation of $\{1, \dots, p\}$ such that setting $f(b_i) = b_{\pi(i)}$ extends f to an isometry of (B, d_B), and setting $f(c_i) = c_{\pi(i)}$ extends f to an isometry of C, then setting $f(b_i) = b_{\pi(i)}$ and $f(c_i) = c_{\pi(i)}$ extends f to an isometry of (X, ρ).*

This pseudometric is such that $\rho(b, c) > 0$ for all $b \in B \setminus A$, $c \in C \setminus A$.

Proof. We use the same idea as in [34]. First, define a partial function ω on X by the following conditions:

- $\forall b, b' \in B$ $\omega(b, b') = d_B(b, b')$.
- $\forall c, c' \in C$ $\omega(c, c') = d_B(c, c')$.
- $\forall i \in \{1, \dots, p\}$ $\omega(b_i, c_i) = \epsilon$.

Note that, if a, a' belong to A then the first two conditions both give $\omega(a, a') = d(a, a')$. If $x, y \in X$, we say that a finite sequence $x_0, \dots, x_k$ of elements of X is a *path from x to y* if $x_0 = x$, $y = x_k$, and $\omega(x_i, x_{i+1})$ is defined for all $i \in \{0, \dots, k-1\}$. Then we set

$$\rho(x, y) = \inf\{\sum_{i=0}^{k-1} \omega(x_i, x_{i+1}) \colon (x_0, \dots, x_k) \text{ is a path from } x \text{ to } y\}$$

It is clear that ρ is a pseudometric, and conditions (2) and (3) follow immediately from the definition. The only fact that remains to be checked is (1); this is straightforward but a bit tedious, reducing to cases. We check why $\rho(b, b') = d_B(b, b')$ for all $b, b' \in B$ (the other case is symmetric). If this were not true, then we could find $b, b' \in B$ and a path $b, x_1, \dots, x_k = b'$ whose length is strictly less than $d_B(b, b')$ and such that k is a minimal integer with this property.

From the definition of ω and the triangle inequality in B, we must have $x_1 \in C \setminus A$ (otherwise the path from x_1 to b' must be of length strictly less than $d_B(b', x_1)$ contradicting the definition of k). From the triangle inequality in C, we see that three consecutive terms of the path cannot belong to C. Thus x_2 or x_3 belongs to B; if x_2 belongs to B then from the fact that $\omega(x_0, x_1)$ and $\omega(x_1, x_2)$ are defined we get that either x_0, x_2 belong to A and apply the triangle inequality in A to find a path of shorter length, or $x_0 = x_2$, which is a contradiction. Thus the path is either of the form x_0, x_1, x_2 with $x_1 \in C \setminus A$ or (by symmetry) (x_0, x_1, x_2, x_3) with $x_0, x_3 \in B$ and $x_1, x_2 \in C \setminus A$. It is easy to check from our hypotheses that the first case yields a length bigger than $d_B(x_0, x_2)$, so only the second case remains.

- If $x_0 \in A$, then the triangle inequality in C shows that the path has length greater than $d_C(x_0, x_2) + \omega(x_2, x_3)$, which is the length of the path (x_0, x_2, x_3), contradicting the definition of k.
- The case $x_3 \in A$ is of course the same, so we may assume $x_0, x_3 \notin A$, $x_1, x_2 \in C \setminus A$. There must exist $i, j \in \{1, \dots, p\}$ such that $x_0 = b_i, x_1 = c_i, x_2 = c_j, x_3 = b_j$. Then the length of the path (x_0, x_1, x_2, x_3) is $\epsilon + d_C(c_i, c_j) + \epsilon \geq d_B(b_i, b_j)$ by assumption.

□

Lemma 7.5. *Let $n \geq 2$ and $\sigma \in \Omega_n(G)$. Then σ has the n-extension property if, and only if, it has the n-approximate extension property.*

Proof. Fix $A = \{a_1, \dots, a_m\}$, (B, d_B, τ) as in the definition of the n-approximate extension property. Using the completeness of $\mathbf{U}$, it is enough to build a sequence (c_q) of elements of $\mathbf{U}$ such that:

1. $\forall q\ d(c_q, c_{q+1}) \leq 2^{1-q}$.
2. $\forall q\ \forall i \in \{1, \dots, m\}\ \forall j \in \{0, \dots, n-1\}\ \left|d(\sigma^j(c_q), a_i) - d_B(\tau^j(b), a_i)\right| \leq 2^{-q}$.
3. $\forall q\ \forall i, j \in \{0, \dots, n-1\}\ \left|d(\sigma^i(c_q), \sigma^j(c_q)) - d_B(\tau^i(b), \tau^j(b))\right| \leq 2^{-q}$.

Use the approximate extension property of σ to define $B^{(0)}$, then assume that c_q has been defined. Let $C_q = A \cup \{c_q, \dots, \sigma^{n-1}(c_q)\}$. Using lemma 7.4 applied to the extensions B, C_q of A, one may find a metric ρ on $X = B \cup C_q$ extending the original metrics, such that $\rho(b, c_q) \leq 2^{-q}$ and the map $f \colon X \to X$ defined by

$$f(x) = \begin{cases} \sigma(x) & \text{if } x \in A \\ \tau^{j+1}(b) & \text{if } x = \tau^j(b) \\ \sigma^{j+1}(c_q) & \text{if } x = \sigma^j(c_q) \end{cases}$$

is an isometry of X. We may see (X, f) as an abstract extension of C_q, and apply the approximate extension property of σ with $\epsilon = 2^{-q-1}$, to find c_{q+1}. All the desired properties are straightforward to check from the definitions and the triangle inequality.

□

There remains one last piece of bookkeeping ahead of us.

Lemma 7.6. *Let $\mathbf{Q}$ be a divisible metric value set, and see $\mathbf{U}_Q$ as a dense subset of $\mathbf{U}$. Let $n \geq 2$ be an integer and σ be a generic element of $\Omega_n(\mathrm{Iso}(\mathbf{U}_Q))$. Then the extension of σ to an element of $\Omega_n(G)$ has the n-approximate extension property.*

Proof. We still denote by σ the extension of σ to $\mathbf{U}$. To see that it has the n-approximate extension property, fix a finite, σ-invariant set $A = \{a_1, \ldots, a_m\}$, a finite enumerated pseudometric space $B = (\{a_1, \ldots, a_m\} \cup \{b, \ldots, \tau^{n-1}(b)\}, d_B)$ containing A, an isometry τ of B that extends $\sigma_{|A}$ and such that $\tau^n = 1$, and $\epsilon > 0$. We may assume that $\epsilon < \mathrm{diam}(B)$. To simplify notation, we write $b_i = \tau^i(b)$ for $i = 0, \ldots, n-1$. Our proof is in three steps.

First step: Let ρ denote the metric on B defined by

- $\forall a, a' \in A\ \rho(a, a') = d(a, a')$.
- $\forall b \in B \setminus A\ \forall a \in A\ \rho(a, b) = \mathrm{diam}(B)$.
- $\forall b \neq b' \in B\ \rho(b, b') = \mathrm{diam}(B)$.

It is easy to check that ρ is a metric on B, and that τ is an isometry of (B, ρ). Define $\tilde{d}$ on B by setting, for all $b, b' \in B$,

$$\tilde{d}(b, b') = (1 - \frac{\epsilon}{\mathrm{diam}(B)})d_B(b, b') + \frac{\epsilon}{\mathrm{diam}(B)}\rho(b, b') .$$

Then $\tilde{d}$ is a metric on B, coinciding with d on A, and we have:

$$\forall i, j \in \{0, \ldots, n-1\} \quad |\rho(b_i, b_j) - d_B(b_i, b_j)| \leq 2\epsilon . \tag{7.1}$$

$$\forall i \in \{1, \ldots, m\}\ \forall j \in \{1, \ldots, p\} \quad |\rho(a_i, b_j) - d_B(a_i, b_j)| \leq 2\epsilon . \tag{7.2}$$

What we have gained with introducing this new metric is that now there exists some $\delta > 0$ such that, for any triple $\{x, y, z\}$ of distinct elements of B not contained in A, one has

$$d(x, y) + d(y, z) \geq d(x, z) + 3\delta . \tag{7.3}$$

We may, and do, assume that $\delta < \epsilon$ and fix such a δ for the remainder of the proof.

Second step: Pick $a'_1, \ldots, a'_n \in \mathbf{U}_Q$ such that $d(a_i, a'_i) \leq \delta$. Define

$$A' = \{\sigma^k(a_i) \colon k \in \{0, \ldots, n-1\},\ i \in \{1, \ldots, m\}\}$$

We may extend $\tilde{d}$ to a metric on $X = B \cup A'$, still denoted by $\tilde{d}$, and extending d on $A \cup A'$ by setting, for all $b \in B, a' \in A'$:

$$\tilde{d}(b, a') = \min\{\tilde{d}(b, a) + d(a, a') \colon a \in A\} .$$

For any triple $\{x, y, z\}$ of distinct elements of X not contained in $A \cup A'$, (7.3) ensures that we still have

$$d(x, y) + d(y, z) \geq d(x, z) + 3\delta . \tag{7.4}$$

The metric $\tilde{d}$ does not need to take its values in Q. We modify it as follows: let $r_1, \ldots, r_j$ denote the values of $\tilde{d}$ which do not belong to Q. For any $i \in \{1, \ldots, j\}$, find $q_i \in Q$ such that $|q_i - r_i| \leq \delta$.

Then define a new metric D on $Y = \{b_0, \ldots, b_{n-1}\} \cup A'$ by setting $D(x, x') = q_i$ whenever $\tilde{d}(x, x') = r_i$, $D(x, x') = \tilde{d}(x, x')$ otherwise. The values taken by D belong to Q, and we claim that it is indeed a metric. To see this, pick a triple $\{x,y,z\}$ of distinct elements of Y. If $\{x,y,z\} \subseteq A'$ then D coincides with $\tilde{d}$ on $\{x,y,z\}$ so the triangle inequality holds. Otherwise, we have:

$$\begin{aligned} D(x,y)+D(y,z) &\geq \tilde{d}(x,y)+\tilde{d}(y,z)-2\delta \\ &\geq \tilde{d}(x,z)+3\delta-2\delta \qquad \text{(by (7.4))} \\ &\geq \tilde{d}(x,z)+\delta \\ &\geq D(x,z)\ . \end{aligned}$$

Also, for any $i,j \in \{0,\dots,n-1\}$ we have

$$|D(b_i,b_j)-d_B(b_i,b_j)| \leq \delta + |\tilde{d}(b_i,b_j)-d_B(b_i,b_j)| \leq \delta+2\epsilon \leq 3\epsilon\ .$$

Define a map ϕ on Y by setting $\phi(b_i)=\tau(b_i)$ for $i \in \{0\dots,n-1\}$, $\phi(a')=\sigma(a')$ for $a' \in A'$. We have $\phi^n=1$; of course, ϕ does not need to be an isometry, and this is what we have to take care of in the last step of the proof.

Third step: For $x,y \in Y$, we set

$$\tilde{D}(x,y)=\frac{1}{n}\sum_{k=0}^{n-1} D(\phi^k(x),\phi^k(y))\ .$$

Then $\tilde{D}$ is a metric on Y, taking its values in Q because Q is divisible, and ϕ is an isometry of $(Y,\tilde{D})$, extending $\sigma_{|A'}$. Also, $\tilde{D}(b_i,b_j)=D(b_i,b_j)$ for all $i,j \in \{0,\dots,n-1\}$. Using the fact that σ is a generic element of $\Omega_n(\mathbf{U}_Q)$, we can find $c \in \mathbf{U}_Q$ such that

$$(A' \cup \{c,\sigma(c),\dots,\sigma^{n-1}(c)\},\sigma) \cong (A' \cup \{b,\tau(b),\dots,\tau^{n-1}(b)\},\phi,\tilde{D})$$

We already know that

$$\forall i,j \in \{0,\dots,n-1\}\ |d(\sigma^i(c),\sigma^j(c))-d_B(\tau^i(b),\tau^j(b))| = |D(b_i,b_j)-d_B(b_i,b_j)| \leq 3\epsilon\ .$$

Letting $c_i=\sigma^i(c)$, the only thing that remains to be checked is whether $|d(c_i,a_j)-d_B(b_i,a_j)|$ is small for all $i \in \{0,\dots,n-1\}, j \in \{1,\dots,m\}$. We have, since $d(c_i,a'_j)=\tilde{D}(b_i,a'_j)$, that

$$|d(c_i,a'_j)-d_B(b_i,a_j)| \leq \frac{1}{n}\sum_{k=0}^{n-1}|D(\tau^k(b_i),\sigma^k(a_j)-d_B(b_i,a_j)|\ . \tag{7.5}$$

By definition of D, for all $k \in \{0,\dots,n-1\}$

$$|D(\tau^k(b_i),\sigma^k(a'_j))-\tilde{d}(\tau^k(b_i),\sigma^k(a'_j))| \leq \delta\ .$$

Since $\tilde{d}$ was a metric on $B \cup A \cup A'$ coinciding with d on $A \cup A'$, we have, for all $k \in \{0,\dots,n-1\}$,

$$|\tilde{d}(\tau^k(b_i),\sigma^k(a'_j)-\tilde{d}(\tau^k(b_i),\sigma^k(a_j)| \leq \tilde{d}(\sigma^k(a'_j),\sigma^k(a_j))=d(a_j,a'_j) \leq \delta\ .$$

Using these inequalities, and the fact that τ is a $\tilde{d}$-isometry exending σ, (7.1) and (7.5) give, for all $i \in \{0,\dots,n-1\}$ and all $j \in \{1,\dots,m\}$,

$$|d(c_i,a'_j)-d_B(b_i,a_j)| \leq \frac{1}{n}\sum_{k=0}^{n-1}(2\delta+|\tilde{d}(b_i,a_j)-d(b_i,a_j)|) \leq 2\epsilon+2\delta \leq 4\epsilon\ .$$

Finally, since $d(a'_j, a_j) \leq \delta \leq \epsilon$, we obtain $|d(c_i, a_j) - d_B(b_i, a_j)| \leq 5\epsilon$ and, since ϵ was arbitrary, this (mercifully) concludes the proof.

□

Essentially the same reasoning as above enables one to check that the n-approximate extension property is equivalent to the following condition:

For any $\epsilon > 0$, any finite σ-invariant subset $A = \{a_1, \ldots, a_m\}$ of $\mathbf{U_Q}$ and any (B, d_B, τ) such that $B = \{a_1, \ldots, a_m, b, \tau(b), \ldots, \tau^{n-1}(b)\}$ is a $\mathbf{Q}$-metric space containing A, τ coincides with σ on A, and $\tau^n = 1$, there exists $\tilde{b} \in \mathbf{U}$ such that :

i. $\forall i \in \{1, \ldots, m\}\ \forall j \in \{0, \ldots, n-1\}\ |d_B(a_i, \tau^j(b)) - d(a_i, \sigma^j(b))| < \epsilon$.

ii. $\forall i, j \in \{0, \ldots, n-1\}\ |d(\sigma^i(\tilde{b}), \sigma^j(\tilde{b})) - d(\tau^i(b), \tau^j(b))| < \epsilon$.

Thus, the set of $\sigma \in \Omega_n(G)$ having the n-approximate extension property is G_δ. It is easy to check from Lemmas 7.1 and 7.6 that this set is dense in $\Omega_n(G)$. Since any two elements of $\Omega_n(G)$ with the n-approximate extension property are conjugate (because they actually have the extension property), we have finally obainted the following result.

Theorem 7.7. *For any integer n there exists an element g_n whose conjugacy class is comeager in $\{g \in \mathrm{Iso}(\mathbf{U})\colon g^n = 1\}$. Any $g \in \mathrm{Iso}(\mathbf{U})$ is a product of four conjugates of g_n.*

Proof. We have already explained why the first sentence is true. The second one follows as in the proof of Theorem 7.2 from the fact that whenever Q is a countable, divisible metric value set and $\mathbf{U}_Q$ is densely embedded in $\mathbf{U}$, the extension to $\mathbf{U}$ of a generic element of $\mathrm{Iso}(\mathbf{U}_Q)$ is a generic element of $\Omega_n(\mathrm{Iso}(\mathbf{U}))$. □

We conclude by discussing an open problem and a partial answer: it is known that there are elements of $\mathrm{Iso}(\mathbf{U})$ without roots of order n for any $n \geq 2$, however it is unknown whether a generic element must have roots of any order (or even, square roots); note that by the $0-1$ topological law the set of elements admitting a n-th root is meager or comeager for all n. It is also unknown whether a stronger condition holds: it is an open question whether a generic element g may be embedded in a *flow*, i.e. whether there exists a continuous homomorphism from $(\mathbf{R}, +)$ to $\mathrm{Iso}(\mathbf{U})$ such that $g = F(1)$. We do not know the anwer to that question, but can answer it in the case of generic elements of order n, and obtain a stronger result than expected in the bounded case.

Definition 7.8. Let X be a bounded metric space. We let d_u denote the uniform metric on $\mathrm{Iso}(X)$, defined by

$$d_u(g, h) = \sup\{d(g(x), h(x))\colon x \in X\}\ .$$

The uniform metric d_u is bi-invariant, complete, and is not separable in general.

Lemma 7.9. *Let $n \geq 2$ be an integer, X a compact metric space of diameter at most 1, and Y be a dense countable $\mathbf{Q}$-metric subspace of X. Assume that $F = (F_t)$ is a flow of isometries of X such that $F_1 = \sigma$ fixes Y, $\sigma^n = 1$, and $d_u(F_t, F_s) \leq n|t-s|$ for all $t, s \in \mathbf{R}$.*

Let $Z = Y \cup A$ be a $\mathbf{Q}$-metric space containing Y with A finite, and τ an isometry of Z extending $\sigma_{|Y}$. Then there exists a compact metric space $\tilde{X}$ containing X, of diameter at most 1, a dense countable $\mathbf{Q}$-metric subspace $\tilde{Y}$ of $\tilde{X}$, and a flow $G = (G_t)$ of isometries of $\tilde{X}$, such that

i. $G_{|X} = F$.

ii. $G_{1|A} = \tau$ *and* $G_1(\tilde{Y}) = \tilde{Y}$.

iii. $\forall t, s \in \mathbf{R}\ d_u(G_t, G_s) \leq n|t-s|$.

Proof. By induction, we may assume that that $A = \{a, \tau(a), \ldots, \tau^{n-1}(a)\}$ for some $a \in A$. For $t, s \in \mathbf{R}$ we set $\delta(t,s) = n|t-s|$; for $t \in \mathbf{R}$ and $x \in X$ we denote $F_t(x) = t \cdot x$. Our assumptions imply that we have, for all $t \in \mathbf{R}$:

$$\delta(t,0) \geq \sup_{x \in X} d(t \cdot x, x) \tag{7.6}$$

We define a map ω on $(\mathbf{R} \times X)^2$ by the following conditions:

- For all $t, s \in \mathbf{R}$, $\omega(t,s) = \min\{d(a, \tau^i(a)) + \delta(t+i, s)\}$
- For all $t \in \mathbf{R}, x \in X$, $\omega(t,x) = d(a, (-t).x)$.
- For all $x, y \in X$, $\omega(x,y) = d(x,y)$.

We let ρ be the pseudometric on $\mathbf{R} \times X$ associated to ω, i.e.

$$\forall a, b \in \mathbf{R} \times X\ \rho(a,b) = \inf\{\sum_{i=0}^{n} \omega(x_i, x_{i+1}) \colon x_0 = a, x_{n+1} = n\}$$

Then one may check the following facts (the verifications, if done in the same order as below, are completely straightforward so we omit them):

i. $\omega_{|\mathbf{R}^2}$ is an invariant pseudometric.

ii. For any triple x, y, z with at least two of its elements in X one has $\omega(x,y) + \omega(y,z) \geq \omega(x,z)$.

iii. For any $s, t \in \mathbf{R}$, for any $x \in X$, $\omega(t,s) + \omega(s,x) \geq \omega(t,x)$.

iv. On X, ρ and d coincide.

v. For all $i \in \mathbf{Z}$, $\rho(0,i) = \omega(0,i) = d(a, \tau^i(a))$.

vi. The infimum in the definition of ρ is actually a minimum, so $\rho(a,b) \in \mathbf{Q}$ for all $a, b \in \mathbf{Q} \cup Y$.

vii. For all t, s, $\rho(t,s) \leq n|t-s|$.

Once all these things are checked, we are essentially done: let $\tilde{X}$ be the metric space obtained by identifying points a, b such that $\rho(a,b) = 0$. Then $\tilde{X}$ is naturally isometric to $X \cup (\mathbf{R}/n\mathbf{Z}, \rho)$, with ρ coinciding with d on X; $\mathbf{R}$ acts isometricaly on $\tilde{X}$, extending the action given by F, the action on $\mathbf{R}/n\mathbf{Z}$ being the quotient of the translation of $(\mathbf{R}, +)$ on itself. We may identify $Y \cup A$ with $Y \cup \{0, \ldots, n-1\}$, and under this identification we have $j \cdot a = \tau^j(a)$ for all $j \in \mathbf{Z}$. Define $\tilde{Y}$ as $Y \cup \mathbf{Q}/n\mathbf{Z}$, which is a dense $\mathbf{Q}$-metric subspace of $\tilde{Y}$ on which ρ takes only rational values. Replacing ρ by $\min(\rho, 1)$, the proof is complete. □

Theorem 7.10. *Let n be an integer. Then a generic element of $\Omega_n(\mathrm{Iso}(\mathbf{U}))$ embeds in a flow. In the case of* $\mathrm{Iso}(\mathbf{U}_1)$, *a generic element embeds in a flow which is n-Lipschitz from* $(\mathbf{R}, +)$ *to* $(\mathrm{Iso}(\mathbf{U}_1), d_u)$.

Before the proof, we need to introduce some notations.

Notation. Let Y be a countable $\mathbf{Q}$-metric space of diameter at most 1, n be an integer and σ an element of $\mathrm{Iso}(Y)$ such that $\sigma^n = 1$. For any finite, σ-invariant subset $A \subseteq Y$, any finite metric space B containing A and any isometry τ of B of order n coinciding with σ on A, one may form the metric amalgam (for spaces of diameter at most 1) Z of Y and B over A and then extend τ to an isometry of Z.

If (Z,τ) has been obtained by the procedure above, we say that (Z,τ) is a *finite extension of* (Y,σ) *attached to* $(A,B,\tau_{|B})$; we denote by $E(Y,\sigma)$ the countable set of all (isomorphism types of) (Z,τ) which can be obtained by this procedure.

Proof of Theorem 7.10. We do not give the proof for $\mathrm{Iso}(\mathbf{U})$, which is based on an easy modification of Lemma 7.9 and the construction below. Fix a bijection $f\colon \mathbf{N}\times\mathbf{N} \to \mathbf{N}$ such that $f(p,q) > p$ for all (p,q).

Then, using lemma 7.9, we may build a sequence of compact metric spaces (X_m), with dense $\mathbf{Q}$-metric subspaces Y_m, and a sequence of isometric flows F_m such that:

i. For all m, $F_m(1) = \sigma_m$ is such that $\sigma_m^n = 1$, $\sigma_m(Y_m) = Y_m$ and σ_m has only finite orbits.

ii. For all m, for all $x \in X_m$, for all $t \in \mathbf{R}$, $d(F_m(t)(x), x) \leq n|t|$.

iii. For all m, X_{m+1} contains X_m, Y_{m+1} contains Y_m, and F_{m+1} extends F_m.

iv. For all m, let $\{(Z_{m,i},\tau_{m,i})\}_{i\in\mathbf{N}}$ be an enumeration of $E(Y_m,\sigma_m)$, and for all i let $A_{m,i}$, $B_{m,i}$, $\tau_{m,i}$ be such that $(Z_{m,i},\tau_{m,i})$ is a finite extension of (Y_m,σ) attached to $(A_{m,i},B_{m,i},\tau_{m,i})$. Then if $m = f(p,q)$, there exists $B \subseteq X_{m+1}$ containing $A_{p,q}$ such that $(B,A_{p,q},\sigma_m) \cong (B_{p,q},A_{p,q},\tau_{p,q})$.

Let $X_\infty = \cup X_m$, $Y_\infty = \cup Y_m$, F_∞ denote the flow on X_∞ produced by the above construction, and $\sigma_\infty = F_\infty(1)$. Then the construction (the last condition in particular) implies that Y_∞ is isometric to $\mathbf{U}_{Q_1}$ and σ_∞ is a generic element of $\mathrm{Iso}(\mathbf{U}_{Q_1})$. The completion of X_∞ contains Y_∞ as a dense subspace, so it is isometric to the completion of $\mathbf{U}_{Q_1}$, i.e. to $\mathbf{U}_1$. Thus σ_∞ extends to a generic element of $\mathrm{Iso}(\mathbf{U}_1)$ and the flow F_∞ extends to a flow such that $F_\infty(1) = \sigma_\infty$, $d_u(F_\infty(t),1) \leq n|t|$ for all $t \in \mathbf{R}$. $\square$

Corollary 7.11. $(\mathrm{Iso}(\mathbf{U}_1), d_u)$ *is path-connected.*

Proof. Let σ be a generic element in $\Omega_2(\mathrm{Iso}(\mathbf{U}_1))$, and $g \in G$. We know that there is a continuous flow $F\colon (\mathbf{R},+) \to (\mathrm{Iso}(\mathbf{U}_1), d_u)$ such that $\sigma = F(1)$, and elements $k_1,\dots,k_4$ such that $g = k_1\sigma k_1^{-1}k_1\cdots k_4\sigma k_4^{-1}$. Define $\phi\colon [0,1] \to \mathrm{Iso}(\mathbf{U}_1)$ by

$$\phi(t) = k_1F(t)k_1^{-1}\cdots k_4F(t)k_4^{-1}\,.$$

Then $\phi(0) = id_{\mathbf{U}}$, $\phi(1) = g$, and ϕ is continuous from $[0,1]$ to $(\mathrm{Iso}(\mathbf{U}_1), d_u)$ (actually, $d_u(\phi(t),\phi(s)) \leq 8|t-s|$ for all t,s). $\square$

Note that it is not known whether $\mathrm{Iso}(\mathbf{U}_1)$ is simple. If that were true then it would immediately imply Corollary 7.11, because the connected component of 1 is a nontrivial normal subgroup.

8 Rigid moieties

When we consider an amalgamation class, we know that it has a Fraïssé limit, but most of the time we have no idea how the structures in the class ”sit” in the limit. When we take an infinite structure whose age lies in the class, there are infinitely many ways to embed it into the limit. In this section we are going to consider a very specific way to embed these structures, as rigid moieties.

A general question was asked by Jaligot in [18] about whether the automorphism groups of Fraïssé limits are universal for the automorphism groups of the infinite structures whose age lies in the class. Here, universal meant that the groups embed into the bigger one. (In general the class of automorphism groups of structures in a Fraïssé class is not itself a Fraïssé class). The answer to this question is positive in the case of free amalgamation classes as shown by Proposition 4.8 and Theorem 4.9 (Consider automorphisms of the limit that respect the tower $E(X_i)$ in the proof of Theorem 4.9). If we could embed an infinite structure as a rigid moiety into the Fraïssé limit, then it would follow that the automorphism group of the structure embeds into the automorphism group of the limit by a much more direct argument (Then we do not need to consider those automorphisms of the limit respecting an intermediate tower). This section is about the amalgamation classes where these kinds of embeddings exist.

Definition 8.1. Given a countably infinite set X, a subset $M \subseteq X$ is called a *moiety* if $|M| = |X - M| = \omega$.

Definition 8.2. Let X be an $\mathcal{L}$-structure. A substructure $Y \subseteq X$ is called *rigid* if any automorphism of Y extends uniquely to an automorphism of X.

There are many examples where rigid moieties do not exist for trivial reasons. Consider the universal structure X in an empty language, which is nothing more than a countably infinite set. And for any moiety, i.e. any infinite and coinfinite subset Y of X, a bijection of Y can be extended in 2^ω many different ways to a bijection of X.

Here are some examples of amalgamation classes where rigid moieties do exist:

(1) Any countably infinite graph can be embedded as a rigid moiety into the random graph. This result was first proved by Henson in [10, Theorem 3.1]. Macpherson and Woodrow gave a slightly different proof in [28, Lemma 2.1] without being aware of Henson's proof.

(2) Any K_n-free graph can be embedded as a rigid moiety into the universal K_n-free graph. This result was stated by Henson in [10, Theorem 3.3], but didn't seem to be fully proved. What was actually proved by Henson was that given a K_n-free graph T, T can be embedded as a moiety into the universal K_n-free graph $\mathbf{K_n}$, such that every automorphism of T extends to an automorphism of $\mathbf{K_n}$. In his argument, the uniqueness of extensions is guaranteed only for those extensions that stabilize a subgraph intermediate between T and $\mathbf{K_n}$. We are going to prove that any countably infinite K_n-free graph can be embedded as a rigid moiety into $\mathbf{K_n}$ in Section 8.2.

(3) Any countably infinite tournament can be embedded as a rigid moiety into the random tournament. This result was proved by Eric Jaligot in [18]. It is one of the few cases where the

Fraïssé class does not admit the free amalgamation property, which is much stronger than the amalgamation property.

(4) In the metric setting, Julien Melleray proved in [29] that, for any Polish group G, there exists a closed set $F \subset \mathbf{U}$ such that G is (topologically) isomorphic to $\mathrm{Iso}(F)$, and every isometry of F is the restriction of a unique isometry of $\mathbf{U}$; in particular, G is isomorphic to $\mathrm{Iso}(\mathbf{U}, F)$.

In this section, we will introduce a slightly different method than the one used by Henson in [10, Theorem 3.3] to prove the result for K_n-free graphs. Then we will gradually generalize this method to prove the same result for the Henson's family of directed graphs in Section 8.3, then for the universal structure in a finite relational language in Section 8.4, and finally in Section 8.5 for the Fraïssé limit of any free amalgamation class which satisfies a very weak property.

8.1 The random graph

In this section we recall the original proof of Henson where he embeds any countably infinite graph H into the random graph $\mathcal{R}$ as a rigid moiety. His method is the main tool for proving similar types of arguments for the different classes of graphs.

Theorem 8.3. *[10, Theorem 3.1] Let H be a countably infinite graph. Then there exists an embedding of H onto an induced subgraph $H' \subset \mathcal{R}$ such that each automorphism of H' extends uniquely to an automorphism of $\mathcal{R}$.*

Proof. Let $n_1 < n_2 ...$ be an increasing sequence of positive integers. Construct a chain of graphs $H_0 \subset H_1 \subset H_2 ...$ by letting $H_0 = H$ and continuing as follows. For $k \geqslant 1$ obtain H_k by adding to H_{k-1} a new vertex $v(A,k)$ for each finite set $A \subset H_{k-1}$ such that $A \cap H_0$ has exactly n_k elements. Each new vertex $v(A,k)$ is connected in H_k to the vertices in A and to no others. Define K to be the union of the chain $\{H_k : k \geqslant 0\}$ so that $H_k \subset K$ for each $k \geqslant 0$ and, in particular, $H \subset K$.

If F_1, F_2 are disjoint, finite subsets of K, choose k large enough so that $F_1 \cup F_2 \subset H_{k-1}$ and $F_1 \cup H_0$ has at most n_k elements. Since H_0 is infinite, there is a set $B \subset H_0$ such that $B \cup F_2 = \emptyset$, $F_1 \cap H_0 \subset B$ and B has exactly n_k elements. Letting $A = F_1 \cup B$, it follows that $v(A,k)$ is a vertex in H_k which is connected in H_k to every vertex in F_1 and to no vertex in F_2. This shows that K satisfies the Alice Restaurant Axiom for the random graph. Since only countably many vertices are added at each stage, it follows that $K \cong \mathcal{R}$.

Any automorphism f of H_{k-1} which satisfies $f(H_0) = H_0$ can be extended to an automorphism of H_k by setting $f(v(A,k) = v(f(A),k)$. Moreover, since $f(v(A,k))$ must be connected in H_k to the vertices in $f(A)$ and no others, this is the only possible way to extend such an f. Therefore, each automorphism of H_0 can be extended to an automorphism of K, and this extension is unique among automorphisms of K which leave each set H_k invariant for $k > 0$. ☐

We remark that, indeed, the proof of Theorem 8.3 yields many rigid moieties.

Theorem 8.4. *Let H be a countably infinite graph. Then there exists 2^ω many embeddings of H into $\mathcal{R}$ as a rigid moiety, which are not conjugate under the automorphism group of $\mathcal{R}$.*

Proof. Notice that the choice of the sequence $n_1 < n_2 ...$ in the proof of Theorem 8.3 is arbitrary. And for two distinct sequences $n_1 < n_2 ...$ and $n'_1 < n'_2 ...$, we get two distinct embeddings of H into

$\mathcal{R}$, call them H_1 and H_2 . Let $n_i \in \{n_k\}_{k\in\mathbb{N}} - \{n'_k\}_{k\in\mathbb{N}}$. Then in $\mathcal{R} - H_1$ we have vertices which are adjacent to exactly n_i many vertices in H_1 but no such vertex exists in $\mathcal{R} - H_2$. Hence, H_1 and H_2 are not conjugate. □

8.2 The universal K_n-free graph

As mentioned before, the main result for this subsection was stated but apparently not proven by Henson in [10, Theorem 3.3]. To fix it, we introduce a revision. We make our structure A bigger by a different method than the one used by Henson, by adding to it a specific countably infinite graph totally disconnected from A, which will be called N, and then build the towers on top of $A \sqcup N$. We will make sure that each newly added vertex during the tower construction will have an edge with at least one element in A, making N the only set of elements in the union with absolutely no edges connecting it to A. Then A will be rigidly embedded into $A \sqcup N$, and $A \sqcup N$ will be rigidly embedded into the limit, yet in this case, A will be rigidly embedded into the limit as well since any automorphism of the limit which fixes A setwise, will have to fix $A \sqcup N$ setwise as well.

This revised method will be the backbone for the proofs in sections 8.3, 8.4 and 8.5. Each of those proofs will introduce new revisions to the method to make it work for the different or eventually, more general classes of structures. The following results in this subsection are scheduled for publication in *Contributions to discrete mathematics* under the title of *"Some rigid moieties of homogeneous graphs"* and is joint work with Jaligot.

Theorem 8.5. *Let T be any countably infinite K_n-free graph. Then T embeds into the universal K_n-free graph $\mathbf{K_n}$ as a rigid moiety.*

Recall that we stated in Section 4.1 that the Fraïssé limit for the class of finite K_n-free graphs for $n \geqslant 3$, denoted $\mathbf{K_n}$, is characterized by the property that for any finite disjoint subsets A and B such that A is K_{n-1}-free, there exists a vertex $v \in \mathbf{K_n}$ such that v is adjacent to every vertex in A and not adjacent to any vertex in B.

Lemma 8.6. *There is a countably infinite graph N, that has a trivial automorphism group and is K_n-free for all $n \geq 3$.*

There are many examples of graphs as in Lemma 8.6, and one is the graph consisting of the set N of natural numbers as vertices, and where edges are exactly of the form $(k, k+1)$ up to symmetry. The fact that it has a trivial automorphism group should be clear because 0 is the only vertex which is connected to only one vertex.

Here starts our proof of Theorem 8.5. Let T be a countably infinite K_n-free graph and let $T' = T \sqcup N$, where we add the specific graph N just described above as a new connected component to T. We are going to construct a supergraph of T'. Let $1 \leq n_1 < n_2 < \cdots$ be any strictly increasing sequence of positive integers. Construct an increasing chain of graphs $T' = T_0 \subset T_1 \subset T_2 \cdots$ as follows. For $k \geq 1$, obtain T_k by adding a new vertex $v(A,k)$ to T_{k-1} for each finite subset $A \subset T_{k-1}$ satisfying the following three conditions:

(1) $T_{k-1}|A$ is K_{n-1}-free.

(2) $A \cap T_0$ has exactly n_k elements.

(3) $A \cap T$ has at least one element.

Then each new vertex $v(A, k)$ in T_k will be adjacent to the vertices in A and to no others.

Let $\mathcal{T}$ be the union of the chain T_k. Notice that $T_k - T_{k-1}$ is infinite for each $k \geq 1$.

Lemma 8.7. *$\mathcal{T}$ is countable and K_n-free.*

Proof. Since at each step we are adding countably many new vertices, the union $\mathcal{T}$ is countable. Notice that $T_0 = T \sqcup N$ is K_n-free. We now show that T_k is K_n-free for each k. By induction on k, assume that T_{k-1} is K_n-free. Each new vertex $v(A, k)$ we add is adjacent only to A inside T_{k-1}, where $T_{k-1}|A$ is K_{n-1}-free, and adjoining a new vertex to all vertices of a K_{n-1}-free graph cannot create a complete graph on n vertices. Since there are no edges in $T_k - T_{k-1}$, we get that T_k is K_n-free. $\square$

Definition 8.8. Let G be a graph. A subset I of vertices of G is said to be an *independent* set if the induced subgraph on I does not have any edge relations. G itself is called *independent* if it does not have any edge relations.

Lemma 8.9. *$T' = T \sqcup N$ satisfies the following condition: if F is any finite subset of vertices of T', then there exists an infinite independent set of vertices $A \subset N - F$ such that no vertex in F is adjacent in T' to any vertex in A.*

Proof. Recall that N has the isomorphism type of the graph described after Lemma 8.6, and is a connected component of T'. Since F is finite, there are infinitely many vertices in $N - F$. Removing the finitely many vertices of $N - F$ connected to some element in F, we get an infinite set of elements of $N - F$ not adjacent to any vertex in F. Now choose A to be an independent set among these vertices, for example all odd numbered or all even numbered remaining vertices. $\square$

Lemma 8.10. *Let A, B be two finite disjoint subsets of vertices of $\mathcal{T}$ such that $\mathcal{T}|A$ is K_{n-1}-free. Then there exists a vertex $v \in \mathcal{T}$ such that v is adjacent to every vertex in A and to none of the vertices in B.*

Proof. Choose k large enough so that $A \sqcup B \subset T_{k-1}$ and $A \cap T_0$ has at most $n_k - 1$ elements. Let $C \subset T_0$ consists of $A \cap T_0$ together with every vertex in T_0 which is connected to some member of $A - T_0$. Since A is finite, and each vertex in $\mathcal{T} - T_0$ is connected to only finitely many members of T_0, it follows that C is finite. Then by Lemma 8.9 there exists an infinite independent set A' in the subgraph N such that $A' \cap C = \emptyset$ and no vertex in C is connected in T_0 to any vertex in A'. Now, $\mathcal{T}|(A \cup A')$ is K_{n-1}-free. Since A' is infinite, we can choose a set $D \subset (A \cup A') \cap T_0$ such that $D \cap B = \emptyset$, $A \cap T_0 \subseteq D$ and D has exactly $n_k - 1$ elements. Now there are two cases:

Case 1: $A \cap T \neq \emptyset$. In this case D has at least one element in T. Also add one more vertex to D to make its size exactly n_k (one may choose a new element in the set A' as above). Letting $E = A \cup D$ it follows that $\mathcal{T}|E$ is K_{n-1}-free and $E \cap T_0 = D$ has n_k elements with at least one element in T. Thus there exists $v(E, k)$ in T_k which is adjacent to every member of E, and in particular to every member of A, and to no member of B.

Case 2: $A \cap T = \emptyset$. Since T is infinite under our standing assumptions for Theorem 8.5, $T - (B \cup C)$ is nonempty. Let x be any vertex in $T - (B \cup C)$. Look at the set $D' = D \sqcup \{x\}$, which is a disjoint union because $D \subset (A \cup A') \subseteq N$. Now D' has exactly n_k vertices with one vertex in T, $D' \cap B = \emptyset$ and $A \cap T_0 \subseteq D \subset D'$. Let $E = A \cup D'$. We need to check that E is K_{n-1}-free. But as in Case 1 we see that $E - \{x\} = A \cup D$ is K_{n-1}-free, and x is chosen in such a way that it is not adjacent to any vertex in $E - \{x\}$ because T and N are two distinct connected components of T_0, so E is K_{n-1}-free. Since $E \cap T_0 = D'$ has exactly n_k elements, with the element x in T, the rest of the proof follows exactly as in Case 1 by considering the vertex $v(E, k)$ of T_k. $\square$

So, $\mathcal{T}$ is a countable K_n-free graph such that for any finite disjoint subsets $A, B \subset \mathcal{T}$ such that A is K_{n-1}-free, there exists a vertex $v \in \mathcal{T}$ such that v is adjacent to each vertex in A and to none of the vertices in B. Hence $\mathcal{T}$ is isomorphic to $\mathbf{K_n}$.

Lemma 8.11. *Every automorphism of T extends uniquely to an automorphism of $\mathcal{T}$.*

Proof. First, to prove the existence of extensions, given an automorphism ϕ of T, extend ϕ to an automorphism of T_0 by setting $\phi(v) = v$ for $v \in N$. Then proceed by induction on k as follows: given that ϕ is extended to an automorphism of T_{k-1}, extend it further to an automorphism of T_k by setting $\phi(v(A, k)) = v(\phi(A), k)$. This takes care of the existence of the extension.

Now, for the uniqueness, let $\phi \in Aut(\mathcal{T})$ such that $\phi(T) = T$. Then it follows that $\phi(N) = N$ since the only vertices in $\mathcal{T}$ which are not adjacent to any vertex in T are the vertices of N. So we have $\phi(T') = \phi(T \sqcup N) = \phi(T) \sqcup \phi(N) = T \sqcup N = T'$. And also notice that ϕ is the identity on N since N only has the trivial automorphism. So ϕ is uniquely determined for vertices in T_0.

Notice that any $\phi \in Aut(\mathcal{T})$ which setwise stabilizes $T_0 = T'$ has to setwise stabilize each $T_k - T_{k-1}$ for $k \geq 1$. This is true because vertices in $T_k - T_{k-1}$ are adjacent to exactly n_k many vertices in T_0 and $n_k \neq n_{k'}$ for $k \neq k'$. Thus, each T_k is setwise stabilized by ϕ as well.

We will proceed by induction. Let $v \in \mathcal{T} - T_0$. Then $v \in T_k - T_{k-1}$ for some k. By induction assume that ϕ is uniquely determined for vertices in T_{k-1}. But $v = v(A, k)$ for some $A \subset T_{k-1}$ such that

(1) $T_{k-1}|A$ is K_{n-1}-free,

(2) $A \cap T_0$ has exactly n_k elements,

(3) $A \cap T$ has at least one element,

and v is adjacent to all vertices of A and to no other vertex in T_{k-1}.

By induction $\phi(A)$ is uniquely determined, and because ϕ is an automorphism which setwise stabilizes both T and T_0, we have

(1) $T_{k-1}|\phi(A)$ is K_{n-1}-free,

(2) $\phi(A) \cap T_0$ has exactly n_k elements,

(3) $\phi(A) \cap T$ has at least one element.

Hence there is a unique vertex $v(\phi(A), k) \in T_k$ which is adjacent to $\phi(A)$ and to no other vertices in T_{k-1}. So $\phi(v) = v(\phi(A), k)$. $\square$

So this concludes the proof of Theorem 8.5. But notice that we actually proved a little more. There are exactly 2^ω such rigid embeddings of T which are not conjugate in $Aut(\mathbf{K_n})$.

Theorem 8.12. *Given a countably infinite K_n-free graph T, there are 2^ω many rigid embeddings of T into $\mathbf{K_n}$ which are pairwise non-conjugate under $Aut(\mathbf{K_n})$.*

Proof. Notice from the above proof that for each strictly increasing sequence $1 \leq n_1 < n_2 < \cdots$ of positive integers we get a rigid embedding of the form

$$X \sqcup X_0 \sqcup X_{n_1} \sqcup \cdots \sqcup X_{n_k} \sqcup \cdots$$

where $X = T, X_0 = T_0 - T, \ldots, X_{n_k} = T_k - T_{k-1}$. Furthermore, X_0 is the set of elements connected to no vertices in T, and X_{n_k} is the set of elements connected to exactly n_k many vertices in $X \sqcup X_0$.

Now for two distinct strictly increasing sequences of natural numbers, we get two rigid embeddings of T into $\mathbf{K_n}$ which are not conjugate by an automorphism of $\mathbf{K_n}$. Since there are 2^ω many such sequences, we conclude that there are 2^ω non-conjugate rigid embeddings of T into $\mathbf{K_n}$. $\square$

Remark 8.2.1. In our proof of Theorem 8.12 we varied the sequence of integers $1 \leq n_1 < n_2 < \cdots$, but we could also have varied the isomorphism type of the graph X_0, there isomorphic to the specific graph N as described after Lemma 8.6, as long as it is a countably infinite K_n-free graph with a trivial automorphism group and satisfying the statement of Lemma 8.9.

8.3 Henson's continuous family of directed graphs

In this section we prove the same result as in the previous section for a class of digraphs first considered by Henson in [11]. We choose a set of tournaments and consider all finite digraphs which omit these tournaments. The resulting class has the free amalgamation property and any countably infinite digraph omitting these tournaments can be embedded as a rigid moiety into the Fraïssé limit of this class. The method is pretty much the same and the only differences are due to the slight complexity of the Alice Restaurant Axiom for this class. In the case of K_n-free graphs, the axiom is pretty straightforward because we know that given disjoint sets A and B, where A is K_{n-1}-free, the graph we construct as $A \sqcup B \sqcup \{v\}$ is K_n-free. But in most cases, we do not know explicitly how we can construct a bigger graph which still belongs to the class. This is one of those cases. Yet we always know that such extensions exists, even if we can't describe them explicitly. We will call them *one-point extensions*.

Definition 8.13. Let $\mathcal{C}$ be a free amalgamation class and let $X \in \mathcal{C}$. Consider $Y \in \mathcal{C}$ such that X embeds into Y and $|Y| = |X| + 1$. Then we say that Y is a one-point extension of X.

Theorem 8.14. *Let $\{T_i\}_{i\in\mathbb{N}}$ be a set of finite tournaments where $|T_i| \geqslant 3$. Let $\mathcal{D}_{T_i}$ be the class of all finite directed graphs which are $\{T_i\}$-free. Let $\mathbf{D_{T_i}}$ be the Fraïssé limit of $\mathcal{D}_{T_i}$. Then, given any countably infinite directed graph D which is $\{T_i\}$-free, D embeds as a rigid moiety into $\mathbf{D_{T_i}}$.*

Definition 8.15. Let $X \in \mathcal{D}_{T_i}$ such that $X = A \sqcup B \sqcup C$. We say that (A, B, C) has a one-point extension if there exists $Y \in \mathcal{D}_{T_i}$ isomorphic to the digraph $X \sqcup \{v\}$ where v has a directed edge going towards every vertex in A, has a directed edge coming from every vertex in B and is not adjacent to any vertices in C. We also say that v extends (A, B, C).

Fact 8.1. $\mathbf{D_{T_i}}$ *is characterized as the countably infinite $\{T_i\}$-free digraph X with the following property (Alice Restaurant Axiom):*

Given finite disjoint subsets $A, B, C \subseteq X$ such that (A, B, C) has a one-point extension in $\mathcal{D}_{T_i}$, there exists a vertex $x \in X$ extending (A, B, C).

Proof. First we will construct a countably infinite directed graph which has a trivial automorphism group and is $\{T_i\}$-free. Let the vertex set V be the natural numbers. Then set a directed edge between each consecutive vertex with the direction going from the small number to the bigger one. Call this digraph $\mathbf{N}$. Notice that N satisfies the following property:

$P1$: Given a finite subset $F \subseteq N$, there exists an infinite independent subset $A' \subseteq N - F$ such that no vertices of F and A are adjacent.

Now let D_0 be the graph $D \sqcup \mathbf{N}$. Notice that D_0 is a countably infinite digraph which is $\{T_i\}$-free. Now we will build a tower of digraphs over D_0. Let $1 \leqslant n_0 < n_1 < n_2 ...$ be a strictly increasing sequence of positive integers. Let $\{(A_i, B_i, C_i) : i \in \mathbb{N}\}$ be an enumeration of all disjoint triples of finite subsets of D_0 which satisfy:

(i) $(A_i \sqcup B_i) \cap D \neq \emptyset$

(ii) $|(A_i \sqcup B_i) \cap D_0| = n_0$

(iii) (A_i, B_i, C_i) has a one-point extension

Now let $v(A_0, B_0, C_0) \in D_1$ extending (A_0, B_0, C_0). After having constructed $v(A_k, B_k, C_k)$, we add $v(A_{k+1}, B_{k+1}, C_{k+1})$ to D_1 only if the tuple $(A_{k+1}, B_{k+1}) \neq (A_i, B_i)$ for every $i < k+1$.

Now, assume D_{k-1} have been constructed. Let $\{(A_i, B_i, C_i) : i \in \mathbb{N}\}$ be an enumeration of all triples of disjoint finite subsets of $D_0 \sqcup D_1 \sqcup ... \sqcup D_{k-1}$. For any finite disjoint subsets of vertices $A, B, C \subseteq D_{k-1}$ satisfying:

(i) $(A_i \sqcup B_i) \cap D \neq \emptyset$

(ii) $|(A_i \sqcup B_i) \cap D_0| = n_{k-1}$

(iii) (A_i, B_i, C_i) has a one-point extension

Now let $v(A_0, B_0, C_0) \in D_k$ extending (A_0, B_0, C_0). After having constructed $v(A_k, B_k, C_k)$, we add $v(A_{k+1}, B_{k+1}, C_{k+1})$ to D_k only if the tuple $(A_{k+1}, B_{k+1}) \neq (A_i, B_i)$ for every $i < k+1$.

Let $D_\omega = \bigcup D_i$. We will show that $D_\omega \cong \mathbf{D_{T_i}}$.

Let A, B, C be a triple of finite disjoint subsets of vertices of D_ω such that (A, B, C) has a one-point extension. We need to find a vertex $v \in D_\omega$ extending (A, B, C).

Choose k large enough so that $(A \sqcup B \sqcup C) \subset D_0 \sqcup D_1 \sqcup ... \sqcup D_k$ and $(A \sqcup B) \cap D_0$ has at most $n_k - 1$ elements. Let $E \subset D_0$ consist of $(A \sqcup B) \cap D_0$ together with every vertex in D_0 which is adjacent to some member of $(A \sqcup B) - D_0$. Since $A \sqcup B$ is finite, and each vertex in $D_\omega - D_0$ is adjacent to only finitely many members of D_0, it follows that E is finite. Then by $P1$, there exists an infinite independent set $A' \subseteq N \subseteq D_0$ such that $A' \cap E = \emptyset$ and no vertex in E is adjacent to any vertex in A'. We can find such A' since no vertices of D and N are adjacent.

Since A' is infinite, we can choose a set $E' \subset (A \sqcup B) \cap D_0 \sqcup A'$ such that $E' \cap C = \emptyset$, $(A \sqcup B) \cap D_0 \subset E'$ and E' has exactly $n_k - 1$ elements.

There are three cases. Either $A \cap D \neq \emptyset$ or $B \cap D \neq \emptyset$ or $(A \cup B) \cap D = \emptyset$. The first two cases are symmetric so we'll only cover one of them.

Case 1: Without loss of generality assume $A \cap D \neq \emptyset$.

Since $A \cap D \neq \emptyset$, $E' - B$ has at least one element in D. Add one more vertex to E' from A' to make its size exactly n_k in a way that E' still satisfies the the same conditions.

Lemma 8.16. *: $(A \cup E' - B, B, C)$ has a one-point extension.*

Proof. : Assume that $(A \cup E' - B) \sqcup B \sqcup C \sqcup \{v\}$ has an induced subgraph isomorphic to one of the constraint tournaments T. Without loss of generality write $T \subseteq (A \cup E' - B) \sqcup B \sqcup C \sqcup \{v\}$. Now since T is a complete digraph and (A, B, C) has a one-point extension, T has to intersect $E' - (A \sqcup B)$ nontrivially. But then T has to be a subgraph of $E' - (A \sqcup B) \sqcup \{v\}$ since there are no directed edges between $E' - (A \sqcup B)$ and $A \sqcup B$. But this is impossible since $E' - (A \sqcup B)$ is an independent set, so $E' - (A \sqcup B) \sqcup \{v\}$ cannot include any complete digraphs of size greater than 2. □

Notice that $|(A \cup (E' - B) \sqcup B) \cap D_0| = |E'| = n_k$. Hence the triple $(A \cup E' - B, B, C)$ satisfies

(i) $((A \cup E' - B) \sqcup B) \cap D \neq \emptyset$

(ii) $|((A \cup E' - B) \sqcup B) \cap D_0| = n_k$

(iii) $(A \cup E' - B, B, C)$ has a one-point extension

So there exists a vertex $v(A \cup E' - B, B, C) \in D_{k+1}$ extending $(A \cup E' - B, B, C)$. But the same v extends (A, B, C) so we are done.

Case 2: $(A \sqcup B) \cap D = \emptyset$. In this case we need to add a vertex to E' from D. Choose an arbitrary vertex $x \in D - E$. To avoid further notation, assume now $x \in E'$ so $|E'| = n_k$. We again claim that $(A \cup E' - B, B, C)$ has a one-point extension. Since $x \notin E$, x is not adjacent to any vertices in $A \sqcup B$ and furthermore $E' - (A \sqcup B)$ is still an independent set. Hence it's the same situation as in Lemma 8.16 and we are done.

So D_ω is the Fraïssé limit of $\mathcal{D}_{T_i}$. Now we show that D indeed embeds rigidly into D_ω.

First, to prove the existence of extensions, given an automorphism ϕ of D, extend ϕ to an automorphism of D_0 by setting $\phi(v) = v$ for $v \in N$. Now assume that ϕ is extended to an automorphism of $D_0 \sqcup D_1 \sqcup ... \sqcup D_{k-1}$. Let $v = v(A, B, C,) \in D_k$ where (A, B, C) is a triple of disjoint finite subsets of $D_0 \sqcup D_1 \sqcup ... \sqcup D_{k-1}$ and define $\phi(v(A, B, C)) = v(\phi(A), \phi(B), \phi(C))$. Notice that since (A, B, C) has an extension in D_k, it has to satisfy the properties i, ii, iii, hence $(\phi(A), \phi(B), \phi(C))$ satisfies them as well by the assumption that $\phi \in Aut(D_0 \sqcup D_1 \sqcup ... \sqcup D_{k-1})$. So there exists a $v(\phi(A), \phi(B), \phi(C)) \in D_k$. Hence, ϕ extends to a homomorphism of D_ω.

Notice that ϕ is a bijection because if $v(A, B, C) \neq v'(A', B', C')$, then $(A, B) \neq (A', B')$ thus $(\phi(A), \phi(B)) \neq (\phi(A'), \phi(B'))$, so $\phi(v) \neq \phi(v')$. And given $v(A, B, C) \in D_k$, we have $v = \phi(v'(\phi^{-1}(A), \phi^{-1}(B), \phi^{-1}(C)))$.

Now to show the uniqueness of extensions, let $\phi \in Aut(D_\omega)$ such that $\phi(D) = D$. Then it follows that $\phi(N) = N$ since the only vertices in $D_\omega - D$ which are not adjacent to any vertex in D, are the vertices of N. So we have $\phi(D_0) = D_0$. And also notice that ϕ is identity on N since N only has the trivial automorphism. So ϕ is uniquely determined for vertices in D_0. Now, any $\phi \in Aut(D_\omega)$ which setwise stabilizes D_0 has to setwise stabilize each D_k for $k \geqslant 1$. This is true because vertices in D_k are adjacent to exactly n_k many vertices in D_0 and $n_k \neq n_{k'}$ for $k \neq k'$. Let $v \in D_\omega - D_0$. Then $v \in D_k$ for some $k \geqslant 1$. By induction assume that ϕ is uniquely determined for vertices in $D_0 \sqcup ... \sqcup D_{k-1}$. But v extends (A, B, C) for some $A \sqcup B \sqcup C \subset D_{k-1}$ such that

(i) (A, B, C) has a one-point extension

(ii) $(A \cup B) \cap D_0$ has exactly n_k elements

(iii) $(A \cup B) \cap D$ has at least one element

By induction ϕ is uniquely determined for A, B, C and because ϕ is an automorphism which setwise stabilizes both D and D_0, we have

(i) $(\phi(A), \phi(B), \phi(C))$ has a one-point extension

(ii) $\phi(A \cup B) \cap D_0$ has exactly n_k elements.

(iii) $\phi(A \cup B) \cap D$ has at least one element.

Hence there is a unique vertex v' extending $(\phi(A), \phi(B), \phi(C))$. So $\phi(v) = v'$. □

8.4 The universal structure in a finite relational language

In this section we are going to take a finite relational language with arbitrarily many relations of any arity and then consider the class of all finite structures in this language. This class does have the free amalgamation property and we will show that any countably infinite structure in this language can be embedded as a rigid moiety into the Fraïssé limit of this class. The proof in this section is similar to Henson's proof for the Random graph given in Section 8.1 because we are not going to use a rigid component similar to the one we used in sections 8.2 and 8.3. The main difference of this proof is that it is the first one where we have more than one relation in our language, and moreover we do not restrict the arity of the relations to 2. Fortunately we do not need to control the number of hyper-edges between the tower and the base structure for each relation one by one. Controlling them for one relation is enough to get the desired result.

Definition 8.17. Let $\mathcal{L}$ be a finite relational language. Consider the class $\mathcal{X}$ of all finite $\mathcal{L}$-structures. Then $\mathcal{X}$ is a free amalgamation class and hence has a Fraïssé limit. Call it $\mathbf{X}$.

Theorem 8.18. *Let T be a countably infinite $\mathcal{L}$-structure. Then T embeds as a rigid moiety into $\mathbf{X}$.*

Proof. Enumerate the relation symbols in $\mathcal{L}$ as $\{R_1, R_2, ..., R_k\}$ and let $\{l_1, l_2, ..., l_k\}$ enumerate respectively their arities. Let $T = T_0$. Let $1 \leqslant n_1 < n_2 < ...$ be a strictly increasing sequence of positive integers. Let $\{A_i\}_{i \in \mathbb{N}}$ be an enumeration of all finite substructures of T_0 with cardinality $\geqslant n_1$. Now let $v_i(A_0) \in T_1$ such that

(i) v_i is R_1-related to exactly n_1 many elements in $T_0 \cap A_0$.

(ii) Given $i \neq j$, $qftp(v_i/A_0) \neq qftp(v_j/A_0)$

(iii) v_i runs over all q.f. types over A_0, which are finitely many for each A_i.

(iv) v_i is not related to any elements in $T_0 - A_0$

Now, assume we are done with A_k. Let $v_i(A_{k+1}) \in T_1$ such that

(i) v_i is R_1-related to exactly n_1 many elements in $T_0 \cap A_{k+1}$.

(ii) $qftp(v_i/T_0) \neq qftp(v/T_0)$ for any $v \in T_1$.

(iii) v_i runs over all q.f. types over A_{k+1}

(iv) v_i is not related to any elements in $T_0 - A_{k+1}$

Now, assume T_k is constructed. Let $\{A_i\}$ be an enumeration of all finite substructures of $T_0 \sqcup T_1 \sqcup ... \sqcup T_k$ with cardinality $\geqslant n_k$. So again, start with A_0 and let $v_i(A_0) \in T_{k+1}$ be such that:

(i) v_i is R_1-related to exactly n_k many elements in $T_0 \cap A_0$.

(ii) Given $i \neq j$, $qftp(v_i/A_0) \neq qftp(v_j/A_0)$

(iii) v_i runs over all q.f. types over A_0

(iv) v_i is not related to any elements in $(T_0 \cup T_1 \cup ... \cup T_k) - A_0$

Assume we are done with A_n. Let $v_i(A_{n+1}) \in T_{k+1}$ such that

(i) v_i is R_1-related to exactly n_k many elements in $T_0 \cap A_{n+1}$.

(ii) $qftp(v_i/T_0 \sqcup T_1 \sqcup ... \sqcup T_k) \neq qftp(v/T_0 \sqcup T_1 \sqcup ... \sqcup T_k)$ for any $v \in T_{k+1}$.

(iii) v_i runs over all q.f. types over A_{n+1}

(iv) v_i is not related to any elements in $(T_0 \cup T_1 \cup ... \cup T_k) - A_{n+1}$

Let $T_\omega = \bigcup T_n$. We claim that $T_\omega \cong \mathbf{X}$. Notice that $\mathbf{X}$ is determined up to isomorphism by the following Axiom:

Given a finite substructure $A \subseteq \mathbf{X}$ and a one-point extension $A \sqcup \{v\} \in \mathcal{X}$, there exists a $v' \in \mathbf{X}$ and an isomorphism $f : A \sqcup \{v\} \to A \sqcup \{v'\}$ where $f(a) = a$ for all $a \in A$.

So let A be a finite substructure of T_ω . Since A is finite and is an $\mathcal{L}$-structure, $A \in \mathcal{X}$. Let $A \sqcup \{v\} \in \mathcal{X}$ be a one-point extension. We want to find a $v' \in T_\omega$ such that $qftp(v/A) = qftp(v'/A)$.

Now, let M be large enough such that $n_M \geqslant |A|$ and $A \subseteq T_0 \sqcup T_1 \sqcup ... \sqcup T_M$. Let $A' = A \sqcup \{a_1, .., a_{n_M}\}$ where $a_1, .., a_{n_M} \in T_0 - A$. So now we have $|A' \cap T_0| \geqslant n_M$ and $A' \subseteq T_0 \sqcup T_1 \sqcup ... \sqcup T_M$. So there exists $v_i(A') \in T_{M+1}$ which is R_1-related to exactly n_M many elements in $A' \cap T_0$ and v_i run over all q.f. types over A'. But since $|A' \cap T_0| = n_M + |A \cap T_0|$ there are enough elements such that v_i's will realize each q.f. type over A, hence for some i, we will have $qftp(v_i/A) = qftp(v/A)$.

So we have $T_\omega \cong \mathbf{X}$. Now, given an automorphism ϕ of T_0 we want to show that it extends uniquely to an automorphism of $\mathbf{X}$.

We will first show the existence of extensions. Let $x \in T_1$. Then $x = x(A)$ for some finite substructure $A \subseteq T_0$ and x is R_1-related to exactly n_1 many elements in T_0. Look at $\phi(A) \subseteq T_0$. There exists a unique $x' \in T_1$ such that $qftp(x/A) = qftp(x'/\phi(A))$. Let $\phi(x) = x'$. This extends ϕ to an automorphism of $T_0 \sqcup T_1$. Similarly we extend ϕ to an automorphism of $T_0 \sqcup T_1 \sqcup ... \sqcup T_n$ for each n.

To see the uniqueness of extensions, given an automorphism ϕ of $\mathbf{X}$ such that $\phi(T_0) = T_0$, ϕ has to fix each T_k setwise because the elements of T_k are R_1-related to exactly n_k many elements of T_0 and for each $k \neq k'$, $n_k \neq n'_k$. So let $v = v(A) \in T_1$ where $A \subseteq T_0$. So we know that $\phi(v) \in T_1$. But there exists a unique $v' \in T_1$ such that $qftp(v/A) = qftp(v'/\phi(A))$. Thus $\phi(v) = v'$. Again, continuing like this we see that ϕ is uniquely determined for each T_k. $\square$

8.5 The general case

This section represents the most general stage for our result on the existence of rigid moieties. We consider an arbitrary free amalgamation class, which satisfies a very weak condition requiring that all one-point sets are isomorphic, or in other words that the automorphism group of the Fraïssé limit is transitive on points. We will prove that rigid moieties exist in this setting if the class satisfies a certain property and they don't exist otherwise. The proof in this section is much more involved compared to the previous ones, even though the method is more or less the same.

Yet here, we introduce a new method to construct rigid infinite structures similar to the ones we called N in the previous proofs. So far, the existence of those structures were evident because the classes were explicitly defined. But in an arbitrary free amalgamation class, it is not immediate that an analogue of N exists, yet fortunately, it does.

So the proof consists of two steps, the existence of N and then the construction of the tower, which does follow more or less the same steps as the proof for the family of digraphs in Section 8.3. We start with a small classification result and say that, for certain free amalgamation classes, rigid moieties do not exist at all, so we focus our attention to the remaining ones.

In cases where we have more than one isomorphism type among one-point sets, there are still revisions we can apply to make the proof work, but then we have to make more assumptions.

In the remaining parts of this section, we are going to use the framework and the notation of Section 4.2. Recall from section Section 4.2 that $\mathcal{K}_\omega$ denotes the class of countable $\mathcal{L}$-structures whose age lies inside $\mathcal{K}$ and let $\mathbf{K}$ denote the Fraïssé limit of the class $\mathcal{K}$.

Definition 8.19. We say that a set $A \in \mathcal{K}$ is *independent*, if no two distinct elements inside A satisfy any relation.

Definition 8.20. Let $A_1, ..., A_k, B \in \mathcal{K}$ such that B embeds into each of the A_i. Then the *k-fold free amalgamation of* $(A_i)_{i\leqslant k}$ *over* B is obtained the following way: Let A^1 be the free amalgamation of A_1 and A_2 over B. Having constructed A^n, let A^{n+1} be the free amalgamation of A^n and A_{n+2} over B. Then the k-fold free amalgamation of $(A_i)_{i\leqslant k}$ over B is the structure A^{k-1}.

Lemma 8.21. *Let $\mathcal{K}$ be a free amalgamation class in a finite relational language $\mathcal{L}$. Assume that for each $R_i \in \mathcal{L}$ and each $x_1, x_2, ..., x_{l_i} \in \mathbf{K}$, if $R_i(x_1, x_2, ..., x_{l_i})$, then $x_1 = x_2 = ... = x_{l_i}$. Then if $T \in K_\omega$, T does not embed as a rigid moiety into $\mathbf{K}$.*

Proof. Let $f(T)$ be an embedding of T into $\mathbf{K}$ such that $\mathbf{K} - f(T)$ is countably infinite. Since there are only finitely many relations in the language, there are finitely many 1-types, say $\{t_1\}, \{t_2\}, ..., \{t_k\}$. Thus there is an $i \in \{1, 2, .., k\}$ and an infinite subset $K' \subseteq K - f(T)$ such that for each $x \in K'$, $\{x\} \cong \{t_i\}$. Hence if $\phi \in Aut(T)$, then ϕ has uncountably many extensions because any permutation of K' will give rise to a different extension. So every automorphism of T has infinitely many extensions to automorphisms of $\mathbf{K}$. Notice that even if K' was not infinite, it would still give rise to as many extensions as $|Perm(K')|$. □

Definition 8.22. Let $\mathcal{K}$ be a free amalgamation class satisfying the conditions in Lemma 8.21. Then $\mathcal{K}$ is called *totally disconnected*.

Lemma 8.21 shows that whenever we have a totally disconnected free amalgamation class, we don't have any rigid moieties. So from now on we are going to focus on free amalgamation classes which are *not totally disconnected*. The next lemma states another formulation for this property.

This formulation of being *not totally disconnected* will be used in the results that follow. We will define a degree for the class, denoted $d(\mathcal{K})$, and whenever this degree is $\geqslant 2$, the class will be *not totally disconnected*.

Lemma 8.23. *Let $\mathcal{K}$ be a free amalgamation class in a finite relational language $\mathcal{L}$. Then $\mathcal{K}$ is not totally disconnected if and only if* $d(\mathcal{K}) = min_{\substack{A\in\mathcal{K} \\ A \text{ is not independent} \\ R_i\in\mathcal{L}}} \{|A| : R_i(a_{i_1}, a_{i_2}, ..., a_{i_s}), a_{i_j} \in A\} \geqslant 2.$

Proposition 8.24. *Let $\mathcal{K}$ be a not totally disconnected free amalgamation class in a finite relational language $\mathcal{L}$. Then there exists an $R_i \in \mathcal{L}$ and an infinite structure $N \in K_\omega$ such that*

(i) *N has a trivial automorphism group*

(ii) *For any finite subset $F \subseteq N$ and any $l \geqslant d(\mathcal{K}) - 1$, there is a finite subset $E \subseteq N - F$ with a one-point extension $\{v\} \sqcup E \in \mathcal{K}$ where v is R_i-related to exactly l many distinct elements in E and no pairs of elements of E and F satisfy any relations.*

Proof. Since $\mathcal{K}$ is not totally disconnected, there exists a set $A = \{a_1, ..., a_{d(\mathcal{K})}\} \in \mathcal{K}$ of cardinality $d(\mathcal{K})$ and a relation $R_i \in \mathcal{L}$ such that $R_i(a_{i_1}, a_{i_2}, ...a_{i_n})$, $a_{i_j} \in A$ where all the elements of A appear inside this relation. Notice that $n \geqslant d(\mathcal{K}) \geqslant 2$ because we may have the same elements of A appearing multiple times inside the relation. Let $B_i = A - \{a_i\}$. Notice that B_i is an independent set for each i since no two elements in a structure of $d(\mathcal{K}) - 1$ elements satisfy any relations in a not totally disconnected class. Now let A^1 be the free amalgamation of two copies of A over B_1. Then $A^1 \cong A \sqcup \{{a_1}^1\}$ where ${a_1}^1$ and a_1 do not satisfy any relations. Now let A^2 be the free amalgamation of A^1 and A over $B_2 \cong \{{a_1}^1, a_3, ...a_{d(\mathcal{K})}\}$ where B_2 embeds into A by ${a_1}^1 \to a_1$, and into A^1 as inclusion. So we have $A^2 \cong A^1 \sqcup \{{a_2}^2\}$. So assume that A^l is constructed by taking the free amalgamation of A^{l-1} and A over B_i. To construct A^{l+1} we take the free amalgamation of A^l and A over B_{i+1} where i runs modulo $d(\mathcal{K})$ over $\{1, 2, ...d(\mathcal{K})\}$. Let $N = \bigcup A^l$.

Now I claim that N has a trivial automorphism group. First of all, for $f \in Aut(N)$, $f(a_1) = a_1$ because a_1 is the only element which is R_i-related to exactly $d(\mathcal{K}) - 1$ many elements, namely $a_2, a_3, ..., a_{d(\mathcal{K})}$. Similarly $f(a_2) = a_2$ because a_2 is the only element which is R_i-related to exactly $d(\mathcal{K})$ many elements, namely $a_1, a_3, ..., a_{d(\mathcal{K})}, {a_1}^1$. Similarly we have that a_i is the only element which is R_i-related to exactly $d(\mathcal{K}) + i - 2$ many elements for $i \in \{1, ..., d(\mathcal{K}) - 1\}$, hence $f(a_i) = a_i$, for $i \in \{1, ..., d(\mathcal{K}) - 1\}$. Starting with $a_{d(\mathcal{K})}$, every other element of N is R_i-related to exactly $2d(\mathcal{K}) - 2$ many elements and $2d(\mathcal{K}) - 2 > d(\mathcal{K}) + i - 2$ for $i \in \{1, ..., d(\mathcal{K}) - 1\}$. But since $a_1, a_2, .., a_{d(\mathcal{K})}$ satisfy R_i, and $a_1, a_2, ..., a_{d(\mathcal{K})-1}$ are fixed by f, $f(a_{d(\mathcal{K})}) = a_{d(\mathcal{K})}$ because for any $x \in N - \{a_{d(\mathcal{K})}\}$ we know that $x, a_1, a_2, ..., a_{d(\mathcal{K})-1}$ do not satisfy R_i. But for any $d(\mathcal{K})$ consecutive elements $\{a_{i_1}, a_{i_2}, ..., a_{i_{d(\mathcal{K})}}\}$ if the first $d(\mathcal{K}) - 1$ are fixed by f, then $f(a_{i_{d(\mathcal{K})}}) = a_{i_{d(\mathcal{K})}}$ because $a_{i_1}, a_{i_2}, ..., a_{i_{d(\mathcal{K})}}$ satisfy R_i and for any $x \in N - \{a_{i_{d(\mathcal{K})}}\}$, $a_{i_1}, a_{i_2}, ..., x$ do not satisfy R_i. Hence f fixes each $x \in N$ and $f = Id_N$.

Now we claim that given a finite subset $F \in N$, and any l, there exists $E \subseteq N - F$ such that no elements of E and F satisfy any relations and $|E| = l$. Let F and l be given. Now, during the construction of N, we have named the second set of a_i in N as ${a_i}^1$. For simplicity let us name the $(n+1)^{th}$ set of a_i as a_i^n. So we have

$$N = \{a_1, a_2, ...a_{d(\mathcal{K})}, {a_1}^1, {a_2}^1, .., {a_{d(\mathcal{K})}}^1, {a_1}^2, {a_2}^2, ..., {a_{d(\mathcal{K})}}^2, ..., {a_1}^n, {a_2}^n, ..., {a_{d(\mathcal{K})}}^n, ...\}$$

where each $d(\mathcal{K})$ consecutive elements satisfy R_i. Since F is a finite subset of N, we can always find an independent set $E \subseteq N - F$ where

$$E = \{a_2{}^{j_1}, a_3{}^{j_2}, ..., a_{d(\mathcal{K})-1}{}^{j_{d(\mathcal{K})-2}}, a_{d(\mathcal{K})}{}^{j_{d(\mathcal{K})-3}}, a_{d(\mathcal{K})}{}^{j_{d(\mathcal{K})-4}}, ..., a_{d(\mathcal{K})}{}^{j_l}\}$$

and no elements of E and F satisfy any relations. Notice that we have not used the assumption that $l \geqslant d(\mathcal{K}) - 1$ so far. The construction we did can be done for any l, but we will need our set E to have a cardinality $\geqslant d(\mathcal{K}) - 1$ for the following Lemma, which will conclude the proof for the proposition.

Lemma 8.25. *If E constructed in Proposition 8.24 is of cardinality $\geqslant d(\mathcal{K}) - 1$, then E has a one-point extension $\{v\} \sqcup E \in \mathcal{K}$ such that v is R_i-related to every element in E.*

Proof. Let $E^0 = A$. Let E^1 be the free amalgamation of two copies of A over $B_{d(\mathcal{K})}$. So $E^1 \cong A \sqcup \{a_k{}^1\}$. Assume E^{n-1} is constructed. Then let E^n be the free amalgamation of E^{n-1} and A over B_k. So we have $E^n \cong A \sqcup \{a_k{}^1\} \sqcup \{a_{d(\mathcal{K})}{}^2\} ... \sqcup \{a_{d(\mathcal{K})}{}^n\}$. Now look at $E^{l+1-d(\mathcal{K})}$:

$|E^{l+1-d(\mathcal{K})}| = l + 1$, $E^{l+1-d(\mathcal{K})} = \{a_1, a_2, ..., a_{d(\mathcal{K})}, a_{d(\mathcal{K})}{}^1, a_{d(\mathcal{K})}{}^2, ..., a_{d(\mathcal{K})}{}^{l+1-d(\mathcal{K})}\}$ and we have that

$a_1, a_2, .., a_{d(\mathcal{K})-1}$ and $a_{d(\mathcal{K})}$ satisfy R_i

$a_1, a_2, .., a_{d(\mathcal{K})-1}$ and $a_{d(\mathcal{K})}{}^1$ satisfy R_i

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$a_1, a_2, .., a_{d(\mathcal{K})-1}$ and $a_{d(\mathcal{K})}{}^{l+1-d(\mathcal{K})}$ satisfy R_i

and there are no other relations satisfied within $E^{l+1-d(\mathcal{K})}$. Thus a_1 is R_i-related to exactly l many elements in $E^{l+1-d(\mathcal{K})}$. Also, the set $E^{l+1-d(\mathcal{K})} - \{a_1\}$ is independent and is isomorphic to E. Hence, E has a one-point extension $E^{l+1-d(\mathcal{K})} \cong E \sqcup \{a_1\}$, where a_1 is R_i-related to every element in E, which is exactly l many. ☐

This concludes the proof for the Proposition 8.24 ☐

Notice that in the proof of Proposition 8.24, the element v in the one-point extension $\{v\} \sqcup E$ is of the same type as a_1. Yet, we can do the same construction for any $a \in A$. This gives us the following:

Corollary 8.26. *Let $A = \{a_1, ..., a_{d(\mathcal{K})}\} \in \mathcal{K}$ with $|A| = d(\mathcal{K})$. Then there exists a structure $N \in K_\omega$ such that*

- (i) *N has a trivial automorphism group*
- (ii) *For any finite subset $F \subseteq N$, any $l \geqslant d(\mathcal{K}) - 1$ and any $a \in A$, there exists a finite subset $E \subseteq N - F$ with a one-point extension $\{v\} \sqcup E \in \mathcal{K}$ where v is R_i-related to exactly l many distinct elements in E and no pairs of elements of E and F satisfy any relations. Moreover $\{v\} \cong \{a\}$.*

Corollary 8.27. *Assume that A given in Corollary 8.26 includes all 1-types in the class $\mathcal{K}$. Then for any independent set $B \in \mathcal{K}$ of cardinality $\geqslant d(\mathcal{K}) - 1$, B has a one-point extension $\{v\} \sqcup B \in \mathcal{K}$ such that v is R_i-related to at least one element in B. Furthermore v can be chosen to be isomorphic to any of the 1-types in the class.*

Proof. Since A includes all 1-types, N includes infinitely many copies of all 1-types. Thus, every independent set $B \in \mathcal{K}$ is isomorphic to an independent subset $B' \subseteq N$ and due to the construction of the one-point extension in the proof of Proposition 8.24, any finite independent subset B' of N of cardinality $\geqslant d(\mathcal{K}) - 1$ has a one-point extension $B' \sqcup \{v\}$ such that v is R_i-related to every element in B', in particular v is related to at least one element in B'. And v can be chosen to be isomorphic to any element of N, which includes all 1-types. $\square$

Lemma 8.28. *Let $A = \{a_0, a_1, ...a_k\} \in \mathcal{K}$ be a finite structure which is not independent and assume that for some $R_j \in \mathcal{L}$ we have $R_j(a_{i_0}, ..., a_{i_l})$, where each element of A appears at least once inside the relation. Then A has a one-point extension $A \sqcup \{v\} \in \mathcal{K}$, where v is R_j-related to at least one element in A. Moreover, v can be chosen to be isomorphic to any element in the set $\{a_0, a_1, ...a_k\}$.*

Proof. Without loss of generality assume that $a_{i_0} = a_0$. Look at the free amalgamation of two copies of A over $A - \{a_0\}$. This generates a structure isomorphic to $A \sqcup \{a_0{}'\}$ where we have $R_j(a_0', a_{i_1}, ..., a_{i_l})$. Since $l \geqslant k \geqslant 1$, we have that $\{a_{i_1}, ..., a_{i_l}\} \neq \emptyset$. Hence $a_0{}'$ is R_j-related to at least one element in A. Notice that the new element is isomorphic to a_0 but can be taken isomorphic to any element in A . $\square$

Corollary 8.29. *Let $B \in \mathcal{K}$ be a finite structure of cardinality $\geqslant d(\mathcal{K}) - 1$. Then B has a one-point extension $B \sqcup \{v\} \in \mathcal{K}$, where v is R_j-related to at least one element in B for some R_j.*

Proof. If B is not independent, then let $B' \subseteq B$ be a subset of cardinality $\geqslant 2$ where all elements of B' satisfy some relation R_j, so we are done by Lemma 8.28. If B is independent, then we are done by Corollary 8.27. $\square$

Now we turn to the proof of our main theorem where we show that rigid moieties exist if all the one-point sets in our class $\mathcal{K}$ are isomorphic. So far we have not used that assumption to prove any of the propositions above. This is because what we have done so far is true even if there is more than one isomorphism type among one-point sets. To prove some of the results, we have assumed a weaker condition stating that there exists a finite structure $A \in \mathcal{K}$, which includes all 1-types. Obviously if there is only one 1-type in the class, then every structure satisfies this condition. But unfortunately, to prove the next theorem, we have to assume this stronger condition that there is only one 1-type in $\mathcal{K}$.

Theorem 8.30. *Let $\mathcal{K}$ be a not totally disconnected free amalgamation class in a finite relational language $\mathcal{L}$ and assume that all the one-point sets in $\mathcal{K}$ are isomorphic. Let T be an infinite structure in $\mathcal{K}_\omega$. Then T embeds as a rigid moiety into $\mathbf{K}$. Moreover, there are 2^ω many such embeddings which are not conjugate in $Aut(\mathbf{K})$.*

Proof. Let N and R_i be given by Proposition 8.24. Without loss of generality assume that $i = 1$. Let $d(\mathcal{K}) < n_1 < n_2 < ... < n_k < ...$ be a strictly increasing sequence of positive integers. Let $T_0 = T \sqcup N$ where no pairs of elements of N and T satisfy any relations. As in the previous proofs, we are going to build the Fraïssé limit as a tower "over T_0" of the form $T_0 \sqcup T_1 \sqcup T_2 \sqcup \cdots$.

Initial step: Let $\{A_i\}_{i \in \mathbb{N}}$ be an enumeration of all finite subsets of T_0. Now, for any one-point extension $\{v\} \sqcup A_0 \in \mathcal{K}$ such that

(i) v is R_1-related to exactly n_1 many distinct elements in A_0

(ii) v is R_j-related to at least one element in $A_0 \cap T$ for some R_j

let $v'(A_0) \in T_1$ such that the q.f. type of v' over A_0 is the same as the q.f. type of v over A_0 and that v' does not satisfy any relations with elements of $T_0 - A_0$.

Assume we have added new elements to T_1 for every one-point extension of A_{k-1}. Now for any one-point extension $A_k \sqcup \{v\} \in \mathcal{K}$ of A_k, such that

(i) v is R_1-related to exactly n_1 many distinct elements in A_k

(ii) v is R_j-related to at least one element in $A_k \cap T$ for some R_j

(iii) the q.f. type of v over T_0 is not the same as as the q.f. type of any previously constructed $v' \in T_1$ over T_0

let $v'(A_k) \in T_1$ such that the q.f. type of v' over A_k is the same as the q.f. type of v over A_k and that v' does not satisfy any relations with elements of $T_0 - A_k$.

Inductive step: Assume that T_{k-1} is constructed. Let $\{A_i\}_{i \in \mathbb{N}}$ be an enumeration of all finite subsets of $T_0 \cup T_1 \cup ... \cup T_{k-1}$. Now, for any one-point extension $\{v\} \sqcup A_0 \in \mathcal{K}$ of A_0 such that

(i) v is R_1-related to exactly n_k many distinct elements in $A_0 \cap T_0$

(ii) v is R_j-related to at least one element in $A_0 \cap T$ for some R_j

let $v'(A_0) \in T_k$ such that the q.f. type of v' over A_0 is the same as the q.f. type of v over A_0 and that v' does not satisfy any relations with elements of $(T_0 \cup T_1 \cup ... \cup T_{k-1}) - A_0$.

Assume we are done with A_{k-1}. Now for any one-point extension $A_k \sqcup \{v\} \in \mathcal{K}$ of A_k, such that

(i) v is R_1-related to exactly n_k many distinct elements in $A_k \cap T_0$

(ii) v is R_j-related to at least one element in $A_k \cap T$ for some R_j

(iii) the q.f. type of v over $T_0 \cup T_1 \cup ... \cup T_{k-1}$ is not the same as as the q.f. type of any previously constructed $v' \in T_k$ over $T_0 \cup T_1 \cup ... \cup T_{k-1}$

let $v'(A_k) \in T_k$ such that the q.f. type of v' over A_k is the same as the q.f. type of v over A_k and that v' does not satisfy any relations with elements of $(T_0 \cup T_1 \cup ... \cup T_{k-1}) - A_k$.

We now let $T_\omega = \bigcup_k (T_k)$ and claim that $T_\omega \cong \mathbf{K}$. There are two things to check:

(I) $Age(T_\omega) \subseteq \mathcal{K}$

(II) Given any finite $A \subseteq T_\omega$ and any $\{v\} \sqcup A \in \mathcal{K}$, there exists $v' \in T_\omega$ such that the q.f. type of v' over A is the same as the q.f. type of v over A.

We first show (I). Let $A \subseteq T_\omega$ be a finite subset. Assume $A \subseteq T_0 \cup T_1$. Then look at $A \cap T_1$. If it's empty, then we are done since $Age(T_0) \subseteq \mathcal{K}$. If it's not empty, then let $\{v_1, ..., v_k\} = A \cap T_1$. So we have $A = A \cap T_0 \cup \{v_1, ..., v_k\}$. But this structure is isomorphic to the k-fold free amalgamation of $(A \cap T_0) \cup \{v_1\}, ..., (A \cap T_0) \cup \{v_k\}$ over $A \cap T_0$ because there are no relations between any of the v_i's. And since each of the $A \cup \{v_i\} \in \mathcal{K}$, the free amalgamation is in $\mathcal{K}$. Hence $A \in \mathcal{K}$. Assume we have shown this for $T_0 \cup ... \cup T_{k-1}$. Let $A \subseteq T_0 \cup ... \cup T_k$. Then $A = (A \cap (T_0 \cup ... \cup T_{k-1})) \sqcup (A \cap T_k)$. Again, if $A \cap T_k = \emptyset$, we are done. So let $A \cap T_k = \{v_1, ..., v_k\}$. Then again, A is isomorphic to the k-fold free amalgamation of $(A \cap (T_0 \cup ... \cup T_{k-1})) \cup \{v_1\}, ..., (A \cap (T_0 \cup ... \cup T_{k-1})) \cup \{v_k\}$ over $(A \cap (T_0 \cup ... \cup T_{k-1}))$ and since we assumed by induction that any finite substructure of $T_0 \cup ... \cup T_{k-1}$ is in $\mathcal{K}$, their k-fold free amalgamation is in $\mathcal{K}$ as well. So we are done.

We now show (II). Let $A \subseteq T_\omega$ be a finite subset. Let M be big enough such that

$$A \subseteq T_0 \cup ... \cup T_{M-1}$$

and $n_M \geqslant max\{2|A|, 2d(\mathcal{K})\}$. Let $\{v\} \sqcup A \in \mathcal{K}$ be a one-point extension of A. We want to find a $v' \in T_\omega$ such that the q.f. type of v' over A is the same as the q.f. type of v over A. Throughout the proof, whenever we say that v is related to some element which sits in T_ω, we always mean the elements in A which are related to v in the abstract one-point extension $A \sqcup \{v\}$ as they are viewed inside T_ω. Remember that these abstract extensions sit in the class $\mathcal{K}$, not in T_ω.

Case 1: $A \cap T \neq \emptyset$ and v is R_j-related to at least one element in T for some R_j.

Let $L = |\{x \in A \cap T_0 : x \text{ is } R_1\text{-related to } v\}|$. Notice that we have $L \leqslant |A| \leqslant n_M$. Let $F = (A \cap N) \cup \{x \in N : x \text{ satisfies some relation with } a \text{ for some } a \in A - N\}$. Now, F is a finite subset of N. Notice that $n_M - L \geqslant n_M - |A| \geqslant \frac{n_M}{2} > d(\mathcal{K})$. Then by Proposition 8.24 there exists an $A' \subseteq N - F$ with a one-point extension $\{v'\} \sqcup A' \in \mathcal{K}$ where v' is R_1-related to exactly $n_M - L$ many distinct elements in A' and no pairs of elements of A' and F satisfy any relations. Hence, no pairs of elements of A' and A satisfy any relations. Now, look at the free amalgamation of $\{v\} \sqcup A$ and $\{v'\} \sqcup A'$ over $v \cong v'$. This structure is isomorphic to $A \sqcup A' \sqcup \{v'\} \in \mathcal{K}$ where v' is R_1-related to L many elements in $A \cap T_0$ and is R_1-related to $n_M - L$ many elements in $A' \subseteq N \subseteq T_0$. So we have that v' is R_1-related to $n_M - L + L = n_M$ many elements in $(A \sqcup A') \cap T_0$. Moreover v' is R_j-related to at least one element in $A \cap T$. And since no pairs of elements of A and A' satisfy any relations, $A \sqcup A' \in \mathcal{K}$ is isomorphic to $A \sqcup A' \subseteq \mathbf{T}_\omega$. Hence the structure $A \sqcup A' \sqcup \{v'\} \in \mathcal{K}$ is a one-point extension of $A \sqcup A' \subseteq T_\omega$. And since v' satisfies the necessary properties, there exists $v''(A \sqcup A') \in T_M$ such that the q.f. type of v'' over $A \sqcup A'$ is the same as the q.f. type of v' over $A \sqcup A'$. But the q.f. type of v' over $A \subseteq A \sqcup A'$ is the same as the q.f. type of v over A. Thus the q.f. type of v'' over $A \subseteq A \sqcup A'$ is the same as the q.f. type of v over A, so we are done.

Case 2: $A \cap T = \emptyset$

Let $L = |\{x \in A \cap T_0 : x \text{ is } R_1\text{- related to } v\}|$ as in the previous case and let $E = \{x \in T : x \text{ is related to some element } a \in A\}$. Notice that E is a finite set. Let $A'' \subseteq T - E$ be a finite set of cardinality $\geqslant d(\mathcal{K}) - 1$. Then by Corollary 8.29 there exists a one-point extension $A'' \sqcup \{v''\} \in \mathcal{K}$ where v'' is R_j-related to at least one element in A'' for some R_j. Let K be the number of elements in A'' which are R_1-related to v''. Now look at the free amalgamation of A and A'' over $v' \cong v''$, which lies in $\mathcal{K}$. This structure is isomorphic to $A \sqcup A'' \sqcup \{v''\}$ where v'' is R_1-related to $L + K$ many distinct elements in $(A \sqcup A'') \cap T_0$. Now, let $K' \geqslant K$ be big enough such that $n_{M+K'} \geqslant max\{2|A \sqcup A''|, 2d(\mathcal{K})\}$ and $A \sqcup A'' \subseteq T_0 \cup ... \cup T_{M+K'-1}$. Now let $F = (A \cap N) \cup \{x \in N : x \text{ satisfies some relation with } a \text{ for some } a \in A - N\}$ as in Case 1 and let $L' = n_{M+K'} - (L + K)$. Notice that since $L \leqslant |A|, K \leqslant |A''|$ and $n_{M+K'} \geqslant 2|A \sqcup A''|$, we have that $L' = n_{M+K'} - (L + K) \geqslant n_{M+K'} - |A \sqcup A''| \geqslant \frac{n_{M+K'}}{2} > d(\mathcal{K}) - 1$. Since F is finite, by Proposition 8.24 there exists an $A' \subseteq N - F$ with a 1-point extension $A' \sqcup \{v'\} \in \mathcal{K}$ where v' is R_1-related to exactly L' many distinct elements in A' and no pairs of elements of A' and F satisfy any relations. Morever, no pairs of elements of A' and $A \sqcup A''$ satisfy any relations. So look at the free amalgamation of $A \sqcup A'' \sqcup \{v''\}$ and $A' \sqcup \{v'\}$ over $v' \cong v''$, which lies in $\mathcal{K}$. This structure is isomorphic to $A \sqcup A' \sqcup A'' \sqcup \{v'\}$ where v' is R_1-related to $n_{M+K'} = (L+K) + (n_{M+K'} - (L+K))$ many distinct elements in $(A \sqcup A' \sqcup A'') \cap T_0$. And also v' is R_j-related to at least one element in $(A \sqcup A' \sqcup A'') \cap T$. And since no pairs of elements of A' and $A \sqcup A''$ satisfy any relations, $A \sqcup A' \sqcup A'' \subseteq T_\omega$ is isomorphic to $A \sqcup A' \sqcup A'' \in \mathcal{K}$. Thus there exists $v'''(A \sqcup A' \sqcup A'') \in T_{M+K'}$ such that the q.f. type of v''' over $A \sqcup A' \sqcup A''$ is the same as the q.f. type of v' over $A \sqcup A' \sqcup A''$. But then the q.f. type of v''' over $A \subseteq (A \sqcup A' \sqcup A'')$ is the same as the q.f. type of v' over $A \subseteq (A \sqcup A' \sqcup A'')$ and that is the same as the q.f. type of v over A. So we are done.

Case 3: $A \cap T \neq \emptyset$ and v does not satisfy any relations with elements of T.

Let $L = |\{x \in A \cap T_0 : x \text{ is } R_1\text{-related to } v\}|$. Let $B = A - T$. Notice that the positive relational formulas v satisfies with elements of A are the same as the ones v satisfies with elements of B. Also, $L = |\{x \in B \cap T_0 : x\ is\ R_1\text{-related to } v\}|$. We are going to find a $v' \in T_\omega$ such that the q.f. type of v' over B is the same as the q.f. type of v over B and that v' does not satisfy any relations with elements of $A - B$. So the q.f. type of v' over A will be the same as the q.f. type of v over A.

Let $E = \{x \in T : x \text{ is related to some element } b \in B\}$. Then E is a finite set. Let $B'' \subseteq T - (E \cup A)$ be a finite set of cardinality $\geqslant d(\mathcal{K}) - 1$. By Corollary 8.29 there exists a one-point extension $B'' \sqcup \{v''\} \in \mathcal{K}$ where v'' is R_j-related to at least one element in B'' for some R_j. Now, no pairs of elements of B and B'' satisfy any relations so look at the free amalgamation of B and B'' over $v \cong v''$, which lies in $\mathcal{K}$. This structure is isomorphic to $B \sqcup B'' \sqcup \{v''\}$ where v'' is R_1-related to $L+K$ many distinct elements in $(B \sqcup B'') \cap T_0$ for some $K \geqslant 0$. Let $K' \geqslant K$ be big enough such that $n_{M+K'} \geqslant max\{2|B \sqcup B''|, 2d(\mathcal{K})\}$ and $B \sqcup B'' \subseteq T_0 \cup ... \cup T_{M+K'-1}$. Similar to previous cases, let

$$F = (B \cap N) \cup \{x \in N : x \text{ satisfies some relation with } b \text{ for some } b \in B - N\}$$

and let $L' = n_{M+K'} - (L+K)$. Since $L \leqslant |B|$, $K \leqslant |B''|$ and $n_{M+K'} \geqslant 2|B \sqcup B''|$, we have $L' = n_{M+K'} - (L+K) \geqslant \frac{n_{M+K'}}{2} > d(\mathcal{K}) - 1$. And since F is a finite subset of N, by Proposition 8.24 there exists a $B' \subseteq N - F$ with a one-point extension $B' \sqcup \{v'\} \in \mathcal{K}$ where v' is R_1-related to exactly L' many distinct elements in B' and no pairs of elements of B' and F satisfy any relations. Moreover, no pairs of elements of B' and $B \sqcup B''$ satisfy any relations. So look at the free amalgamation of $B \sqcup B'' \sqcup \{v''\}$ and $B' \sqcup \{v'\}$ over $v' \cong v''$, which lies in $\mathcal{K}$. This structure is isomorphic to $B \sqcup B' \sqcup B'' \sqcup \{v'\}$ where v' is R_1-related to $n_{M+K'} = (L+K) + (n_{M+K'} - (L+K))$ many distinct elements in $(B \sqcup B' \sqcup B'') \cap T_0$. And also v' is R_j-related to at least one element in $(B \sqcup B' \sqcup B'') \cap T$. But since $(B \sqcup B' \sqcup B'') \subseteq T_\omega$ is isomorphic to $(B \sqcup B' \sqcup B'') \in \mathcal{K}$, there exists a $v'''(B \sqcup B' \sqcup B'') \in T_{M+K'}$ such that the q.f. type of v''' over $B \sqcup B' \sqcup B''$ is the same as the q.f. type of v' over $B \sqcup B' \sqcup B''$. Hence the q.f. type of v''' over B is the same as the q.f. type of v over B. So the only thing to check is that whether v''' satisfies any relations with elements in $A - B \subseteq T$. But the only elements in T which are related to v''' are the ones in B'', which does not intersect $A - B$. Hence the q.f. type of v''' over A is the same as the q.f. type of v over A and this finishes the proof of (II).

Now we will prove that given $\phi \in Aut(T)$, ϕ extends uniquely to an automorphism of T_ω. First we show the existence of extensions. For $x \in N$, let $\phi(x) = x$. We also have that $\phi(T_0) = T_0$. As can be seen clearly, we have extended ϕ to an automorphism of T_0. Now, assume that ϕ has been extended to an automorphism from $(T_0 \cup ... \cup T_{k-1})$ to itself. Let $v \in T_k$. Then look at the set $A = \{x \in T_0 \cup .... \cup T_{k-1} : x \text{ is related to } v\}$. Then A is a finite set and $v = v(A)$. Since $\phi(A) \subseteq T_0 \cup .... \cup T_{k-1}$ by the inductive assumption, there exists a unique $v' \in T_k$ such that v' has the same q.f. type over $\phi(A)$ as the q.f. type of v over A. Let $\phi(v) = v'$. We extend ϕ this way to a map from T_ω to itself.

Now we show that extending ϕ this way, we get a bijection. To see surjectivity, notice that ϕ is surjective (and injective) on T_0. Assume that ϕ is surjective on $T_0 \cup .... \cup T_{k-1}$. Now, given $v \in T_k$, let $v = v(A)$ for $A \subseteq T_0 \cup .... \cup T_{k-1}$. So let $A' = \phi^{-1}(A) \subseteq T_0 \cup .... \cup T_{k-1}$. There exists a unique $v' \in T_k$ such that v' has the same q.f. type over A' as v over A. Thus $\phi(v') = v$.

For injectivity, assume that for some $v \neq y \in T_k$ we have $\phi(v) = \phi(y)$. But the q.f. type of v over $T_0 \cup ... \cup T_{k-1}$ cannot be the same as y because each q.f. type over $T_0 \cup ... \cup T_{k-1}$ appears only once in T_k. But the q.f. type of their images over $T_0 \cup ... \cup T_{k-1}$ have to be the same (since their images are the same), so we get a contradiction.

Now we are going to show that ϕ is indeed an automorphism of T_ω. We need to show that $R_i(v_1, v_2, ..., v_{l_i}) \iff R_i(\phi(v_1), \phi(v_2), ..., \phi(v_{l_i}))$ for any $R_i \in \mathcal{L}$ and any $v_1, v_2, ..., v_{l_i} \in T_\omega$. We are going to proceed by induction. We already know that $\phi\restriction_{(T_0)} \in Aut(T_0)$, so we are done if $v_1, v_2, ..., v_{l_i} \in T_0$. Now assume that we have shown $\phi\restriction_{(T_0 \cup T_1 \cup ... \cup T_k)} \in Aut(T_0 \cup T_1 \cup ... \cup T_k)$ and let $v_1, v_2, ..., v_{l_i} \in T_0 \cup T_1 \cup ... \cup T_{k+1}$ with $R_i(v_1, v_2, ..., v_{l_i})$. Without loss of generality, assume that at least one $v_j \in T_{k+1}$ and notice that at most one v_j can be in T_{k+1} since elements of T_{k+1} don't satisfy any relations inside $T_0 \cup T_1 \cup ... \cup T_{k+1}$ for $k \geqslant 0$. So by reordering the elements we can assume that $v_0 \in T_{k+1}$ and $v_1, ..., v_{l_i} \in T_0 \cup ... \cup T_k$. So we have that $\{v_0, ...v_{l_i}\}$ is a one-point extension of $\{v_1, ..., v_{l_i}\}$. By the construction of ϕ we know that $\phi(v_0)$ is the unique element in T_{k+1} which has the same q.f. type over $\{\phi(v_1), ..., \phi(v_{l_i})\}$ as v_0 has over $\{v_1, ..., v_{l_i}\}$. But since that complete q.f type includes the formula $R_i(v_0, ..., v_{l_i})$, we have that $R_i(\phi(v_0), \phi(v_1), ..., \phi(v_{l_i}))$. But this proves the converse as well: If $R_i(\phi(v_0), \phi(v_1), ..., \phi(v_{l_i}))$, since the q.f. type of $\phi(v_0)$ over $\{\phi(v_1), ..., \phi(v_{l_i})\}$ is the same as the q.f. type of v_0 over $\{v_1, ..., v_{l_i}\}$, we have $R(v_0, v_1, ..., v_{l_i})$.

Finally, to see uniqueness of extensions, let $\phi \in Aut(T_\omega)$ such that $\phi(T) = T$. We will show that ϕ is uniquely determined for T_ω. First of all, since N consists of all the elements in T_ω which do not satisfy any relations with elements of T, $\phi(N) = N$, but N has a trivial automorphism group, so ϕ has to be the identity on N. Thus ϕ is uniquely determined for T_0. Notice that we have for each $i \in \mathbb{N}$, $\phi(T_i) = T_i$ since the elements of T_i are R_1-related to n_i many distinct elements in T_0 and $n_i \neq n_j$ for $i \neq j$.

Assume that ϕ is uniquely determined for $T_0 \cup ... \cup T_{k-1}$. Let $x \in T_k$. Then $x = x(A)$ for some $A \subseteq T_0 \cup ... \cup T_{k-1}$. But since $x' = \phi(x) \in T_k$, there's only one $x' \in T_k$ such that the q.f. type of x over A is the same as the q.f. type of x' over $\phi(A)$. So ϕ is uniquely determined for T_k, and we are done.

The last thing to prove is that there exist 2^ω such embeddings which are not conjugate. Recall that the sequence $d(\mathcal{K}) < n_1 < n_2 < ... < n_k < ...$ is an arbitrary one, so there exist 2^ω such sequences, and for each one, we get a different embedding. And two such embeddings are never conjugate. $\square$

Definition 8.31. If $\mathcal{X}$ is an amalgamation class in a finite relational language $\mathcal{L}$, we can define a new amalgamation class $\overline{\mathcal{X}}$ in $\mathcal{L}$ the following way:

Given a finite structure $A \in \mathcal{X}$, let $\overline{A} \in \overline{\mathcal{X}}$ be such that

(i) $|A| = |\overline{A}|$

(ii) for any relation $R_i \in \mathcal{L}$ with arity n_i, any $\{a_1, a_2, ..., a_{n_i}\} \in A$ and any $\{\overline{a_1}, \overline{a_2}, ..., \overline{a_{n_i}}\} \in \overline{A}$, we have

$$R_i(\overline{a_1}, \overline{a_2}, ..., \overline{a_{n_i}}) \iff \neg R_i(a_1, a_2, ..., a_{n_i})$$

We call $\overline{\mathcal{X}}$ *the complement* of $\mathcal{X}$.

One can easily check the following.

Proposition 8.32. *If $\mathcal{X}$ is an amalgamation class in a finite relational language and $X \in X_\omega$ embeds as a rigid moiety into the Fraïssé limit $\mathbf{X}$ of $\mathcal{X}$, then $\overline{X}$ embeds as a rigid moiety into the Fraïssé limit $\overline{\mathbf{X}}$ of $\overline{\mathcal{X}}$.*

So by Proposition 8.32, Fraïssé limits of free amalgamation classes as in Theorem 8.30 have a complement that also have many rigid moieties. Notice that most of the time such complements do not have the free amalgamation property.

8.6 Counter examples in amalgamation classes

Now we do know that rigid moieties exist for the limits of almost all free amalgamation classes. Yet if we don't have free amalgamation, it's easy to find nontrivial examples where they don't exist.

Proposition 8.33. *$\langle \mathbb{Q}, \leqslant \rangle$ does not have any rigid moieties.*

Proof. Let $X \subseteq \mathbb{Q}$ be a moiety, which is not dense. Then there exists a rational interval (p, q) which intersects X trivially. Let ϕ be the identity on X. Then we can clearly extend ϕ to an automorphism of $\mathbb{Q}$ such that ϕ is not the identity on (p, q). So X is not rigid.

Now assume X is dense. Let $q \in \mathbb{Q} - X$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $\{x_n\}$ is strictly increasing, $\{y_n\}$ is strictly decreasing and they both converge to q. Now let ϵ be an irrational number bigger than y_0. Take two sequences $\{x'_n\}$ and $\{y'_n\}$ in X as before, which converge to ϵ. Since $y_0 < \epsilon$ one can take these sequences bigger than y_0. Now we are going to construct an automorphism ϕ of X which carries $\{x_n\}$ onto $\{x'_n\}$ and $\{y_n\}$ onto $\{y'_n\}$. Let $\phi(x_i) = x'_i$ and $\phi(y_j) = y'_j$. Now we proceed with back and forth. Let $\{X_i\}$ be an enumeration of $X - (\{x_n\} \bigcup \{y_n\} \bigcup \{x'_n\} \bigcup \{y'_n\})$. X can be viewed as a countable disjoint union of $(-\infty, x_0) \sqcup [x_0, x_1) \sqcup ... \sqcup [x_n, x_{n+1}) \sqcup ... \sqcup [y_{m+1}, y_m) \sqcup ... \sqcup [y_1, y_0) \sqcup [y_0, x'_0) \sqcup ... \sqcup [x'_i, x'_{i+1}) \sqcup ... \sqcup [y'_{j+1}, y'_j) \sqcup ... \sqcup [y'_1, y'_0) \sqcup [y'_0, +\infty)$.

Look at X_0. It lies in one of these intervals, call it I_0, and the images of the endpoints of I_0 is determined by ϕ. So pick the smallest m such that X_m is in the interval $\phi(I_0)$, and set $\phi(X_0) = X_m$. Now we partition X into disjoint union of intervals one more time by adding X_0 into the list and do the same. Let X_{i_0} be the element with the smallest index such that it does not have a preimage under ϕ. Let $\phi^{-1}(X_{i_0})$ be the element X_{j_0} with the smallest index such that it does not have an image under ϕ. Since X is dence, we can always find elements in every interval, so this process continues and ϕ extends to an automorphism of X.

And now, ϕ cannot extend to an automorphism $\phi'$of $\mathbb{Q}$ because if it did, then $\phi'(q)$ would have to be ϵ, which is not in $\mathbb{Q}$. $\square$

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