1

On the geometry of Urysohn's universal metric space

Julien Melleray

Abstract

In recent years, much interest was devoted to the Urysohn space $\mathbb{U}$ and its isometry group; this paper is a contribution to this field of research. We mostly concern ourselves with the properties of isometries of $\mathbb{U}$, showing for instance that any Polish metric space is isometric to the set of fixed points of some isometry φ. We conclude the paper by studying a question of Urysohn, proving that compact homogeneity is the strongest homogeneity property possible in $\mathbb{U}$.

1 Introduction

In a paper published posthumously ([12]), P.S Urysohn constructed a complete separable metric space $\mathbb{U}$ that is *universal*, i.e contains an isometric copy of every complete separable metric space. This seems to have been forgotten for a while, perhaps because around the same time Banach and Mazur proved that $\mathcal{C}([0,1])$ is also universal.
Yet, the interest of the Urysohn space $\mathbb{U}$ does not lie in its universality alone: as Urysohn himself had remarked, $\mathbb{U}$ is also *ω-homogeneous*, i.e for any two finite subsets A, B of $\mathbb{U}$ which are isometric (as abstract metric spaces), there exists an isometry φ of $\mathbb{U}$ such that $\varphi(A) = B$. Moreover, Urysohn proved that $\mathbb{U}$ is, up to isometry, the only universal ω-homogeneous Polish metric space.
In the case of Polish metric spaces, it turns out that universality and ω-homogeneity can be merged in one property, called *finite injectivity*: a metric space (X,d) is finitely injective iff for any pair of finite metric spaces $K \subseteq L$

[*] 2000 Mathematical Subject Classification: 51F99 ; 03E15
Keywords: Urysohn universal space, homogeneity, Polish group, definable equivalence relation

and any isometric embedding $\varphi\colon K \to X$, there exists an isometric embedding $\tilde{\varphi}\colon L \to X$ such that $\tilde{\varphi}_{|_K} = \varphi$.
Then one can prove that a Polish metric space is universal and ω-homogeneous if, and only if, it is finitely injective; this is also due to Urysohn, who was the first to use finite injectivity (using another definition of it). [1]
This point of view highlights the parallel between $\mathbb{U}$ and other universal objects, such as the universal graph for instance; the interested reader can find a more detailed exposition of this and references in [2].
The interest in $\mathbb{U}$ was revived in 1986 when Katětov, while working on analogues of the Urysohn space for metric spaces of a given density character, gave in [7] a new construction of $\mathbb{U}$, which enables one to naturally "build" an isometric copy of $\mathbb{U}$ "around" any separable metric space X. In [13] Uspenskij remarked that this construction (which we detail a bit more in section 2) enables one to keep track of the isometries of X, and used that to obtain a continuous embedding of the group of isometries of X into $Iso(\mathbb{U})$, the group of isometries of $\mathbb{U}$ (both groups being endowed with the product topology, which turns $Iso(\mathbb{U})$ into a Polish group). Since any Polish group G continuously embeds in the isometry group of some Polish space X (actually, Gao and Kechris proved in [3] that any Polish group is topologically isomorphic to the isometry group of some Polish metric space), this shows that any Polish group is isomorphic to a (necessarily closed) subgroup of $Iso(\mathbb{U})$.
This result spurred interest for the study of $\mathbb{U}$; in [16], Vershik showed that generically (for a natural Polish topology on the sets of distances on $\mathbb{N}$) the completion of a countable metric space is finitely injective, and thus isometric to $\mathbb{U}$; in [15] Uspenskij completely characterized the topology of $\mathbb{U}$ by showing, using Torunczyk's criterion, that $\mathbb{U}$ is homeomorphic to $l^2(\mathbb{N})$.
During the same period, Gao and Kechris used $\mathbb{U}$ to study the complexity of the equivalence relation of isometry between certain classes of Polish metric spaces (viewed as elements of $\mathcal{F}(\mathbb{U})$). For instance, they proved that the relation of isometry between Polish metric spaces is Borel bi-reducible to the translation action of $Iso(\mathbb{U})$ on $\mathcal{F}(\mathbb{U})$, given by $\varphi.F = \varphi(F)$, and that this relation is universal among relations induced by a continuous action of a Polish group (see [3] for a detailed exposition of their results and references about the theory of Borel complexity of definable equivalence relations).
Despite all the recent interest in $\mathbb{U}$, not much work has yet been done on its geometric properties, with the exception of [2], where the authors build interesting examples of subgroups of $Iso(\mathbb{U})$.

[1] About finite injectivity, Urysohn stated in [12] "Voici la propriété fondamentale de [cet] espace dont, malgré son caractère auxiliaire, les autres propriétés de cet espace sont des conséquences plus ou moins immédiates".

As Urysohn himself had understood, finite injectivity has remarkable consequences on the geometry of $\mathbb{U}$, some of which we study in section 3; we begin with the easy fact that any isometric map which coincides with $id_{\mathbb{U}}$ on a set of non-empty interior must actually be $id_{\mathbb{U}}$. We then go on to study a bit the isometric copies of $\mathbb{U}$ contained in $\mathbb{U}$, e.g we show that $\mathbb{U}$ is isometric to $\mathbb{U} \setminus B$, where B is any open ball in $\mathbb{U}$.
We also use similar ideas to study the sets of fixed points of isometries, proving in particular that any Polish metric space is isometric to the set of fixed points of some isometry of $\mathbb{U}$.
The remainder of the article is devoted to the study of a question of Urysohn, who asked in [12] whether $\mathbb{U}$ had stronger homogeneity properties than ω-homogeneity [2]; we build on known results to solve that problem. Most importantly, we use the tools introduced by Katětov in [7]. Let us state precisely the problems we concern ourselves with:

Question 1. Characterize the Polish metric spaces (X, d) such that whenever $X_1, X_2 \subseteq \mathbb{U}$ are isometric to X, there is an isometry φ of $\mathbb{U}$ such that $\varphi(X_1) = X_2$.
As it turns out, we will not directly study that question, but another related one, which can be thought of as looking if one can extend finite injectivity:

Question 2. Characterize the Polish metric spaces (X, d) such that, whenever $X' \subseteq \mathbb{U}$ is isometric to X and $f \in E(X')$, there is $z \in \mathbb{U}$ such that $\forall x \in X'$, $d(x, z) = f(x)$.
($E(X)$ denotes the set of Katětov maps on X, see section 2).

It is rather simple, as we will see in section 4, to show that Property 1 implies Property 2, and it is a well-known fact (see [5] or [4]) that the answer to both questions is positive whenever X is compact:

Theorem 1.1. *(Huhunaišvili) If $K \subseteq \mathbb{U}$ is compact and $f \in E(K)$, then there is $z \in \mathbb{U}$ such that $d(z, x) = f(x)$ for all $x \in K$.*

Corollary 1.2. *If $K, L \subseteq \mathbb{U}$ are compact and $\varphi \colon K \to L$ is an isometry, then there is an isometry $\tilde{\varphi} \colon \mathbb{U} \to \mathbb{U}$ such that $\tilde{\varphi}_{|K} = \varphi$.*

The corollary is deduced from the theorem by the standard back-and-forth method (So, in that case, a positive answer to question 2 enables one to answer positively question 1; we will see that it is actually always the case).
Remarking that if X is such that $E(X)$ is not separable then X can have

[2] "On demandera, peut-être, si [cet] espace ne jouit pas d'une propriété d'homogénéité plus précise que celle que nous avons indiquée au n. 14".

neither property (1) nor property (2), we provide a characterization of the spaces X such that $E(X)$ is separable; these turn out to be exactly the spaces with the *collinearity property*, defined independently and simultaneously by N. Kalton in [6]. Afterwards, we show that, if X is not compact but has the collinearity property then X does not have property 2 either.
Therefore, our results enable us to deduce that a space has property 1 (or 2) if, and only if, it is compact, thus answering Urysohn's question: compact homogeneity is the strongest homogeneity property possible in $\mathbb{U}$.
Note. After submission of this article, I learnt that E. Ben Ami has independently proved the above result about the homogeneity properties of the Urysohn space, using a different method.
Acknowledgements. I would not have written this article if not for many conversations with Thierry Monteil; he introduced me to the Urysohn space, and the results in section 3 are answers to questions we asked during these conversations. I have learnt much from these talks, and for that I am extremely grateful.
I also would like to seize the opportunity to thank Vladimir Pestov, whose kindness and insights while at the CIRM in september 2004 were much appreciated; Alekos Kechris, for numerous remarks and suggestions; and Nigel Kalton for sending me a copy of [6], which helped me better understand the collinearity property. The anonymous referee also made interesting remarks and suggestions, and found several mistakes of mine; I am very grateful for his nice work.

2 Notations and definitions

If (X,d) is a complete separable metric space, we say that it is a *Polish metric space*, and often write it simply X.
If X is a topological space and there is a distance d on X which induces the topology of X and is such that (X,d) is a Polish metric space, we say that the topology of X is Polish.
If (X,d) is a metric space, $x \in X$ and $r > 0$, we use the notation $B(x,r[$ (resp. $B(x,r]$) to denote the open (resp. closed) ball of center x and radius r; $S(x,r)$ denotes the sphere of center x and radius r.
To avoid confusion, we say, if (X,d) and (X',d') are two metric spaces and f is a map from X into X', that f is an *isometric map* if $d(x,y) = d'(f(x),f(y))$ for all $x,y \in X$. If additionally f is onto, then we say that f is an *isometry*.
A *Polish group* is a topological group whose topology is Polish; if X is a separable metric space, then we denote its isometry group by $Iso(X)$, and endow it with the pointwise convergence topology, which turns it into a

second countable topological group, and into a Polish group if X is Polish (see [1] or [8] for a thorough introduction to the theory of Polish groups).
If (X,d) is a metric space, we say that $f: X \to \mathbb{R}$ is a *Katětov map* if

$$\forall x,y \in X \ |f(x)-f(y)| \leq d(x,y) \leq f(x)+f(y) \quad .$$

These maps correspond to one-point metric extensions of X. We denote by $E(X)$ the set of all Katětov maps on X; we endow it with the sup-metric, which turns it into a complete metric space.
That definition was introduced by Katětov in [7], and it turns out to be pertinent to the study of finitely injective spaces, since one can see by induction that a metric space X is finitely injective if, and only if,

$$\forall A \text{ finite } \subset X \ \forall f \in E(A) \ \exists z \in X \ \forall a \in A \ d(z,a) = f(a) \ .$$

This is the form under which Urysohn used finite injectivity in his original article.
If $Y \subseteq X$ and $f \in E(Y)$, define $k(f)\colon X \to \mathbb{R}$ (the Katětov extension of f) by

$$k(f)(x) = \inf\{f(y) + d(x,y) \colon y \in Y\}.$$

Then $k(f)$ is the greatest 1-Lipschitz map on X which is equal to f on Y; one checks easily (see for instance [7]) that $k(f) \in E(X)$ and that $f \mapsto k(f)$ is an isometric embedding of $E(Y)$ into $E(X)$.
To simplify future definitions, if $f \in E(X)$ and $S \subseteq X$ are such that $f(x) = \inf\{f(s) + d(x,s) \colon s \in S\}$ for all $x \in X$, we say that S is a *support of* f, or that S *controls* f.
Notice that if S controls $f \in E(X)$ and $S \subseteq T$, then T controls f.
Similarly, X isometrically embeds in $E(X)$ via the Kuratowski map $x \mapsto f_x$, where $f_x(y) = d(x,y)$. A crucial fact for our purposes is that

$$\forall f \in E(X) \ \forall x \in X \ d(f, f_x) = f(x).$$

Thus, if one identifies X to a subset of $E(X)$ via the Kuratowski map, $E(X)$ is a metric space containing X and such that all one-point metric extensions of X embed isometrically in $E(X)$.

We now go on to sketching Katětov's construction of $\mathbb{U}$; we refer the reader to [3], [4], [7] or [13] for a more detailed presentation and proofs of the results we will use below.
Most important for the construction is the following

Theorem 2.1. *(Urysohn) If X is a finitely injective metric space, then the completion of X is also finitely injective.*

Since $\mathbb{U}$ is, up to isometry, the unique finitely injective Polish metric space, this proves that the completion of any separable finitely injective metric space is isometric to $\mathbb{U}$.
The basic idea of Katětov's construction works like this: if one lets $X_0 = X$, $X_{i+1} = E(X_i)$ then, identifying each X_i to a subset of X_{i+1} via the Kuratowski map, let $Y = \cup X_i$.
The definition of Y makes it clear that Y is finitely injective, since any $\{x_1, \dots, x_n\} \subseteq Y$ must be contained in some X_m, so that for any $f \in E(\{x_1, \dots, x_n\})$ there exists $z \in X_{m+1}$ such that $d(z, x_i) = f(x_i)$ for all i.
Thus, if Y were separable, its completion would be isometric to $\mathbb{U}$, and one would have obtained an isometric embedding of X into $\mathbb{U}$.
The problem is that $E(X)$ is in general not separable (see section 4).
At each step, we have added too many functions; define then

$$E(X, \omega) = \{f \in E(X) \colon f \text{ is controlled by some finite } S \subseteq X\} .$$

Then $E(X, \omega)$ is separable if X is, and the Kuratowski map actually maps X into $E(X, \omega)$, since each f_x is controlled by $\{x\}$. Notice that, if $\{x_1, \dots, x_n\} \subseteq X$ and $f \in E(\{x_1, \dots, x_n\})$, then its Katětov extension $k(f)$ is in $E(X, \omega)$, and $d(k(f), f_{x_i}) = f(x_i)$ for all i.
Thus, if one defines this time $X_0 = X$, $X_{i+1} = E(X_i, \omega)$, and assume again that $X_i \subseteq X_{i+1}$ then $Y = \cup X_i$ is separable and finitely injective, hence its completion Z is isometric to $\mathbb{U}$, and $X \subseteq Z$.
For definiteness, we henceforth denote by $\mathbb{U}$ the space obtained by applying this construction to $X_0 = \{0\}$.
The most interesting property of this construction is that it enables one to keep track of the isometries of X: indeed, any $\varphi \in Iso(X)$ is the restriction of a unique isometry $\tilde{\varphi}$ of $E(X, \omega)$, and the mapping $\varphi \mapsto \tilde{\varphi}$ from $Iso(X)$ into $Iso(E(X, \omega))$ is a continuous group embedding (see [7]).
That way, we obtain for all $i \in \mathbb{N}$ continuous embeddings $\Psi^i \colon Iso(X) \to Iso(X_i)$, such that $\Psi^{i+1}(\varphi)_{|X_i} = \Psi^i(\varphi)$ for all i and all $\varphi \in Iso(X)$.
This in turns defines a continuous embedding from $Iso(X)$ into $Iso(Y)$, and since extension of isometries defines a continuous embedding from the isometry group of any metric space into that of its completion (see [14]), we actually have a continuous embedding of $Iso(X)$ into the isometry group of Z, that is to say $Iso(\mathbb{U})$ (and the image of any $\varphi \in Iso(X)$ is actually an extension of φ to $\mathbb{U}$).
In the remainder of the text, we follow [11] and say that a metric space X is *g-embedded* in $\mathbb{U}$ if X is embedded in $\mathbb{U}$, and there is a continuous morphism $\Phi \colon Iso(X) \to Iso(\mathbb{U})$ such that $\Phi(\varphi)$ extends φ for all $\varphi \in Iso(X)$.

3 Finite injectivity and the geometry of $\mathbb{U}$

3.1 First results

The following result, though easy to prove, is worth stating on its own, since it gives a good idea of the kind of problems we concern ourselves with in this section:

Theorem 3.1. *If $\varphi\colon \mathbb{U} \to \mathbb{U}$ is an isometric map, and $\varphi_{|_B} = id_B$ for some nonempty ball B, then $\varphi = id_{\mathbb{U}}$.*

Proof. Say that $A \subseteq \mathbb{U}$ is a *set of uniqueness* iff

$$\forall x, y \in \mathbb{U}\Big(\big(\forall z \in A\ d(x,z) = d(y,z)\big) \Rightarrow x = y\Big).$$

To prove theorem 3.1, we only need to prove that nonempty balls of $\mathbb{U}$ are sets of uniqueness. Of course, if $A \subset B$ and A is a set of uniqueness, then B is one too; therefore, the following proposition is more than what is needed to prove theorem 3.1:

Proposition 3.2. *Let $x_1, \dots, x_n \in \mathbb{U}$; assume that $f \in E(\{x_1, \dots, x_n\})$ is such that*

$$\forall i \neq j\ |f(x_i) - f(x_j)| < d(x_i, x_j) \text{ and } f(x_i) + f(x_j) > d(x_i, x_j)\ .$$

Then $K = \{x_1, \dots, x_n\} \cup \{z \in \mathbb{U} \colon \forall i\ d(z, x_i) = f(x_i)\}$ is a set of uniqueness.

Proof of Proposition 3.2.
Let $x \neq y \in \mathbb{U}$; we want to prove that there is some $z \in K$ such that $d(x,z) \neq d(y,z)$.
We may of course assume that $d(x, x_i) = d(y, x_i)$ for all i. Let now $g \in E(\{x_1, \dots, x_n\} \cup \{x\} \cup \{y\})$ be the Katětov extension of f; notice that $g(x) = g(y)$.
Now, pick $\alpha > 0$ and define a map g_α by:

- $g_\alpha(x_i) = g(x_i)$ for all i,
- $g_\alpha(y) = g(y)$, and $g_\alpha(x) = g(x) - \alpha$.

Our hypothesis on f ensures that, if $\alpha > 0$ is small enough, then $g_\alpha \in E(\{x_1, \dots, x_n\} \cup \{x\} \cup \{y\})$.
Hence there is some $z \in \mathbb{U}$ which has the prescribed distances to $x_1, \dots, x_n, x, y$, so that $z \in K$ and $d(z,x) \neq d(z,y)$. $\Diamond$

Remark: Geometrically, this means that if $S_1, \dots, S_n$ are spheres of center $x_1, \dots, x_n$, no two of which are tangent (inwards or outwards), and $\cap S_i \neq \emptyset$, then $\cap S_i \cup \{x_1, \dots, x_n\}$ is a set of uniqueness.
One may also notice that actually any nonempty sphere is a set of uniqueness.
Other examples of sets of uniqueness include the sets $Med(a, b) \cup \{a, b\}$, where $Med(a, b) = \{z \in \mathbb{U} \colon d(z, a) = d(z, b)\}$ (the proof is similar to the one above); in fact $Med(a, b) \cup \{a\}$ is a set of uniqueness, whereas $Med(a, b)$ obviously is not!
Also, one may wonder whether the condition in the statement of Proposition 3.2 is necessary; to see that one needs a condition of that kind, consider the following example: let x_0, x_1 be any two points such that $d(x_0, x_1) = 1$, and let f be defined by $f(x_1) = 1$, $f(x_2) = 2$. Then, for any point x such that $d(x, x_0) = d(x, x_1) = \frac{1}{2}$, one necessarily has $f(x) = \frac{3}{2}$, which proves that the result of Proposition 3.2 is not true in that case.

Theorem 3.1 shows that elements of $Iso(\mathbb{U})$ have some regularity properties; in particular, if an isometric map φ coincides on an open ball with an isometry ψ, then actually $\varphi = \psi$. One might then wonder, if $\varphi, \psi \colon \mathbb{U} \to \mathbb{U}$ are two isometric maps such that $\varphi_{|B} = \psi_{|B}$ for a nonempty ball B, whether one must have $\varphi = \psi$. It is easy to see that this is the case if $\varphi(B) = \psi(B)$ is a set of uniqueness; on the other hand, it is not true in general, which is the content of the next proposition.

Proposition 3.3. *Let B be any nonempty closed ball in $\mathbb{U}$.*
There are two isometric maps $\varphi, \psi \colon \mathbb{U} \to \mathbb{U}$ such that $\varphi(x) = \psi(x)$ for all $x \in B$, and $\varphi(\mathbb{U}) \cap \psi(\mathbb{U}) = \varphi(B) = \psi(B)$.

Proof of Proposition 3.3.
This result is a consequence of the universality of $\mathbb{U}$: let X denote the metric amalgam of two copies of $\mathbb{U}$ (say, X_1 and X_2) over $B(0, 1]$, and let φ_0 be an isometry of $X = X_1 \cup X_2$ such that $\varphi_0(X_1) = X_2$, $\varphi_0^2 = id$ and $\varphi_0(x) = x$ for all $x \in B(0, 1]$.
Pick an isometric embedding $\varphi_1 \colon X \to \mathbb{U}$, and let $y_0 = \varphi_1(0)$; also, let η be an isometry from $\mathbb{U}$ onto X_1, and let $x_0 = \eta^{-1}(0)$.
Now let $\varphi = \varphi_1 \circ \eta$, and $\psi = \varphi_1 \circ \varphi_0 \circ \eta$; by definition of φ_0, φ and ψ are equal on $\eta^{-1}(B(0, 1]) = B(x_0, 1]$.
Also, one has that
$\varphi(\mathbb{U}) = \varphi_1(X_1)$ and $\psi(\mathbb{U}) = \varphi_1(X_2)$, so $\varphi(\mathbb{U}) \cap \psi(\mathbb{U}) = \varphi_1(X_1 \cap X_2) = \varphi_1(B(0, 1]) = \varphi(B(x_0, 1]) = \psi(B(x_0, 1])$. ◊

In a way, the preceding proposition illustrates the fact that $\mathbb{U}$ contains many non-trivial isometric copies of itself (other examples include the sets $Med(x_1, \ldots x_n) = \{z \in \mathbb{U}\colon \forall i,j \ d(z,x_i) = d(z,x_j)\}$).
Still, all the isometric copies of $\mathbb{U}$ which we have seen so far are of empty interior. The next theorem (the proof of which is based on an idea of Pestov) shows that this is not always the case:

Theorem 3.4. *If $X \subseteq \mathbb{U}$ is closed and Heine-Borel (with the induced metric), $M > 0$, then $\{z \in \mathbb{U}\colon d(z,X) \geq M\}$ is isometric to $\mathbb{U}$.*

(Recall that a Polish metric space X is *Heine-Borel* iff closed bounded balls in X are compact).
In particular, $\mathbb{U}$ and $\mathbb{U} \setminus B(0,1[$ are isometric.

Proof . We first prove the result supposing that X is compact.
Let $Y = \{z \in \mathbb{U}\colon d(z,X) \geq M\}$; Y is a closed subset of $\mathbb{U}$, so to show that it is isometric to $\mathbb{U}$ we only need to prove that Y is finitely injective.
Let $y_1, \ldots, y_n \in Y$ and $f \in E(\{y_1, \ldots, y_n\})$. There exists a point $c \in \mathbb{U}$ such that $d(c,y_i) = f(y_i)$ for all i; the problem is that we cannot be sure a priori that $d(c,X) \geq M$.
To achieve this, define first $\varepsilon = \min\{f(y_i)\colon 1 \leq i \leq n\}$.
We may of course assume $\varepsilon > 0$.
X is compact, so we may find $x_1, \ldots, x_p \in X$ such that

$$\forall x \in X \, \exists j \leq p \ \ d(x,x_j) \leq \varepsilon \ .$$

Let then g be the Katětov extension of f to $\{y_1, \ldots, y_n\} \cup \{x_1, \ldots, x_p\}$.
By the finite injectivity of $\mathbb{U}$, there is $c \in \mathbb{U}$ such that $d(c,y_i) = g(y_i)$ for all $i \leq n$ and $d(c,x_j) = g(x_j) = d(x_j, y_{i_j}) + f(y_{i_j}) \geq M + \varepsilon$ for all $j \leq p$.
Since for all $x \in X$, there is $j \leq p$ such that $d(x,x_j) \leq \varepsilon$, the triangle inequality shows that $d(c,x) \geq d(c,x_j) - d(x_j,x) \geq M$, hence $c \in Y$. This proves that Y is finitely injective.
Suppose now that X is Heine-Borel but not compact, and let again $Y = \{z \in \mathbb{U}\colon d(z,X) \geq M\}$.
As before, we only need to show that Y is finitely injective; to that end, let $y_1, \ldots, y_n \in Y$ and $f \in E(\{y_1, \ldots, y_n\})$.
Let also $x \in X$ and $m = f(y_1) + d(y_1,x)$.
Since $B(x, M+m] \cap X$ is compact, there exists $c \in \mathbb{U}$ such that $d(c,y_i) = f(y_i)$ for all $i \leq n$, and $d(c, B(x, M+m] \cap X) \geq M$.
Then we claim that for all $x' \in X$ we have $d(c,x') \geq M$: if $d(x',x) \leq M+m$ then this is true by definition of c, and if $d(x',x) > M+m$ then one has $d(c,x') \geq d(x,x') - d(c,x) > M$ (since $d(c,x) \leq f(y_1) + d(y_1,x) = m$). $\Diamond$

From the combination of theorems 3.1 and 3.4, one can easily deduce that:

Corollary 3.5. *If B is any nonempty closed ball in $\mathbb{U}$, then there is an isometry φ of B such that no isometry of $\mathbb{U}$ coincides with φ on B.*

To derive corollary 3.5 from the previous results, let $\varphi\colon \mathbb{U} \to \mathbb{U} \setminus B(0,1[$ be an isometry, and choose $x \notin B(0,2]$. There exists, because of the homogeneity of $\mathbb{U} \setminus B(0,1[$, an isometry ψ of $\mathbb{U} \setminus B(0,1[$ such that $\psi(\varphi(x)) = x$. Thus, composing if necessary φ with ψ, we may suppose that x is a fixed point of φ. But then φ must send the ball of center x and radius 1 (in $\mathbb{U}$) onto the ball of center x and radius 1 (in $\mathbb{U} \setminus B(0,1[$).
Since by choice of x both balls are the same, we see that $\varphi_{|_{B(x,1]}}$ is an isometry of $B(x,1]$, yet theorem 3.1 shows that no isometry of $\mathbb{U}$ can coincide with φ on $B(x,1]$. $\Diamond$

(Using finite injectivity and automatic continuity of Baire measurable morphisms between Polish groups, one can give a direct, if somewhat longer, proof of corollary 3.5).

3.2 Fixed points of isometries

Here we use the tools introduced above - most notably Katětov maps and the compact injectivity of $\mathbb{U}$ - in order to study some properties of the sets of fixed points of elements of $Iso(\mathbb{U})$. For all $\varphi \in Iso(\mathbb{U})$, we let $Fix(\varphi) = \{x \in \mathbb{U}\colon \varphi(x) = x\}$.
Since the isometry class of $Fix(\varphi)$ is an invariant of the conjugacy class of φ, one may hope to glean some information about the conjugacy relation by the study of fixed points.
Clemens, quoted by Pestov in [11], conjectured that this invariant was the weakest possible: the exact content of his conjecture was that, if $\varphi \in Iso(\mathbb{U})$, then the set of fixed points of φ is either empty or isometric to $\mathbb{U}$.
This turns out to be false in general, as we will see below; this will enable us to compute the complexity of the conjugacy relation in $Iso(\mathbb{U})$.
First, we prove the rather surprising fact that the conjecture holds for all isometries of finite order (and even for isometries with totally bounded orbits); so, studying their fixed points will tell us nothing about, say, conjugacy of isometric involutions.
We wish to attract the attention of the reader to a consequence of the triangle inequality, which, though obvious, is crucial in the following constructions:

$$\forall \varphi \in Iso(\mathbb{U})\ \forall z \in \mathbb{U}\ \forall x \in \mathbb{U}\ d(z,\varphi(z)) \leq d(z,x) + d(z,\varphi(x)).$$

The interest of this is that it enables us to control the diameter of the orbits of the points in our constructions. If $\varphi\colon \mathbb{U} \to \mathbb{U}$ is an isometry, and $x \in \mathbb{U}$, we let $\rho_\varphi(x)$ denote the diameter of $\{\varphi^n(x)\}_{n\in\mathbb{Z}}$; when there is no risk of confusion we simply write it $\rho(x)$.
Then, if y and x are such that $d(y,\varphi^n(x)) = \frac{\rho(x)}{2}$ for all $n \in \mathbb{Z}$, the above remark implies that $\rho(y) \leq \rho(x)$. This is what enables us to build better and better approximations of a Katětov map on the set of fixed points, while ensuring that the points realizing the approximations have orbits of small diameter. This is the first step of the proof, for which we use Lemma 3.6 below; once it is done, we will need to see whether one can find a fixed point "close enough" to any point whose orbit has a small diameter.

Lemma 3.6. *Let $\varphi \in Iso(\mathbb{U})$, $x_1,\ldots,x_m \in Fix(\varphi)$, $f \in E(\{x_1,\ldots,x_m\})$, and $z \in \mathbb{U}$. Assume that $min\{f(x_i)\} \geq \rho_\varphi(z) > 0$.*
Then define
$A = \{1 \leq i \leq m\colon d(z,x_i) < f(x_i) - \frac{\rho_\varphi(z)}{2}\}$, $B = \{1 \leq i \leq m\colon d(z,x_i) > f(x_i) + \frac{\rho_\varphi(z)}{2}\}$, and $C = \{1 \leq i \leq m\colon |d(z,x_i) - f(x_i)| \leq \frac{\rho_\varphi(z)}{2}\}$.
These equations define a Katětov map on $\{\varphi^n(z)\}_{n\in\mathbb{Z}} \cup \{x_i\}_{1\leq i\leq n}$:
- $\forall n \in \mathbb{Z}\ g(\varphi^n(z)) = \frac{\rho_\varphi(z)}{2}$,
- $\forall i \in A\ g(x_i) = d(z,x_i) + \frac{\rho_\varphi(z)}{2}$,
- $\forall i \in B\ g(x_i) = d(z,x_i) - \frac{\rho_\varphi(z)}{2}$,
- $\forall i \in C\ g(x_i) = f(x_i)$.

Hence, if the orbit of z is totally bounded, there exists $z' \in \mathbb{U}$ with the prescribed distances to $\{\varphi^n(z)\}_{n\in\mathbb{Z}} \cup \{x_i\}_{1\leq i\leq n}$; then $\rho_\varphi(z') \leq \rho_\varphi(z)$.

Proof of lemma 3.6.
To simplify notation, we let $\rho = \rho_\varphi(z)$. To check that the above equations define a Katětov map, we begin by checking that g is 1-Lipschitz:
First, we have that $|g(x_i) - g(\varphi^n(z))| = |d(z,x_i) + \alpha - \frac{\rho}{2}|$, where $|\alpha| \leq \frac{\rho}{2}$. If $\alpha = \frac{\rho}{2}$ there is nothing to prove, otherwise it means that $d(z,x_i) \geq f(x_i) - \frac{\rho}{2}$, so that $d(z,x_i) \geq \frac{\rho}{2}$, which is enough to show that $|d(z,x_i) + \alpha - \frac{\rho}{2}| \leq d(z,x_i) = d(\varphi^n(z),x_i)$.
We now let $1 \leq i,j \leq m$ and assume w.l.o.g that $|g(x_i) - g(x_j)| = g(x_i) - g(x_j)$; there are three nontrivial cases.
(a) $g(x_i) = d(z,x_i) + \alpha$, $g(x_j) = d(z,x_j) + \beta$, with $\alpha > \beta \geq 0$.
Then one must have $g(x_j) = f(x_j)$, and also $g(x_i) \leq f(x_i)$, so that $g(x_i) - g(x_j) \leq f(x_i) - f(x_j) \leq d(x_i,x_j)$.
(b) $g(x_i) = d(z,x_i) + \alpha$, $g(x_j) = d(z,x_j) - \beta$, $0 \leq \alpha,\beta \leq \frac{\rho}{2}$. Then the definition of g ensures that $g(x_i) \leq f(x_i)$ and $g(x_j) \geq f(x_j)$, so that $g(x_i) - g(x_j) \leq f(x_i) - f(x_j) \leq d(x_i,x_j)$.

(c)$g(x_i) = d(z,x_i) - \alpha$, $g(x_j) = d(z,x_j) - \beta$, $0 \leq \alpha < \beta$.
Then we have $g(x_i) = f(x_i)$, and $g(x_j) \geq f(x_j)$, so $g(x_i) - g(x_j) \leq f(x_i) - f(x_j)$.
We proceed to check the remaining inequalities:
- $g(\varphi^n(z)) + g(\varphi^m(z)) = \rho \geq d(\varphi^n(z), \varphi^m(z))$ by definition of ρ;
- $g(\varphi^n(z)) + g(x_i) = \frac{\rho}{2} + d(z,x_i) + \alpha$, where $|\alpha| \leq \frac{\rho}{2}$, so $g(\varphi^n(z)) + g(x_i) \geq d(z,x_i) = d(\varphi^n(z), x_i)$.
The last remaining inequalities to examine are that involving x_i, x_j; we again break the proof in subcases, of which only two are not trivial:
(a) $g(x_i) = d(z,x_i) + \alpha$ and $g(x_j) = d(z,x_j) - \beta$, where $0 \leq \alpha < \beta$. Then $g(x_i) = f(x_i)$, and $g(x_j) \geq f(x_j)$, so that $g(x_i) + g(x_j) \geq d(x_i, x_j)$.
(b) $g(x_i) = d(z,x_i) - \alpha$, $g(x_j) = d(z,x_j) - \beta$: then we have both that $g(x_i) \geq f(x_i)$ and $g(x_j) \geq f(x_j)$, so we are done. $\diamondsuit$

This technical lemma enables us to prove the following result, which is nearly enough to prove that $Fix(\varphi)$ is finitely injective:

Lemma 3.7. *Let φ be an isometry of $\mathbb{U}$ with totally bounded orbits, $x_1, \ldots, x_m \in Fix(\varphi)$, $f \in E(\{x_1, \ldots, x_m\})$, and $\varepsilon > 0$. Then one (or both) of the following assertions is true:*
- There exists $z \in \mathbb{U}$ such that $\rho_\varphi(z) \leq \varepsilon$ and $d(z,x_i) = f(x_i)$ for all i.
- There is $z \in Fix(\varphi)$ such that $|f(x_i) - d(z,x_i)| \leq \varepsilon$ for all i.

Proof of lemma 3.7:
Let $x_1, \ldots, x_m \in Fix(\varphi)$, $f \in E(\{x_1, \ldots, x_m\})$, and $\varepsilon > 0$, which we assume w.l.o.g to be strictly smaller than $\min\{f(x_i) \colon i = 1, \ldots, m\}$
We may assume that

$$\gamma = \inf \Big\{ \sum_{i=1}^{m} |f(x_i) - d(x,x_i)| \colon x \in Fix(\varphi) \Big\} > 0 \ .$$

Let $x \in Fix(\varphi)$ be such that $\sum_{i=1}^m |f(x_i) - d(x,x_i)| \leq \gamma + \frac{\varepsilon}{4}$.
We let z be any point such that
- $d(z,x) = \frac{\varepsilon}{2}$;
- $\forall i = 1, \ldots, m \ |d(x,x_i) - f(x_i)| \leq \frac{\varepsilon}{2} \Rightarrow d(z,x_i) = f(x_i)$;
- $\forall i = 1, \ldots, m \ d(x,x_i) \geq f(x_i) + \frac{\varepsilon}{2} \Rightarrow d(z,x_i) = d(x,x_i) - \frac{\varepsilon}{2}$;
- $\forall i = 1, \ldots, m \ d(x,x_i) \leq f(x_i) - \frac{\varepsilon}{2} \Rightarrow d(z,x_i) = d(x,x_i) + \frac{\varepsilon}{2}$.
One checks as above that these equations indeed define a Katětov map; z cannot be a fixed point of φ since it would contradict the definition of γ, or the fact that $\gamma > 0$.

We use lemma 3.6 to build a sequence (z_n) of points of $\mathbb{U}$ such that:
(0) $z_0 = z$;
(1) $0 < \rho(z_n) \leq \varepsilon$;
(2) $\forall p \in \mathbb{Z}\ d(z_{n+1}, \varphi^p(z_n)) = \frac{\rho(z_n)}{2}$;
(3) $\forall i \in A_n\ d(z_{n+1}, x_i) = d(z_n, x_i) + \frac{\rho(z_n)}{2}$;
(4) $\forall i \in B_n\ d(z_{n+1}, x_i) = d(z_n, x_i) - \frac{\rho(z_n)}{2}$;
(5) $\forall i \in C_n\ d(z_{n+1}, x_i) = f(x_i)$.
(Where A_n, B_n, C_n are defined as in the statement of Lemma 3.6)

Suppose now that the sequence has been constructed up to rank n.
Since $\{x_1, \ldots, x_m\}, z_n, f$ satisfy the hypothesis of lemma 3.6, we may find a point z' with the prescribed distances to $\{\varphi^p(z_n)\} \cup \{x_1, \ldots, x_m\}$. As before, z' cannot be fixed, since it would contradict the definition of γ; we let $z_{n+1} = z'$, and the other conditions are all ensured by lemma 3.6.
If we do not obtain in finite time a z_n such that $\rho(z_n) \leq \varepsilon$ and $d(z_n, x_i) = f(x_i)$ for all i, then either A_n or B_n is nonempty for all n; hence (3) and (4) imply that $\sum \rho(z_n)$ converges. Therefore, z_n converges to some fixed point z_∞. Necessarily, there was some i such that
$|d(z_0, x_i) - f(x_i)| \leq |d(x, x_i) - f(x_i)| - \frac{\varepsilon}{2}$, so $\sum_{i=1}^m |f(x_i) - d(z_0, x_i)| \leq \gamma - \frac{\varepsilon}{4}$.
By construction, $\sum_{i=1}^m |f(x_i) - d(z_\infty, x_i)| \leq \sum_{i=1}^m |f(x_i) - d(z_0, x_i)|$, which contradicts the definition of γ. $\Diamond$

This is not quite enough to produce fixed points with prescribed distances to some finite set of fixed points; the following lemma ensures that it is indeed possible:

Lemma 3.8. *Let φ be an isometry of $\mathbb{U}$ with totally bounded orbits, $x \in \mathbb{U}$ be such that $\rho_\varphi(x) \leq 2\varepsilon$, and assume that $Fix(\varphi) \neq \emptyset$.*
Then for any $\delta > 0$ there exists $y \in \mathbb{U}$ such that :
- $\forall n \in \mathbb{Z}\ d(y, \varphi^n(x)) = d(y, x) \leq \varepsilon + \delta$;
- $\rho_\varphi(y) \leq \varepsilon$.

Proof of lemma 3.8.
Let x, φ be as above; let also

$$E = \{y \in \mathbb{U} \colon \forall n \in \mathbb{Z}\ d(y, \varphi^n(x)) = d(y, x) \text{ and } \rho(y) \leq \varepsilon\}$$

Notice that E is nonempty, since any fixed point of φ belongs to E.
Now let $\alpha = \inf\{d(y, x) \colon y \in E\}$; we want to prove that $\alpha \leq \varepsilon$. Assume that it is not, let $\delta > 0$ and pick $y \in E$ such that $d(y, x) < \alpha + \delta$.
Let now $\rho(y) = \rho \leq \varepsilon$; one checks as above that the following map g belongs to $E(\{\varphi^n(x)\} \cup \{\varphi^n(y)\})$:

- $\forall n \in \mathbb{Z}\ g(\varphi^n(x)) = \max(\varepsilon, d(y,x) - \frac{\rho}{2})$.
- $\forall n \in \mathbb{Z}\ g(\varphi^n(y)) = \frac{\rho}{2}$.

Since the orbits of φ are totally bounded, there exists $z \in \mathbb{U}$ with the prescribed distances; consequently $z \in E$, and we see that necessarily $\rho < 2\delta$.
Letting δ go to 0, there are only two cases to consider:
(1) For all $p \in \mathbb{N}^*$ there is a fixed point y_p such that $\alpha \leq d(y_p, x) < \alpha + \frac{1}{p}$. If so, let p be big enough that $\frac{1}{p} < \frac{\varepsilon}{2}$, and consider the following map:
- $g(y_p) = \frac{1}{p}$
- $\forall n \in \mathbb{Z}\ g(\varphi^n(x)) = d(y_p, x) - \frac{1}{p}$.

A direct verification shows that $g \in E(\{\varphi^n(x)\} \cup \{y_p\})$, therefore there is $z \in \mathbb{U}$ with the desired distances; to conclude, notice that $z \in E$ and $d(z,x) < \alpha$, which is absurd.
(2) If we are not in case (1), we may pick a point $y \in E$ such that no fixed point is as close as y to x. Then, starting with such a point, we may iterate the construction at the beginning of the lemma. This yields a sequence of points $y_i \in E$ such that $\rho(y_{i+1}) \leq \rho(y_i)$, and $d(y_{i+1}, y_i) = \frac{\rho(y_i)}{2}$. If $\sum \rho(y_i)$ converges, then the sequence y_i converges to a fixed point which is closer to x than y, and this is impossible by definition of y. Since $d(x, y_{i+1}) > \varepsilon \Rightarrow d(x, y_{i+1}) \leq d(x, y_i) - \frac{\rho(y_i)}{2}$, we see that if $\sum \rho(y_i)$ does not converge there must be some i such that $d(x, y_i) = \varepsilon$. ◊

We have finally done enough to obtain the following result:

Theorem 3.9. *If $\varphi\colon \mathbb{U} \to \mathbb{U}$ is an isometry whose orbits are totally bounded, and $Fix(\varphi)$ is nonempty, then $Fix(\varphi)$ is isometric to $\mathbb{U}$.*

Proof. Recall that a nonempty metric space X is said to have the *approximate extension property* iff

$$\forall A \subset X \text{ finite } \forall f \in E(A)\ \forall \varepsilon > 0\ \exists z \in X\ \forall a \in A\, |d(z,a) - f(a)| \leq \varepsilon\ .$$

It is a classical result (see e.g [11]) that, up to isometry, $\mathbb{U}$ is the only Polish metric space with the approximate extension property. So, to prove Theorem 3.9, it is enough to prove that $Fix(\varphi)$ has the approximate extension property. To prove this, notice first that lemma 3.8 implies that, for all $x \in X$ such that $\rho_\varphi(x) \leq \varepsilon$, there is a fixed point y such that $d(y,x) \leq 3\varepsilon$ (take $\delta = \frac{\varepsilon}{2}$ in the lemma above, and iterate).
Let now $x_1, \ldots, x_n \in Fix(\varphi)$, $f \in E(\{x_1, \ldots, x_n\})$, and $\varepsilon > 0$.
Lemma 3.7 tells us that :
- there exists a point z such that $\rho_\varphi(z) \leq \frac{\varepsilon}{3}$, and $d(z, x_i) = f(x_i)$ for all $i = 1, \ldots, n$, or

- there exists $z \in Fix(\varphi)$ such that $|d(z,x_i) - f(x_i)| \leq \varepsilon$ for all $i = 1, \dots, n$.
In the second case, we have what we wanted; so suppose we are dealing with the first case, and pick any fixed point y such that $d(y,z) \leq \varepsilon$. Then $y \in Fix(\varphi)$, and $|d(y,x_i) - f(x_i)| \leq \varepsilon$ for all $i = 1, \dots, n$. $\diamondsuit$

Actually, looking carefully at the proof of theorem 3.9 enables one to see that a more general result is true:

Theorem 3.10. *Let G be a group acting on $\mathbb{U}$ by isometries, and assume that for all $x \in \mathbb{U}$ its orbit $G \cdot x = \{g \cdot x \colon g \in G\}$ is totally bounded. Then the set $Fix(G) = \{x \in \mathbb{U} \colon \forall g \in G \ g \cdot x = x\}$ is either empty or isometric to $\mathbb{U}$.*
In particular, if a compact group G acts continuously on $\mathbb{U}$ by isometries, then $Fix(G)$ is either empty or isometric to $\mathbb{U}$.

To show this, just repeat the proof of theorem 3.9, replacing the orbits under the action of φ by the orbits under the action of G.
Notice that no condition on the topology of G is given: knowing that the orbits are totally bounded is enough to make the proof work.

It turns out that the situation is very different when it comes to studying isometries with non totally bounded orbits; one may still prove, using the same methods as above, that if φ is an isometry with a fixed point x, then on any sphere S centered in x and for any $\varepsilon > 0$ there is $z \in S$ such that $d(z, \varphi(z)) \leq \varepsilon$. This is not enough to ensure the existence of other fixed points than x.

Theorem 3.11. *Let X be a Polish metric space.*
There exists an isometric copy $X' \subset \mathbb{U}$ of X, and an isometry φ of $\mathbb{U}$, such that $Fix(\varphi) = X'$.

Proof .
We may assume that $X \neq \emptyset$ (it is not hard to build isometries of $\mathbb{U}$ without fixed points; in [2], Cameron and Vershik actually prove the existence of isometries of $\mathbb{U}$ with dense orbits) .
We first need a few definitions: if X is a metric space, we denote by $E(X, \omega, \mathbb{Q})$ the set of functions $f \in E(X, \omega)$ which take rational values on some finite support (This set is countable if X is).
Also, if $X_0 \subset X$ are two countable metric spaces, and φ is an isometry of X, we want to find a condition on (X, X_0, φ) which expresses the idea that "φ fixes all the points of X_0, and for each $x \in X \setminus X_0$, $\varphi^n(x)$ gets to be as

far away from x as possible". The following definition is a possible way to translate this naive idea into formal mathematical language:

We say that (X, X_0, φ) has property (*) if:
- $\forall x \in X_0 \ \varphi(x) = x$.
- $\forall x_1, x_2 \in X \ \liminf_{|p| \to +\infty} d(x_1, \varphi^p(x_2)) \geq d(x_1, X_0) + d(x_2, X_0)$.

The following lemma, which shows that this property is suitable for an inductive construction similar to Katětov's, is the core of the proof.

Lemma 3.12. *Let (X, X_0, φ) have property (*).*
Then there exists a countable metric space X' and an isometry φ' of X' such that :
- X embeds in X', and φ' extends φ.
- $\forall f \in E(X, \omega, \mathbb{Q}) \ \exists x' \in X' \ \forall x \in X \ d(x', x) = f(x)$.
- (X', X_0, φ') has property (*) (identifying X_0 to its image via the isometric embedding of X in X').

Admit this lemma for a moment; now, let X_0 be any dense countable subset of X, and $\varphi_0 = id_{X_0}$. Then (X_0, X_0, φ_0) has property (*), so lemma 3.12 shows that we may define inductively countable metric spaces X_i and isometries $\varphi_i \colon X_i \to X_i$ such that:
-X_i embeds isometrically in X_{i+1}, φ_{i+1} extends φ_i;
-(X_i, X_0, φ_i) has property (*);
-$\forall f \in E(X_i, \omega, \mathbb{Q}) \ \exists z \in X_{i+1} \ \forall x \in X_i \ d(z, x) = f(x)$.
Let Y denote the completion of $\cup X_i$, and φ be the extension to Y of the map defined by $\varphi(x) = \varphi_i(x)$ for all $x \in X_i$.
By construction, Y has the approximate extension property; since Y is Polish, this shows that Y is isometric to $\mathbb{U}$.
The construction also ensures that all points of X_0 are fixed points of φ, and

$$\forall y_1, y_2 \in Y, \ \liminf_{|p| \to +\infty} d(y_1, \varphi^p(y_2)) \geq d(y_1, X_0) + d(y_2, X_0) \ .$$

Therefore, $Fix(\varphi)$ is the closure of X_0 in $\mathbb{U}$; hence it is isometric to the completion of X_0, so it is isometric to X. $\lozenge$

Proof of Lemma 3.12.
First, let $f \in E(X, \omega, \mathbb{Q})$; we let $X(f) = X \cup \{y_i^f\}_{i \in \mathbb{Z}}$ and define a distance on $X(f)$, which extends the distance on X, by:
-$d(x, y_i^f) = f(\varphi^{-i}(x))$;
-$d(y_i^f, y_j^f) = \inf_{x \in X}(d(y_i^f, x) + d(y_j^f, x))$.
(In other words, $X(f)$ is the metric amalgam of the spaces $X \cup \{f \circ \varphi^i\}$ over

X.)
Let φ_f be defined by $\varphi_f(y_i^f) = y_{i+1}^f$, $\varphi_f(x) = \varphi(x)$ for $x \in X$.
Notice that, by definition of d, φ_f is an isometry of $X(f)$, which extends φ.
We claim that $(X(f), X_0, \varphi_f)$ has property (*).
To prove this, let $y, y' \in X(f)$; we want to prove that

$$\liminf_{|p|\to+\infty} d((\varphi_f)^p(y'), y) \geq d(y, X_0) + d(y', X_0) \ .$$

If both y and y' are in X, there is nothing to prove. Two cases remain:
(1) $y \in X$, $y' = y_j^f$. Without loss of generality, we may assume that $j = 0$.
By definition, we know that there are $x_1, \dots, x_n \in X$ such that

$$d((\varphi_f)^p(y_0^f), y) = f(\varphi^{-p}(y)) = \min_{i=1,\dots,n} (f(x_i) + d(y, \varphi^p(x_i))) \ .$$

Let $\varepsilon > 0$; for $|p|$ big enough, $d(y, \varphi^p(x_i)) \geq d(y, X_0) + d(x_i, X_0) - \varepsilon$.
We then have

$$d((\varphi_f)^p(y_0^f), y) \geq \min_{i=1,\dots,n} (f(x_i) + d(y, X_0) + d(x_i, X_0) - \varepsilon) \ ,$$

so $d((\varphi_f)^p(y_0^f), y) \geq d(y, X_0) + \min_{i=1,\dots,n}(f(x_i) + d(x_i, X_0)) - \varepsilon$.
Hence $d((\varphi_f)^p(y_0^f), y) \geq d(y, X_0) + d(y_0^f, X_0) - \varepsilon$, and we are done.
(2) $y = y_i^f$ and $y' = y_j^f$; we may assume that $i = 0$.
Then we have $d(\varphi_f^p(y'), y) = \inf_{x\in X}(f(x) + f(\varphi^{-p-j}(x))$.
Therefore, we need to show that

$$\liminf_{|p|\to+\infty} \inf_{x\in X} (f(x) + f(\varphi^{-p}(x)) \geq 2 \inf_{x\in X_0} f(x) \ .$$

Assume again that f is controlled by $\{x_1, \dots, x_n\}$, choose $\varepsilon > 0$, and let $|p|$ be big enough that $d(x_i, \varphi^p(x_j)) \geq d(x_i, X_0) + d(x_j, X_0) - \varepsilon$ for all i, j.
Then we have, for all $x \in X$:
$f(x) + f(\varphi^{-p}(x)) = f(x_i) + d(x, x_i) + f(x_j) + d(x, \varphi^p(x_j))$ for some i, j. Since $d(x, x_i) + d(x, \varphi^p(x_j)) \geq d(x_i, \varphi^p(x_j))$, we see that there is some (i, j) such that

$$\inf_{x\in X} (f(x) + f(\varphi^{-p}(x)) = f(x_i) + d(x_i, \varphi^p(x_j)) + f(x_j).$$

We know that $d(x_i, \varphi^p(x_j)) \geq d(x_i, X_0) + d(x_j, X_0) - \varepsilon$, so

$$\inf_{x\in X} (f(x) + f(\varphi^{-p}(x)) \geq f(x_i) + d(x_i, X_0) + d(x_j, X_0) + f(x_j) - \varepsilon \geq 2 \inf_{x\in X_0} f(x) - \varepsilon.$$

This is enough to prove that $(X(f), X_0, \varphi_f)$ has property (*).

Now, let X' denote the metric amalgam of the spaces $X(f)$ over X, where f varies over $E(X,\omega,\mathbb{Q})$. It is countable, and letting $\varphi'(x)=\varphi_f(x)$ for all $x \in X^f$ defines an isometry of X' which extends φ.
The same arguments as above are enough to show that (X', X_0, φ') has property (*). $\Diamond$

3.3 The complexity of conjugacy in $Iso(\mathbb{U})$ and $Iso(\mathbb{QU})$.

This construction has an additional interest, since it enables one to compute the complexity of conjugacy between isometries of the *rational Urysohn metric space* $\mathbb{QU}$ (actually, a variation on this construction also works to compute the complexity of the relation of conjugacy in $Iso(\mathbb{U})$, see the remark at the end of this section).
We will not detail here the theory of complexity of definable equivalence relations; see [1] or [9] for details and a bibliography on the subject.
We let $GRAPH$ denote the (Borel) set of countable graphs, see [8].
Recall that $\mathbb{QU}$ is, up to isometry, the only countable metric space whose distance takes its values in $\mathbb{Q}$ and such that:

$$\forall f \in E(X,\omega,\mathbb{Q})\ \exists z \in \mathbb{QU}\ \forall x \in \mathrm{supp}(f)\ d(z,x)=f(x)\ .$$

It is the Fraïssé limit of the class of finite metric spaces with rational distances; $\mathbb{U}$ is the completion of $\mathbb{QU}$. For more information about this space, see for instance [11].
We endow its isometry group $Iso(\mathbb{QU})$ with the pointwise convergence topology on $\mathbb{QU}$ endowed with the discrete distance, which turns $Iso(\mathbb{QU})$ into a Polish group, isomorphic to a closed subgroup of S_∞.
We may endow any countable graph with the graph distance, turning it into a countable Polish metric space; two graphs are isomorphic if, and only if, the corresponding metric spaces are isometric.
Now, let X and X' denote two isometric countable Polish metric spaces. Let $X_\infty=\cup X_i$ and $X'_\infty=\cup X'_i$ denote the spaces obtained by our construction, and $\varphi_\infty, \varphi'_\infty$ the corresponding isometries. By construction, both X_∞ and X'_∞ are isometric to $\mathbb{QU}$.
Also, one sees that the isometry between X and X' extends to an isometry $\psi\colon X_\infty \to X'_\infty$ such that $\psi \circ \varphi_\infty = \varphi'_\infty \circ \psi$.
Since $\mathbb{QU}$ has the rational extension property, we may thus, using methods similar to those of [3], assume that $X_\infty = X'_\infty = \mathbb{QU}$, and that the map $\Psi\colon X \mapsto \varphi_\infty$ is Borel (from $GRAPH$ into $Iso(\mathbb{QU})$).

What we have seen above implies that $\Psi(G)$ and $\Psi(G')$ are conjugate if G and G' are isomorphic.
Conversely, assume that there is $\varphi \in Iso(\mathbb{QU})$ such that $\varphi \circ \Psi(G) = \Psi(G') \circ \varphi$; this implies that $\varphi(Fix(\Psi(G))) = Fix(\Psi(G'))$, and this proves that G and G', when endowed with the graph distance, are isometric, and hence G and G' are isomorphic.
Thus, Ψ is a Borel reduction of graph isomorphism to conjugacy in $Iso(\mathbb{QU})$. Since $Iso(\mathbb{QU})$ is a closed subgroup of S_∞, and graph isomorphism is universal for relations induced by Borel actions of S_∞ on Polish spaces (see [3]), we have obtained the following result:

Theorem 3.13. *Graph isomorphism is Borel bi-reducible to the conjugacy relation in $Iso(\mathbb{QU})$.*

With a very similar proof, albeit fraught with more technicalities, one may also show that:

Theorem 3.14. *Isometry between Polish metric spaces is Borel bireducible to conjugacy of isometries in $\mathbb{U}$.*

Sketch of proof.
To show this, one may use a slightly more complicated version of the proof above: if X is a Polish metric space, we let $E'(X)$ denote the metric amalgam over X of a countably infinte set of copies of $E(X, \omega)$. Then we again use an inductive construction: we start with $\varphi_0 =$ the identity of $X_0 = X$. Then, we let $X_{i+1} = E'(X_i) = \cup Y_n$ (where each Y_n is a copy of $E(X_i)$), and define φ_{i+1} as the isometry which maps each $f \in Y_n$ to $f \circ \varphi^{-1}$ in Y_{n+1}. This enables one to assign to each X an isometry φ_X of $\mathbb{U}$ (identified with the completion of $\cup X_i$), in such a way that the set of fixed points of φ_X is isometric to X, and φ_X and $\varphi_{X'}$ are conjugate if X and X' are isometric. The construction above can be done uniformly (though the details are very cumbersome, which is why we don't give the proof in its entirety), so the mapping $X \mapsto \varphi_X$ is a Borel reduction of isometry between Polish metric spaces to conjugacy in $Iso(\mathbb{U})$. $\diamondsuit$

4 Trying to extend finite homogeneity

4.1 Reformulating the problem

The remainder of this article will be devoted to proving the following result:

Theorem 4.1. *Let X be a Polish metric space. The following assertions are equivalent:*
(a) X is compact.
(b) If $X_1, X_2 \subseteq \mathbb{U}$ are isometric to X and $\varphi \colon X_1 \to X_2$ is an isometry, then there exists $\tilde{\varphi} \in Iso(\mathbb{U})$ which extends φ.
(c) If $X_1, X_2 \subseteq \mathbb{U}$ are isometric to X, then there exists $\varphi \in Iso(\mathbb{U})$ such that $\varphi(X_1) = X_2$.
(d) If $X_1 \subseteq \mathbb{U}$ is isometric to X and $f \in E(X_1)$, there exists $z \in \mathbb{U}$ such that $d(z, x) = f(x)$ for all $x \in X_1$.

$(a) \Rightarrow (b)$ is well-known, as explained in the introduction (see [5] for a proof); $(b) \Rightarrow (c)$ is trivial.
To see that $(c) \Rightarrow (d)$, let a space X having property (c) be embedded in $\mathbb{U}$. Notice that, since there exists a copy of X which is g-embedded in $\mathbb{U}$, and all isometric copies of X are isometric by an isometry of the whole space, all the isometric copies of X are necessarily g-embedded in $\mathbb{U}$. Therefore, any isometry between copies of X extends to an isometry of $\mathbb{U}$. Let now $f \in E(X)$; the metric space $X_f = X \cup \{f\}$ embeds in $\mathbb{U}$, so that there exists an isometric copy $Y = X' \cup \{z\} \subset \mathbb{U}$ of X_f, where X' is an isometric copy of X.
By definition, there exists an isometry φ from X onto X' such that $d(z, \varphi(x)) = f(x)$ for all $x \in X$; pick some $\psi \in Iso(\mathbb{U})$ which extends φ. Then we have $d(\psi^{-1}(z), x) = f(x)$ for all $x \in X$, which shows that X has property (d) of theorem 4.5.
It only remains to show that $(d) \Rightarrow (a)$; this turns out to be the hard part of the proof.
If $X \subseteq \mathbb{U}$ is closed, define $\Phi^X \colon \mathbb{U} \to E(X)$ by $\Phi^X(z)(x) = d(z, x)$.
Notice that Φ^X is 1-Lipschitz. Property (d) in theorem 4.1 is equivalent to Φ^{X_1} being onto for any isometric copy $X_1 \subseteq \mathbb{U}$ of X; but $\Phi^{X_1}(\mathbb{U})$ is necessarily separable since $\mathbb{U}$ is, so we see that for X to have property (d) it is necessary that $E(X)$ be separable.

4.2 Spaces with the collinearity property

The next logical step is to determine the Polish metric spaces X such that $E(X)$ is separable. One can rather easily narrow the study:

Proposition 4.2. *If X is Polish and not Heine-Borel, then $E(X)$ is not separable.*

Proof.
The hypothesis tells us that there exists $M, \varepsilon > 0$ and $(x_i)_{i\in\mathbb{N}}$ such that

$$\forall i \neq j \ \varepsilon \leq d(x_i, x_j) \leq M.$$

If $A \subseteq \mathbb{N}$, define $f_A : \{x_i\}_{i\geq 0} \to \mathbb{R}$ by $f_A(x_i) = \begin{cases} M & \text{if } i \in A \\ M+\varepsilon & \text{else} \end{cases}$.
It is easy to check that for all $A \subseteq \mathbb{N}$, $f_A \in E(\{x_i\}_{i\geq 0})$, and if $A \neq B$ one has $d(f_A, f_B) = \varepsilon$ (where d is the distance on $E(\{x_i\})$).
Hence $E(\{x_i\}_{i\geq 0})$ is not separable; since it is isometric to a subspace of $E(X)$ (see section 2), this concludes the proof. $\diamondsuit$

So we know now that, to have property (d) of theorem 4.1, a metric space X has to be Heine-Borel; at this point, one could hope that either only compact sets are such that $E(X)$ is separable, or all Heine-Borel Polish spaces have this property. Unfortunately, the situation is not quite so simple, as the following two examples show:

Example 4.3. *If $\mathbb{N}$ is endowed with its usual distance, then $E(\mathbb{N}) = \overline{E(\mathbb{N}, \omega)}$.*

Indeed, let $f \in E(\mathbb{N})$; then one has for all n that $|f(n) - n| \leq f(0)$, and also $f(n+1) \leq f(n)+1$. This last inequality can be rewritten as $f(n+1)-(n+1) \leq f(n) - n$.
So $f(n) - n$ converges to some $a \in \mathbb{R}$; let $\varepsilon > 0$ and choose M big enough that $n \geq M \Rightarrow |f(n) - n - a| \leq \varepsilon$.
Then, for all $n \geq M$, one has

$$0 \leq f(M) + n - M - f(n) = (f(M) - M - a) - (f(n) - n - a) \leq 2\varepsilon.$$

If one lets, for all i, f_i be the Katětov extension of $f_{|_{[0,i]}}$, then
$f_i \in E(\mathbb{N}, \omega)$ and we have just shown that (f_i) converges to f in $E(\mathbb{N})$.
Replacing the sequence $(f(n) - n)$ by the function $x \mapsto f(x) - x$, one would have obtained the same result for any subset of $\mathbb{R}$ (endowed with its usual metric, of course); actually, one may use the same method to prove that $E(\mathbb{R}^n, ||.||_1)$ and $E(\mathbb{R}^n, ||.||_\infty)$ are separable for all n.

The situation turns out to be very different when $\mathbb{R}^n$ is endowed with other norms, as the following example shows.

Example 4.4. *If $n \geq 2$ and $\mathbb{R}^n$ is endowed with the euclidian distance, then $E(\mathbb{R}^n)$ is not separable.*

We only need to prove this for $n = 2$, since $E(\mathbb{R}^2, ||.||_2)$ is isometric to a closed subset of $E(\mathbb{R}^n, ||.||_2)$ for any $n \geq 2$.
Remark first that it is easy to build a sequence (x_i) of points in $\mathbb{R}^2$ such that $d(x_{i+1}, 0) \geq d(x_i, 0) + 1$ for all i, and

$$\forall i > j \in \mathbb{N},\ d(x_i, 0) \leq d(x_i, x_j) + d(x_j, 0) - 1 \qquad (*)$$

One can assume that $d(x_i, 0) \geq 1$ for all i; now define $f\colon \{x_i\}_{i\geq 0} \to \mathbb{R}$ by $f(x_i) = d(x_i, 0)$. Obviously, f is a Katětov map.
If $A \subseteq \mathbb{N}$ is nonempty, define $f_A\colon \{x_i\}_{i\geq 0} \to \mathbb{R}$ as the Katětov extension of $f_{|\{x_i\colon i\in A\}}$.
Suppose now that $A \neq B$ are nonempty subsets of $\mathbb{N}$, let m be the smallest element of $A\Delta B$, and assume without loss of generality that $m \in A$.
Then one has $f_A(x_m) = d(x_m, 0)$, and $f_B(x_m) = d(x_m, x_i) + d(x_i, 0)$ for some $i \neq m$.
If $i < m$, then $(*)$ shows that $f_B(x_m) - f_A(x_m) \geq 1$; if $i > m$, then $f_B(x_m) - f_A(x_m) \geq d(x_i, 0) - d(x_m, 0) \geq 1$.
In any case, one obtains $d(f_A, f_B) \geq 1$ for any $A \neq B$, which shows that $E(\{x_i\}_{i\geq 0})$ is not separable.
Hence $E(\mathbb{R}^2, ||.||_2)$ cannot be separable either.

These two examples have something in common: in the first case, the fact that all points lie on a line gives us that $\overline{E(X,\omega)} = E(X)$; in the second case, the existence of an infinite sequence of points on which the triangle inequality is always far from being an equality enables us to prove that $E(X)$ is not separable.
It turns out that this is a general situation, and we can now characterize the spaces X such that $E(X)$ is separable:

If (X, d) is a nonempty metric space and $\varepsilon > 0$, we say that a sequence $(u_n)_{n\in\mathbb{N}}$ in X is *ε-inline* if for every $r \geq 0$ we have $\sum_{i=0}^{r} d(u_i, u_{i+1}) \leq d(u_0, u_{r+1}) + \varepsilon$.
A sequence $(u_n)_{n\in\mathbb{N}}$ in X is said to be *inline* if for every $\varepsilon > 0$ there exists $N \geq 0$ such that $(u_0, u_N, u_{N+1}, \ldots,)$ is ε-inline.

Theorem 4.5. *Let X be a Polish metric space.*
The following assertions are equivalent:
(a)$E(X) = \overline{E(X,\omega)}$.
(b)$E(X)$ is separable.
(c)$\forall\delta > 0\, \forall(x_n)\, \exists N\, \forall n \geq N\, \exists i \leq N\ d(x_0, x_n) \geq d(x_0, x_i) + d(x_i, x_n) - \delta$.
(d)Any sequence of points of X admits an inline subsequence.

Proof of Theorem 4.5.

$(a) \Rightarrow (b)$ is obvious; the proof of $\neg(c) \Rightarrow \neg(b)$ is similar to Example 4.4, so we leave it as an exercise for the interested reader.

To see that $(c) \Rightarrow (d)$, notice first that property (c) implies that, for any $\varepsilon > 0$ and any sequence $(x_n) \in X^{\mathbb{N}}$, one may extract a subsequence $(x_{\varphi(n)})$ with $\varphi(0) = 0$ such that

$$\forall n \le m \ d(x_{\varphi(0)}, x_{\varphi(n)}) + d(x_{\varphi(n)}, x_{\varphi(m)}) \le d(x_{\varphi(0)}, x_{\varphi(m)}) + \varepsilon \quad .$$

Then a diagonal process enables one to build the desired inline subsequence of (x_i).

It remains to prove that $(d) \Rightarrow (a)$.

For that, suppose by contradiction that some Polish metric space X has property (d), but not property (a).

Notice first that this implies that X is Heine-Borel. Indeed, assume by contradiction that there exist ε, $M > 0$ and a sequence $(x_n) \subset X^{\mathbb{N}}$ such that $\varepsilon \le d(x_n, x_m) \le M$ for all $n < m$. Then this sequence cannot have an inline subsequence.

Choose now $f \in E(X) \setminus \overline{E(X,\omega)}$, and let f_n be the Katětov extension to X of $f_{|_{B(z,n]}}$ (where z is some point in X).

Then for all $x \in X$, $n \le m$ one has $f_n(x) \ge f_m(x) \ge f(x)$; hence the sequence $(d(f_n, f))$ converges to some $a \ge 0$.

Notice that, since closed balls in X are compact, each f_n is in $\overline{E(X,\omega)}$: this proves that $a > 0$, and one has $d(f_n, f) \ge a$ for all n.

One can then build inductively a sequence $(x_i)_{i \ge 1}$ of elements of X, such that for all $i \ge 1$ $d(x_{i+1}, z) \ge d(x_i, z) + 1$ and

$$f(x_i) \le \min_{j<i} \{f(x_j) + d(x_i, x_j)\} - \frac{3a}{4}$$

Since $|f(x_i) - d(x_i, z)| \le f(z)$, one can assume, up to some extraction, that $(f(x_i) - d(x_i, z))$ converges to some $l \in \mathbb{R}$.

Now, let $\delta = \frac{a}{4}$. Property (d) tells us that we can extract from the sequence (x_i) a subsequence $(x_{\varphi(i)})$ having the additional property that

$$\forall 1 \le j \le i, \qquad d(z, x_{\varphi(i)}) \ge d(z, x_{\varphi(j)}) + d(x_{\varphi(i)}, x_{\varphi(j)}) - \delta$$

To simplify notation, we again call that subsequence (x_i).

Choose then $M \in \mathbb{N}$ such that $n \ge M \Rightarrow |f(x_n) - d(x_n, z) - l| \le \frac{\delta}{2}$.

For all $n \ge M$, we have

$f(x_M) + d(x_M, x_n) - f(x_n) = (f(x_M) - d(x_M, z) - l) - (f(x_n) - d(x_n, z) - l) + (d(x_M, z) - d(x_n, z) + d(x_M, x_n))$, so that

$f(x_M) + d(x_M, x_n) - f(x_n) \leq 2\delta = \frac{a}{2} < \frac{3a}{4}$.
This contradicts the definition of the sequence (x_i), and we are done. $\Diamond$

For future proofs, it is worth pointing out here that in the course of the proof of theorem 4.5, we proved that, if $E(X)$ is separable and $f \in E(X)$, then for any $\varepsilon > 0$ there exists a compact $K \subseteq X$ such that $d(f, k(f_{|K})) < \varepsilon$.
There is another equivalent definition of this property, which was introduced simultaneously (and independently) by N. Kalton in [6].
To explain it, we follow Kalton and say that an ordered triple of points $\{x_1, x_2, x_3\}$ are *ε-collinear* $(\varepsilon > 0)$ if $d(x_1, x_3) \geq d(x_1, x_2) + d(x_2, x_3) - \varepsilon$.
We say that a Polish space X has the *collinearity property* if for every infinite subset $A \subset X$ and every $\varepsilon > 0$ there are $x_1, x_2, x_3 \in A$ (pairwise distinct) such that $\{x_1, x_2, x_3\}$ is ε-collinear.
Using the infinite Ramsey theorem, Kalton proved in [6] that a space has the collinearity property if, and only if, every sequence admits an inline subsequence.
Therefore, we have the following result:

Corollary 4.6. *$E(X)$ is separable if, and only if, X has the collinearity property.*

It is certainly curious that, for quite different reasons, Kalton and us were led to consider the same notion. This might mean that this notion has more depth than it seems, in any case it should be investigated more thoroughly.
Therefore, it is not uninteresting to mention that in [6] Kalton provides a characterization of normed vector spaces (necessarily finite-dimensional) with the collinearity property:
First, recall that a finite-dimensional metric space is *polyhedral* if it (linearly isometrically) embeds in l_n^∞ for some n.
Then, since one proves as in example 1 that l_n^∞ has the collinearity property, we see that any polyhedral space has it .
The converse is a direct consequence of the following result of Lindenstrauss [10], quoted in [6]:
If a finite-dimensional normed space X is not polyhedral, then there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of points in X such that

$$\forall k < j \ ||x_k - x_j|| + ||x_k|| \leq ||x_j|| - 1 \ .$$

It may be worth pointing out that the Theorem 4.5(c) shows that

$$Col = \{F \in \mathcal{F}(\mathbb{U})\colon F \text{ has the collinearity property}\}$$

is a coanalytic subset of $\mathcal{F}(\mathbb{U})$(endowed with the Effros Borel structure). We do not know if it is Borel.

4.3 End of the proof of theorem 4.1

Now we are ready to finish the proof of theorem 4.1; we need to study the case of noncompact spaces with the collinearity property.
Let X be such a space; we wish to build a copy $X' \subset \mathbb{U}$ of X such that $\Phi^{X'}(\mathbb{U}) \neq E(X')$.
So, it is natural to try to build an isometric copy $X' \subset \mathbb{U}$ of X such that $\Phi^{X'}(\mathbb{U})$ is as small as possible.
To do this, we need a definition:
If X is a metric space and $\varepsilon > 0$, we say that $f \in E(X)$ is *ε-saturated* if there exists a compact $K \subset X$ such that, for any $g \in E(X)$, $g_{|_K} = f_{|_K} \Rightarrow d(f,g) \leq \varepsilon$. For convenience, we say that such a compact K *witnesses* the fact that f is ε-saturated.
We say that f is *saturated* if it is ε-saturated for all $\varepsilon > 0$; the definition is linked to our problem, since a saturated map on X is necessarily contained in $\Phi^X(\mathbb{U})$ whenever X is embedded in $\mathbb{U}$.
Simple examples of saturated maps are given by maps of the form $z \mapsto d(x,z)$, where $x \in X$ (since for any $\varepsilon > 0$ one can take $K = \{x\}$).
A more interesting example is the following: let $X = \mathbb{N}$, and $f \in E(\mathbb{N})$ be such that $f(0) = f(1) = 1/2$.
Then the triangle inequality implies that $f(n+2) = n + 3/2$ for all $n \in \mathbb{N}$, which shows that f is saturated.

The interest of those maps comes from the fact that, if X is a noncompact metric space, then there is $f \in E(X)$ which is not in the closure of all saturated maps in $E(X)$. Let us explain how to show this when X is not bounded (which is all we need to do, since we already know that X is Heine-Borel).
We pick $x_0 \in X$, and let $f(x) = d(x,x_0) + 2$. Then $f \in E(X)$, and we claim that all maps g such that $d(f,g) \leq 1$ are not 1-saturated.
Indeed, let g be such a map, and let $K \subset X$ be a compact set. Let M be some big enough constant (in a sense which we specify below), and pick some $x \in X$ such that $d(x,x_0) \geq M$. Let now $h(k) = g(k)$ for all $k \in K$, and $h(x) = g(x) - 1$. We have $|h(x) - h(k)| = h(x) - h(k)$ for all $k \in K$ if M is big enough, so $|h(x) - h(k)| \leq g(x) - g(k) \leq d(x,k)$. Also, $h(x) + h(k) = g(x) + g(k) - 1 \geq f(x) + f(k) - 3 \geq d(x,k)$. Therefore, the Katětov extension of h to X witnesses that g is not 1-saturated. Consequently, the following proposition is enough to finish the proof of Theorem 4.1:

Proposition 4.7. *Let X be a Polish metric space with the collinearity property. Then there exists an isometric copy $X' \subseteq \mathbb{U}$ of X such that $\{z \in$*

$\mathbb{U}$: $\Phi^{X'}(z)$ *is saturated* } *is dense in* $\mathbb{U}$.

We will use in the proof of Proposition 4.7 some simple properties of ϵ-saturated maps in Polish spaces with the collinearity property, which we regroup in the following technical lemma in the hope of making the proof itself clearer:

Lemma 4.8. *Let X be a Polish metric space with the collinearity property.*
(1) If $\varepsilon > 0$ and $f \in E(X)$ is not ε-saturated, then for any compact $K \subseteq X$ there is some $x \in X$ such that $f(x) + f(k) > d(x,k) + \varepsilon$ for all $k \in K$.
(2) If $f \in E(X)$ is saturated, then for any $\varepsilon > 0$ there exists a compact $K \subseteq X$ such that

$$\exists M\, \forall x \in X \;\; d(x,K) \geq M \Rightarrow \exists z \in K \;\; f(z) + f(x) \leq d(z,x) + \varepsilon.$$

(3) Let $f_n \in E(X)$ be ε_n-saturated maps such that :
- For any n there exists a compact K_n which witnesses the fact that f_n is $2\varepsilon_n$-saturated, and such that $m \geq n \Rightarrow f_{m|_{K_n}} = f_{n|_{K_n}}$.
- $\varepsilon_n \to 0$.
- $\cup K_n = X$
Then f_n converges uniformly to a saturated Katětov map f.

Proof of Lemma 4.8

(1)Since X has the collinearity property, there exists a compact set L such that $d(k(f_{|_L}), f) \leq \frac{\varepsilon}{2}$; we may assume that $K \supseteq L$.
Since f is not ε-saturated, we know that there is $g \in E(X)$ such that $g_{|_K} = f_{|_K}$ and $d(g,f) > \varepsilon$.
Thus there exists x such that $|f(x) - g(x)| > \varepsilon$.
Yet, by definition of a Katětov extension, we necessarily have that $g \leq k(f_{|_K}) \leq k(f_{|_L}) \leq f + \frac{\varepsilon}{2}$, so that $|f(x) - g(x)| > \varepsilon$ is only possible if $f(x) - g(x) > \varepsilon$, i.e $g(x) < f(x) - \varepsilon$. We must have $g(x) + g(k) \geq d(x,k)$ for all $k \in K$,which implies that $f(x) + f(k) > d(x,k) + \varepsilon$, and we are done.

(2)Let f, $\varepsilon > 0$ be as above, and K be a compact witnessing the fact that f is $\frac{\varepsilon}{2}$-saturated.
Now, pick any x such that $d(x,K) \geq M = 2\max\{f(x) \colon x \in K\} + \varepsilon$.
Suppose by contradiction that one has $f(x) + f(z) > d(z,x) + \varepsilon$ for any $z \in K$, and let g be defined on $K \cup \{x\}$ by $g_{|_K} = f_{|_K}$ and $g(x) = f(x) - \varepsilon$.
Then for any $z \in K$ we have
$|g(x) - g(z)| = |f(x) - f(z) - \varepsilon| = f(x) - f(z) - \varepsilon \leq f(x) - f(z) \leq d(x,z)$.
Also, for any $z \in K$ one has $g(x) + g(z) = f(x) + f(z) - \varepsilon > d(z,x)$.

Finally, it is obvious that $|g(z_1) - g(z_2)| = |f(z_1) - f(z_2)| \leq d(z_1, z_2) \leq f(z_1) + f(z_2) = g(z_1) + g(z_2)$ for all $z_1, z_2 \in K$.
Consequently, the Katětov extension $k(g)$ of g to X is such that $k(g)_{|K} = f_{|K}$ and $d(f, k(g)) \geq \varepsilon$, which contradicts the definition of K.

(3)Let X, f_n, ε_n and K_n be as in the statement of 4.8(3).
Then (f_n) obviously converges pointwise to some Katětov map f, and we have to show that f is saturated and the convergence is actually uniform.
To that end, let $\varepsilon > 0$ and choose N such that $2\varepsilon_N \leq \frac{\varepsilon}{2}$.
Then we have, for all $n \geq N$, that $f_{n|_{K_N}} = f_{N|_{K_N}}$, which by definition of K_N implies that $d(f_n, f_N) \leq \varepsilon_N$. But then one gets $d(f_n, f_m) \leq \varepsilon$ for any $n, m \geq N$, which proves that the convergence is uniform.
To show that f is saturated, let again $\varepsilon > 0$ and find n such that $2\varepsilon_n \leq \frac{\varepsilon}{2}$ and $d(f_n, f) \leq \frac{\varepsilon}{2}$.
Then any Katětov map g such that $g_{|K_n} = f_{|K_n} = f_{n|_{K_n}}$ has to satisfy
$d(f, g) \leq d(f, f_n) + d(f_n, g) \leq \varepsilon$. $\Diamond$

Proof of Proposition 4.7.
We again use a variation on Katětov's construction; for this we need to introduce a new definition.
If $Y \subset X$ are metric spaces, we let $E(X, Y, \omega)$ denote the set of maps $f \in E(X)$ which have a support which is contained in $Y \cup \{F\}$, where F is some finite subset of X. for instance, $E(X, \emptyset, \omega) = E(X, \omega)$ and $E(X, X, \omega) = E(X)$. The interest for us is that $E(X, Y, \omega)$ is separable if $E(Y)$ is.
We can now detail our construction: we let $X_0 = X$, and define

$$X_{i+1} = \{f \in E(X_i, X_0, \omega) \colon f_{|X_0} \text{ is saturated } \} .$$

(This makes sense since, as in section 2, we may assume, using the Kuratowski map, that $X_i \subseteq X_{i+1}$).
As usual, we let Y denote the completion of $\cup X_i$, and need only prove that Y is finitely injective to conclude the proof.
For that, it is enough to show that $\cup X_i$ is finitely injective; take then $\{x_1, \dots, x_n\} \subseteq X_p$ (for some $p \geq 0$) and $f \in E(\{x_1, \dots, x_n\})$.
We need to find a map $f \in E(X_p, X_0, \omega)$ which takes the prescribed values on $x_1, \dots, x_n$ and whose restriction to X_0 is saturated, since this will belong to X_{p+1} and have the desired distances to $x_1, \dots, x_n$.
To achieve this, we use the following lemma:

Lemma 4.9. *Let $x_1, \dots, x_n \in X_p$, $f \in E(\{x_1, \dots, x_n\})$.*
Let also $f' \in E(X_p, X_0, \omega)$ and $\varepsilon > 0$ be such that $f'(x_i) = f(x_i)$ for all i,

and $f'_{|X_0}$ *is not* ε*-saturated.*
Then, for any compact $K \subset X_0$*, there exists* $g \in E(X_p, X_0, \omega)$ *such that*

$$\forall i = 1, \ldots, n \; g(x_i) = f(x_i), \; g_{|K} = f'_{|K} \text{ and } \exists x \in X_0 \backslash K \; g(x) \le f'(x) - \frac{\varepsilon}{2}. \qquad (*)$$

Proof.
To simplify notation below, fix some point $z_0 \in K$.
Since $f'_{|X_0}$ is not ε-saturated, lemma 4.8(1) show that we can find $y_1 \in X_0 \setminus K$ such that $f'(y_1) + f'(z) > d(y, z) + \varepsilon$ for all $z \in K \cap X_0$. Letting $K_1 = B(z_0, 2d(z_0, y_1))$ we can apply the same process and find y_2, and so on.
It is not hard to see that one can indefinitely continue this process, and one can thus build a sequence (y_n) of elements of X_0 such that $d(y_n, z_0) \to +\infty$, an increasing sequence of compact sets (K_i) such that $K_0 = K$, $\cup K_i = X_p$, and

$$\forall i \ge 1 \; \forall z \in K_{i-1} \cap X_0 \; f'(y_i) + f'(z) > d(y_i, z) + \varepsilon \; .$$

Claim: If one cannot find a map g as in $(*)$, then there exists I such that

$$\forall i \ge I \; \exists k_i \; f'(y_i) + f(x_{k_i}) < d(x_{k_i}, y_i) + \frac{\varepsilon}{2} \; . \qquad (**)$$

Proof. By contradiction, assume that for all I there exists $i \ge I$ such that $f'(y_i) + f'(x_k) \ge d(x_k, y_i) + \frac{\varepsilon}{2}$ for all $k = 1, \ldots, n$.
Choose I such that $d(y_I, z_0) \ge \max\{f'(z) \colon z \in K_0\} + \frac{\varepsilon}{2}$, $f'(y_i) \ge f'(z)$ for all $z \in K_0$ and $i \ge I$, $K_I \supseteq B(z_0, 2\mathrm{diam}(K_0)]$, then find $i \ge I$ as above.
Define a map g on $\{x_k\}_{k=1,\ldots,n} \cup K \cup \{y_i\}$ by $g(y_i) = f'(y_i) - \frac{\varepsilon}{2}$, $g(x) = f'(x)$ elsewhere.
By choice of i and since $f'(y_i) + f'(z) \ge d(y, z) + \frac{\varepsilon}{2}$ for all $z \in K_0$, we see that g is Katětov, and that its Katětov extension $k(g)$ to X_p is such that $k(g)(x_i) = f(x_i)$, $k(g)_{|K} = f'_{|K}$ and $k(g)(y_i) \le f'(y_i) - \frac{\varepsilon}{2}$.
This concludes the proof of the claim.

Up to some extraction, we may assume that $k_i = k$ for all $i \ge I$. By definition of X_p, we know that the restriction to X_0 of the map $d(x_k, .)$ is saturated, so lemma 4.8(2) shows that there exists J such that

$$\forall j > J \, \exists z \in K_J \cap X_0 \;\; d(x_k, z) + d(x_k, y_j) \le d(z, y_j) + \frac{\varepsilon}{4} \; .$$

Combining this with $(**)$, we obtain, for $j > \max(I, J)$, that there exists $z \in K_J \cap X_0 \subseteq K_{j-1} \cap X_0$ such that $f'(y_j) + f(x_k) + d(x_k, z) \le d(z, y_j) + \frac{\varepsilon}{2} + \frac{\varepsilon}{4}$.

This in turn implies that $f'(y_j) + f'(z) < d(z, y_j) + \varepsilon$, which contradicts the definition of the sequence (y_i). $\Diamond$

We are now ready to move on to the last step of the proof of proposition 4.7: First, pick $\{x_1, \ldots, x_n\} \subseteq X_p$ (for some $p \geq 0$) and $f \in E(\{x_1, \ldots, x_n\})$.
We wish to obtain $g \in E(X_p, X_0, \omega)$ such that $g(x_i) = f(x_i)$ for all i, and $g_{|X_0}$ is saturated.
Letting $\varepsilon_0 = \inf\{\varepsilon > 0\colon k(f)_{|X_0} \text{ is } \varepsilon-\text{saturated }\}$, we only need to deal with the case $\varepsilon_0 > 0$.

Let $L_0 \subset X_0$ be a compact set witnessing the fact that $k(f)_{|X_0}$ is $2\varepsilon_0$-saturated, and choose $z_0 \in L_0$; lemma 4.9 shows that there exists $f_1 \in E(X_p, X_0, \omega)$ such that $f_{1_{|L_0}} = k(f)_{|L_0}$, $f_1(x_i) = f(x_i)$ for $i = 1, \ldots, n$ and $z_1 \in X_0 \setminus L_0$ such that $f_1(z_1) \leq \min\{k(f)(z) + d(z, z_1)\colon z \in L_0\} - \frac{\varepsilon_0}{2}$.
Again, let $\varepsilon_1 = \inf\{\varepsilon > 0\colon f_{1_{|X_0}} \text{ is } \varepsilon-\text{saturated }\}$: if $\varepsilon_1 = 0$ we are finished, so assume it is not, let $X_0 \supseteq L_1 \supseteq B(z_0, \operatorname{diam}(L_0) + d(z_0, z_1)) \cap X_0$ be a compact set witnessing the fact that $f_{1_{|X_0}}$ is $2\varepsilon_1$-saturated and apply the same process as above to $(f_1, L_1, \varepsilon_1)$.
Then we obtain $z_2 \notin L_1$ and $f_2 \in E(X_p, X_0, \omega)$ such that $f_2(x_i) = f(x_i)$ for $i = 1, \ldots, n$, $f_{2_{|L_1}} = f_{1_{|L_1}}$ and $f_2(z_2) \leq \min\{f_1(z) + d(z, z_2)\colon z \in L_1\} - \frac{\varepsilon_1}{2}$.

We may iterate this process, thus producing a (finite or infinite) sequence $(f_m) \in E(X_p, X_0, \omega)$ who has (among others) the property that
$f_m(x_i) = f(x_i)$ for all m and $i = 1, \ldots, n$; the process terminates in finite time only if some $f_{m_{|X_0}}$ is saturated, in which case we have won.
So we may focus on the case where the sequence is infinite: then the construction produces a sequence of Katětov maps (f_m) whose restriction to X_0 is ε_m-saturated, an increasing sequence of compact sets (L_m) such that $\cup L_m = X_0$ and witnessing that $f_{m_{|X_0}}$ is $2\varepsilon_m$-saturated, and points $z_m \in L_m \setminus L_{m-1}$ such that
$f_m(z_m) \leq \min\{f_{m-1}(z) + d(z, z_m)\colon z \in L_{m-1}\} - \dfrac{\varepsilon_{m-1}}{2}$.
If 0 is a cluster point of (ε_n), passing to a subsequence if necessary, we may apply lemma 4.8(3) and thus obtain a map $h \in E(X_0 \cup \{x_1, \ldots, x_n\})$ such that $h(x_i) = f(x_i)$ for all $i = 1, \ldots, n$ and $h_{|X_0}$ is saturated; then its Katětov extension to X_p has the desired properties.
Therefore, we only need to deal with the case when there exists $\alpha > 0$ such that $\varepsilon_n \geq 2\alpha$ for all n; we will show by contradiction that this never happens.
To simplify notation, let $A = \{x_1, \ldots, x_n\} \cup X_0$. Since the sequence (L_m) exhausts X_0, $(f_n)_{|A}$ converges pointwise to some $h \in E(A)$ such that $h(z_m) = f_m(z_m)$ for all m.

Up to some extraction, we may assume, since X has the collinearity property, that $d(z_0, z_m) + d(z_m, z_{m+1}) \leq d(z_0, z_{m+1}) + \frac{\alpha}{2}$ for all m.
Also we know that $h(z_{m+1}) \leq h(z_m) + d(z_m, z_{m+1}) - \alpha$.
The two inequalities combined show that $h(z_{m+1}) - d(z_{m+1}, z_0) \leq h(z_m) - d(z_m, z_0) - \frac{\alpha}{2}$.
This is clearly absurd, since if it were true the sequence $(h(z_m) - d(z_m, z_0))$ would have to be unbounded, whereas we have necessarily $h(z_m) - d(z_m, z_0) \geq -h(z_0)$.
This is enough to conclude the proof. $\diamondsuit$

Remark. If one applies the construction above to $X_0 = (\mathbb{N}, |.|)$, one obtains a countable set $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathbb{U}$ such that $d(x_n, x_m) = |n - m|$ for all $n,\ m$ and

$$\forall z \in \mathbb{U}\, \forall \varepsilon > 0\, \exists n, m \in \mathbb{N}\ d(x_n, z) + d(z, x_m) \leq |n - m| + \varepsilon.$$

In particular, $\{x_n\}$ is an isometric copy of $\mathbb{N}$ which is not contained in any isometric copy of $\mathbb{R}$.

References

[1] H.Becker and A.S Kechris, *The Descriptive Set Theory of Polish Group Actions*, London Math. Soc. Lecture Notes Series, 232, Cambridge University Press (1996).

[2] P.J Cameron and A.M Vershik, *Some isometry groups of Urysohn space*, preprint (2004).

[3] S. Gao and A.S Kechris, *On the classification of Polish metric spaces up to isometry*, Memoirs of Amer. Math. Soc., 766, Amer. Math. Soc. (2003).

[4] M. Gromov, *Metric Structures for Riemannian and Non-Riemannian spaces.* Birkauser, pp 78-85(1998).

[5] G.E Huhunaišvili, *On a property of Uryson's universal metric space*(Russian), Dokl. Akad. Nauk. USSR (N.S), 101, pp 332-333 (1955).

[6] N. Kalton, *Extending Lipschitz maps in $C(K)$ spaces*, preprint (2005).

[7] M. Katětov, *On universal metric spaces*, Proc. of the 6th Prague Topological Symposium (1986), Frolik (ed). Helderman Verlag Berlin, pp 323-330 (1988).

[8] A.S Kechris, *Classical descriptive set theory*, Springer-Verlag (1995).

[9] A.S Kechris, *Actions of Polish groups and Classification Problems*, Analysis and Logic, London Math. Soc. Lecture Notes Series,Cambridge University Press (2000).

[10] J. Lindenstrauss, *Extensions of compact operators* , Mem. Amer. Math. Soc. 48, p 112 (1964).

[11] V. Pestov, *Dynamics of infinite-dimensional groups and Ramsey-type phenomena*, Publicações dos Colóquios de Matemática, IMPA, Rio de Janeiro, to appear (2005).

[12] P.S Urysohn, *Sur un espace métrique universel*, Bull. Sci. Math 51 (1927), pp 43-64 and 74-96.

[13] V.V Uspenskij, *On the group of isometries of the Urysohn universal metric space*, Comment. Math. Univ. Carolinae, 31(1) (1990), pp 181-182.

[14] V.V Uspenskij, *Compactifications of topological groups*, Proc. of the 9th Prague Topological Symposium (2001), pp 331-346.

[15] V.V Uspenskij *The Urysohn Universal Metric Space is homeomorphic to a Hilbert Space* , Topology Appl. 139(1-3) (2004) pp 145-149.

[16] A.M Vershik, *The universal Urysohn space, Gromov metric triples and random metrics on the natural numbers*, Russ. Math. Surveys, 53(5) (1998), pp 921-928.

Equipe d'Analyse Fonctionnelle, Université Paris 6
Boîte 186, 4 Place Jussieu, Paris Cedex 05, France.
e-mail: melleray@math.jussieu.fr
Tel : (+33) 1 44 27 61 50
Fax: (+33) 1 44 27 25 55