BAIRE CATEGORY METHODS : SOME APPLICATIONS TO THE STUDY OF AUTOMORPHISM GROUPS OF COUNTABLE HOMOGENEOUS STRUCTURES

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ABSTRACT. These notes are an expanded write-up of two lectures given in Lyon in the spring of 2011, during a special semester on model theory. We discuss some applications of Baire category methods to the study of automorphism groups of homogeneous countable first-order structures.

1. BACKGROUND ON POLISH SPACES AND BAIRE CATEGORY

We begin by discussing quickly some basic notions from descriptive set theory. A very good general reference is the book by Kechris [K]. For the theory of Polish group actions, the curious reader may also consult [G]. For the parallels between measure and category, I cannot recommend Oxtoby's gem of a book [O] too highly.

Definition 1.1. A *Polish space* is a topological space (X, τ) whose topology admits a countable basis and is induced by a complete metric.

Note that, by definition, a Polish space has a countable dense subset. I will often use in the following the fact that any topological space with a countable basis has the *Lindelöff* property, i.e from any open cover of X one may extract a countable subcover.

Example. $\mathbf{R}$, $\mathbf{R}^n$ are examples of Polish spaces, as is any compact metric space, and any locally compact separable metric space.

Example. Let X be a Polish space; then $X^{\mathbf{N}}$, endowed with the product topology, is a Polish metric space. Indeed, if (U_i) is a basis of the topology of X, sets of the form $\{\bar{x} \in X^{\mathbf{N}} \colon \forall i \leq N \ x_i \in U_{n_i}\}$ form a countable basis of the topology of $X^{\mathbf{N}}$. Also, if d is a complete metric compatible with the topology of X, then the metric d_∞ defined by

$$d_\infty(\bar{x}, \bar{y}) = \sum_{i=0}^{\infty} 2^{-i} \min(d(x_i, y_i), 1)$$

is a complete metric compatible with the product topology on $X^{\mathbf{N}}$.

A sequence $\bar{x}_n$ converges to $\bar{x}$ in $X^{\mathbf{N}}$ if and only if $x_n(i)$ converges to $x(i)$ for all i.

The spaces $\{0,1\}^{\mathbf{N}}$ and $\mathbf{N}^{\mathbf{N}}$, where $\{0,1\}$ and $\mathbf{N}$ are endowed with the discrete topology, are of particular importance in descriptive set theory.

Theorem 1.2 (Baire). *Let X be a Polish space, and (O_n) be a sequence of dense open subsets of X. Then $\cap O_n$ is dense in X.*

Notation. Let X be a Polish space. Then $A \subseteq X$ is G_δ if $A = \cap O_n$ where each O_n is open. If each (O_n) is dense, we say that A is dense G_δ; note that the Baire category theorem says that a dense G_δ set is indeed dense.

We say that $A \subseteq X$ is *comeager* if A contains a dense G_δ; A is meager if the complement of A is comeager, i.e if A is contained in a countable union of closed subsets of X with empty interior.

Remark. If (X, τ) is a Polish space, O is an open subset of X, then O, with the induced topology, is Polish. Indeed, if d is a complete metric on X, then d_O defined on O by

$$d_O(x,y) = d(x,y) + \left| \frac{1}{d(x, X \setminus O)} - \frac{1}{d(y, X \setminus O)} \right|$$

is a complete metric compatible with the topology induced by τ on O.

This is particularly useful, since one may "localize " the Baire category theorem, based on the observation that if X is Polish, $A \subseteq X$ is comeagre and O is open in X, then $A \cap O$ is comeagre in O. We will use this frequently without mention.

Notation. If X is Polish, we will use the notation

$$\forall^* x \in X \; A(x)$$

to signify that $\{x \colon x \text{ satisfies } A(x)\}$ is comeager in X. Baire's category theorem says that the two assertions

$$\forall n \in \mathbf{N} \, \forall^* x \in X \; A_n(x)$$

and

$$\forall^* x \in X \; \forall n \in \mathbf{N} \; A_n(x)$$

are equivalent in a Polish space.

The notation $\forall^* x$ must be understood as meaning "for almost all x"; thus the Baire category notions give us a notion of largeness - a subset A of a Polish space X is large if almost every element of X belongs to A, i.e if A is comeager - and, dually, a notion of smallness (meager sets). This should remind the reader of full measure sets/nullsets in measure theory; as in that context, a countable intersection of large sets is large, dually, a countable union of small sets is small. Just as in the measure-theoretic context, not every set need behave nicely with regard to this notion of largeness: one has to introduce a concept of measurability adapted to our context.

Definition 1.3. Let X be a Polish space. A subset A of X is *Baire-measurable* if there exists an open subset O of X such that $A \Delta O$ is meager.

An important point here is that, if A is Baire-measurable and not meagre, then A is comeagre in some nonempty open subset O of X.

Proposition 1.4. *Let X be a Polish space. Baire-measurable subsets of X form a σ-algebra.*

An immediate corollary of this is that any Borel set is Baire-measurable, since any open set is obviously Baire-measurable and Borel sets form the smallest σ-algebra containing the open sets.

We will need something stronger.

Proposition 1.5. *Let X, Y be Polish spaces, $f \colon X \to Y$ a continuous map, and $B \subseteq X$ a Borel subset. Then $f(B)$ is Baire-measurable.*

Let us illustrate the analogy between measure and category further with the following analogue of the Fubini theorem.

Theorem 1.6 (Kuratowski–Ulam)**.** *Let X, Y be Polish spaces, and $A \subseteq X \times Y$ be a Baire-measurable subset. Then the following properties are equivalent.*

(1) *A is comeagre in $X \times Y$.*
(2) *$\{x\colon A_x$ is comeagre in $Y\}$ is comeagre in X, where $A_x = \{y \in Y\colon (x,y) \in A\}$.*

Note that, symbolically, this theorem says that the assertions

$$\forall^*(x,y) \in X \times Y\; A(x,y)$$

and

$$\forall^* x \in X\, \forall^* y \in Y\; A(x,y)$$

are equivalent, for a Baire-measurable subset of $X \times Y$.

Proof. To familiarize ourselves with Baire-category proofs, we sketch the proof of the fact that (1) implies (2) in the particular case when A is open (the general case then follows rather easily, and the converse implication is also a good exercise on Baire category). So, let A be a dense open subset of $X \times Y$. Then A_x is open in Y for all $x \in X$, so we need to prove that

$$\{x \in X\colon A_x \text{ is dense in } Y\}$$

is comeagre in X. To that end, fix a countable basis (U_n) of the topology of Y, and assume w.l.o.g that each U_n is nonempty. Saying that A_x is dense in Y is the same as saying that $A_x \cap U_n$ is nonempty for all n, thus we need to prove that

$$\forall^* x\; \forall n\; A_x \cap U_n \neq \emptyset.$$

Via the Baire theorem, this is equivalent to

$$\forall n\; \forall^* x\; A_x \cap U_n \neq \emptyset.$$

Equivalently, denoting by $\pi\colon X \times Y \to X$ the projection map, we need to show that $\pi(A \cap (X \times U_n))$ is comeager in X for all n. It is not hard to check that the projection of any open subset of $X \times U_n$ is open in X, and the projection of any dense subset of $X \times U_n$ is dense in X. Since $A \cap (X \times U_n)$ is open dense in $X \times U_n$, we obtain that $\pi(A \cap (X \times U_n))$ is open dense in X, finishing the proof in that particular case.

□

2. Polish groups

Definition 2.1. A *topological group* is a pair (G, τ) where G is a group, τ is a topology on G, and the group operations $(g,h) \mapsto gh$ and $g \mapsto g^{-1}$ are continuous (this is equivalent to simply requiring that $(g,h) \mapsto gh^{-1}$ is continuous).

Definition 2.2. A *Polish group* is a topological group whose topology is Polish.

Example. $(\mathbf{R}, +)$, $(\mathbf{R}^n, +)$, Lie groups, locally compact separable metrizable groups are Polish groups.

The following is the main example of Polish group we will be discussing.

Example. Denote by S_∞ tthe group of all bijections of $\mathbf{N}$. The topology is given by the pointwise convergence topology for the discrete topology on $\mathbf{N}$; explicitly, a basis of the topology is given by subsets of the form

$$\{\sigma \in S_\infty \colon \sigma(i_1) = j_1, \dots, \sigma(i_n) = j_n\}.$$

Of particular importance is the fact that a basis of open neighborhoods of *id* is given by the family (V_n) where

$$V_n = \{\sigma \in S_\infty \colon \forall i \leq n \; \sigma(i) = i\}.$$

These subsets are actually subgroups, so that S_∞ has a basis of neighborhoods of *id* consisting of open subgroups.

To see that there is a complete metric inducing the topology of S_∞, one may for instance set, for all $\sigma, \tau \in S_\infty$,

$$\delta(\sigma,\tau) = 2^{-n} \text{ where } n = \min\{i \colon \sigma(i) \neq \tau(i)\}$$

(with $\min(\emptyset) = +\infty$) and then check that the metric d defined by $d(\sigma,\tau) = \delta(\sigma,\tau) + \delta(\sigma^{-1},\tau^{-1})$ works.

This leads to plenty of other examples: indeed, any closed subgroup of S_∞ is a Polish group in its own right. This class of groups is intimately related to model theory: if $\mathcal{M}$ is a first-order countable structure, with universe $\mathbf{N}$, its automorphism group $\mathrm{Aut}(\mathcal{M})$ is naturally a subgroup of S_∞ (here and below, I always assume that languages contain $=$, so that any automorphism of a first-order structure induces a bijection of its universe).

Actually, $\mathrm{Aut}(\mathcal{M})$ is a *closed* subgroup of S_∞. Let us for instance check that, if R is a n-ary relation symbol of the language of $\mathcal{M}$, then $\{\sigma \in S_\infty \colon \sigma \text{ preserves } R\}$ is a closed subgroup of S_∞. This is the same as proving that the set of all σ that do not preserve R is open; to show this, pick σ in that set, and let $i_1, \dots, i_n, j_1, \dots, j_n$ be such that $\sigma(i_k) = j_k$, $\mathcal{M} \models R(i_1, \dots, i_n)$ and $\mathcal{M} \models \neg R(j_1, \dots, j_n)$ (the other case is dealt with in the same way). Then, the set of τ such that $\tau(i_k) = j_k$ for all $k \in \{1, \dots, n\}$ is open in S_∞, contains σ, and any element of that set does not preserve R.

Using a similar proof for function symbols, we see that $\mathrm{Aut}(\mathcal{M})$ is a closed subgroup of S_∞ for any first-order structure with universe $\mathbf{N}$. Allow us to mention that the converse is true without giving a proof.

Theorem 2.3. *Let G be a closed subgroup of S_∞. Then, there exists a first-order countable homogeneous relational structure $\mathcal{M}$ such that G is isomorphic (as a topological group) to* $\mathrm{Aut}(\mathcal{M})$.

Now, the plan is to address the following type of question: when $\mathcal{M}$ is a "nice" structure, what does $\mathrm{Aut}(\mathcal{M})$ (as an abstract group) remember about $\mathcal{M}$? Is $\mathcal{M}$ encoded in $\mathrm{Aut}(\mathcal{M})$? Can we reconstruct $\mathcal{M}$ from $\mathrm{Aut}(\mathcal{M})$?

A good reference for the rest of these notes is MacPherson's survey [M]. Let us recall the following theorem.

Theorem 2.4 (Ahlbrandt–Ziegler). *Let $\mathcal{M}$, $\mathcal{N}$ be ω-categorical countable structures. Then $\mathcal{M}$ and $\mathcal{N}$ are bi-interpretable if and only if* $\mathrm{Aut}(\mathcal{M})$ *and* $\mathrm{Aut}(\mathcal{N})$ *are isomorphic as topological groups.*

Thus, our focus shifts to the following issue: knowing that $\mathrm{Aut}(\mathcal{M})$ and $\mathrm{Aut}(\mathcal{N})$ are isomorphic as abstract groups, under what assumptions can we obtain that

$\mathrm{Aut}(\mathcal{M})$ and $\mathrm{Aut}(\mathcal{N})$ are isomorphic as topological groups? Can we recover the topological structure of $\mathrm{Aut}(\mathcal{M})$ from its algebraic structure?

We now discuss Baire-category conditions that ensure that a homomorphism from a Polish group to another is continuous. These should make apparent some ideas that we will use later, if time permits.

3. Continuity of morphisms between Polish groups

The method of proof of the following theorem, due to Pettis, will be essential for the rest of this lecture.

Theorem 3.1 (Banach). *Let G, H be Polish groups, and $\varphi\colon G \to H$ a homomorphism. Assume that φ is Baire-measurable, i.e $\varphi^{-1}(O)$ is Baire-measurable for any open subset O of H. Then φ is continuous.*

Note that the assumption of the theorem is true when φ is Borel measurable. We postpone the proof of this theorem for the moment. Let us make note of a nice corollary.

Corollary 3.2. *Let G, H be Polish groups, $\varphi\colon G \to H$ an abstract group isomorphism, and assume that φ is Baire-measurable. Then φ is a topological group isomorphism.*

Proof. By Banach's theorem, we know that $\varphi\colon G \to H$ is continuous. Let $\psi\colon H \to G$ denote the inverse of φ. Then, for any open subset O of G, we have $\psi^{-1}(O) = \varphi(O)$, so by Proposition 1.5, $\psi^{-1}(O)$ is Baire-measurable, and then Banach's theorem implies that ψ is continuous. Hence φ is a homeomorphism and we are done. $\square$

To prove Banach's theorem, we will first establish an important lemma.

Notation. When G is a group and A is a subset of G, we set $A^{-1} := \{a^{-1}\colon a \in A\}$. Similarly, if $A, B \subseteq G$ we let $A \cdot B := \{ab\colon a \in A, b \in B\}$.

Lemma 3.3 (Pettis). *Let G be a Polish group. For $A \subseteq G$, define*

$$O(A) = \bigcup \{O \text{ open in } X\colon A \text{ is comeagre in } O\}.$$

Then, for any $A, B \subseteq G$, one has

$$O(A) \cdot O(B) \subseteq A \cdot B.$$

Note that $O(A)$ is open, and by Lindelöff's property it is actually a *countable* union of open sets in which A is comeagre, so A is comeagre in $O(A)$. The converse is true exactly when A is Baire-measurable.

We also note that the definition of $O(A)$ makes it straightforward to check that $O(A^{-1}) = O(A)^{-1}$ and, for all $g \in G$, $O(g \cdot A) = g \cdot O(A)$ (because inversion and left-translation are homeomorphisms of G).

Proof. Pick $g \in O(A) \cdot O(B)$. Then we have $O(A) \cap g \cdot O(B)^{-1} \neq \emptyset$, equivalently, $O(A) \cap O(g \cdot B^{-1}) \neq \emptyset$. Note that A is comeagre in $O(A)$, hence in $O(A) \cap O(g \cdot B^{-1})$; similarly, $g \cdot B^{-1}$ is comeagre in $O(g \cdot B^{-1})$, hence in $O(A) \cap O(g \cdot B^{-1})$.

Hence $A \cap g \cdot B^{-1}$ is comeagre in the nonempty open set $O(A) \cap O(g \cdot B^{-1})$, thus $A \cap g \cdot B^{-1}$ is nonempty, so $g \in A \cdot B$. $\square$

Theorem 3.4. *Let G be a Polish group, and $A \subseteq G$ be a Baire-measurable, non-meagre subset of G. Then 1_G belongs to the interior of $A \cdot A^{-1}$.*

Proof. By assumption, $O(A)$ is nonempty, and it is open by definition; hence $O(A^{-1}) = O(A)^{-1}$ is also nonempty open, and $O(A) \cdot O(A^{-1})$ is also open, and contains 1_G. Pettis' lemma asserts that $O(A) \cdot O(A^{-1}) \subseteq A \cdot A^{-1}$, and we are done. □

Corollary 3.5. *Let G be a Polish group, and H be a Baire-measurable, non-meager subgroup. Then H is clopen in G.*

Proof. By Pettis' theorem, 1 belongs to the interior of $H \cdot H^{-1} = H$. Hence H has nonempty interior, which easily implies since H is a subgroup that H is open. Then $G \setminus H = \bigcup_{g \notin H} gH$ is also open, hence H is closed, and we are done. □

Proof of Banach's theorem. Since φ is a homomorphism, we only need to show that φ is continuous at 1_G, i.e that for any open set $V \subseteq H$ containing 1_H there is an open set $U \subseteq G$ containing 1_G and such that $\varphi(U) \subseteq V$.

Pick such a V, and use the continuity of group operations to find an open neighborhood W of 1_H such that $W \cdot W^{-1} \subseteq V$. Note that $\varphi(G)$ is covered by translates of $\varphi(G) \cap W$, which is open in $\varphi(G)$, hence the Lindelöff property implies that there exist a sequence (g_n) of elements of G such that

$$\varphi(G) = \bigcup_n \varphi(g_n) \cdot (\varphi(G) \cap W)$$

From this we obtain

$$G = \bigcup_n g_n \cdot \varphi^{-1}(W)$$

and this implies that $\varphi^{-1}(W)$ is not meagre in G (otherwise G itself would be meagre, contradicting the Baire category theorem).

Thus we may apply Pettis' theorem to $\varphi^{-1}(W)$, and obtain an open set $U \ni 1_G$ such that $U \subseteq \varphi^{-1}(W) \cdot (\varphi^{-1}(W))^{-1}$. We are done, since

$$\varphi(U) \subseteq W \cdot W^{-1} \subseteq V.$$

□

Recall that we only really care, in our setting, about homomorphisms with values in S_∞. A way to ensure that such homomorphisms are continuous is given by the following property.

Definition 3.6. Let G be a Polish group. We say that G has the *small index property* if any subgroup of G with at most countable index is open.

Remark. Usually, for the small index property one asks that groups of index stricly less than the continuum be open. Given the restricted scope of these notes, the property above is sufficient for our purposes and will make some proofs simpler, so we use nonstandard terminology to simplify the exposition (also, to be completely honest, I know of no example where one property is satisfied and the other is not!).

Theorem 3.7. *Let G be a Polish group with the small index property. Then any homomorphism from G to S_∞ is continuous.*

Proof. Let $\varphi \colon G \to S_\infty$. As above, we need to show that φ is continuous at 1_G; given that *id* has a basis of open neighborhoods made up of open subgroups, it is enough to show that $\varphi^{-1}(V)$ is open for any open subgroup V of S_∞.

By the Lindelöff property, V must have at most countable index (S_∞ is covered by left-translates of V, which are open, so S_∞ is covered by countably many left-translates of V).

Hence $\varphi^{-1}(V)$ is a subgroup of G with at most countable index ($G/\varphi^{-1}(V)$ embeds in S_∞/V), so $\varphi^{-1}(V)$ is open since G has the small index property, and we are done. □

An immediate corollary of this, and Ahlbrandt-Ziegler's theorem 2.4, is that if $\mathcal{M}, \mathcal{N}$ are ω-categorical structures, $\mathrm{Aut}(\mathcal{M})$ has the small index property and $\mathrm{Aut}(\mathcal{M}), \mathrm{Aut}(\mathcal{N})$ are isomorphic as abstract groups, then $\mathcal{M}$ and $\mathcal{N}$ are bi-interpretable.

Remark. If you believe in the axiom of choice, it should be pointed out that there exist discontinuous homomorphisms between Polish groups. For instance, $(\mathbf{R}, +)$ and $(\mathbf{R}^2, +)$ are vector spaces of the same dimension (continuum) over $\mathbf{Q}$, hence they are isomorphic as abstract groups. However, they are not homeomorphic, hence any isomorphism between $(\mathbf{R}, +)$ and $(\mathbf{R}^2, +)$ must be discontinuous.

4. Small index property and ample generics

We refer to MacPherson [M] for information about Fraïssé classes and the related terminology. Whenever $\mathcal{K}$ is a Fraïssé class, we will denote by $\mathbf{K}$ its limit. To simplify the exposition we assume from now on that all our Fraïssé classes are defined using a relational language.

Definition 4.1. A Fraïssé class $\mathcal{K}$ has the *free amalgamation property* if, whenever $B_1, B_2 \in \mathcal{K}$, $f_i \colon A \to B_i$ are embeddings, there exists $D \in \mathcal{K}$ and embeddings $g_i \colon B_i \to D$ such that $g_1 \circ f_1 = g_2 \circ f_2$ and, in addition, for each relation symbol R of the language of $\mathcal{K}$, no tuple of D which satisfies R meets both $g_1(B_1) \setminus g_1 f_1(A)$ and $g_2(B_2) \setminus g_2 f_2(A)$ (since the language contains $=$, this implies that $g_1(B_1) \cap g_2(B_2) = g_1 f_1(A)$, the so-called *disjoint amalgamation property*).

Definition 4.2. A Fraïssé class $\mathcal{K}$ has the *extension property* if, for all $A \in \mathcal{K}$, there exists $B \in \mathcal{K}$ such that $A \leq B$ and any *partial* automorphism of A extends to a *global* automorphism of B.

Example. Hrushovski [H] showed that the class of finite graphs has the extension property. This was the start of investigations of this property, which is typically very hard to establish (well, not for the class of all finite sets, for instance...). A more general version of this is Herwig's result [H] implying that, if $\mathcal{K}$ is a Fraïssé class in a finite relational language, $\mathcal{K}$ has the free amalgamation property and $\mathcal{K}$ is closed under weak substructures, then $\mathcal{K}$ has the extension property.

In the spirit of these notes, it is particularly interesting to analyze how the extension property of a Fraïssé class $\mathcal{K}$ is reflected in the properties of the automorphism group of $\mathbf{K}$. The following proposition is more or less implicit in [HHLS], and explicitly stated in [H] in a slightly different way. As stated below, I believe it appeared for the first time in [KR].

Proposition 4.3. *Let $\mathcal{K}$ be a Fraïssé class with limit $\mathbf{K}$, and $G = \mathrm{Aut}(\mathcal{K})$. Then the following conditions are equivalent:*

(1) *$\mathcal{K}$ has the extension property.*
(2) *There exists a countable chain of* compact *subgroups $G_1 \leq G_2 \leq \dots$ whose union is dense in G.*

Notation. If G is a group and $\bar{g} \in G^n$, we denote by $\langle \bar{g} \rangle$ the subgroup generated by $g_1, \dots, g_n$.

Proof. Let us first show that (1) implies (2). Fix $n < \omega$. Consider

$$F_n = \{\bar{g} \in G^n \colon \forall x \in \mathbf{K}\ \langle \bar{g} \rangle \cdot x \text{ is finite}\}$$

Note that $F_n = \{\bar{g} \in G^n \colon \langle \bar{g} \rangle \text{ is relatively compact}\}$ (if $\bar{g} \in F_n$ then $\langle \bar{g} \rangle$ embeds in a product of finite permutation groups, which is compact; conversely, the orbit of any point under a compact subgroup of S_∞ must be a compact subset of $\mathbf{N}$, hence finite). We claim that F_n is dense G_δ.

To see that F_n is G_δ, it is enough to show that for all $x \in \mathbf{K}$ the set $\Omega_x = \{\bar{g} \in G^n \colon \langle \bar{g} \rangle \cdot x \text{ is finite}\}$ is open; indeed, if $\bar{g} \in \Omega_x$, set $A = \langle \bar{g} \rangle \cdot x$. Then A is finite, hence the set of $\bar{h}$ coinciding with $\bar{g}$ on A is open, contains $\bar{g}$, and any such $\bar{h}$ belongs to Ω_x.

Next, we need to show that each Ω_x is dense. To that end, pick a nonempty open subset O of G^n, and assume that there exist partial automorphisms $p_1, \dots, p_n$ with domains $A_1, \dots, A_n$, range $B_1, \dots, B_n$ such that O is made up of all $\bar{g}$ such that g_i extends p_i for all i. Let C be the union of all the $A_i's$ and $B_i's$ and $\{x\}$; this is a structure of $\mathcal{K}$ (recall that the language is relational). Apply the extension property of $\mathcal{K}$ to find $D \in K$ in which C embeds and automorphisms $g_1, \dots, g_n$ of D extending $p_1, \dots, p_n$. Then use homogeneity to assume that $C \subseteq D \subseteq \mathbf{K}$ and to extend $g_1, \dots, g_n$ to automorphisms of $\mathbf{K}$ (still denoted by $g_1, \dots, g_n$). Then $\bar{g} \in O$ by construction, and $\langle \bar{g} \rangle \cdot x \subseteq D$ is finite, so $\bar{g} \in O \cap \Omega_x$, showing that Ω_x is dense.

Thus F_n is dense G_δ; it follows that

$$\{\bar{g} \in G^{\mathbf{N}} \colon \forall n (g_1, \dots, g_n) \in F_n\}$$

is dense G_δ in $G^{\mathbf{N}}$. An easy exercise on Baire category is that

$$\{\bar{g} \in G^{\mathbf{N}} \colon \{g_i\} \text{ is dense in } G\}$$

is also dense G_δ. Hence we may pick $\bar{g}$ in the intersection of these two sets, and define $G_n = \overline{\langle g_1, \dots, g_n \rangle}$. Then each G_n is compact and their union is dense, finishing the proof that (1) implies (2).

The converse implication is easier: pick $A \in \mathcal{K}$, and let $p_1, \dots, p_n$ denote the partial automorphisms of A. Viewing A in $\mathbf{K}$, p_i defines a nonempty open subset of G for all $i \in \{1, \dots, n\}$. Hence, we may find m such that G_m contains an element extending p_i for all i. As G_m is compact, $G_m \cdot x$ is finite for all x, so $B = G_m \cdot A$ is finite, and invariant by any element of G_m. □

The relevance of the extension property in our context is that it often implies the following property, which in turn will be seen to imply the small index property.

Definition 4.4. Let G be a Polish group. We say that G has *ample generics* if for all n there exists $\bar{g} \in G^n$ such that the *diagonal conjugacy class*

$$\{(gg_1g^{-1}, \dots, gg_ng^{-1}) \colon g \in G\}$$

is comeagre in G^n.

If such a $\bar{g}$ exists, we call it a *generic* element of G^n.

The previous definition was introduced by Hodges, Hodkinson, Lascar and Shelah [HHLS] in the context of automorphism groups of countable structures, and then studied by Kechris and Rosendal [KR] in the more general context of

Polish groups. The approach we discuss below was initiated in [HHLS] and refined in [KR].

Let us point out two easy facts about ample generics before moving on:

- If $\bar{g}$ and $\bar{h}$ are generic in G^n, then there exists $k \in G$ such that $k\bar{g}k^{-1} = \bar{h}$. Indeed, both sets $\{k\bar{g}k^{-1} \colon k \in G\}$ and $\{k\bar{h}k^{-1} \colon k \in G\}$ are comeager, hence they must intersect, so $\bar{g}$ and $\bar{h}$ have the same diagonal conjugacy class.
- If $\bar{g}$ is generic in G^n and G has ample generics, then $\forall^* h$ $(\bar{g}, h)$ is generic in G^{n+1}. Indeed, the Kuratowski-Ulam theorem applied to the set of generic elements of G^{n+1} implies that

$$\forall^* \bar{x} \in G^n \ \forall^* h \in G \ (\bar{x}, h) \text{ is generic.}$$

Thus $\{\bar{x} \in G^n \colon \forall^* h \in G \ (\bar{x}, h) \text{ is generic}\}$ is comeagre in G^n; this set is also invariant under diagonal conjugacy, so it must contain $\bar{g}$, and we are done.

Theorem 4.5. *Assume $\mathcal{K}$ is a Fraïssé class with the free amalgamation property and the extension property. Denote by $\mathbf{K}$ the Fraïssé limit of $\mathcal{K}$. Then $\mathrm{Aut}(\mathbf{K})$ has ample generics.*

Proof. Fix an integer n. Consider the set Ω of all $\bar{g} \in G^n$ such that

(1) $\forall x \in \mathbf{K}$ $\langle \bar{g} \rangle \cdot x$ is finite.

(2) For any $B \in \mathcal{K}$, any $C \in \mathcal{K}$ such that $B \leq C$ and any automorphisms $p_1, \ldots, p_n$ of C leaving B invariant and coinciding with g_i on B, there exists $\tilde{C} \subseteq \mathbf{K}$ such that $(\tilde{C}, g_1|\tilde{C}, \ldots, g_n|\tilde{C}) \cong (C, p_1, \ldots, p_n)$.

We already saw that the extension property implies that the set of elements satisfying condition (1) is dense G_δ. It is easy to see that condition (2) also defines a G_δ set. To show that it is dense, fix $B \leq C$ and automorphisms $p_1, \ldots, p_n$ of C such that $p_i(B) = B$ for all i.

Let O be a nonempty open subset of G, and assume without loss of generality that there exists $D \subseteq \mathbf{K}$ and automorphisms $q_1, \ldots, q_n$ of D such that O consists of the set of all $(g_1, \ldots, g_n)$ extending $(q_1, \ldots, q_n)$. We need to find automorphisms $g_1, \ldots, g_n$ of G extending $q_1, \ldots, q_n$ and such that either some g_i does not coincide with p_i on B, or there exists $\tilde{C}$ such that $(\tilde{C}, g_1|\tilde{C}, \ldots, g_n|\tilde{C}) \cong (C, p_1, \ldots, p_n)$.

If some q_i does not coincide with p_i on $E := D \cap B$, then any automorphisms extending $q_1, \ldots, q_n$ willl do. Otherwise, notice that E must be fixed by $q_1, \ldots, q_n$. Then, we may amalgamate freely D and C over E, and glue together p_i and q_i to obtain automorphisms g_i of E. Call this free amalgam D; using the homogeneity of $\mathbf{K}$, we may realize F inside $\mathbf{K}$ in such a way that $D \subseteq F$, and extend $g_1, \ldots, g_n$ to automorphisms of $\mathbf{K}$ extending $q_1, \ldots, q_n$. We ensured that there exists $\tilde{C}$ such that $(\tilde{C}, g_1|\tilde{C}, \ldots, g_n|\tilde{C}) \cong (C, p_1, \ldots, p_n)$, and we are done.

This shows that Ω is dense G_δ in G^n. It is not hard, using a back-and-forth argument, to show that for $\bar{g}, \bar{h} \in \Omega$ there exists $k \in G$ such that $k\bar{g}k^{-1} = \bar{h}$, proving that Ω is a single diagonal conjugacy class, hence that G has ample generics. $\square$

Note that the assumptions of the previous theorem are satisfied for example when $\mathbf{K}$ is the class of finite sets (then $G = S_\infty$), $\mathcal{K}$ is the class of finite graphs (then G is the automorphism group of the random graph), and more generally when $\mathcal{K}$ has the free amalgamation property, is closed under weak substructures and is defined over a finite relational language.

To conclude these notes, we'll apply the techniques of [HHLS] and [KR] to show that any Polish group with ample generics must have the small index property.

Below, to simplify the notation a little bit, we denote the diagonal conjugacy action of G on G^n by $*$.

Lemma 4.6. [KR] *Let G be a Polish group with ample generics, and $\bar{g} \in G^n$ a generic element. Assume that $A, B \subseteq G$ are such that A is not meager, and B is not meager in any nonempty open set.*

Then, for any open neighborhood V of 1_G, there exist $g_1 \in A$, $g_2 \in B$ such that $(\bar{g}, g_1)$ and $(\bar{g}, g_2)$ are generic, and $h \in V$ such that $h * (\bar{g}, g_1) = (\bar{g}, g_2)$.

Proof. Denote by Ω the comeagre diagonal conjugacy class in G^{n+1}. We saw, right after the definition of ample generics, that the set $C = \{h \colon (\bar{g}, h) \in \Omega\}$ is comeagre in G. Hence we may pick $h_1 \in A \cap C$.

Note that $C = \mathrm{Stab}(\bar{g}) * h_1$, where $\mathrm{Stab}(\bar{g})$ is the stabilizer of $\bar{g}$ for the diagonal conjugacy action.

Hence $\mathrm{Stab}(\bar{g}) * h_1$ is comeagre in G; as $\mathrm{Stab}(\bar{g})$ is the union of countably many translates of $\tilde{V} = V \cap \mathrm{Stab}(\bar{g})$, this implies that $\tilde{V} * h_1$ is not meagre.

As $\tilde{V} * h_1$ is Baire-measurable by Proposition 1.5, this means that $\tilde{V} * h_1$ is comeagre in some nonempty open set O, and the same must then be true of $(\tilde{V} * h_1) \cap C$. Since B is not meagre in O, we may pick $h_2 \in B \cap (\tilde{V} * h_1) \cap C$, which concludes the proof. □

Lemma 4.7 (Fundamental lemma for ample generics [KR]). *Let G be a Polish group with ample generics, (A_n), (B_n) be two sequences of subsets of X such that for all n A_n is not meager and B_n is not meager in any nonempty open set. Then there exists a continuous map $a \mapsto h_a$ from $2^{\mathbf{N}}$ to G such that, for any $a, b \in 2^{\mathbf{N}}$, $n \in \mathbf{N}$ such that $a|n = b|n$, $a(n) = 0$, $b(n) = 1$, one has $(h_a * A_n) \cap (h_b * B_n) \neq \emptyset$.*

Sketch of proof. Pick a complete metric d inducing the topology of G. Then, using Lemma 4.6, it is not hard to build sequences $(g_s)_{s \in 2^{<\omega}}$ and $(f_s)_{s \in 2^{<\omega}}$ of elements of G such that (denoting $\bar{g}_s = (g_{s|1}, \ldots, g_s)$ and $h_s = f_{s|1} \ldots f_s$, $h_\emptyset = 1_G$):

(1) $\bar{g}_s$ is generic in $G^{|s|}$, where $|s|$ denotes the length of s.
(2) $g_{s \frown 0} \in A_{|s|}$ and $g_{s \frown 1} \in B_{|s|}$.
(3) $f_{s \frown 0} = 1_G$ and $d(h_s, h_s f_{s \frown 1}) < 2^{-|s|}$.
(4) $f_{s \frown 1} * \bar{g}_{s \frown 1} = \bar{g}_{s \frown 0}$.

From the third condition above, it is easy to see that for all $a \in 2^\omega$, $h_{a|n}$ converges to an element of G, denoted by h_a.

The last two conditions imply that, for any s and any extension t of $s \frown 0$ one has $h_t * g_{s \frown 0} = h_s g_{s \frown 0}$, and for any extension t of $s \frown 1$ one has $h_t * g_{s \frown 1} = h_{s \frown 1} * g_{s \frown 1} = h_s * g_{s \frown 0}$.

Hence, for any a, b, n as in the statement of the lemma, we have, letting $s = a|n = b|n$, that $h_a * g_{s \frown 0} = h_s * g_{s \frown 0} = h_b * g_{s \frown 1}$. Since the second condition ensured that $g_{s \frown 0} \in A_n$ and $g_{s \frown 1} \in B_n$, we are done. □

Theorem 4.8 ([HHLS], [KR]). *Let G be a Polish group with ample generics. Then G has the small index property.*

Proof. Let H be a subgroup of G with at most countable index. Since G is the union of at most countably many left-translates of H, H cannot be meager. If $G \setminus H$ is meager in some nonempty open set O, then H is comeager in O, so Pettis' Lemma implies that $H \cdot H^{-1} = H$ has nonempty interior, thus H is open and we are done.

In other words, we know that H is not meagre in G, and we may assume that $(G \setminus H)$ is not meagre in any nonempty open subset of G; thus, we may apply Lemma 4.7 with $A_n = H$ and $B_n = G \setminus H$ or all n. This gives us a map $a \mapsto h_a$ from $2^{\mathbf{N}}$ to G as in the statement of Lemma 4.7.

Let now a, b be distinct elements of $2^{\mathbf{N}}$, let n be the smallest integer such that $a(n) \neq b(n)$ and assume w.l.o.g that $a(n) = 0$, $b(n) = 1$. Then Lemma 4.7 ensures that $(h_a * H) \cap (h_b * (G \setminus H)) \neq \emptyset$, in other words $(h_b^{-1} h_a * H) \cap (G \setminus H) \neq \emptyset$. Hence $h_b^{-1} h_a$ does not belong to H, so the map $a \mapsto h_a H$ is an injection from $2^{\mathbf{N}}$ to G/H, contradicting the fact that H has at most countable index.

□

Before ending these notes, we should point out that, using the fundamental lemma for ample generics above, Kechris and Rosendal [KR] actually prove the stronger result that any Polish group G with ample generics must have the *automatic continuity property*, i.e any homomorphism from G to a Polish group is necessarily continuous. They also discuss various other consequences of ample generics, and present examples of groups with ample generics beyond those that were mentioned here. Finally, they provide a characterization of Fraïssé classes $\mathcal{K}$ such that the automorphism group of $\mathbf{K}$ has ample generics.

Also, to toot my own horn a little bit and to make a link with continuous logic, which was discussed in Itaï ben Yaacov's talks, it may be worth pointing out that at the moment no example of a Polish group having ample generics and which is *not* a closed subgroup of S_∞ is known. To tackle more general Polish groups, ideas from continuous logic adapt relatively well (e.g any Polish group is isomorphic, as a topological group, to the automorphism group of a homogeneous Polish metric structure), leading to the notion of a "Polish topometric group" and a related notion of ample generics, which are discussed in [BYBM] and [BYM].

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