UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN Fall 2006
Math 380 Group G1

Graded Homework X .
Due Friday, November 17.

1. Compute the surface integral $\iint_S x^2y^2z\,d\sigma$, where S is the portion of the cone of equation $x^2+y^2=z^2$ where $0 \leq z \leq 1$.
Correction. Let's use cartesian coordinates here; one has $z=\sqrt{x^2+z^2}$, and the projection of our cone onto the (x,y)-plane is the disk D of equation $x^2+y^2=1$.
We have $d\sigma = \sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}\,dxdy = \sqrt{2}dxdy$, and we get

$$\iint_S x^2y^2z\,d\sigma = \iint_D x^2y^2\sqrt{x^2+z^2}\sqrt{2}\,dxdy = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^4\sin^2(\theta)\cos^2(\theta)\sqrt{2}.r.r\,drd\theta =$$

$$\int_{\theta=0}^{2\pi}\frac{\sin^2(2\theta)}{4}.\frac{\sqrt{2}}{7}d\theta = \int_0^{4\pi}\frac{\sqrt{2}\sin^2(u)}{8.7}\,du = \frac{2\pi\sqrt{2}}{56} = \frac{\pi\sqrt{2}}{28}\ .$$

2. Compute the surface integral $\iint_S xz\,d\sigma$, where S is the surface parameterized by $\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = \theta \end{cases}$ $0 \leq r \leq R$, $0 \leq \theta \leq \pi$.
Correction. This time, we are using a parametric description; using the same notations as in class, we have $\frac{\partial P}{\partial r} = \begin{pmatrix}\cos(\theta)\\ \sin(\theta)\\ 0\end{pmatrix}$, and $\frac{\partial P}{\partial \theta} = \begin{pmatrix}-r\sin(\theta)\\ r\cos(\theta)\\ 1\end{pmatrix}$, which yields $\frac{\partial P}{\partial r}\times\frac{\partial P}{\partial \theta} = \begin{pmatrix}\sin(\theta)\\ -\cos(\theta)\\ r\end{pmatrix}$, $\|\frac{\partial P}{\partial r}\times\frac{\partial P}{\partial \theta}\| = \sqrt{1+r^2}$; so

$$\iint_S xz\,d\sigma = \int_{r=0}^{R}\int_{\theta=0}^{\pi} r\cos(\theta)\theta\sqrt{1+r^2}\,drd\theta = \int_{r=0}^{R} r\sqrt{1+r^2}\,dr\int_{\theta=0}^{\pi}\theta\cos(\theta)\,d\theta =$$

$$\left[\frac{(1+r^2)^{3/2}}{3}\right]_{r=0}^{R}\Big[\theta\sin(\theta)+\cos(\theta)\Big]_0^{\pi} = \frac{2}{3}\big(1-(1+R^2)^{3/2}\big)\ .$$

3. Compute the surface integral $\iint_S (x+y^2+z^3)d\sigma$, where S is the boundary of the cube given by the inequalities $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.
Correction. This cube (as any other) has six faces, so we actually need to compute six different surface integrals. The choice of normal doesn't matter here since we are computing the integral of a scalar function; the integrals are :

$$(\text{on the face } x=0,\ 0 \leq y \leq 1,\ 0 \leq z \leq 1)\ \int_{y=0}^{1}\int_{z=0}^{1}(y^2+z^3)dzdy = \int_{y=0}^{1}(y^2+\frac{1}{4})dy = \frac{7}{12}\ ;$$

$$(\text{on the face } x=0,\ 0 \leq y \leq 1,\ 0 \leq z \leq 1)\ \int_{y=0}^{1}\int_{z=0}^{1}(1+y^2+z^3)dzdy = 1+\frac{7}{12} = \frac{19}{12}\ ;$$

$$(\text{on the face } y=0,\ 0 \leq x \leq 1,\ 0 \leq z \leq 1)\ \int_{x=0}^{1}\int_{z=0}^{1}(x+z^3)dzdx = \int_{x=0}^{1}(x+\frac{1}{4})dx = \frac{3}{4}\ ;$$

$$(\text{on the face } y=1,\ 0 \leq x \leq 1,\ 0 \leq z \leq 1)\ \int_{x=0}^{1}\int_{z=0}^{1}(x+1+z^3)dzdx = 1+\frac{3}{4} = \frac{7}{4}\ ;$$

(on the face $z=0$, $0\le x\le 1$, $0\le y\le 1$) $\displaystyle\int_{x=0}^{1}\int_{y=0}^{1}(x+y^2)dydx = \int_{x=0}^{1}(x+\frac{1}{3})dx = \frac{5}{6}$;

(on the face $z=1$, $0\le x\le 1$, $0\le y\le 1$) $\displaystyle\int_{x=0}^{1}\int_{y=0}^{1}(x+y^2+1)dydx = 1+\frac{5}{6}=\frac{11}{6}$.

Finally, we get that the value of the integral is $\displaystyle\frac{7}{12}+\frac{19}{12}+\frac{3}{4}+\frac{7}{4}+\frac{5}{6}+\frac{11}{6}=\frac{22}{3}$.

4. Let H be the portion of hyperboloid parameterized by $\begin{cases} x=u\cos(v)-\sin(v) \\ y=u\sin(v)+\cos(v) \\ z=u \end{cases}$, $0\le u\le 1$, $0\le v\le 2\pi$.

(a) Show that the surface area of H is $\displaystyle 2\pi\int_0^1\sqrt{2u^2+1}\,du$.

(b) Define $\mathrm{sh}(t)=\dfrac{e^t-e^{-t}}{2}$, $\mathrm{ch}(t)=\dfrac{e^t+e^{-t}}{2}$. Show that $1+\mathrm{sh}^2(t)=\mathrm{ch}^2(t)$. Use this to compute the area of H.

Correction.(a) Using again the same notations as in class, we have $\dfrac{\partial P}{\partial u}=\begin{pmatrix}\cos(v)\\ \sin(v)\\ 1\end{pmatrix}$ and $\dfrac{\partial P}{\partial v}=\begin{pmatrix}-u\sin(v)-\cos(v)\\ u\cos(v)-\sin(v)\\ 0\end{pmatrix}$.

This yields $\dfrac{\partial P}{\partial u}\times\dfrac{\partial P}{\partial v}=\begin{pmatrix}\sin(v)-u\cos(v)\\ -u\sin(v)-\cos(v)\\ u\cos^2(v)-\sin(v)\cos(v)+u\sin^2(v)+\sin(v)\cos(v)\end{pmatrix}=\begin{pmatrix}\sin(v)-u\cos(v)\\ -u\sin(v)-\cos(v)\\ u\end{pmatrix}$. We finally obtain

$$\|\frac{\partial P}{\partial u}\times\frac{\partial P}{\partial v}\|=\sqrt{u^2+\sin^2(v)-2u\sin(v)\cos(v)+u^2\cos^2(v)+u^2\sin^2(v)+2u\sin(v)\cos(v)+\cos^2(v)}=\sqrt{1+2u^2}\ .$$

Thus, the surface area of H is $\displaystyle\int_{u=0}^{1}\int_{v=0}^{2\pi}\sqrt{1+2u^2}dvdu=2\pi\int_{u=0}^{1}\sqrt{1+2u^2}du$.

(b) One has $1+\mathrm{sh}^2(t)=1+\dfrac{e^{2t}+e^{-2t}-2}{4}=\dfrac{e^{2t}+e^{-2t}+2}{4}=\left(\dfrac{e^t+e^{-t}}{2}\right)^2=\mathrm{ch}^2(t)$.

The point of this is that one has $\sqrt{1+\mathrm{sh}^2(t)}=\mathrm{ch}(t)$. Given the integral that we wish to compute, it is then tempting to set $\sqrt{2}u=\mathrm{sh}(t)$, so that $\sqrt{2}du=\mathrm{ch}(t)dt$. The bounds of our integral under this change of variable become : α such that $\mathrm{sh}(\alpha)=0$, i.e $\alpha=0$, and β such that $\mathrm{sh}(\beta)=\sqrt{2}$. We will compute β later on ; for now, just remember how it is defined. Our formula for the surface area of H becomes

$$\mathrm{Area}(H)=2\pi\int_{u=0}^{\beta}\sqrt{1+\mathrm{sh}^2(t)}\frac{\mathrm{ch}(t)}{\sqrt{2}}dt=\frac{2\pi}{\sqrt{2}}\int_0^{\beta}\mathrm{ch}^2(t)dt=\frac{2\pi}{\sqrt{2}}\int_0^{\beta}\frac{\mathrm{ch}(2t)+1}{2}dt;$$

$$\mathrm{Area}(H)=\frac{\pi}{\sqrt{2}}\left[\frac{\mathrm{sh}(2t)}{2}+t\right]_{t=0}^{\beta}=\frac{\pi}{\sqrt{2}}(\frac{\mathrm{sh}(2\beta)}{2}+\beta)=\frac{\pi}{\sqrt{2}}(\mathrm{sh}(\beta)\mathrm{ch}(\beta)+\beta)=\frac{\pi}{\sqrt{2}}(\sqrt{2}\sqrt{3}+\beta)=\pi\sqrt{3}+\frac{\pi\beta}{\sqrt{2}}\ .$$

Remark To compute the integral above, we used the formulas $\mathrm{ch}(2u)=2\mathrm{ch}^2(u)-1$, and $\mathrm{sh}(2u)=2\mathrm{sh}(u)\mathrm{ch}(u)=2\mathrm{sh}(u)\sqrt{1+\mathrm{sh}^2(u)}$ (these may be checked using the definitions of the functions, and are part of the so-called hyperbolic geometry)

Now, we still need to compute β ; in general, given a real number u, how can we find u such that $\mathrm{sh}(t)=u$? In mathematical terms, we need to compute the inverse function of sh ; of course you can use the corresponding touch on your hi-tech calculator/mobile phone (?), but it would be good to be able to compute it using a pen and paper (how quaint). We want to solve $e^t-e^{-t}=2u$; setting $T=e^t$, this equation becomes $T-\dfrac{1}{T}=2u$, or $T^2-2uT-1=0$, which is pretty easily solvable : the solutions are $T=u\pm\sqrt{1+u^2}$. Since the T we're looking for is equal to e^t, it must be a positive number ; so the solution we are looking for is $T=u+\sqrt{1+u^2}$, and we eventually obtain $t=\ln(T)=\ln(u+\sqrt{1+u^2})$. Thus, $\beta=\ln(\sqrt{2}+\sqrt{1+2})=\ln(\sqrt{2}+\sqrt{3})$, and we finally obtain

$$\mathrm{Area}(H)=\pi\sqrt{3}+\frac{\pi}{\sqrt{2}}\ln(\sqrt{2}+\sqrt{3})\ .$$