Graded Homework XI.
Correction.

1. Compute the following surface integrals :
(a) $\iint_S \vec{F}\cdot\vec{n}\,d\sigma$, where S is the triangle with vertices $(1,0,0)$, $(0,2,0)$, $(0,0,3)$, $F(x,y,z)=(xy,y+z,z-x)$ and the normal vector is pointing away from the origin.
(b) $\iint_S (x^2+y-z)\,d\sigma$, where S is the portion of the cylinder of equation $x^2+y^2=1$ that is below the plane $z=1$, and above the plane $x+z=0$.

Correction. First, we need to find a description of the surface S ; for that, we need to find an equation of the plane in which S lives. The vectors $(-1,2,0)$ and $(0,-2,3)$ are parallel to S, hence S is normal to their cross product $(6,3,2)$. Thus an equation for the plane containing S is given by $6(x-1)+3y+2z=0$ (because this plane is normal to $(6,3,2)$ and contains the point $(1,0,0)$), in other words $6x+3y+2z=6$. If one uses the equation $z=-3x-\frac{3}{2}y+3$, then one obtains $\vec{F}\cdot\vec{n}d\sigma=(xy,y+z,z-x)\cdot(3,\frac{3}{2},1)dxdy$ (note the choice of normal vector). Hence

$$\vec{F}(x,y,z)\cdot\vec{n}\,d\sigma=(3xy+\frac{3}{2}(y+z)+z-x)dxdy=(3xy-\frac{17}{2}x-\frac{9}{4}y+\frac{15}{2})dxdy\ .$$

Now we still need to determine the projection of the triangle onto the (x,y) plane ; the good thing is that the projection of a triangle is still a triangle, whose edges are the projections of the edges of the edges of the original triangle (this is due to the fact that the projection is linear). Thus the corresponding domain in the (x,y)-plane is the triangle with edges $(1,0)$, $(0,2)$ and $(0,0)$. Thus our integral I eventually reduces to

$$I=\int_{x=0}^{1}\int_{y=0}^{2-2x}(3xy-\frac{17}{2}x-\frac{9}{4}y+\frac{15}{2})dxdy=\int_{x=0}^{1}\big(\frac{3}{2}x(2-2x)^2-\frac{17}{2}x(2-2x)-\frac{9}{8}(2-2x)^2+\frac{15}{2}(2-2x)\big)dx$$

$$I=\int_{x=0}^{1}\big(6x(1-x)^2-17x(1-x)-\frac{9}{2}(1-x)^2+15(1-x)\big)dx=\int_{x=0}^{1}\big(6x^4-12x^3-11x+17x^2\big)dx-\frac{3}{2}+\frac{15}{2}$$

$$I=\frac{6}{5}-3-\frac{11}{2}+\frac{17}{3}+6=\frac{36+90-165+170}{30}=\frac{131}{30}\ .$$

Remark. Pay attention to the fact that, since we want to use a representation of the surface by an equation $z=f(x,y)$ then one has

$$\iint_S \vec{F}\cdot\vec{n}\,d\sigma=\pm\iint_{R_{x,y}}\vec{F}\cdot(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},-1)\,dxdy\ .$$

The choice of sign ± 1 depends of the orientation chosen for the problem (here our normalvector must point outwards) ; also, notice that instead of computing $\vec{F}\cdot n$ by himself, we conisdered $\vec{F}\cdot\vec{n}\,d\sigma$ (doing things this way avoids common mistakes).

(b) The top of our surface is the disk of equation $z=1$, $x^2+y^2\le 1$, thus our integral on that portion of S is

$$I_1=\iint_{x^2+y^2\le 1}(x^2+y-1)\,dxdy=\int_{r=0}^{1}\int_{\theta=0}^{2\pi}r(r^2\cos^2(\theta)+r\sin(\theta)-1)d\theta dr=\int_{r=0}^{1}(\pi r^3-2\pi r)\,dr=\frac{\pi}{4}-\pi=-\frac{3\pi}{4}\ .$$

The bottom part is obtained when $x+z=0$ (or $z=-x$), $x^2+y^2\le 1$. We then have $d\sigma=\sqrt{1+0+1}dxdy=\sqrt{2}dxdy$ (pay attention to this, it is very common to make a mistake at that step...). The projection of this set on the (x,y) plane is the disk $x^2+y^2\le 1$, and the integral on the bottom surface is

$$I_2=\sqrt{2}\iint_{x^2+y^2\le 1}(x^2+y+x)\,dxdy=\sqrt{2}\int_{r=0}^{1}\int_{\theta=0}^{2\pi}r(r^2\cos^2(\theta)+r\sin(\theta)+r\cos(\theta))d\theta dr=\sqrt{2}\int_{r=0}^{1}\pi r^3\,dr=\frac{\pi\sqrt{2}}{4}\ .$$

To finish our computation, we need to find a way to describe the portion of cylinder that lies between the two planes. Well, we may take θ and z as parameters ; as usual for a cylinder we obtain $d\sigma=d\theta dz$. The problem is now to determine the domain for θ and z. We have $-x \leq z \leq 1$ on this cylinder, which yields $-\cos(\theta) \leq z \leq 1$. Looking at a picture, we see that θ can take any value between 0 and 2π; for any given θ, z can vary between $-\cos(\theta)$ and 1. Thus the surface integral on the side part of the surface is

$$I_3 = \int_{\theta=0}^{2\pi} \int_{z=-\cos(\theta)}^{1} (\cos^2(\theta) + \sin(\theta) - z)\, dz d\theta = \int_{\theta=0}^{2\pi} \Big((\cos^2(\theta) + \sin(\theta))(1 + \cos(\theta)) - \frac{1}{2} + \frac{\cos^2(\theta)}{2} \Big) d\theta$$

$$I_3 = 3\frac{\pi}{2} - \pi = \frac{\pi}{2} \ .$$

Eventually, we obtain that $\displaystyle\iint_S (x^2 + y - z)\, d\sigma = \frac{\pi}{4}(\sqrt{2} - 1)$.

2. Compute the following line integrals :
(a) $\displaystyle\int_C yzdx + zxdy + xydz$, where C is the arc of helix $x = R\cos(t)$, $y = R\sin(t)$, $z = \dfrac{t}{2\pi}$, with $0 \leq t \leq 2\pi$ and C is oriented in the direction of increasing t.
(b) $\displaystyle\int_C xdx + ydy + (x + y - 1)dz$ where C is the straight line segment from $(1,1,1)$ to $(2,3,4)$.
Correction. (a) The curve is already parametrized and oriented, so we obtain

$$I = \int_{t=0}^{2\pi} R^2\Big(-\sin^2(t)\frac{t}{2\pi} + \frac{t}{2\pi}\cos^2(t) + \frac{\cos(t)\sin(t)}{2\pi} \Big) dt = \int_{t=0}^{2\pi} \frac{R^2}{2\pi} \Big(t\cos(2t) + \frac{\sin(2t)}{2} \Big) dt \ .$$

An integration by parts (for the left-hand part, the integal of the right-hand part is clearly 0) yields

$$I = \frac{R^2}{2\pi} \left(\Big[-\frac{t\sin(2t)}{2} \Big]_{t=0}^{2\pi} + \int_0^{2\pi} \frac{\sin(2t)}{2} dt \right) = 0 \ .$$

(b) The curve can be parametrized by taking $x = 1 + t$, $y = 1 + 2t$, $z = 1 + 3t$, where $0 \leq t \leq 1$. Using this parametrization, our integral J becomes

$$\int_{t=0}^{1} (1 + t + (1 + 2t)2 + (1 + 3t).3)dt = \int_{t=0}^{1} (6 + 14t)dt = 6 + 7 = 13 \ .$$

3. Compute in two different ways the integral $\displaystyle\iint_S (\vec{F} \cdot \vec{n})\, d\sigma$ (following the definition of a surface integral, and using the Divergence theorem) :
(a) $\vec{F}(x,y,z) = (x,y,z)$ and S is the surface of the cube of equation $0 \leq x \leq l$, $0 \leq y \leq l$, $0 \leq z \leq l$.
(b) $\vec{F}(x,y,z) = (x^2, y^2, z^3)$ and S is the surface of the quarter-cylinder of equation $x^2 + y^2 = R^2$, $0 \leq x, y$, and $0 \leq z \leq H$.
(c) $\vec{F}(x,y,z) = (xz, yz, 3z^2)$ and S is the surface bounded by the paraboloid of equation $z = x^2 + y^2$ and the plane $z = 1$.
Correction. (a) Using the divergence theorem, we have, since $\operatorname{div}(\vec{F}) = 3$, that $\displaystyle\iint_S (\vec{F} \cdot \vec{n})\, d\sigma$ is equal to 3 times the volume of the cube, i.e $\displaystyle\iint_S (\vec{F} \cdot \vec{n})\, d\sigma = 3l^3$.
Of course, it is a bit longer to compute the integral using the definition of a surface integral; as usual, one divides the surface cube into 6 faces and computes each surface integral, enjoying the fact that each face is parallel to a coordinate axis, which makes computations easier.

$$(x = 0, 0 \leq y \leq l, 0 \leq z \leq l). \text{ Then } \vec{n} = (-1,0,0), so \iint_S (\vec{F} \cdot \vec{n})\, d\sigma = -\int_{y=0}^{l} \int_{z=0}^{l} 0\, dydz = 0 \ ;$$

$$(x = l, 0 \leq y \leq l, 0 \leq z \leq l). \text{ Then } \vec{n} = (1,0,0), so \iint_S (\vec{F} \cdot \vec{n}) d\sigma = \int_{y=0}^{l} \int_{z=0}^{l} l dydz = l^3 \ ;$$

Similarly, when $y = 0$, the unit normal is $(0,-1,0)$ and the corresponding integral is 0; when $y = l$ the unit normal is $(0,1,0)$ and the integral is l^3; when $z = 0$ the integral is 0, and when $y = l$ the integral is l^3. Overall,

we obtain $\iint_S (\vec{F}\cdot\vec{n})\, d\sigma = 3l^3$, and this is what we expected.
(b) This time we have $\text{div}(\vec{F}) = 2x+2y+3z^2$; in cylindrical coordinates the quarter cylinder corresponds to the domain $0 \le r \le R$, $0 \le \theta \le \frac{\pi}{2}$, $0 \le z \le H$, so our integral is

$$I = \int_{z=0}^{H}\int_{r=0}^{R}\int_{\theta=0}^{\pi/2} (2r\cos(\theta)+2r\sin(\theta)+3z^2) r d\theta dr dz = \int_{z=0}^{H}\int_{r=0}^{R} (4r^2 + \frac{3z^2 r\pi}{2}) dr dz$$

$$I = \int_{z=0}^{H} (\frac{4}{3}R^2 + \frac{3\pi z^2 R^2}{4}) dz = \frac{4}{3}HR^3 + \frac{\pi}{4}R^2H^3 .$$

To apply the divergence theorem, one has to be careful and remember that it only applies to **closed** surfaces, meaning that one has to take into account the top, bottom and **side** parts of the cylinder. On the top one has $\vec{n} = (0,0,1)$, hence $\vec{F}\cdot\vec{n} = z^3 = H^3$, so the integral on that part is equal to $3H^2$ multiplied by the area of a quarter disk of radius R, i.e $\frac{\pi H^3 R^2}{4}$.
On the bottom part of the cylinder one has $\vec{n} = (0,0,-1)$ hence $\vec{F}\cdot\vec{n} = 0$ and the integral on that part is 0. Well, we still need to describe the side part ; there are two rectangles (we're only dealing with a quarter-cylinder !) : one of them is $x=0$, $0 \le y \le R$, $0 \le z \le H$, and one sees easily that $\vec{F}$ is parallel to the surface here so the contribution to the integral is 0. Similarly the side $y=0$, $0 \le x \le R$, $0 \le z \le H$ doesn't contribute to the integral. Still, **these sides must be taken into account** : for another vector field they could contribute to the integral. You have to learn how to make rough sketches of the volumes/surfaces you are dealing with.
The last remaining part is cylindrical, and a parametric description $x = R\cos(\theta)$, $y = R\sin(\theta)$, $z = z$ (with parameters z, θ) is the way to go. We obtain $\frac{\partial P}{\partial \theta} \times \frac{\partial P}{\partial z} = (R\cos(\theta), R\sin(\theta), 0)$; this is pointing outwards so we don't have to change the sign, and we get $\vec{F}\cdot\vec{n}\, d\sigma = (R^3\cos^2(\theta) + R^3\sin^2(\theta)) d\theta dz$. Eventually, we get that the integral on that side of our surface is

$$R^3 \int_{\theta=0}^{\pi/2}\int_{z=0}^{H} (\cos^3(\theta)+\sin^3(\theta))\, dz d\theta = R^3 H \int_{\theta=0}^{\pi/2} \big(\cos(\theta) - \cos(\theta)\sin^2(\theta) + \sin(\theta) - \sin(\theta)\cos^2(\theta)\big) d\theta = \frac{4}{3}R^3 H$$

Overall, we obtain that the surface integral we were looking for is $\frac{4}{3}R^3H + \frac{\pi}{4}R^2H^3$, and this is the same result as what the Divergence Theorem gave us. (at this point it is considered acceptable to celebrate).
(c) Here we have $\text{div}(\vec{F}) = 8z$, so the divergence theorem gives

$$\iint_S (\vec{F}\cdot\vec{n})\, d\sigma = \int_{z=0}^{1} \Big(\iint_{x^2+y^2 \le z} 8z\, dxdy \Big) dz = \int_{z=0}^{1} 8\pi z^2\, dz = \frac{8\pi}{3} .$$

The surface is simpler that the one in the question before ; the side part is given by the formula $z = x^2+y^2$, which gives a normal in the direction of $(2x, 2y, -1)$ (this is the outer normal), hence $\vec{F}\cdot\vec{n} d\sigma = (2x^2z + 2y^2z - 3z^2)dxdy = -(x^2+y^2)^2 dxdy$ (recall that on this part of the surface we have $z = x^2+y^2$). Going to polar coordinates, this integral is $\int_{r=0}^{1}\int_{\theta=0}^{2\pi} -r^4 r d\theta dr$ (don't forget the jacobian determinant !), thus the integral one the side part of the surface is $-\frac{\pi}{3}$. The top part of the surface is kind enough to be parallel to a coordinate plane ; there one has $\vec{F}\cdot\vec{n} = 3z^2 = 3$, so the integral there is 3 times the area of the top part (a disk of radius 1), i.e 3π. Overall, we get $\iint_S (\vec{F}\cdot\vec{n})\, d\sigma = 3\pi - \frac{\pi}{3} = \frac{8\pi}{3}$ (yay !).

4. (a) Verify Stokes's theorem for the vector field $F(x,y,z) = (z^2+x, -y^2, z-y)$, if C is the boundary of the square $0 \le x \le 1$, $0 \le y \le 1$ oriented counterclockwise, and the capping surface of C is a cube.
(b) Let C be the intersection of the hyperbolic paraboloid of equation $z = y^2 - x^2$ and of the cylinder of equation $x^2+y^2 = 1$. Find a parametrization of C, and verify Stokes's Theorem for the vector field $F(x,y,z) = (x^2y, \frac{1}{3}x^3, xy)$ and the curve C (you will need to produce your own surface !)
Correction. (a) One can parametrize C using four different parametrizations, one for each straight segment :

$x = t, y = 0$, $0 \le t \le 1$, $x = 1$, $y = t$, $0 \le t \le 1$, $x = 1 - t$, $y = 1$, $0 \le t \le 1$ and $x = 0$, $y = 1 - t$, $0 \le t \le 1$; on the whole curve one has $z = 0$. Thus the circulation of $\vec{F}$ along the curve is

$$\int_{t=0}^{1} (t - 0 + 0)dt + \int_{t=0}^{1} (0 - t^2 + 0)dt + \int_{t=0}^{1} ((1-t).(-1) + 0 + 0)dt + \int_{t=0}^{1} (0 + (1-t)^2 + 0)dt = 0 \ .$$

The capping surface of C is, say, the cube above C (or rather, the five faces of it that are not the face $z = 0$) ; the curl of $\vec{F}$ is $\text{curl}(\vec{F}) = (-1, 2z, 0)$ and the five integrals are rather easily computed : on the face $x = 1$ $\text{curl}(\vec{F}) \cdot \vec{n} = -1$, one the face $x = 0$ one has $\text{curl}(\vec{F}) \cdot \vec{n} = -1$, so these faces cancel out. On the face $y = 1$ on has $\text{curl}(\vec{F}) \cdot \vec{n} = 2z$, and on the face $y = 0$ one has $\text{curl}(\vec{F}) \cdot \vec{n} = -2z$, so these two faces cancel out too. Finally, on the face $z = 1$ one has $\text{curl}(\vec{F}) \cdot \vec{n} = 0$, so the total surface integral of $\text{curl}(\vec{F}) \cdot \vec{n}$ is indeed 0.
(b) Let's first find a parametrization of the curve : since it is the intersection of the cylinder of equation $x^2 + y^2 = 1$ and the paraboloid $z = y^2 - x^2$, we get $z = 1 - 2x^2$ and we see that we could use x as a parameter if we accepted to divide the curve in two parts ($y \ge 0$ and $y < 0$; the problem is that y is not a function of x) ; however this would lead to ugly computations, and most likely a change of variables to compute the integrals. A more clever way of parametrizing the curve here is to use $x = \cos(t)$, $y = \sin(t)$, $z = y^2 - x^2 = \sin^2(t) - \cos^2(t) = -\cos(2t)$; for instance $0 \le t \le 2\pi$ (notice that we have chosen an orientation of the curve).
The line integral is then not too hard to compute, provided that one remembers how to deal with integrals of trigonometric functions :

$$I = \int F_x dx + F_y dy + F_z dz = \int_{t=0}^{2\pi} \big(\cos^2(t)\sin(t)(-\sin(t)) + \frac{\cos^3(t)}{3} \cdot \cos(t) + \cos(t)\sin(t)(4\sin(2t))\big)dt$$

$$I = \int_0^{2\pi} \big(\frac{3}{4}\sin^2(t) + \cos^2(t)(1 - \sin^2(t))\big)dt = \int_0^{2\pi} \big(\frac{2}{3}\sin^2(2t) + \frac{1}{3}\cos^2(t)\big)dt = \int_0^{4\pi} \frac{1}{3}\sin^2(u)du + \int_0^{2\pi} \frac{1}{3}\cos^2(t)dt \ .$$

Eventually, we get $I = \dfrac{2\pi}{3} + \dfrac{\pi}{3} = \pi$. Pay attention to how important it was to know basic trigonometric formulas to compute the line integral (calculators will not be allowed for the final....).
Now, let's try to check Stokes's theorem for this line integral ; we need to come up with a surface of which C is the boundary ; the simplest choice is probably the portion of cylinder that lies below C and above the plane $z = -1$. Given the orientation that we chose for C, we need to orient that surface with the outer normal. An easy computation yields $\text{curl}(\vec{F}) = (x, -y, 0)$ so on the bottom part of the surface $\text{curl}(\vec{F})$ is parallel to the surface, thus the flow of F outside that part is 0. We need to parametrize the side of the surface : given what we did for the curve, it is natural to set again $x = \cos(t)$, $y = \sin(t)$, $z = z$; the domain is $0 \le t \le 2\pi$ and $-1 \le z \le y^2 - x^2 = -\cos(2t)$. We obtain (as usual for this parametrization) $\dfrac{\partial P}{\partial t} \times \dfrac{\partial P}{\partial z} = (\cos(t), \sin(t), 0)$ and this is pointing outwards (hence this is the correct choice of orientation given the orientation of C that we used above). One has $\text{curl}(\vec{F}) \cdot \vec{n}\, d\sigma = (\cos^2(t) - \sin^2(t))dzdt = -\cos(2t)dzdt$. Hence,

$$\iint_S \text{curl}(\vec{F}) \cdot \vec{n}\ d\sigma = \int_{t=0}^{2\pi} \left(\int_{z=-1}^{-\cos(2t)} -\cos(2t)\, dz \right) dt = \int_{t=0}^{2\pi} (\cos^2(2t) - \cos(2t))dt = \pi \ .$$

We do recover the result we were expecting (which is again nothing short of miraculous).