UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN — Fall 2006
Math 380 — Group G1

Graded Homework IV
Due Friday, October 6.

1. Let $F\colon \mathbb{R}^3\backslash\{(0,0,0)\} \to \mathbb{R}^3$ be the mapping defined by $F(x,y,z)=(\frac{x}{x^2+y^2+z^2},\frac{y}{x^2+y^2+z^2},\frac{z}{x^2+y^2+z^2})$.
Let (x,y,z) be on the sphere of center 0 and radius 1 ; show in two different ways that the Jacobian matrix of F at (x,y,z) is equal to its inverse matrix.
(Hint : compute $F\circ F(x,y,z)$ and use the Chain Rule)

Correction. Set $F(x,y,z)=(X,Y,Z)$. Then $X^2+Y^2+Z^2=\dfrac{1}{x^2+y^2+z^2}$, so that
$F(F(x,y,z))=F(X,Y,Z)=(X(x^2+y^2+z^2),Y(x^2+y^2+z^2),Z(x^2+y^2+z^2))=(x,y,z)$. Thus, F is equal to its inverse function ; the Chain Rule then tells us that $JF(F(x,y,z))$ is the inverse matrix of $JF(x,y,z)$. If (x,y,z) is in the unit sphere, then $F(x,y,z)=(x,y,z)$, in other words the inverse matrix to $JF(x,y,z)$ is $JF(x,y,z)$.
To prove it in a different way, one may "simply" compute $JF(x,y,z).JF(x,y,z)$ for (x,y,z) on the unit sphere ; one has

$$JF(x,y,z)\begin{pmatrix} \frac{1}{x^2+y^2+z^2}-\frac{2x^2}{(x^2+y^2+z^2)^2} & -\frac{2xy}{(x^2+y^2+z^2)^2} & -\frac{2xz}{(x^2+y^2+z^2)^2} \\ -\frac{2xy}{(x^2+y^2+z^2)^2} & \frac{1}{x^2+y^2+z^2}-\frac{2y^2}{(x^2+y^2+z^2)^2} & -\frac{2yz}{(x^2+y^2+z^2)^2} \\ -\frac{2xz}{(x^2+y^2+z^2)^2} & -\frac{2yz}{(x^2+y^2+z^2)^2} & \frac{1}{x^2+y^2+z^2}-\frac{2z^2}{(x^2+y^2+z^2)^2} \end{pmatrix}.$$

Since on the sphere one has $x^2+y^2+z^2=1$, this becomes $JF(x,y,z)=\begin{pmatrix}1-2x^2 & -2xy & -2xz\\ -2xy & 1-2y^2 & -2yz \\ -2xz & -2yz & 1-2z^2\end{pmatrix}$.

A direct, if long, computation yields that $JF(x,y,z).JF(x,y,z)$ is equal to

$$\begin{pmatrix} (1-2x^2)^2+4x^2y^2+4x^2z^2 & -2xy(1-2x^2)-2xy(1-2y^2)+4xyz^2 & -2xz(1-2x^2)+4xy^2z-2xz(1-2z^2) \\ -2xy(1-2x^2)-2xy(1-2y^2)+4xyz^2 & 4x^2y^2+(1-2y^2)^2+4y^2z^2 & 4x^2yz-2yz(1-2y^2)-2yz(1-2z^2) \\ -2xz(1-2x^2)+4xy^2z-2xz(1-2z^2) & 4x^2yz-2yz(1-2y^2)-2yz(1-2z^2) & 4x^2z^2+4y^2z^2+(1-2z^2)^2 \end{pmatrix}$$

Let us explain how to simplify the terms in the first row (the other ones are similar) :
$(1-2x^2)^2+4x^2y^2+4x^2z^2=1+4x^2(x^2-1+y^2+z^2)=1$ because $x^2+y^2+z^2=1$;
$-2xy(1-2x^2)-2xy(1-2y^2)+4xyz^2=-2xy(1-2x^2+1-2y^2-2x^2)=0$ for the same reason ;
$-2xz(1-2x^2)+4xy^2z-2xz(1-2z^2)=-2xz(1-2x^2-2y^2+1-2z^2)=0$.

Checking all the other values, one eventually gets that $JF(x,y,z).JF(x,y,z)=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$; this shows that $JF(x,y,z)$ indeed is equal to its own inverse matrix if (x,y,z) lies on the unit sphere.

2. Assume $F\colon (u,v)\mapsto F(u,v)$ is a continuously differentiable function from $\mathbb{R}^2$ to $\mathbb{R}$ such that $F(0,0)=0$ and $\dfrac{\partial F}{\partial v}(0,0)\neq 0$. Let $\varphi\colon \mathbb{R}^3\to\mathbb{R}^2$ be defined by $\varphi(x,y,z)=(xy,x^2-y^2-z)$, and define $f=F\circ\varphi$.
Show that the equation $f(x,y,z)=0$ implicitly defines z as a function of (x,y) near $(0,0,0)$, and that one has $x\dfrac{\partial z}{\partial x}-y\dfrac{\partial z}{\partial y}=2(x^2+y^2)$.

Correction. By the Chain Rule, the Jacobian matrix of $F\circ\varphi$ at $(0,0,0)$ is $JF(\varphi(0,0,0)).J\varphi(0,0,0)=JF(0,0).J\varphi(0,0,0)$. One has $J\varphi(x,y,z)=\begin{pmatrix} y & x & 0\\ 2x & 2y & -1\end{pmatrix}$, so

$$Jf(0,0,0)=\begin{pmatrix}\frac{\partial F}{\partial u}(0,0) & \frac{\partial F}{\partial v}(0,0)\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&-1\end{pmatrix}=\begin{pmatrix}0 & 0 & -\frac{\partial F}{\partial v}(0,0)\end{pmatrix}.$$

Since $\frac{\partial F}{\partial v}(0,0) \neq 0$, the Implicit Function Theorem enables us to assert that the equation $F(x,y,z)=0$ defines implicitly z as a function of (x,y) near $(0,0,0)$.
Then, implicit differentiation yields $(y\frac{\partial F}{\partial u} + 2x\frac{\partial F}{\partial v})dx + (x\frac{\partial F}{\partial u} - 2y\frac{\partial F}{\partial v})dy - \frac{\partial F}{\partial v}dz = 0$. From this, one recovers that $\frac{\partial z}{\partial x} = y\frac{\frac{\partial F}{\partial u}}{\frac{\partial F}{\partial v}} + 2x$, and $\frac{\partial z}{\partial y} = x\frac{\frac{\partial F}{\partial u}}{\frac{\partial F}{\partial v}} - 2y$. Thus, one does indeed have $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2(x^2+y^2)$.

3. Recall that we saw in class that, if a system of two equations $F(x,y,z)=0$ and $G(x,y,z)=0$ defines implicitly two of the variables as a function of the third one near a point $P \in \mathbb{R}^3$, then that system of equations defines a curve in the neighborhood of P.
Prove that the system of equations $\begin{cases} 4xy + 2xz + 4y - z &= 0 \\ xy + xz + yz + 2x + zy - z = 0 \end{cases}$ defines a curve near $(0,0,0)$. What is the tangent line to this curve at that point ?
Correction. The Jacobian matrix associated to this system is $\begin{pmatrix} 4y+2z & 4x+4 & 2x-1 \\ y+z+2 & x+2z & x+2y-1 \end{pmatrix}$. At $(0,0,0)$ this matrix is $\begin{pmatrix} 0 & 4 & -1 \\ 2 & 0 & -1 \end{pmatrix}$. To define (y,z) implicitly as functions of x near $(0,0,0)$, one needs the determinant of $\begin{pmatrix} 4 & -1 \\ 0 & -1 \end{pmatrix}$ to be nonzero. Since this determinant is equal to -4, the equations define implicitly (y,z) as functions of n near $(0,0,0)$.
To find the tangent line to this curve at $(0,0,0)$, we can use the fact that it is orthogonal to the normal vectors to both planes, which, reading from the Jacobian matrix, are $\begin{pmatrix} 0 \\ 4 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$. Thus, the tangent line is paralel to the cross product of these two vectors, which is the vector $\begin{pmatrix} -4 \\ -2 \\ -8 \end{pmatrix}$. So, the tangent line to the curve at this point is the line parallel to $\begin{pmatrix} -4 \\ -2 \\ -8 \end{pmatrix}$ and going through $(0,0,0)$, in other words the set of $(x,y,z) \in \mathbb{R}^3$ such that $x = 2y$ and $z = 4y$ (You may recover these equations by saying that the tangent line is the set of (x,y,z) which lie on both of the tangent planes of the surfaces whose intersection defines the curve).

4. Consider the application from $\mathbb{R}^3 \times \mathbb{R}^3$ to $\mathbb{R}$ that maps (u,v) to $u.v$. Identifying $\mathbb{R}^3 \times \mathbb{R}^3$ with $\mathbb{R}^6$ (the first three variables giving the coordinates of u, and the last three giving the coordinates of v), compute the Jacobian matrix of this application. Use this, and the Chain Rule, to show that, if $u = u(t)$ and $v = v(t)$, then $(u.v)'(t) = u'(t).v(t) + u(t).v'(t)$.
Similarly, one may consider the cross product $(u,v) \mapsto u \times v$ as a function from $\mathbb{R}^6 \to \mathbb{R}^3$. Write the Jacobian matrix of this application. Use it to show that again $(u \times v)'(t) = u'(t) \times v(t) + u(t) \times v'(t)$.
Correction. In this setting, the dot product is the mapping $(u_x, u_y, u_z, v_x, v_y, v_z) \mapsto u_x v_x + u_y v_y + u_z v_z$. Thus, its Jacobian matrix at a point $(u,v) = \begin{pmatrix} u_x & u_y & u_z & v_x & v_y & v_z \end{pmatrix}$ is simply $\begin{pmatrix} v_x & v_y & v_z & u_x & u_y & u_z \end{pmatrix}$. Now, if $t \mapsto (u(t), v(t))$ is a differentiable mapping, the Chain Rule tells us that the derivative of the mapping $t \mapsto u(t).v(t)$ is equal to

$$\begin{pmatrix} v_x(t) & v_y(t) & v_z(t) & u_x(t) & u_y(t) & u_z(t) \end{pmatrix} \begin{pmatrix} u'_x(t) \\ u'_y(t) \\ u'_z(t) \\ v'_x(t) \\ v'_y(t) \\ v'_z(t) \end{pmatrix},$$

which is equal to $v_x(t)u'_x(t) + v_y(t)u'_y(t) + v_z(t)u'_z(t) + u_x(t)v'_x(t) + u_y(t)v'_y(t) + u_z(t)v'_z(t) = u'(t).v(t) + u(t).v'(t)$.

To do the same for the cross product, we need to use the fact that $\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} \times \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} u_y v_z - u_z v_y \\ -u_x v_z + v_x u_z \\ u_x v_y - v_x u_y \end{pmatrix}$. From this, we find that the Jacobian matrix of the cross product is

$$\begin{pmatrix} 0 & v_z & -u_y & 0 & -u_z & u_y \\ -v_z & 0 & v_x & u_z & 0 & -u_x \\ v_y & -v_x & 0 & -u_y & u_x & 0 \end{pmatrix} .$$

Using the same method as for the dot product, we obtain that $(u \times v)'(t)$ is equal to

$$\begin{pmatrix} 0 & v_z(t) & -v_y(t) & 0 & -u_z(t) & u_y(t) \\ -v_z(t) & 0 & v_x(t) & u_z(t) & 0 & -u_x(t) \\ v_y(t) & -v_x(t) & 0 & -u_y(t) & u_x(t) & 0 \end{pmatrix} \begin{pmatrix} u'_x(t) \\ u'_y(t) \\ u'_z(t) \\ v'_x(t) \\ v'_y(t) \\ v'_z(t) \end{pmatrix} \text{, which yields}$$

$$(u \times v)'(t) = \begin{pmatrix} v_z(t)u'_y(t) - v_y(t)u'_z(t) + u_y(t)v'_z(t) - u_z(t)v'_y(t) \\ v_x(t)u'_z(t) - u'_x(t)v_z(t) + u_z(t)v'_x(t) - v'_z(t)u_x(t) \\ v_y(t)u'_x(t) - v_x(t)u'_y(t) + u_x(t)v'_y(t) - u_y(t)v'_x(t) \end{pmatrix} = u'(t) \times v(t) + u(t) \times v'(t) .$$

Of course, this is *not* an efficient way of computing the derivative of a dot product or a cross product : it would be much simpler to compute derivatives directly, coordinate by coordinate.