University of Illinois at Urbana-Champaign — Fall 2006
Math 380 — Group G1

Graded Homework VII .
Due Friday, November 3.

1. Use two different methods to compute the circulation of the vector field V on the curve C, in the following cases :
(a) $V(x,y)=(xy,x-y)$, C is the triangle with vertices $(0,0)$, $(0,3)$, $(1,-1)$ and is oriented clockwise;
(b) $V(x,y)=(xy,e^y)$, and R is the circle of center $(0,0)$ and radius 3, oriented counterclockwise.
Correction. (a) There are three line integrals to compute. The first is given by the parameterization $x(t)=0$, $y(t)=3t$, $0\le t\le 1$; the second is $x(t)=t$, $y(t)=3-4t$, $0\le t\le 1$; and the third is $x(t)=1-t$, $y(t)=t-1$, $0\le t\le 1$ (Of course, there are other possible parameterizations!).
The first line integral is then $\displaystyle\int_0^1 \big(0+(0-3t).3\big)dt=-\frac{9}{2}$; the second is
$\displaystyle\int_0^1 \big(t(3-4t).1+(t-(3-4t)).(-4)\big)dt=\int_0^1\big(-4t^2-17t+12\big)dt=-\frac{4}{3}-\frac{17}{2}+12=\frac{13}{6}$.
Finally, the third integral is $\displaystyle\int_0^1\big((1-t).(t-1).(-1)+(2-2t).1\big)dt=\int_0^1\big(t^2-4t+3\big)=\frac{1}{3}-2+3=\frac{4}{3}$. Thus, we eventually obtain that $\displaystyle I=\int_C x^2y\,dx+(x-y)\,dy=-\frac{9}{2}+\frac{13}{6}+\frac{4}{3}=-1$.
Green's theorem tells us that (denoting by T the interior of the triangle)

$$I=-\iint_T (1-x)dxdy=-\int_{x=0}^1\left(\int_{y=-x}^{3-4x}(1-x)dy\right)dx=-\int_{x=0}^1(1-x)\big((3-4x)-(-x)\big)dx\ ,\ \text{so}$$

$$I=\int_0^1(1-x)\big(-3x^2+6x-3\big)dx=-1+3-3=-1\ .$$

(b) First, using a line integral, we have, using the usual parameterization for a circle :

$$J=\int_C xydx+e^ydy=\int_0^{2\pi}\big((9\cos(t)\sin(t))(-\cos(t)+e^{3\sin(t)}(3\cos(t))\big)dt=\int_0^{2\pi}\big(-9\cos^2(t)\sin(t)+3\cos(t)e^{3\sin(t)}\big)dt$$

$$J=\Big[3\cos^3(t)+e^{3\sin(t)}\Big]_0^{2\pi}=0$$

Denoting by D the disk delimited by C, Green's theorem gives

$$J=\iint_D (0-y)dxdy=-\iint_D ydxdy=0\ .$$

2. For each of the following "differential forms" $P(x,y)dx+Q(x,y)dy$, determine whether there exists a function f such that $P(x,y)dx+Q(x,y)dy=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$; is it exists, find such a function.
(a) $P(x,y)=x^2+y$, $Q(x,y)=2y$.
(b) $P(x,y)=xy^2$, $Q(x,y)=x^2y$.
(c) $P(x,y)=2xy\cos(x^2y)+1$, $Q(x,y)=x^2\cos(x^2y)+e^y$
Correction (a) One has $\frac{\partial P}{\partial y}=1$, and $\frac{\partial Q}{\partial x}=0$; thus, $\frac{\partial P}{\partial y}\ne\frac{\partial Q}{\partial x}$, and this shows that there exists no function f such that $P(x,y)dx+Q(x,y)dy=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$.
(b) This time one can see that $\frac{\partial P}{\partial y}=2xy=\frac{\partial Q}{\partial x}$; $f(x,y)=\frac{x^2y^2}{2}$ is easily seen to be such that $P(x,y)dx+Q(x,y)dy=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$.

(c) This time we see that $\frac{\partial P}{\partial y} = 2x\cos(x2y) - 2x^3 y \sin(x^2)y = \frac{\partial Q}{\partial x}$, so we know that there exists a function f with the desired property. To find it, we integrate with respect to x with y constant, and with respect to y with x constant ; this yields $f(x,y) = \sin(x^2y) + x + g(y)$, and $f(x,y) = \sin(x^2y) + e^y + h(x)$; thus $f(x,y) = \sin(x^2y) + x + e^y$ is a solution, which is easily verified by computing the partial derivatives of f.
Remark. Pay attention to the fact that in (b) and (c) f is not the only solution ; any function of the form $f(x,y) + c$, where c is a constant, is also a solution.

3. (a) Prove that the integral $\int_\gamma (6x+2y)dx + (6y+2x)dy$ has the same value whenever γ is a positively oriented curve from $A = (0,0)$ to $B = (1,1)$. Check this by computing this integral in the case where γ is a straight line segment, and γ is an arc of the parabola of equation $y = x^2$.
(b) Find a function f such that its gradient at the point (x,y) is equal to $(6x+2y, 6y+2x)$; explain why this function enables one to compute easily the integrals of the preceding question.
Correction. (a) The functions $P(x,y) = 6x+2y$ and $Q(x,y) = 6y+2x$ are continuously differentiable in $\mathbb{R}^2$ and $\frac{\partial P}{\partial y} = 2 = \frac{\partial Q}{\partial x}$. Thus the integral $\int Pdx + Qdy$ is independent of path.
For the first example, a parameterization of the curve γ_1 is $x(t) = y(t) = t$, $0 \le t \le 1$, so one obtains $\int_{\gamma_1} (6x+2y)dx + (6y+2x)dy = \int_{t=0}^{1} \big((6t+2t)1 + 6t+2t).1\big)dt = 8$.
For the second example, the curve γ_2 may be parameterized by setting $x = t$, $y = t^2$, $0 \le t \le 1$, so one gets

$$\int_{\gamma_1} (6x+2y)dx + (6y+2x)dy = \int_{t=0}^{1} \big((6t+2t^2)1 + (6t^2+2t).2t\big)dt = \int_0^1 \big(12t^3+6t^2+6t\big)dt = 3+2+3 = 8\ .$$

(b) To find f , one first considers y as a constant, and computes an integral in terms of x, which gives that $f(x,y) = 3x^2 + 2yx + g(y)$, where g is some function of y ; doing the same for y, one obtains $f(x,y) = 3y^2 + 2xy + h(x)$. These two identities indicate that the function $f(x,y) = 3y^2 + 2xy + 3y^2$ should work, and a direct verification shows that it is indeed the case.
Since the line integrals that we computed above are of the form $\int \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$ (where the line integral is computed on some curve between $(0,0)$ and $(1,1)$, their value is $f(1,1) - f(0,0) = 3+2+3-0 = 8$.

4. A *cardioid* is a curve of equation (in polar coordinates) $r = (1+\cos(\theta))$, $0 \le \theta \le 2\pi$. Compute the area of the domain delimited by a cardioid ; for this, use θ as a parameter, and use trigonometric relations to show that $x(\theta)y'(\theta) - y(\theta)x'(\theta) = 1 + 2\cos(\theta) + \cos^2(\theta)$ (why does this help ?).
Correction. Denoting the cardioid, oriented counterclockwise, by γ, Green's theorem gives that the area A of the domain delimited by the cardioid is such that $2A = \int_\gamma xdy - ydx$. Since we are given equations in polar coordinates, our parameterization here is $x = r\cos(\theta) = (1+\cos(\theta))\cos(\theta)$, and $y = (1+\cos(\theta))\sin(\theta)$. Thus, $x'(\theta) = -\sin(\theta) - 2\sin(\theta)\cos(\theta)$, and $y'(\theta) = \cos(\theta) - \sin^2(\theta) + \cos^2(\theta)$. This gives

$$x(\theta)y'(\theta) = \cos^2(\theta) + 2\cos^3(\theta) - \sin^2(\theta)\cos(\theta) + \cos^4(\theta) - \cos^2(\theta)\sin^2(\theta)\ , \text{ and}$$

$$y(\theta)x'(\theta) = -\sin^2(\theta) - 2\sin^2(\theta)\cos(\theta) - \sin^2(\theta)\cos(\theta) - 2\sin^2(\theta)\cos^2(\theta)\ .$$

This yields

$$x(\theta)y'(\theta) - y'(\theta)x(\theta) = 1 + 2\cos(\theta)(\cos^2(\theta) + \sin^2(\theta)) + \cos^2(\theta)(\sin^2(\theta) + \sin^2(\theta)) = 1 + 2\cos(\theta) + \cos^2(\theta)\ .$$

We then get that

$$2A = \int_{\theta=0}^{2\pi} (1 + 2\cos(\theta) + \cos^2(\theta))d\theta = 2\pi + 0 + \pi = 3\pi\ .$$

Eventually, we obtain that the area of the cardioid is $\frac{3\pi}{2}$.