University of Illinois at Urbana-Champaign — Fall 2006
Math 380 — Group G1

Graded Homework IX .
Due Friday, November 3.

1. Evaluate the following integrals by reversing the order of integration :
$\int_0^1\int_{3y}^3 e^{x^2}\,dxdy\,;\ \int_0^1\int_{\sqrt{y}}^1 \sqrt{x^3+1}\,dxdy\,;\ \int_0^1\int_{x^2}^1 x^3\sin(y^3)\,dydx.$

Correction. For the first integral, the domain of integration is the triangle $0 \le y \le 1$, $3y \le x \le 3$. One can see that these equations are the same as $0 \le x \le 3$, $0 \le y \le \frac{x}{3}$. Thus,

$$\int_0^1\int_{3y}^3 e^{x^2}\,dxdy = \int_0^3\int_0^{x/3} e^{x^2}\,dydx = \int_0^3 e^{x^2}.\frac{x}{3}dx = \left[\frac{1}{6}e^{x^2}\right]_0^3 = \frac{e^9-1}{6}\ .$$

Similarly, the domain of all (x,y) such that $0 \le y \le 1$, $\sqrt{y} \le x \le 1$ is also the domain of all (x,y) such that $0 \le x \le 1$, $0 \le y \le x^2$. Our integral then becomes

$$\int_0^1\int_{\sqrt{y}}^1 \sqrt{x^3+1}\,dxdy = \int_0^1\int_0^{x^2} \sqrt{x^3+1}\,dydx = \int_0^1 x^2\sqrt{x^3+1}\,dx = \left[\frac{2}{9}(x^3+1)^{3/2}\right]_0^1 = \frac{2}{9}(2\sqrt{2}-1)\ .$$

Finally, the domain for the third integral is given by $0 \le y \le 1$, $0 \le x \le \sqrt{y}$; thus the last integral is

$$\int_0^1\int_{x^2}^1 x^3\sin(y^3)\,dydx = \int_0^1\int_0^{\sqrt{y}} x^3\sin(y^3)\,dxdy = \int_0^1 \left[\frac{x^4}{4}\right]_0^{\sqrt{y}}\sin(y^3)\,dy = \int_0^1 \frac{y^2}{4}\sin(y^3)\,dy = \left[-\frac{\cos(y^3)}{12}\right]_0^1 = \frac{1-\cos(1)}{12}\ .$$

2. (a) Evaluate $\iiint_R x^2\,dxdydz$, where R is the region bounded by the planes $x+y+z=1$, $x=0$, $y=0$ and $z=0$.
(b) Compute the volume of the intersection of the paraboloid of equation $x^2+y^2 \le \frac{3z}{2}$ and the sphere of equation $x^2+y^2+z^2=1$.

Correction. (a) x is between 0 and y; if x is fixed, y can vary between 0 and $1-x$ and if x,y are fixed then z can vary between 0 and $1-x-y$. Thus,

$$\iiint_R x^2\,dxdydz = \int_{x=0}^1\left(\int_{y=0}^{1-x}\left(\int_{z=0}^{1-x-y} dz\right)dy\right)x^2dx = \int_{x=0}^1\left(\int_{y=0}^{1-x}(1-x-y)dy\right)x^2\,dx =$$

$$\int_0^1 x^2\big((1-x)^2-\frac{(1-x)^2}{2}\big)dx = \int_0^1 \frac{x^2-2x^3+x^4}{2}dx = \frac{1}{6}-\frac{1}{4}+\frac{1}{10} = \frac{1}{60}\ .$$

(b) There are two conditions on x^2+y^2; we first remark that on the paraboloid one has $z \ge 0$, so in the intersection one has $z \ge 0$ too. Also, for $z \ge 0$ one has $\frac{3z}{2} \le 1-z^2$ if, and only if, $z \le \frac{1}{2}$. Thus, it is the first condition (paraboloid) that applies when $z \le \frac{1}{2}$, and the second (sphere) that applies when $z \ge \frac{1}{2}$. Hence, the volume V of the domain we are interested in is

$$V = \int_{z=0}^{1/2}\left(\iint_{x^2+y^2\le 3z/2} dxdy\right)dz + \int_{z=1/2}^1\left(\iint_{x^2+y^2\le 1-z^2} dxdy\right)dz = \int_{z=0}^{1/2}\frac{3\pi z}{2}dz + \int_{z=1/2}^1 \pi(1-z^2)dz =$$

$$\frac{3\pi}{16}+\frac{\pi}{3}\left(\frac{1}{2}-\frac{1}{3}+\frac{1}{24}\right) = \frac{19\pi}{48}\ .$$

3. Let γ be the boundary of the region bounded by $y=x$ and $y=x^2$, oriented clockwise; use two different methods to compute $\int_\gamma y^2dx - xdy$.

Correction. Using Green's theorem, we see that our integral I is given by

$$I = \int_{x=0}^{1} \int_{x^2}^{x} (1+2y)\,dydx = \int_0^1 (x - x^2 + x^2 - x^4)dx = \frac{1}{2} - \frac{1}{5} = \frac{3}{10}\ .$$

To compute I using a line integral, we need to parameterize our two curves; for the first one one may set $x = t$, $y = y$, and the second one is equal to minus the integral obtained with the parameterization, $x = t$, $y = t^2$ (pay attention to the orientation!). Thus,

$$I = \int_{t=0}^{1} (t^2.1 - t.1)dt - \int_0^1 (t^4.1 - t.(2t))dt = \frac{1}{3} - \frac{1}{2} - \frac{1}{5} + \frac{2}{3} = \frac{3}{10}\ .$$

4. Set $\omega = e^{(x+2y)^2}dx + 2e^{(x+2y)^2}dy$. Show that there exists a function f such that $df = \omega$. Is it possible to give a simple expression for the function f?
Correction. One has $\dfrac{\partial P}{\partial y} = 4(x+2y)e^{(x+2y)^2} = \dfrac{\partial Q}{\partial x}$; since the functions P, Q are defined on the whole plane, which is simply connected, there exists a function f such that $\dfrac{\partial f}{\partial x} = P$, $\dfrac{\partial f}{\partial y} = Q$. However, one cannot give a simple expression for f, because the integrals involved cannot be expressed using usual functions.

5. Compute the area of the surface of equation $z = x^2 + y^2$ where $0 \leq z \leq h$ (where h is some positive number).
Correction. Setting $f(x,y) = x^2 + y^2$, this surface is of equation $z = f(x,y)$; the partial derivatives of f are $\dfrac{\partial f}{\partial x} = 2x$ and $\dfrac{\partial f}{\partial y} = 2y$. The projection of the surface on the (x,y) plane is the disk of center 0 and radius $\sqrt{h}$; denoting this disk by D, the formula seen in class gives us that the area A of our surface is

$$A = \iint_D \sqrt{1+4x^2+4y^2}\,dxdy = \int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{h}} r\sqrt{1+4r^2}\,drd\theta = 2\pi \left[\frac{1}{12}(1+4r^2)^{3/2}\right]_{r=0}^{\sqrt{h}} = \frac{\pi}{6}\left((1+4h)^{3/2} - 1\right)\ .$$