University of Illinois at Urbana-Champaign | Fall 2006
Math 444 | Group E13

Final Exam : Answer Key

You are allowed to use your textbook, but no other kind of documentation.
Calculators, mobile phones and other electronic devices are prohibited.

NAME ____

SIGNATURE ____

1. (20 points)
Define a function $f\colon [0,+\infty) \to \mathbb{R}$ by setting $f(x) = \sin(\sqrt{x})$. Show that f is continuous on $[0,+\infty)$ and differentiable on $(0,+\infty)$; give a formula for $f'(x)$ for all $x > 0$. Is f differentiable at 0 ?
(You may use without demonstration the fact that the function $x \mapsto \sin(x)$ is differentiable on $\mathbb{R}$ and that $\sin'(x) = \cos(x)$, and the fact that $x \mapsto \sqrt{x}$ is differentiable on $(0,+\infty)$ and $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$)

Answer. The function $g\colon x \mapsto \sin(x)$ is continuous on $\mathbb{R}$, and the function $h \mapsto \sqrt{x}$ is continuous on $\mathbb{R}^+$. Hence $f = g \circ h$ is continuous on $\mathbb{R}^+$, since it is a obained by composition of two continuous functions. Similarly, the Chain Rule ensures that f is differentiable on $(0,+\infty)$, and $f'(x) = \frac{1}{2\sqrt{x}} \cos(\sqrt{x})$.
To see whether f is differentiable at 0, the simplest thing is to go back to the definition ; one has $f(0) = 0$, so $\frac{f(x)-f(0)}{x-0} = \frac{f(x)}{x} = \frac{\sin(\sqrt{x})}{x}$. Since we know that $\lim_{x\to 0} \frac{\sin(x)}{x} = \cos(0) = 1$, we see that $\frac{f(x)}{x}$ is not bounded in the neighborhood of 0 (it is the product of a function with limit 1 and a function which is not bounded). Hence $\frac{f(x)-f(0)}{x-0}$ doens't have a limit at 0, and this shows that f is not differentiable at 0.

2. (30 points)
Let $0 < \alpha < 1$.
(a) Show that for all $x > 0$ one has

$$\frac{\alpha}{(x+1)^{1-\alpha}} \le (x+1)^{\alpha} - x^{\alpha} \le \frac{\alpha}{n^{1-\alpha}} \ .$$

(b) Define a sequence (u_n) by the formula

$$u_n = \sum_{k=1}^{n} \frac{1}{k^{\alpha}} \ .$$

Use the inequality above (applied to $\alpha' = 1 - \alpha$) to show that this sequence is not convergent.

Answer. (a) The Mean Value Theorem, applied to the function $x \mapsto x^{\alpha}$ (which is differentiable on $(0, +\infty)$) on the interval $[x, x+1]$, yields that there exists $c \in (x, x+1)$ such that

$$(x+1)^{\alpha} - x^{\alpha} = \frac{\alpha}{c^{1-\alpha}} \ .$$

Since $0 < \alpha < 1$ one has $1 - \alpha > 0$ and the fact that $x < c < x + 1$ yields

$$\frac{1}{(x+1)^{1-\alpha}} < \frac{1}{c^{1-\alpha}} < \frac{1}{x^{1-\alpha}} \ .$$

This gives us

$$\frac{\alpha}{(x+1)^{1-\alpha}} \ge (x+1)^{\alpha} - x^{\alpha} \ge \frac{\alpha}{n^{1-\alpha}} \ .$$

(b) The inequality above (applied to $\alpha' = 1 - \alpha$, which is such that $0 < \alpha' < 1$) gives in particular that for all $k \ge 1$ one has $(k+1)^{1-\alpha} - k^{1-\alpha} \le \frac{1-\alpha}{k^{\alpha}}$. Summing these inequalites for $k = 1, \ldots, n$ we get

$$\sum_{k=1}^{n} (k+1)^{1-\alpha} - k^{1-\alpha} \le \sum_{k=1}^{n} \frac{1-\alpha}{k^{\alpha}} \ .$$

Given the cancellations, this is equivalent to

$$(n+1)^{1-\alpha} - 1 \le (1-\alpha) u_n \ .$$

Given that the sequence $(n+1)^{1-\alpha} - 1$ is not bounded above, and that $1 - \alpha > 0$, this shows that (u_n) is not bounded above, hence it isn't convergent.

3. (30 points)

Let f be continuous on $[0, +\infty)$; for all $x > 0$, set $g(x) = \dfrac{1}{x}\displaystyle\int_0^x f(t)dt$.

(a) Show that g is continuous on $(0, +\infty)$, and that g has a limit at 0 ; give the value of this limit.

(b) Show that g is differentiable on $(0, +\infty)$ and that for all $x > 0$ one has

$$g'(x) = \frac{f(x) - g(x)}{x} \ .$$

Answer. (a) By the fundamental theorem of integration we know, since f is continuous on $[0, +\infty)$, that the function $x \mapsto F(x) = \int_0^x f(t)dt$ is differentiable on $[0, +\infty)$; hence it is continuous on $[0, +\infty)$. So on $(0, +\infty)$ g is the product of two continuous functions, which shows that it is continuous on this interval. Also, with our notations one has $g(x) = \frac{F(x)}{x}$. Since $F(0) = 0$, and F is differentiable at 0, we know that $\lim_{x \to 0} \frac{F(x)}{x}$ exists and is equal to $F'(0) = f(0)$. This is equivalent to saying $\lim_{x \to 0} g(x) = f(0)$.

(b) The product of two differentiable functions is a differentiable function, so (since F is differentiable, $F'(x) = f(x)$ and $g(x) = \frac{1}{x} \cdot F(x)$) we see that g is differentiable on $(0, +\infty)$ and

$$g'(x) = -\frac{1}{x^2}F(x) + \frac{1}{x}F'(x) = -\frac{g(x)}{x} + \frac{f(x)}{x} = \frac{f(x) - g(x)}{x} \ .$$

4. (30 points)
Pick two real numbers a, b such that $a < b$ and let $f\colon [a,b] \to \mathbb{R}$ be continuous. We want to show that

$$\sup\{f(x)\colon x \in (a,b)\} = \sup\{f(x)\colon x \in [a,b]\} \ .$$

(a) Explain why $\sup\{f(x)\colon x \in (a,b)\}$ and $\sup\{f(x)\colon x \in [a,b]\}$ exist.
(b) Show that $\sup\{f(x)\colon x \in (a,b)\} \leq \sup\{f(x)\colon x \in [a,b]\}$.
(c) Assume $f(a) = \sup\{f(x)\colon x \in [a,b]\}$. Show that one also has $f(a) = \sup\{f(x)\colon x \in (a,b)\}$ (look at the sequence $(a + \frac{1}{n})$). Can you prove a similar result when $f(b) = \sup\{f(x)\colon x \in [a,b]\}$?
(d) Prove the equality $\sup\{f(x)\colon x \in (a,b)\} = \sup\{f(x)\colon x \in [a,b]\}$.

Answer. (a) Since f is a continuous function on the closed bounded interval $[a,b]$, the Boundedness Theorem ensures that there exists m, M such that $m \leq f(x) \leq M$ for all $x \in [a,b]$. This shows that the set $\{f(x)\colon x \in [a,b]\}$ is bounded, so it has a supremum (because of the completeness property of the real numbers). Since $\{f(x)\colon x \in (a,b)\}$ is a subset of $\{f(x)\colon x \in [a,b]\}$ and the latter set is bounded, we see that $\sup\{f(x)\colon x \in (a,b)\}$ exists too.
(b) For any $x \in (a,b)$ one has $f(x) \leq \sup\{f(x)\colon x \in [a,b]\}$, so $\sup\{f(x)\colon x \in [a,b]\}$ is an upper bound of $\{f(x)\colon x \in (a,b)\}$. By definition of a supremum (least upper bound), this implies that $\sup\{f(x)\colon x \in [a,b]\} \leq \sup\{f(x)\colon x \in [a,b]\}$.
(c) There exists N such that $a_n = a + \frac{1}{n} \leq b$ for all $n \geq N$. Thus we can consider the sequence $(f(a_n))_{n \geq N}$; since f is continuous at a, this sequence converges to $f(a)$. Since one has $f(a_n) \leq \sup\{f(x)\colon x \in (a,b)\}$, we also have $\lim f(a_n) = f(a) \leq \sup\{f(x)\colon x \in (a,b)\}$. Thus if $f(a) = \sup\{f(x)\colon x \in [a,b]\}$ then we get $\sup\{f(x)\colon x \in [a,b]\} \leq \sup\{f(x)\colon x \in (a,b)\}$, and the result of question (b) ensures that in factsup$\{f(x)\colon x \in [a,b]\} = \sup\{f(x)\colon x \in (a,b)\}$ in that case.
Considering the sequence $b_n = b - \frac{1}{n}$, we obtain in the same way that if $f(b) = \sup\{f(x)\colon x \in [a,b]\}$ then $\sup\{f(x)\colon x \in [a,b]\} = \sup\{f(x)\colon x \in (a,b)\}$.
(d) If the sup for f on $[a,b]$ is obtained at either a or b then the result of question (c) shows that $\sup\{f(x)\colon x \in [a,b]\} = \sup\{f(x)\colon x \in (a,b)\}$. There must be a sup for the continuous function f on $[a,b]$, and actually it is a maximum ; so if we are not in the case above then there exists $c \in (a,b)$ such that $f(c) = \sup\{f(x)\colon x \in [a,b]\}$. By definition, one has $f(c) \leq \sup\{f(x)\colon x \in (a,b)\}$, so we again obtain

$$\sup\{f(x)\colon x \in [a,b]\} \leq \sup\{f(x)\colon x \in (a,b)\}$$

Since the maximum for f must be attained at either a, b, or some $c \in (a,b)$, the reasoning above shows that if f is continuous on a closed bounded interval $[a,b]$ then

$$\sup\{f(x)\colon x \in [a,b]\} = \sup\{f(x)\colon x \in (a,b)\} \ .$$

5. (30 points)

Let $0 < \lambda < 1$ and $f : \mathbb{R} \to \mathbb{R}$ be such that $f(\lambda x) = \lambda f(x)$ for all $x \in \mathbb{R}$.

(a) Prove that $f(0) = 0$.

(b) Assume that f is differentiable at 0. Show that there exists $a \in \mathbb{R}$ such that $f(x) = ax$ for all $x \in \mathbb{R}$.

Hint. What can you say of the sequence $\frac{f(\lambda^n x)}{\lambda^n x}$? show that $a = f'(0)$ works).

(c) Is the result above still true if one no longer assumes that f is differentiable at 0 ?

Answer. (a) One has $f(0) = \lambda f(0)$ and $\lambda \neq 1$, so one must have $f(0) = 0$.

(b) Notice that one has, for all $x \neq 0$, that the assumption on f is the same as $\dfrac{f(\lambda x)}{\lambda x} = \dfrac{f(x)}{x}$. An easy induction yields that, for all $x \neq 0$ and all $n \in \mathbb{N}$, one has $\dfrac{f(\lambda^n x)}{\lambda^n x} = \dfrac{f(x)}{x}$.

Now, notice that since $0 < \lambda < 1$ the sequence $(\lambda^n x)$ converges to 0. Since f is differentiable at 0 and $f(0) = 0$, one has $\displaystyle\lim_{y \to 0} \frac{f(y)}{y} = f'(0)$ by definition of a derivative. But then we get that $\displaystyle\lim_{n \to +\infty} \frac{f(\lambda^n x)}{\lambda^n x} = f'(0)$. Since on the other hand we proved that under the assumptions on f this sequence is constant, equal to $\dfrac{f(x)}{x}$, this shows that $\dfrac{f(x)}{x} = f'(0)$ for all $x \in \mathbb{R}$ different from 0. This gives us $f(x) = xf'(0)$ for all $x \in \mathbb{R}$.

(c) Set $f(x) = 0$ for all $x \in \mathbb{Q}$, $f(x) = x$ for all $x \in \mathbb{R} \setminus \mathbb{Q}$. Then one has that $f(\dfrac{x}{2}) = \dfrac{f(x)}{2}$ for all $x \in \mathbb{R}$, yet there doesn't exist any real number a such that $f(x) = a \cdot x$ for all $x \in \mathbb{R}$. Indeed, if $x \in \mathbb{Q} \setminus \{0\}$ this would yield $a = 0$, and if $x \in \mathbb{R} \setminus \mathbb{Q}$ this would yield $a = 1$.

6. (30 points)
Let $f\colon [0,1] \to [0,1]$ be an increasing function (not necessarily continuous). Show that there exists $x \in [0,1]$ such that $f(x) = x$.
Hint. Consider the set $E = \{x \in [0,1]\colon f(x) > x\}$; show that one can assume that $0 \in E$. Show that $x = \sup(E)$ works.

Answer. If $f(0) = 0$ there is nothing to prove, so we may assume that $f(0) > 0$, and this gives

$$0 \in E = \{x \in [0,1]\colon\ f(x) > x\}\ .$$

Thus we may assume that E is nonempty. Since E is bounded (it is a subset of $[0,1]$), it has a supremum S, which is larger than 0 (because $0 \in E$) and smaller than 1 (because 1 is an upper bound for E).
For any $\varepsilon > 0$ there exists $x \in E$ such that $S - \varepsilon < x < S$. Since f is increasing, $f(S) \geq f(x) > x > S - \varepsilon$, so $f(S) > S - \varepsilon$ for all $\varepsilon > 0$. This yields $f(S) \geq S$. If $f(S) = S$ we are done; assume that it is not true and $f(S) > S$. Then pick $a \in [0,1]$ such that $S < a < f(S)$. Since f is increasing one has $f(a) \geq f(S) > a$, so $a \in E$, which is impossible because S is the supremum of E. Hence $f(S) = S$, and we are done.

7. (30 points)
Recall that if X is a set, one denotes by $\mathcal{P}(X)$ the set whose elements are the subsets of X ; in other words, $\mathcal{P}(X) = \{A \colon A \subset X\}$. Let now X, Y be sets and $f \colon X \to Y$ be a function.
(a) Define a function $\hat{f} \colon \mathcal{P}(X) \to \mathcal{P}(Y)$ by setting $\hat{f}(A) = f(A)$ for all $A \subset X$. Show that $\hat{f}$ is injective if, and only if, f is injective.
(b) Similarly, define a function $\tilde{f} \colon \mathcal{P}(Y) \to \mathcal{P}(X)$ by setting $\tilde{f}(B) = f^{-1}(B)$ for all $B \subset Y$. Compute $\tilde{f}(\emptyset)$; show that $\tilde{f}$ is injective if, and only if, f is surjective.

Answer. (a) Assume that $\hat{f}$ is injective, and let $x, y \in X$ be such that $f(x) = f(y)$. Then $\hat{f}(\{x\}) = \{f(x)\} = \hat{f}(\{y\})$, so since $\hat{f}$ is injective we obain $\{x\} = \{y\}$, in other words $x = y$. Thus if $\hat{f}$ is injective then f is injective. Converesly, assume that f is injective and $A, B \subset X$ are such that $f(A) = f(B)$. Then pick $a \in A$. One has $f(a) \in f(A) = f(B)$, so there exists $b \in B$ such that $f(b) = f(a)$. Since f is injective, this is only possible if $b = a$, hence $a \in B$. Thus $A \subset B$; similarly, one sees that if $B \subset A$. This shows that $A = B$; hence if f is injective then $\hat{f}$ is injective too.
We have just proved that f is injective if, and only if, $\hat{f}$ is injective.

(b) This one is perhaps a bit more complicated. Assume that $\tilde{f}$ is injective ; one has $\tilde{f}(\emptyset) = \emptyset$, so for all $y \in Y$ one has $\tilde{f}(\{y\}) \neq \emptyset$. This exactly means that for all $y \in Y$ $f^{-1}(\{y\}) = \{x \in X \colon f(x) = y\}$ is nonempty, in other words that f is surjective.
Converesly, assume that f is surjective, and $A, B \subset Y$ are such that $\tilde{f}(A) = \tilde{f}(B)$. Then pick $a \in A$; since f is surjective, there exists x such that $f(x) = a$. By definition, $x \in f^{-1}(A)$, and since $f^{-1}(A) = f^{-1}(B)$ we also have $x \in f^{-1}(B)$, which means that $f(x) = a \in B$. This is true for all $a \in A$, so $A \subset B$. Since A, B play symmetric roles here, one obtains similarly that $B \subset A$. Hence $A = B$, hence $\tilde{f}$ is injective. This shows that is f is surjective then $\tilde{f}$ is injective.
We have thus proved that $\tilde{f}$ is injective if, and only if, f is surjective.