University of Illinois at Urbana-Champaign
Math 444

Fall 2006
Group E13

Graded Homework X.
Due Friday, November 17.

1.(a) Define a function f on $\mathbb{R}$ by setting $f(x) = \begin{cases} f(x) = x\sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{else} \end{cases}$. Is this function continuous on $\mathbb{R}$? (you may use without proof the fact the the function $x \mapsto \sin(x)$ is continuous).
(b) Let $g\colon \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 1-x & \text{else} \end{cases}$. At which points in $\mathbb{R}$ is g continuous?

2. Let $f\colon [a,b] \to [a,b]$ $(a < b)$ be a function such that $|f(x) - f(x')| < |x - x'|$ for all $x \neq x' \in [a,b]$.
(a) Using the ε, δ definition of continuity, show that f is continuous on $[a,b]$.
(b) Prove that there exists a unique point $x \in [a,b]$ such that $f(x) = x$ (introduce a suitable auxiiary function).

3. Let $f\colon [0,1] \to [0,1]$ be a continuous function such that $f(0) = f(1)$.
Show that for all $n \in \mathbb{N}$ there exists $x \in [0, 1 - \frac{1}{n}]$ such that $f(x) = f(x + \frac{1}{n})$.
(Hint : is it possible that $f((k+1)/n) - f(k/n)$ keeps a constant sign for all $k = 0, \ldots, n-1$?)

4.(a) Show that if f is a continuous function on a closed bounded interval $[a,b]$ such that $f(x) > 0$ for all $x \in [a,b]$ then there exists $m > 0$ such that $f(x) \geq m$ for all $x \in [a,b]$.
In the following, we pick two continuous functions f, g from $[0,1] \to [0,1]$ such that $f(x) < g(x)$ for all $x \in [0,1]$.
(b) Show that there exists $m > 0$ such that $f(x) + m < g(x)$ for all $x \in [0,1]$.
(c) Show that there exists $M > 1$ such that $Mf(x) < g(x)$ for all $x \in [0,1]$.

5. Let f be a function from $\mathbb{R}$ to $\mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Recall that we proved in the first midterm that one has then $f(q) = qf(1)$ for all $q \in \mathbb{Q}$. Use this to show that $f(x) = xf(1)$ for all $x \in \mathbb{R}$.