University of Illinois at Urbana-Champaign — Fall 2006
Math 444 — Group E13

Graded Homework XI.
Due Wednesday, November 29.

1. (a) Give an example of a function $f\colon \mathbb{R} \to \mathbb{R}$ which is not constant and satisfies $f(x) = f(x^2)$ for all $x \in \mathbb{R}$.
(b) Assume now that f is continuous at 0 and 1 and $f(x) = f(x^2)$ for all $x \in \mathbb{R}$. Show that f must be constant.
Hint : assume that $|x| < 1$; then what is the limit of the sequence (x_n) defined by $x_1 = x$, $x_2 = x^2, \ldots, x_{n+1} = x_n^2 \ldots$? How about the sequence $(f(x_n))$? Can you use a similar idea when $|x| > 1$?

2. Let $f\colon [0,1] \to [0,1]$ be a continuous function such that $f \circ f = f$ $(*)$. Set

$$E_f = \{x \in [0,1]\colon f(x) = x\} \ .$$

Show that E_f is nonempty, then that it is an interval.
Hint : what is the link between E_f and $f([0,1])$?
Can you describe (accurately and using as few words as possible) the functions that satisfy $(*)$?

3. (a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^2 \sin(\frac{1}{x}) & \text{else} \end{cases}$. Prove that f is continuous, and even differentiable, on $\mathbb{R}$, but that f' is not continuous at 0.
(b) Is it true that any function satisfying the conclusion of the intermediate value theorem must be continuous ?

4. Determine $a, b \in \mathbb{R}$ such that the function $f\colon [0,+\infty) \to \mathbb{R}$ defined by $f(x) = \begin{cases} \sqrt{x} & \text{if } 0 \le x \le 1 \\ ax^2 + bx + 1 & \text{else} \end{cases}$ is differentiable on $(0,+\infty)$.

5. Show that a polynomial function of the form $f(x) = x^n + ax + b$ has at most three distinct real roots (here a, b are reals, and n is a natural integer).
Hint : How many zeros can f' have ? What must happen to f' between any two zeros of f ?

6. Pick a function $f\colon \mathbb{R}^+ = [0,+\infty) \to \mathbb{R}$, and $l \in \mathbb{R}$. One says that f *has limit* l *at* $+\infty$, and one writes $\lim\limits_{x\to+\infty} f(x) = l$, if for any $\varepsilon > 0$ there exists $M \in \mathbb{R}^+$ such that $x \ge M \Rightarrow |f(x) - l| \le \varepsilon$.
(a) Show that, for any continuous function f, one has the following implication : if $f\colon \mathbb{R}^+ \to \mathbb{R}$ has a limit at $+\infty$ then f is bounded on $\mathbb{R}^+$. What is the converse of this assertion ? Is it true ?
(b) Let $f\colon \mathbb{R}^+ \to \mathbb{R}$ be such that $f(0) = 1$ and $\lim\limits_{x\to+\infty} f(x) = 0$. Show that f admits a global maximum on $\mathbb{R}^+$. Must it also admit a global minimum on $\mathbb{R}^+$?
(c) Let $f\colon \mathbb{R}^+ \to \mathbb{R}$ be differentiable on $\mathbb{R}^+$, and suppose that $\lim\limits_{x\to+\infty} f'(x) = l$, where l is some real number. Using the mean value theorem, show that $\lim\limits_{x\to+\infty} \dfrac{f(x)}{x} = l$.

Hint : First prove that for any $\varepsilon > 0$, there exists $a > 0$ such that for any $x > a$ one has $\left|\dfrac{f(x) - f(a)}{x - a} - l\right| \le \varepsilon$. How can you prove this ? Why does question 6(c) help ?)