University of Illinois at Urbana-Champaign | Fall 2006
Math 444 | Group E13

Graded Homework III
Correction of the exercises.

1. Compute, if they exist, $\sup(A)$ and $\inf(A)$ in the following cases. In each case, state whether A admits a maximal element, and do the same for minimal elements.
$A = \{\frac{1}{n} : n \in \mathbb{N}\}$; $A = \{x \in \mathbb{Q} : x^2 < 2\}$; $A = \{(-1)^n + \frac{1}{n} : n \in \mathbb{N}\}$.
Correction. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then for all $a \in A$ we have $a = \frac{1}{n}$ for some $n \in \mathbb{N}$, so that $0 \le a \le 1$. This shows that 0 is a lower bound of A and that 1 is a upper bound of A. So both $\sup(A)$ and $\inf(A)$ exist and, since $1 \in A$, this immediately implies that $\sup(A) = 1$. Furthermore, by the archimedean property of $\mathbb{R}$, we know that for any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $\frac{1}{n} \le \varepsilon$, so that for any $\varepsilon > 0$ there exists some $a \in A$ such that $a \le 0 + \varepsilon$. Since 0 is a lower bound of A this proves that $0 = \inf(A)$. Thus we see that A admits a maximal element (because $\sup(A) \in A$) but no minimal element (because $\inf(A) \notin A$).
Let now $A = \{x \in \mathbb{Q} : x^2 < 2\}$. We have $a^2 < 2$ for all $a \in A$, so that $-\sqrt{2} \le a \le \sqrt{2}$ for all $a \in A$, and this proves that both $\sup(A)$ and $\inf(A)$ exist.
To compute $\sup(A)$, recall that we saw in class that for any $x \in \mathbb{R}$ and any $\varepsilon > 0$ there exists $q \in \mathbb{Q}$ such that $x - \varepsilon \le q \le x$. Applying this to $x = \sqrt{2}$, we see that for any $\varepsilon > 0$ there exists $a \in A$ such that $a \ge x - \varepsilon$; since we saw that $\sqrt{2}$ is a upper bound of A this is enough to prove that $\sup(A) = \sqrt{2}$. The same idea works to prove that $\inf(A) = -\sqrt{2}$; notice that one could also use the fact that $A = -A$, so that $\inf(A) = -\sup(A)$ (why ?), which shows $\inf(A) = -\sqrt{2}$. We saw in class that $\sqrt{2} \notin \mathbb{Q}$, so that $\sup(A) \notin A$, $\inf(A) \notin A$, and this proves that A has neither a maximal element nor a minimal element.
Let this time $A = \{(-1)^n + \frac{1}{n} : n \in \mathbb{N}\}$. We have $-1 \le (-1)^n + \le 1 + \frac{1}{n} \le 2$ for all $n \in \mathbb{N}$, so that -1 is a lower bound of A, 2 is a upper bound of A, and both $\sup(A)$ and $\inf(A)$ exist. Let us now compute $\inf(A)$: for all $\varepsilon > 0$, we have $\frac{1}{n} \le \varepsilon$ for all big enough n. Therefore, if n is odd, we have $(-1)^n + \frac{1}{n} \le -1 + \varepsilon$ and this, added to the fact that -1 is a lower bound of A, yields $-1 = \sup(A)$. Therefpre A doesn't have a minimal element ($-1 \notin A$). To compute $\sup(A)$, notice that for all $n \ge 2$ we have $(-1)^n + \frac{1}{n} \le 1 = \frac{1}{2} = \frac{3}{2}$. Since the element obtained for $n = 1$ is 0, this proves that $a \le \frac{3}{2}$ for all $a \in A$; and since $\frac{3}{2} \in A$, this is enough to conclude that $\sup(A) = \frac{3}{2}$ and that A has a maximal element.

2. Let $A = \{x^2 + y^2 : x, y \in \mathbb{R} \text{ and } xy = 1\}$. Prove that A is bounded below, but not bounded above. Compute $\inf(A)$.
Correction. For all $a \in A$, we have $a = x^2 + y^2$ for some $x, y \in \mathbb{R}$. It implies that $a \ge 0$, so 0 is a lower bound of A, which proves that A is bounded below.
To prove that A is not bounded above, notice that $n^2 + \frac{1}{n^2} \in A$ for all $n \in \mathbb{N}$ (because $n.\frac{1}{n} = 1$ for all n), so any upper bound u of A would have to satisfy $u \ge n^2$ for all $n \in \mathbb{N}$. We saw in class that $\mathbb{N}$ is not bounded above, so this is impossible, and this proves that A has no upper bound.
To compute $\inf(A)$, let x and y be such that $xy = 1$. Then $x^2 + y^2 = x^2 + \frac{1}{x^2}$. Looking at a picture of a circle and a hyperbola (what is the link here ?), one can guess that $inf(A)$ is attained for $x = y = 1$. To prove this, notice that since the equation for A depends only on $|x|$ and $|y|$ one may assume that both are positive ; furthermore, one of $|x|$ and $|y|$ has to be bigger than 1 (why ?), so one may assume without loss of generality

that $x \geq 1$. We have then $x = 1+\varepsilon$, with $\varepsilon > 0$. Then one can write

$$x^2+\frac{1}{x^2} = (1+\varepsilon)^2+\frac{1}{(1+\varepsilon)^2} = 1+2\varepsilon+\varepsilon^2+\frac{1}{1+2\varepsilon+\varepsilon^2} = 1+\frac{(2\varepsilon+\varepsilon^2)(1+2\varepsilon+\varepsilon^2)+1}{1+2\varepsilon+\varepsilon^2} = 2+\frac{2\varepsilon^3+\varepsilon^4}{1+2\varepsilon+\varepsilon^2} \geq 2\,.$$

Therefore 2 is a lower bound of A; since one has $2 \in A$, this proves that $\inf(A) = 2$.

3. Let $A, B \subset \mathbb{R}$ be bounded subsets of $\mathbb{R}$. We define $A+B = \{a+b \colon a \in A, b \in B\}$.
Show that $\sup(A)$, $\sup(B)$, $\sup(A+B)$ exist and that $\sup(A+B) = \sup(A)+\sup(B)$.
Correction.
By definition of a bounded set, there exist M, N such that $a \leq M$ for all $a \in A$, and $b \leq N$ for all $b \in B$. This implies that $a+b \leq M+N$ for all $(a,b) \in A \times B$; in other words, $x \leq M+N$ for all $x \in A+B$, which proves that $A+B$ is bounded above, so that $\sup(A+B)$ exists. The fact that $\sup(A)$, $\sup(B)$ exist is a direct consequence of the fact that A, B are bounded.
Notice that above we could have taken $M = \sup(A)$, $N = \sup(B)$, so that the preceeding inequality implies that $x \leq \sup(A)+\sup(B)$ for all $x \in A+B$; in other words, $\sup(A)+\sup(B)$ is an upper bound of $A+B$, so that $\sup(A+B) \leq \sup(A)+\sup(B)$.
To show the converse inequality, we need to find, for all $\varepsilon > 0$, some $x \in A+B$ such that $x \geq \sup(A)+\sup(B)-\varepsilon$. We know that, for all $\delta > 0$, there exists $a \in A$ such that $a \geq \sup(A)-\delta$, and $b \geq \sup(B)-\delta$; this implies $a+b \geq \sup(A)+\sup(B)-2\delta$. Thus, if we now let $\delta = \frac{\varepsilon}{2}$, the above inequality becomes $a+b \geq \sup(A)+\sup(B)-\varepsilon$. Therefore there does exist, for all $\varepsilon > 0$, some $x \in A+B$ such that $x \geq \sup(A)+\sup(B)-\varepsilon$. This concludes the proof of the fact that $\sup(A+B) = \sup(A)+\sup(B)$.

4. Let $A \subset \mathbb{R}$ be a bounded set containing at least two elements, and $x \in A$.
(a) Prove that $\sup(A \setminus \{x\})$ exists (remember that $A \setminus \{x\} = \{a \in A \colon a \neq x\}$.
(b) Prove that if $x < \sup(A \setminus \{x\})$ then $\sup(A \setminus \{x\}) = \sup(A)$. (c) Prove that if $\sup(A \setminus \{x\}) < \sup(A)$ then $x = \sup(A)$.
Correction. (a) $A \setminus \{x\} \subset A$, so any upper bound of A is also an upper bound of $A \setminus \{x\}$); since A is bounded, this proves that the set of upper bounds of $A \setminus \{x\}$) is nonempty, so that $\sup(A \setminus \{x\}))$ exists (and is $\leq \sup(A)$).
(b) The fact that $A \setminus \{x\}) \subset A$ implies that $\sup(A \setminus \{x\})) \leq \sup(A)$. To see that the converse inequality is true in our case, let $\varepsilon > 0$ be small enough that $\sup(A)-\varepsilon > x$. By definition of the sup, there exists $a \in A$ such that $a > \sup(A)-\varepsilon$. This implies that $a \neq x$, so that we actually proved that for all $\varepsilon > 0$ there is $a \in A \setminus \{x\}$ such that $a > \sup(A)-\varepsilon$. This shows that $\sup(A)-\varepsilon$ is not an upper bound of $A \setminus \{x\}$, so $\sup(A)-\varepsilon \leq \sup(A \setminus \{x\})$, for all $\varepsilon > 0$, so that $\sup(A) \leq \sup(A \setminus \{x\}$. This concludes the proof of the fact that $\sup(A) = \sup(A \setminus \{x\}$.
(c) Assume that $\sup(A \setminus \{x\}) < \sup(A)$, and pick $\varepsilon > 0$ small enough that $\sup(A)-\varepsilon > \sup(A \setminus \{x\})$. By definition of the sup, there exists $a \in A$ such that $a \geq \sup(A)-\varepsilon$; in particular $a > \sup(A \setminus \{x\})$, so $a \notin A \setminus \{x\}$. Since $a \in A$, this implies that $a = x$. Thus we obtained $x \geq \sup(A)-\varepsilon$ for all ε. This implies that $x \geq \sup(A)$, and $x \leq \sup(A)$ is also true because $x \in A$. We finally obtained $x = \sup(A)$.

5. Let A, B be bounded subsets of $\mathbb{R}$. Prove that $A \cup B$ is also bounded and that $\sup(A \cup B) = \max(\sup(A), \sup(B))$, $\inf(A \cup B) = \min(\inf(A), \inf(B))$.
Correction. Let M (resp. M') be an upper bound of A (resp B), and m (resp. m') be a lower bound of A (resp. M'). Then, for all $x \in A$ we have $m \leq x \leq M$, and for all $x \in B$ we have $m' \leq x \leq M'$. Thus, for all $x \in A \cup B$ we have $\min(m, m') \leq x \leq \max(M, M')$.
This shows that $\min(m, m')$ is a lower bound for $A \cup B$, and $\max(M, M')$ is an upper bound for $A \cup B$. Thus, $A \cup B$ is bounded. Notice that we could have take $M = \sup(A)$, $M' = \sup(B)$, $m = \inf(A)$, $m' = \inf(B)$; thus the above reasoning implies that $\min(\inf(A), \inf(B))$ is a lower bound of $A \cup B$, and $\max(\sup(A), \sup(B))$ is an upper bound of $A \cup B$.
For any $\varepsilon > 0$ there exist $a \in A$ and $b \in B$ such that $a \leq \inf(A)+\varepsilon$ and $b \leq \inf(B)+\varepsilon$; therefore, $a \leq \min(\inf(A), \inf(B))+\varepsilon$ (if $\inf(A) \leq \inf(B)$) or $b \leq \min(\inf(A), \inf(B))+\varepsilon$ (if $\inf(B) \leq \inf(A)$). This means that for any $\varepsilon > 0$ there exists $x \in A \cup B$ such that $x \leq \min(\inf(A), \inf(B))+\varepsilon$. This, added to the fact that $\min(\inf(A), \inf(B))$ is a lower bound of $A \cup B$, implies that $\min(\inf(A), \inf(B)) = \inf(A \cup B)$.
The proof for the least upper bound is similar.