University of Illinois at Urbana-Champaign Fall 2006
Math 444 Group E13

Graded Homework V

Correction.

1 Using the definition of the limit, show that the following sequences are convergent and compute their limit : $x_n = \frac{1}{n+1} - \frac{1}{n}$; $y_n = \sqrt{n+1} - \sqrt{n}$.

Correction. We have $x_n = \frac{1}{n+1} - \frac{1}{n} = \frac{1}{n(n+1)}$, so it seems that $\lim x_n = 0$. To prove it, pick $\varepsilon > 0$. By the Archimedean Property of the reals, we know that there exists a natural integer $K(\varepsilon)$ such that $K(\varepsilon) \geq \frac{1}{\varepsilon}$. Then we get, for any $n \geq K(\varepsilon)$, that

$$x_n = \frac{1}{n(n+1)} \leq \frac{1}{n} \leq \frac{1}{K(\varepsilon)} \leq \varepsilon .$$

This concludes the proof that the sequence (x_n) converges to 0 ; to deal with the sequence y_n, we use the usual trick for square roots to get

$$y_n = \sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} .$$

Therefore, we guess that here again the limit of (y_n) exists and is 0 ; to prove it, picking $\varepsilon > 0$, we first notice that, as above, there exists a natural integer $K(\varepsilon)$ such that $K(\varepsilon) \geq \frac{1}{\varepsilon^2}$. Thus, for any $n \geq K(\varepsilon)$, we have

$$y_n = \frac{1}{\sqrt{n+1} + \sqrt{n}} \leq \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{K(\varepsilon)}} \leq \varepsilon .$$

2. Using the theorems that we saw in class (Squeeze Theorem, algebraic manipulations of limits), determine whether the following sequences are convergent and, if they are, compute their limit.
$x_n = \frac{(n+1)^3}{n^3}$; $y_n = \frac{\sin(n)}{\sqrt{n}}$; $z_n = \frac{\sqrt{n}}{n + \sin(n)}$.

Correction. The first sequence is a quotient of two divergent sequences, so it seems at first that we cannot use the theorems seen in class. But, dividing numerator and denominator by n^3, we get that
$x_n = (1 + \frac{1}{n})^3 = (1 + \frac{1}{n})(1 + \frac{1}{n})(1 + \frac{1}{n})$. Since the sequence $(\frac{1}{n})$ converges to 0 (archimedean property of the reals), we see that (x_n) is the product of three sequences that converge to 1. Since the product of a finite number of convergent sequences converges to the product of the limits, this proves that (x_n) is convergent and that $\lim x_n = 1.1.1 = 1$.
For the second sequence, we use the Squeeze Theorem : since $-1 \leq \sin(n) \leq 1$ for all $n \in \mathbb{N}$, we have $\frac{-1}{\sqrt{n}} \leq y_n \leq \frac{1}{\sqrt{n}}$. Thus (y_n) is squeezed between two sequences that converge to 0, which implies that (y_n) is convergent and $\lim y_n = 0$.
For the third one, we divide numerator and denominator by $\sqrt{n}$ and obtain $z_n = \frac{1}{\sqrt{n} + \frac{\sin(n)}{n}}$. Thus, using the fact that $-1 \leq \frac{\sin(n)}{n} \leq 1$ for all $n \in \mathbb{N}$, one has $\frac{1}{\sqrt{n}+1} \leq z_n \leq \frac{1}{\sqrt{n}-1}$ for any $n \geq 2$. This proves that (z_n) is squeezed between two sequences which converge to 0, which implies that (z_n) is convergent and $\lim z_n = 0$.

3.. Using the definition of $E(x)$ given in the last homework, and the fact that $E(x) \leq x < E(x)+1$, prove that, for any $x \in \mathbb{R}$, one has $\lim(\frac{E(nx)}{n}) = x$.
(Optional) Can you use this to prove the Density Theorem ?

Correction. One has $E(nx) \leq nx < nx+1$ which, dividing by n, yields $\dfrac{E(nx)}{n} \leq x < \dfrac{E(nx)}{n} + \dfrac{1}{n}$, and thus $0 \leq x - \dfrac{E(nx)}{n} < \dfrac{1}{n}$. Thanks to the Squeeze Theorem, we can conclude that the sequence $(\dfrac{E(nx)}{n} - x)$ converges to 0, which is equivalent to saying that $\lim \dfrac{E(nx)}{n} = n$.

To use this to prove the Density Theorem, notice that each $\dfrac{E(nx)}{n}$ is a rational number, so the above proof shows that any real is the limit of a sequence of rationals (q_n) such that $q_n \leq x$. Pick now $x < y \in \mathbb{R}$, and let $\varepsilon > 0$ be such that $y - \varepsilon > x$. We know that there exists a sequence of rationals (q_n) such that $q_n \leq y$ and $\lim q_n = y$; pick such a sequence, and let n be big enough that $y - q_n \leq \varepsilon$. Then we have $y \geq q_n \geq y - \varepsilon > x$, which shows that q_n is a rational number such that $x < q_n \leq y$. If $y \notin \mathbb{Q}$ then we are done, otherwise notice that $y - \dfrac{1}{n} \in \mathbb{Q}$ for all $n \in \mathbb{N}$, and for n big enough one has $x < y - \dfrac{1}{n} < y$, so the desired result is proved in that case too.

4. Recall that $n!$ is defined by induction by $1! = 1$, $(n+1)! = (n+1)n!$. Said differently, one has $n! = 1.2.3 \dots n$. Define now $u_n = \dfrac{n!}{n^3}$.
(a) Prove that there exists some $N \in \mathbb{N}$ such that, for all $n \geq N$, one has $\dfrac{u_{n+1}}{u_n} \geq 2$.
(b) Prove by induction that, for all $n \geq N$, one has $u_n \geq u_N.2^{n-N}$.
(c) Use this to show that the sequence (u_n) is not convergent.

Correction. (a) By definition of u_n, one has, for any $n \in \mathbb{N}$, that

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)!}{n!}\frac{n^3}{(n+1)^3} = n\frac{n^3}{(n+1)^3} .$$

We saw in the second exercise that $\dfrac{n^3}{(n+1)^3}$ converges to 1 (it is the inverse of a sequence that converges to 1), so there exists an integer $K(\frac{1}{2})$ such that, for any $n \in \mathbb{N}$, one has $n \geq K(\frac{1}{2}) \Rightarrow \dfrac{n^3}{(n+1)^3} \geq 1 - \dfrac{1}{2} = \dfrac{1}{2}$. Thus, for any $n \geq K(\frac{1}{2})$, one has $\dfrac{u_{n+1}}{u_n} \geq \dfrac{n}{2}$, which proves that, for any $n \geq \max(K(\frac{1}{2}), 4) = N$ one has $\dfrac{u_{n+1}}{u_n} \geq 2$.
Before going on to question (b), it is worth explaining a bit what's going on here : we have a sequence whose terms become arbitrarily large (n), and one whose terms become increasingly close to 1 ($\frac{(n+1)^3}{n^3}$). To *prove* that the terms of the product of these two sequences become arbitrarily large (here, larger than 2, but you should convince yourself that 2 may be replaced by any real number), we do not exactly use the fact that the second sequence converges to 1 : instead, we use this fact to deduce that, for n big enough, the sequence only takes values that are greater than $\frac{1}{2}$, and deduce the desired result from it. Think about this proof!
(b) The desired statement clearly holds for $n = N$; assume now that $n \geq N$ is such that $u_n \geq u_N.2^{n-N}$. Then, by (a), one has $u_{n+1} = \dfrac{u_{n+1}}{u_n}.u_n \geq 2u_n \geq 2.2^{n-N} = 2^{n+1-N}$. This proves that the property is hereditary ; since it is true for $n = N$, it must hold for all $n \geq N$.
(c) The sequence (2^{n-N}) is not bounded, so the inequality of question (b) shows that (u_n) is not bounded either ; since any convergent sequence has to be bounded, this is enough to conclude that (u_n) is not convergent.