University of Illinois at Urbana-Champaign | Fall 2006
Math 444 | Group E13

Graded Homework VI
Due Friday, October 20.

1. Let A be a bounded subset of $\mathbb{R}$. Show that there exists a sequence (a_n) of elements of A such that $\lim(a_n) = \sup(A)$.

Correction. We build by induction a sequence (a_n) of elements of A such that $a_n \geq \sup(A) - \dfrac{1}{n}$ for all $n \in \mathbb{N}$. To see that we can do this, first pick for a_1 any element in A such that $a_1 \geq \sup(A) - 1$. Then, assume that $a_1, \ldots, a_n$ have been defined. By definition of $\sup(A)$, there is some $a \in A$ such that $\sup(A) - \dfrac{1}{n+1} \leq a$. Then, set $a_{n+1} = a$.
This shows that, using an inductive definition, we can indeed build a sequence a_n of elements of A such that $a_n \geq \sup(A) - \dfrac{1}{n}$; since $a_n \leq \sup(A)$ because $a_n \in A$, this implies that $|a_n - \sup(A)| \leq \dfrac{1}{n}$, and this inequality ensures that (a_n) converges to $\sup(A)$.

2. Define, for $n \in \mathbb{N}$, $u_n = 1 + \dfrac{1}{2!} + \ldots \dfrac{1}{n!}$ and $v_n = u_n + \dfrac{1}{n.n!}$.
(a) Show that (u_n) is increasing, (v_n) is decreasing, $u_n \leq v_n$ and $\lim(u_n - v_n) = 0$.
(b) Prove that both sequences converge to the same limit (called e.).
(c) Show that, for any n, one has the inequality $u_n < e < u_n + \dfrac{1}{n.n!}$.
(d) Use this to show that e is irrational.

Correction. $u_{n+1} - u_n = \dfrac{1}{(n+1)!} > 0$, so (u_n) is stricly increasing. To see that (v_n) is decreasing, compute

$$v_{n+1} - v_n = \frac{1}{(n+1)!} + \frac{1}{(n+1)(n+1)!} - \frac{1}{n.n!} = \frac{n(n+1) + n - (n+1)^2}{n(n+1)(n+1)!} = \frac{-1}{n(n+1)(n+1)!} < 0 .$$

This proves that (v_n) is strictly ecreasing; the fact that $u_n \leq v_n$ is a direct consequence of the definition of the two sequences, since $v_n - u_n = \dfrac{1}{n.n!} \geq 0$. As for $\lim(u_n - v_n)$, we have $0 \leq u_n - v_n = \dfrac{1}{n.n!} \leq \dfrac{1}{n}$, so the Squeeze Theorem (and the fact that the Archimedean property of the reals ensures that $\lim \dfrac{1}{n} = 0$) yields $\lim(u_n - v_n) = 0$.
(b) (u_n) is increasing and is bounded above (any v_m is an upper bound of $\{u_n : n \in \mathbb{N}\}$), so (u_n) is convergent. Similarly, (v_n) is decreasing and bounded below, so it is convergent. We may then write that $\lim u_n - \lim v_n = \lim(u_n - v_n) = 0$ (proved in question (a)), so $\lim(u_n) = \lim(v_n) = e$.
(c) Recall that the limit of an increasing sequence (u_n) (if it exists) is equal to $\sup(u_n)$, so $u_n \leq e$ for all n. To prove the strict inequality, notice that $u_n < u_{n+1} \leq e$, so that actually $u_n < e$ for all $n \in \mathbb{N}$. Similarly, v_n is strictly decreasing so $v_n > \lim v_n = e$. Put together, these two inequalities, yield $u_n < e < u_n + \dfrac{1}{n.n!}$.
(d) Assume that $e \in \mathbb{Q}$, that is $e = \dfrac{p}{q}$ for some $q \in \mathbb{Q}$. Then, using the inequality from question (c) with $n = q$, we get $q!u_q < q!e < q!u_q + \frac{1}{q}$. This means that $|q!u_q - q!e| < \dfrac{1}{q}$. Yet, $q!u_q \in \mathbb{N}$, and $q!e = p(q-1)! \in \mathbb{N}$. Thus, the only way for $|q!u_q - q!e| < \dfrac{1}{q}$ to be true is if $q!u_q = q!e$, and it's impossible since $q!u_q < q!e$ according to question (c).

3. Let (u_n) be a sequence such that $\lim(u_n) = u \in \mathbb{R}$, and $\varphi\colon \mathbb{N} \to \mathbb{N}$ be a bijection (not necessarily increasing!). Show that $\lim(u_{\varphi(n)}) = u$.

Correction. Pick $\varepsilon > 0$. Since $\lim(u_n) = u$, there exists $K(\varepsilon)$ such that $n \geq K(\varepsilon) \Rightarrow |u_n - u| \leq \varepsilon$. Thus, we only need to prove that there exists some $N(\varepsilon)$ such that $n \geq N(\varepsilon) \Rightarrow \varphi(n) \geq K(\varepsilon)$. This would be easy if φ was increasing ($N(\varepsilon) = K(\varepsilon)$ would work in that case). Here, one can notice that, since $\varphi \colon \mathbb{N} \to \mathbb{N}$ is a surjection, there exist $n_1, \ldots, n_{K(\varepsilon)-1}$ such that $\varphi(n_1) = 1, \ldots, \varphi(n_{K(\varepsilon)-1}) = K(\varepsilon) - 1$. But then, since φ is injective, as soon as $n > \max(n_1, \ldots, n_{K(\varepsilon)-1})$, we know that $\varphi(n)$ has to be different from $\varphi(n_1) = 1, \ldots, \varphi(n_{K(\varepsilon)-1}) = K(\varepsilon) - 1$. This exactly means that, if $n \geq N(\varepsilon) = \max(n_1, \ldots, n_{K(\varepsilon)-1}) + 1$ then $\varphi(n) \geq K(\varepsilon)$. In particular, $n \geq N(\varepsilon) \Rightarrow |u_{\varphi(n)} - u| \leq \varepsilon$. This proves that $u_{\varphi(n)}$ is convergent, and $\lim(u_{\varphi(n)}) = u$.

4. Let (x_n) be a monotone sequence such that a subsequence of (x_n) is convergent. Show that (x_n) is convergent.
Correction. Assume first that (x_n) is increasing, and let $(x_{\varphi(n)})$ denote a convergent subsequence of (x_n); call its limit l. Fix $\varepsilon > 0$. We know that there exists $K_\varphi(\varepsilon)$ such that

$$n \geq K_\varphi(\varepsilon) \Rightarrow l - \varepsilon \leq x_{\varphi(n)} \leq l + \varepsilon \ .$$

Then, set $K(\varepsilon) = \varphi(K_\varphi(\varepsilon))$. For any $n \geq K(\varepsilon)$, we have $x_n \geq x_{K(\varepsilon)}$ (because (x_n) is increasing) , and $x_{K(\varepsilon)} = x_{\varphi(K_\varphi(\varepsilon))} \geq l - \varepsilon$. We also have, since $\varphi(n) \geq n$ (this was done in class), that $x_n \leq x_{\varphi(n)} \leq l$. The two inequalities together show that $l - \varepsilon \leq x_n \leq l$ for any $n \geq K(\varepsilon)$, and this proves that $\lim(x_n) = l$.
If (x_n) is decreasing, then $(-x_n)$ is increasing; if a subsequence $(x_{\varphi(n)})$ of (x_n) converges, then the corresponding subsequence $(-x_{\varphi(n)})$ of $(-x_n)$ is convergent, so the reasoning above proves that $(-x_n)$ is convergent, and thus that (x_n) is convergent.

5. Show that the sequences (u_n) and (v_n) defined by $u_n = (-1)^n + \dfrac{2}{n}$ and $v_n = \cos(\pi n^2)$ are not convergent.
Correction. One has $u_{2n} = 1 + \dfrac{1}{n}$, so that $\lim(u_{2n}) = 1$; similarly, $u_{2n+1} = 1 + \dfrac{2}{2n+1}$, so $\lim(u_{2n+1}) = -1$. Therefore (u_n) have two subsequences which converge to different limits, so (u_n) doe not converge.
Similarly, one has $v_{2n} = \cos(4\pi n^2) = 1$, and $v_{2n+1} = \cos(4\pi n^2 + 4\pi n + \pi) = -1$, so, for the same reason, (v_n) does not converge.