University of Illinois at Urbana-Champaign | Fall 2006
Math 444 | Group E13

Graded Homework VIII
Due Friday, November 3.

1. Let q be an integer larger than or equal to 2. For all $n \in \mathbb{N}$, define u_n by the formula $u_n = \cos(\frac{2n\pi}{q})$. Compute u_{nq}, u_{nq+1} ; is the sequence (u_n) convergent ?

2. Let $A \subset \mathbb{R}$. A function $f\colon A \to A$ is said to be *increasing* if $x \leq y \Rightarrow f(x) \leq f(y)$ for all $x, y \in A$. Similarly, one may define what a *decreasing* function is : f is decreasing if $x \leq y \Rightarrow f(x) \geq f(y)$ for all $x, y \in A$.
1. Prove that if f is decreasing then $f \circ f$ is increasing.
2. Let now (u_n) be a sequence such that $u_{n+1} = f(u_n)$, where $u_1 \in [0,1]$ and $f\colon [0,1] \to [0,1]$ is a function.
2.a. Prove that if f is increasing then (u_n) is monotone.
2.b. Prove that if f is decreasing then (u_{2n}) and (u_{2n+1}) are monotone.

3.Prove that a subset A of $\mathbb{R}$ is dense if, and only if, for any real number x there exists a sequence (a_n) of elements of A such that $\lim(a_n) = x$.

4. Given a sequence of real numbers (x_n), we say that $\lim(u_n) = +\infty$ if, and only if, for any $M \in \mathbb{R}$ there exists a naturel number N such that for any $n \in \mathbb{N}$ one has $n \geq N \Rightarrow u_n \geq M$.
1.a. Prove that a if sequence (x_n) is such that $\lim(x_n) = +\infty$ then all of its subsequences $x_{\varphi(n)})$ are such that $\lim(x_{\varphi(n)}) = +\infty$.
1.b. Prove that if (x_n) is a sequence of positive reals such that $\lim(x_n) = +\infty$ is not true then (x_n) has a bounded subsequence.
1.c. Prove that a sequence of positive reals (x_n) is such that $\lim(x_n) = +\infty$ if, and only if, it doesn't have a convergent subsequence.
2. We wish to prove that, if $\alpha > 0$ is an irrational number and (p_n), (q_n) are sequence of natural integers such that $\lim\left(\frac{p_n}{q_n}\right) = \alpha$ then $\lim(p_n) = +\infty$ and $\lim(q_n) = +\infty$.
2.a. Pick an irrational number $\alpha > 0$; explain why there exist sequences (p_n), (q_n) as above.
In the following questions we assume we have picked α, (p_n), (q_n) as above.
2.b Prove that if $\lim(q_n) = +\infty$ then $\lim(p_n) = +\infty$.
2.c. Prove that if (q_n) is not such that $\lim(q_n) = +\infty$ then (q_n) admits a constant subsequence $(q_{\psi(n)})$ (use 1.c ; what can you tell about a convergent sequence of integers ?).
2.d. Prove that $(p_{\psi(n)})$ is such that for n, m big enough one has $p_{\psi(n)} = p_{\psi(m)}$.
2.e. Conclude.