University of Illinois at Urbana-Champaign — Fall 2006
Math 444 — Group E13

Graded Homework IX

Due Monday, November 13.

1. Let (u_n) be a sequence of real numbers. We say that $a \in \mathbb{R}$ is an *accumulation point* of (u_n) if there exists a subsequence of (u_n) which converges to a.
(a) What are the accumulation points of a convergent sequence?
(b) What are the accumulation points of the sequence $u_n = \cos(n\frac{\pi}{3})$?
(c) Let (u_n) be a bounded, divergent sequence. Prove that it has at least two (distinct) accumulation points (Hint : why does there exist one accumulation point? Can you use the fact that this point is not the limit of (u_n)?)
Correction. (a) If a sequence is convergent to a limit l, then all its subsequences are convergent to that same limit, so a convergent sequence has exactly one accumulation point : its limit.
(b) One can see that for all $n \in \mathbb{N}$ one has $u_{6n} = 1$, $u_{6n+1} = cos(\frac{\pi}{3}) = \frac{1}{2}$, $u_{6n+2} = \cos(\frac{2\pi}{3}) = -\frac{1}{2}$, $u_{6n+3} = \cos(\pi) = -1$, $u_{6n+4} = \cos(\frac{4\pi}{3}) = -\frac{1}{2}$, and $u_{6n+5} = \cos(\frac{5\pi}{3}) = \frac{1}{2}$. Thus, 1, $\frac{1}{2}$, -1 $-\frac{1}{2}$ are accumulation points of (u_n). Conversely, if $\varphi\colon \mathbb{N} \to \mathbb{N}$ is any strictly increasing map, there exists at least one $k = 0, 1, \ldots, 5$ and infinitely many n such that $\varphi(n) = 6m+k$ (because $\mathbb{N}$ is infinite and for any $n \in \mathbb{N}$ the remainder of its euclidean division by 6 is in the set $\{0, 1, 2, 3, 4, 5\}$, and if an infinite set is the union of six subsets then at least one of these subsets is infinite). This implies that any subsequence of (u_n) has a further subsequence which is also a subsequence of (u_{n+k}) for some $k = 0, 1, \ldots, 5$ (there may be more than one such k). But then, if $(u_{\varphi(n)})$ is convergent, its limit has to be the same as that of any of its subsequences, so that $\lim(u_{\varphi(n)}) \in \{1, -1, \frac{1}{2}, -\frac{1}{2}\}$; this shows that these are the only accumulation points of (u_n).
(c) The Bolzano-Weierstrass theorem tells us that (u_n) has a convergent subsequence, with limit l (so it has a least one accumulation point) because it is a bounded sequence. Since we are told that (u_n) is not convergent, it cannot be convergent to l : this means that there exists $\varepsilon > 0$ such that for any $K \in \mathbb{N}$ there is $n \geq K$ such that $|u_n - l| \geq \varepsilon$. One can then use this to build a subsequence (x_{n_k}) of (x_n) with the property that $|x_{n_k} - l| \geq \varepsilon$. Indeed, one can pick any n_1 such that $|x_{n_1} - l| \geq \varepsilon$; assume now that $n_1, \ldots, n_k$ have been defined. Applying the above property for $K = n_k + 1$, we get that there exists some $n \geq K$ such that $|x_n - l| \geq \varepsilon$; pick some such n, set $n_{k+1} = n$, and go on to the next step.
So, we just proved that there exists a subsequence $(x_{\varphi(n)})$ of (x_n) with the property that $|x_{\varphi(n)} - l| \geq \varepsilon$; since it is bounded, $(x_{\varphi(n)})$ has a convergent subsequence, and its limit l' has to satisfy $|l' - l| \geq \varepsilon$. But l' is an accumulation point of (x_n), so we proved that a bounded divergent sequence of reals has at least two accumulation points.
Remark. The sequence may have just two accumulation points, as shown by the sequence defined by $x_n = (-1)^n$, or it may have any finite number of accumulation points (can you build an example?), or countably many accumulation points, or even uncountably many (for instance, a whole interval of accumulation points). That being said, not just any set can be a set of accumulation points for a given sequence; sets with this property are the *closed* subsets of the real line, and are an important class of subsets of the real line.

2. We define a sequence by setting $u_1 = \frac{1}{2}$, $u_{n+1} = 1 - u_n^2$. Show that (u_{2n}) is increasing, u_{2n+1} is decreasing and both sequences are convergent. Show that (u_n) is divergent.
Correction. The function $f\colon [0,1] \to [0,1]$ defined by $f(x) = 1 - x^2$ is decreasing. So $f \circ f$ is increasing, and we know (it was in the last homework assignment) that the sequences u_{2n}, u_{2n-1} are monotone. One has $u_1 = \frac{1}{2}$, $u_2 = \frac{3}{4}$, $u_3 = \frac{7}{16}$ and $u_4 = \frac{207}{256}$. We then see that $u_2 < u_4$, $u_1 > u_3$. Since (u_{2n-1}), (u_{2n}) are known to be monotone, this proves that (u_{2n}) is increasing, and (u_{2n-1}) (hence (u_{2n+1})) is decreasing. To compute the

limits l, l' of (u_{2n}), (u_{2n+1}), we use the inductive definition of the sequences to obtain $l = 1-(1-l^2)^2 = 2l^2-l^4$. Thus we get $l(l^3-2l+1)=0$, or $l(l-1)(l^2+l-1)=0$. Using the fact that l is between 0 and 1, we obtain that l is equal to 0, 1 or $\frac{\sqrt{3}-1}{2}$. Since u_{2n} is increasing, and $u_2 = \frac{3}{4}$, the only possible limit for (u_{2n}) is $l=1$. But then, since $u_{2n+1} = 1-u_{2n}^2$, algebraic manipulation of limits yields $l' = 1-l^2 = 0$. Since two sequences of (u_n) converge to different limits, we see that (u_n) is not convergent.

3. Let a, b be two reals different from 0. We define a sequence (u_n) by setting $u_1 = u \neq 0$, $u_{n+1} = a + \frac{b}{u_n}$. We assume that u is chosen in such a way that $u_n \neq 0$ for all $n \in \mathbb{N}$.
(a) What are the possible limits for (u_n) ?
(b) We suppose that the equation $x^2 = ax+b$ has two distinct solutions $\alpha, \beta \in \mathbb{R}$ and that $\alpha < \beta$. Prove that the sequence defined by $v_n = \frac{u_n-\alpha}{u_n-\beta}$ is geometric (i.e $\frac{v_{n+1}}{v_n}$ is constant) and use this to determine the limit of (u_n) (depending on u).
Correction. (a) If (u_n) was convergent to a limit l, then u_{n+1} would be convergent to the same limit. Thus one would have $l = a + \frac{b}{l}$, in other words a possible limit l of (u_n) has to be a solution of the equation $l^2 = al + b$.
(b) If $\alpha < \beta$ are solutions of the equation $x^2 = ax + b$, then they are different from 0 (because $b \neq 0$) and one has $a = \alpha - \frac{b}{\alpha} = \beta - \frac{b}{\beta}$. Thus,

$$v_{n+1} = \frac{u_{n+1}-\alpha}{u_{n+1}-\beta} = \frac{a+\frac{b}{u_n}-\alpha}{a+\frac{b}{u_n}-\beta} = \frac{\alpha-\frac{b}{\alpha}+\frac{b}{u_n}-\alpha}{\beta-\frac{b}{\beta}+\frac{b}{u_n}-\beta} = \frac{b\frac{\alpha-u_n}{u_n\alpha}}{b\frac{\beta-u_n}{u_n\beta}} = \frac{\beta}{\alpha}\cdot\frac{u_n-\alpha}{u_n-\beta} = \frac{\beta}{\alpha}v_n \ .$$

Thus, we obtain $v_n = \left(\frac{\beta}{\alpha}\right)^n v_1$.
Now, there are several possibilities :
• $|\frac{\beta}{\alpha}| < 1$; then we get that (v_n) converges to 0 no matter what v_1 is, thus $\frac{u_n-\alpha}{u_n-\beta}$ converges to 0, and this is only possible if (u_n) converges to α ; since (v_n) is defined only if $u \neq \beta$, in which case $u_n = \beta$ for all n, we get that either (u_n) is constant equal to β (if $u = \beta$), or it converges to α (in all the other cases).
• $|\frac{\beta}{\alpha}| > 1$; then we see that $|v_n|$ diverges to $+\infty$ (provided that $v_1 \neq 0$), and this is possible only if u_n converges to β (this is the same argument as before : indeed $\frac{1}{v_n}$ converges to 0, so (u_n) converges to β). Thus either (u_n) is constant equal to α (if $u = \alpha$) or is convergent to β (in all the other cases).
• $\alpha = -\beta$; then we get $v_n = (-1)^{n+1}v_1$, so (u_n) is not convergent unless one has $u = \alpha$ or $u = \beta$, in which case the sequence (u_n) is constant ; in all the other cases the sequence (u_n) is not convergent.

4. Pick $0 < x_1 < y_1$ and define two sequences $(x_n), (y_n)$ by setting $\begin{cases} x_{n+1} = \frac{x_n^2}{x_n+y_n} \\ y_{n+1} = \frac{y_n^2}{x_n+y_n} \end{cases}$. Show that these sequences are convergent and compute their liimit.
Correction. First, one can prove by induction that for all n one has $x_n > 0$, $y_n > 0$: it is true for $n = 1$, and if it is true for some n then $x_{n+1} = \frac{x_n^2}{x_n+y_n} > 0$, $y_{n+1} = \frac{y_n^2}{x_n+y_n} > 0$. But then for all $n \in \mathbb{N}$ one has $x_n + y_n > x_n$ and $x_n + y_n > y_n$, hence $x_{n+1} = \frac{x_n^2}{x_n+y_n} < \frac{x_n^2}{x_n} = x_n$, and similarly $y_{n+1} = \frac{y_n^2}{x_n+y_n} < y_n$. This shows that the two sequences (x_n), (y_n) are decreasing ; since they are bounded below by 0, we know that the sequences are convergent (let us call their respective limits x, y). The problem is now to find enough information from the definition of the sequences to determine x, y. First, we can write that $x_n(x_n+y_n) = x_n^2$, which yields $x(x+y) = x^2$, or $xy = 0$. Thus, $x = 0$ or $y = 0$. Going back to the definition of our sequences, we see (again, an easy induction proof works) that $x_n < y_n$ for all $n \in \mathbb{N}$; this implies that $x \leq y$. Since both $x, y \geq 0$, $xy = 0$ is only possible if $x = 0$. We have now determined one of the limits (this was the easy one) ; the problem is to find some information about the other one. The definition of y_{n+1} only yields again that $x = 0$, so we need to do something more, using the definition of the sequences ; here, one has, for all $n \in \mathbb{N}$, that

$x_{n+1} - y_{n+1} = \dfrac{x_n^2 - y_n^2}{x_n + y_n} = x_n - y_n$. Thus, the sequence $(x_n - y_n)$ is a constant sequence, so $x_n - y_n = x_1 - y_1$ for all $n \in \mathbb{N}$. But then, the algebraic theorems about limits give us $x - y = x_1 - y_1$, and since $x = 0$ we get $y = y_1 - x_1$.

Remark. Actually one could have proved that $\dfrac{x_n}{y_n} = \left(\dfrac{x_1}{y_1}\right)^{2^n}$ (why ?) and then obtained, using the fact that $x_n - y_n$ is constant, a formula for x_n, y_n.