Some geometric and dynamical properties of the Urysohn space

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Abstract

This is a survey article about the geometry and dynamical properties of the Urysohn space. Most of the results presented here are part of the author's Ph.D thesis and were published in the articles [Me1], [Me2] and [Me3]; a few results are new, most notably the fact that Iso($\mathbb{U}$) is not divisible.

Introduction.

This paper has two main objectives: presenting the author's results about the Urysohn space and its isometry group in a unified setting, and providing an introduction to the techniques and methods that are commonly used to study this space. Hopefully the paper is sufficiently self-contained to be of use to people who haven't worked on the Urysohn space before, and still covers basic material fast enough not to bore people who already know it.

Urysohn's universal metric space $\mathbb{U}$ was characterized in Urysohn's original paper [Ur] as being, up to isometry, the unique Polish metric space with the following two properties:

- Given any two isometric finite metric subsets $A, A' \subset \mathbb{U}$, and any isometry $\varphi\colon A \to A'$, there exists an isometry $\tilde{\varphi}$ of $\mathbb{U}$ which extends φ;
- Any separable metric space is isometric to a subspace of $\mathbb{U}$.

The first property is now called *ω-homogeneity* (or *ultrahomogeneity*) the second is called *universality*. There are other examples of ω-homeogenous Polish metric spaces, for example the Hilbert space; similarly, there are other universal Polish metric spaces, the best-known example being perhaps $\mathcal{C}([0,1])$. It was universality which interested Urysohn when he built $\mathbb{U}$, but it's the combination of both properties that makes it an important and fascinating geometric object, which may be thought of as an analogue of the random graph (within the more general setting of Polish metric spaces). Remarkably, this space was constructed more than 30 years before the random graph was!

There was little interest in this space during the first 50 years after its construction; Katětov's work in [Kat], and the way Uspenskij applied it in [Usp2] to prove that its group of isometries is a universal Polish group, finally piqued the curiosity of mathematicians. Most notably, A. Vershik lobbied to generate interest in the Urysohn space, and since 10 years or so it is actively studied. In this article, we focus on the geometry of the Urysohn space, and some of its dynamical properties (by which we mean properties of isometries and the way they act on the space); most of the results presented here have already been published and were part of the author's Ph.D thesis (and of the articles [Me1], [Me3]). Some results are new, including the construction of translations (section 4), and the proof that $\mathrm{Iso}(\mathbb{U})$ is not a divisible group (section 5).
The paper is organized as follows: after setting the basic notations and definitions, we introduce and study Katětov maps, then recall Katětov's construction of the Urysohn space. Then we try to give the reader a feel for the geometry of this space via several examples and exercises. Over the remainder of the article, we study some properties of $\mathrm{Iso}(\mathbb{U})$ as a topological group, dynamical properties of isometries, and discuss quickly the embeddings of the Urysohn space into Banach spaces. Throughout the text, we propose exercises to the reader, the purpose of which is to help understand the geometry of $\mathbb{U}$ and the techniques that are used to study it; some exercises consist in verifying a technical lemma used in the proof of a theorem. Hints and references for the exercises are given at the end of the paper. We attribute, to the extent possible, each theorem to its author, and provide a reference to the article in which it was originally published. Hopefully, this should help the reader determine which results are due to the author, and which among those are presented here for the first time.

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1 Notations and definitions.

Throughout this paper, we'll be dealing with metric spaces (X,d). When there is no risk of confusion, we don't mention the metric on a metric space (X,d) and simply denote it as X (this will lead to statements such as "let X be a metric space ...").
A map $\varphi\colon (X,d) \to (Y,d')$ is said to be an *isometric map* if $d(x,x') = d'(\varphi(x),\varphi(x'))$ for all $x,x' \in X$. We say that φ is an *isometry* from (X,d) onto (Y,d') if it is a bijective isometric map.
Also, if X is a metric space and $x \in X$, we denote the closed (resp. open) ball with center d and radius r by $B(x,r]$ (resp. $B(x,r[$). The sphere $\{x' \in X\colon d(x,x') = r\}$ is denoted by $S(x,r)$.
A *Polish metric space* is a separable, complete metric space.
If X is a topological space such that there is a distance turning X into a Polish metric space, we say that the topology of X is Polish. We only use this notion in the setting of topological groups: a *Polish group* is defined as a topological group whose topology is Polish. For an introduction to the theory of Polish groups, see [BK] or [Ke2].
The reason we are focusing on Polish groups here is that they are the groups of transformations corresponding to isomorphisms of Polish spaces. To make this clear, define $\mathrm{Iso}(X)$ as the group of isometries of a metric space X, endowed with the pointwise convergence topology (i.e the topology it inherits as a subset of X^X endowed with the product topology). Then, $\mathrm{Iso}(X)$ is a Polish group if X is a Polish metric space. Conversely, Gao and Kechris proved in [GK] that, for any Polish group G, there exists a Polish metric space X such that G is isomorphic (as a topological group) to $\mathrm{Iso}(X)$.
Several constructions below will be based on the notion of *amalgam* of two metric spaces X,Y over a common metric subspace Z; we only use it in the case when Z is closed in X. To define it properly, assume that $Z \subset X$ is closed and $i\colon Z \to Y$ is an isometric embedding. Let A denote the disjoint union of X and Y; define a pseudo-distance d on A that extends the distances on X and Y by setting, for all $x \in X$ and all $y \in Y$, $d(x,y) = \inf\{d(x,z) + d(y,i(z))\colon z \in Z\}$. The metric amalgam of X and Y over Z is then defined as the metric space obtained by quotienting the pseudo-metric space (A,d) by the zeroset of d (in other words, "sticking" the two copies of Z together).
There is another essential definition to introduce; we discuss it in detail in the next section.

2 One-point metric extensions: Katětov maps.

Before we turn our attention to the real subject matter of this article, it seems worthwile to take the time to detail some properties of the so-called Katětov maps. These are the essential tool to study the Urysohn space. The reason these maps are of interest for us is that they appear naturally when one tries to build isometries, as we will see below.

Definition 2.1. A map $f\colon X \to \mathbb{R}$ is a *Katětov map* if

$$\forall x, y \in X \ |f(x) - f(y)| \leq d(x, y) \leq f(x) + f(y) \ .$$

We denote by $E(X)$ the set of Katětov maps on X.

These maps correspond to one-point metric extensions $X \cup \{z\}$ of X in the following way: f is a Katětov map if, and only if, setting $d(x, z) = f(x)$ defines an extension to $X \cup \{z\}$ of the distance d on X (in other words, the triangle inequality is still satisfied). This correspondence was known well before Katětov's time; his contribution was to introduce a distance between these maps, defined by

$$\forall f, g \in E(X) \ d(f, g) = \sup\{|f(x) - g(x)| \colon x \in X\} \ .$$

It is well-defined because, for any $x, x_0 \in X$ one has $|f(x) - d(x, x_0)| \leq f(x_0)$ and $|g(x) - d(x, x_0)| \leq g(x_0)$, so that necessarily $|f(x) - g(x)| \leq f(x_0) + g(x_0)$. An equivalent way of defining $d(f, g)$ is saying that it is equal to the smallest distance $d(z, z')$, where $X \cup \{z, z'\}$ is a two-point metric extension of X such that $d(z, x) = f(x)$ and $d(z', x) = g(x)$ for all $x \in X$ (the above majoration of $d(f, g)$ may then be seen as a consequence of the triangle inequality in any two-point metric extension of X).
Endowed with this distance, $E(X)$ is a complete metric space; it has several nice properties, which make it very useful for the type of problems we concern ourselves with here.

Proposition 2.2. *X embeds isometrically in $E(X)$ via the* Kuratowski map *$x \mapsto d(x, .)$; identifying X with its image under this embedding, one has $d(f, x) = f(x)$ for all $f \in E(X)$ and all $x \in X$, and each isometry of X admits a unique extension to an isometry of $E(X)$.*

Proof. The first two statements of the proposition are a direct consequence of the triangle inequallity. To see that isometries of X extend uniquely to isometries of $E(X)$, pick $\varphi \in \mathrm{Iso}(X)$, let $\tilde{\varphi}$ be an extension of φ (if there exists one) and pick $f \in E(X)$. Then one must have, for all $x \in X$, that

$d(\tilde{\varphi}(f), \varphi(x)) = d(f, x)$, which yields $\tilde{\varphi}(f)(\varphi(x)) = f(x)$. Hence $\tilde{\varphi}(f)(x) = f(\varphi^{-1}(x))$ for all $x \in X$. This shows that the only possible isometric extension of φ to $E(X)$ is defined by $\tilde{\varphi}(f)(x) = f(\varphi^{-1}(x))$. Conversely, if $\tilde{\varphi}$ is defined by the preceding equation, it is clearly onto, is indeed an extension of φ, and one has, for all $f, g \in E(X)$: $d(\tilde{\varphi}(f), \tilde{\varphi}(g)) = \sup_X |\tilde{\varphi}(f)(x) - \tilde{\varphi}(g)(x)| = \sup_X |f(\varphi^{-1}(x)) - g(\tilde{\varphi}^{-1}(x))| = d(f, g)$. $\Diamond$

Unfortunately for our purposes, $E(X)$ is not separable in general, and the extension morphism from $\mathrm{Iso}(X)$ to $\mathrm{Iso}(E(X))$ does not have to be continuous: there are two many possible one-point metric extensions of X, and they are too complicated. There does exist a remarkable separable subset of $E(X)$, which plays a fundamental role in the forthcoming constructions. To introduce it, we need to explain the notion of *Katětov extension*.

Definition 2.3. If $Y \subset X$ and $f \in E(Y)$, then its *Katětov extension* to X $\hat{f}$ is defined by

$$\forall x \in X \ \hat{f}(x) = \inf\{f(y) + d(x, y) \colon y \in Y\}$$

Geometrically, $\hat{f}$ corresponds to the one-point extension of X obtained by amalgamating $Y \cup \{f\}$ and X over Y. Thus $\hat{f}$ coincides with f on Y, and belongs to $E(X)$.
If $f \in E(X)$ and $Y \subset X$ are such that f is the Katětov extension of $f_{|Y}$, we say that Y is a *support* for f (notice that then any $Z \supset Y$ is also a support for f).

Definition 2.4. We let $E(X, \omega) = \{f \in E(X) \colon f \text{ has a finite support}\}$.

Exercise 1. Let $Y \subset X$. Prove that the Katětov extension from Y to X induces an isometric embedding of $E(Y)$ into $E(X)$, and of $E(Y, \omega)$ into $E(X, \omega)$.

We will often use an equivalent version of this statement: if $f, g \in E(X)$ have a common support $S \subset X$, then $d(f, g) = \sup\{|f(s) - g(s)| \colon s \in S\}$.

Exercise 2. Prove that X embeds isometrically in $E(X, \omega)$ via the Kuratowski map, and that the embedding is such that any isometry of X uniquely extends to an isometry of $E(X, \omega)$.

Proposition 2.5. *The extension morphism from Iso(X) to Iso$(E(X, \omega))$ is continuous.*

Proof. We have to show that, given any $f \in E(X, \omega)$, the map from $\mathrm{Iso}(X)$ to $E(X, \omega)$ defined by $\varphi \mapsto \tilde{\varphi}(f)$ is continuous. By definition, there are

$x_1, \ldots, x_m$ such that $f(x) = \inf\{f(x_i) + d(x, x_i)\}$ for all $x \in X$. Pick some $\varphi \in \mathrm{Iso}(X)$ and a sequence $(\varphi_n) \in \mathrm{Iso}(X)^{\mathbb{N}}$ that converges to φ in $\mathrm{Iso}(X)$. Then, given any $\varepsilon > 0$, there exists N such that $d(\varphi_n(x_i), \varphi(x_i)) \leq \varepsilon$ for all $n \geq N$ and all $i = 1, \ldots, m$. Given that $\tilde{\psi}(f)(x_i) = f(\psi^{-1}(x_i))$ for all $\psi \in \mathrm{Iso}(X)$, the triangle inequality implies that $\tilde{\varphi_n}(f)$ and $\tilde{\varphi}(f)$ differ by at most ε on their common support $\{\varphi(x_1), \ldots, \varphi(x_m)\} \cup \{\varphi_n(x_1), \ldots, \varphi_n(x_m)\}$; therefore, one must have $d(\tilde{\varphi_n}, \tilde{\varphi}) \leq \varepsilon$ for all $n \geq N$. $\Diamond$

We are now ready to move on to the study of the Urysohn space; before we do this, however, we wish to unearth a necessary and sufficient condition for $E(X)$ to be separable, which will be useful below when we study the homogeneity properties of the Urysohn space. The reader uninterested in this problem may safely skip the remainder of this section for the time being.

Proposition 2.6. [Me3] *If X is Polish and not Heine-Borel, then $E(X)$ is not separable.*

Proof. Recall that a metric space has the *Heine-Borel property* if closed bounded subsets of X are compact. If X doesn't have this property, then there exist $M, \varepsilon > 0$ and $(x_i)_{i\in\mathbb{N}}$ such that $\varepsilon \leq d(x_i, x_j) \leq M$ for all $i \neq j$.

For $A \subseteq \mathbb{N}$, define $f_A : \{x_i\}_{i\geq 0} \to \mathbb{R}$ by $f_A(x_i) = \begin{cases} M & \text{if } i \in A \\ M + \varepsilon & \text{else} \end{cases}$.

It is easy to check $f_A \in E(\{x_i\}_{i\geq 0})$, and if $A \neq B$ one has $d(f_A, f_B) = \varepsilon$. Hence $E(\{x_i\}_{i\geq 0})$ is not separable; since it is isometric to a subspace of $E(X)$, this concludes the proof. $\Diamond$

Definition 2.7. If (X, d) is a nonempty metric space and $\varepsilon > 0$, we say that a sequence $(u_n)_{n\in\mathbb{N}}$ in X is *ε-inline* if $\sum_{i=0}^{r} d(u_i, u_{i+1}) \leq d(u_0, u_{r+1}) + \varepsilon$ for every $r \geq 0$. A sequence $(u_n)_{n\in\mathbb{N}}$ in X is said to be *inline* if for every $\varepsilon > 0$ there exists $N \geq 0$ such that $(u_0, u_N, u_{N+1}, \ldots)$ is ε-inline.

Theorem 2.8. [Me3] *Let X be a Polish metric space.*
The following assertions are equivalent:
(a) $E(X) = \overline{E(X, \omega)}$.
(b) $E(X)$ is separable.
(c) For any $\delta > 0$, for any sequence (x_n) of elements of X, there exists an integer N such that

$$\forall n \geq N \, \exists i \leq N \;\; d(x_0, x_n) \geq d(x_0, x_i) + d(x_i, x_n) - \delta \; .$$

(d)Any sequence of elements of X admits an inline subsequence.

Proof of Theorem 2.8.
$(a) \Rightarrow (b)$ is obvious; let us show that $\neg(c) \Rightarrow \neg(b)$. Assume that, for some $\delta > 0$, X contains a sequence (x_n) such that

$$\forall N\, \exists n \geq N\, \forall i \leq N\, d(x_0, x_n) + \delta \leq d(x_0, x_i) + d(x_i, x_n) \ .$$

Then one may extract a subsequence of (x_n), which we still denote by (x_n), such that for all $i < j$ one has

$$d(x_0, x_j) + \delta \leq d(x_0, x_i) + d(x_i, x_j) \qquad (*)$$

This sequence cannot have a convergent subsequence, so if it is bounded then X is not Heine-Borel, hence $E(X)$ is not separable. If the sequence is unbounded, we may make the additional assumption that $d(x_0, x_{i+1}) \geq d(x_0, x_i) + 1$. Assume also for simplicity that $\delta = 1$. Then let $f\colon \{x_i\}_{i \geq 0} \to \mathbb{R}$ be defined by $f(x_i) = d(x_i, x_0)$. Obviously, f is a Katětov map. If $A \subseteq \mathbb{N}$ is nonempty, we let $f_A\colon \{x_i\}_{i\geq 0} \to \mathbb{R}$ be the Katětov extension of $f_{|\{x_i \colon i \in A\}}$.
Suppose now that $A \neq B$ are nonempty subsets of $\mathbb{N}$, let m be the smallest element of $A \Delta B$, and assume without loss of generality that $m \in A$. Then one has $f_A(x_m) = d(x_m, 0)$, and $f_B(x_m) = d(x_m, x_i) + d(x_i, 0)$ for some $i \neq m$. If $i < m$, then $(*)$ shows that $f_B(x_m) - f_A(x_m) \geq 1$; if $i > m$, then $f_B(x_m) - f_A(x_m) \geq d(x_i, 0) - d(x_m, 0) \geq 1$.
In any case, one obtains $d(f_A, f_B) \geq 1$ for any $A \neq B$, which shows that $E(\{x_i\}_{i\geq 0})$ is not separable. Hence $E(X)$ cannot be separable either.
To see that $(c) \Rightarrow (d)$, notice first that property (c) implies that, for any $\varepsilon > 0$ and any sequence $(x_n) \in X^{\mathbb{N}}$, one may extract a subsequence $(x_{\varphi(n)})$ with $\varphi(0) = 0$ such that

$$\forall n \leq m\ d(x_{\varphi(0)}, x_{\varphi(n)}) + d(x_{\varphi(n)}, x_{\varphi(m)}) \leq d(x_{\varphi(0)}, x_{\varphi(m)}) + \varepsilon \quad .$$

Then a diagonal process enables one to build the desired inline subsequence of (x_i).
It remains to prove that $(d) \Rightarrow (a)$. For that, suppose by contradiction that some Polish metric space X has property (d), but not property (a). Notice first that this implies that X is Heine-Borel. Indeed, assume by contradiction that there exist ε, $M > 0$ and a sequence $(x_n) \in X^{\mathbb{N}}$ such that $\varepsilon \leq d(x_n, x_m) \leq M$ for all $n < m$. Then this sequence cannot have an inline subsequence.
Choose now $f \in E(X) \setminus \overline{E(X, \omega)}$, and let f_n be the Katětov extension to X of $f_{|B(z,n]}$ (where z is some point in X). Then for all $x \in X$, $n \leq m$, one has $f_n(x) \geq f_m(x) \geq f(x)$; hence the sequence $(d(f_n, f))$ converges to some $a \geq 0$.

Notice that, since closed balls in X are compact, each f_n is in $\overline{E(X,\omega)}$: this proves that $a>0$, and one has $d(f_n,f)\geq a$ for all n.
One can then build inductively a sequence $(x_i)_{i\geq 1}$ of elements of X, such that for all $i\geq 1$ $d(x_{i+1},z)\geq d(x_i,z)+1$ and

$$f(x_i)\leq \min_{j<i}\{f(x_j)+d(x_i,x_j)\}-\frac{3a}{4}$$

Since $|f(x_i)-d(x_i,z)|\leq f(z)$, one can assume, up to some extraction, that $(f(x_i)-d(x_i,z))$ converges to some $l\in\mathbb{R}$.
Now, let $\delta=\frac{a}{4}$. Property (d) tells us that we can extract from the sequence (x_i) a subsequence $(x_{\varphi(i)})$ having the additional property that

$$\forall 1\leq j\leq i,\qquad d(z,x_{\varphi(i)})\geq d(z,x_{\varphi(j)})+d(x_{\varphi(i)},x_{\varphi(j)})-\delta$$

To simplify notation, we again call that subsequence (x_i).
Choose then $M\in\mathbb{N}$ such that $n\geq M\Rightarrow |f(x_n)-d(x_n,z)-l|\leq\frac{\delta}{2}$.
For all $n\geq M$, we have $f(x_M)+d(x_M,x_n)-f(x_n)=(f(x_M)-d(x_M,z)-l)-(f(x_n)-d(x_n,z)-l)+(d(x_M,z)-d(x_n,z)+d(x_M,x_n))$, so that $f(x_M)+d(x_M,x_n)-f(x_n)\leq 2\delta=\frac{a}{2}<\frac{3a}{4}$.
This contradicts the definition of the sequence (x_i), and we are done. $\Diamond$

Notice that in the course of the proof of theorem 2.8 we proved that, if $E(X)$ is separable and $f\in E(X)$, then for any $\varepsilon>0$ there exists a compact $K\subseteq X$ such that $d(f,\widehat{f_{|K}})<\varepsilon$ (This fact will be used later on).

We may add yet another line to the list of equivalent conditions in Theorem 2.8; to explain it, we follow Kalton ([Kal]) and say that an ordered triple of points $\{x_1,x_2,x_3\}$ is *ε-collinear* ($\varepsilon>0$) if $d(x_1,x_3)\geq d(x_1,x_2)+d(x_2,x_3)-\varepsilon$. We say that a metric space X has the *collinearity property* if for every infinite subset $A\subset X$ and every $\varepsilon>0$ there are $x_1,x_2,x_3\in A$ (pairwise distinct) such that $\{x_1,x_2,x_3\}$ is ε-collinear.
Using the infinite Ramsey theorem, Kalton proved in [Kal] that a space X has the collinearity property if, and only if, any sequence of elements of X admits an inline subsequence. Therefore, we have the following corollary.

Corollary 2.9. [Me3] *Let X be a Polish metric space. Then $E(X)$ is separable if, and only if, X has the collinearity property.*

3 Construction of the Urysohn space.

As explained in the introduction, the Urysohn space $\mathbb{U}$ is characterized as being, up to isometry, the only Polish metric space which is both universal and ω-homogeneous. It turns out that having these two properties is equivalent to a universal property, which is the starting point of our study of the geometry of $\mathbb{U}$.

Definition 3.1. A space X has the *approximate extension property* if

$$\forall A \text{ finite} \subset X \, \forall f \in E(A) \, \forall \varepsilon > 0 \, \exists z \in X \, \forall a \in A \, |d(z,a) - f(a)| \leq \varepsilon \, .$$

We say that X has the *extension property* if one can take $\varepsilon = 0$ in the above definition; in other words, X has the extension property iff any one-point metric extension of any finite subset of X is realized in X.

Spaces with the extension property are also commonly called *finitely injective* metric spaces. The reason is that a space X has the extension property if, and only if, given any two finite metric subsets A, A' such that $A \subset A'$ and any isometric embedding $\varphi \colon A \to X$ there is an isometric embedding $\varphi' \colon A' \to X$ which extends φ. In the remainder of the text, we'll often use this terminology.

Exercise 3. Prove that a metric space is finitely injective if, and only if, it has the extension property.

Theorem 3.2. *(Urysohn* [Ur2]*) A Polish metric space is finitely injective if, and only if, it is both universal and ω-homogeneous.*

Proof of Theorem 3.2.
Assume that P is a finitely injective Polish metric space, and let $X = \{x_i\}_{i \in \mathbb{N}}$ be a countable metric space. One may build by induction isometric maps $\varphi_i \colon \{x_0, \ldots, x_i\} \to P$ such that φ_{i+1} extends φ_i for all i. To do this, begin by picking any element $y_0 \in P$, and set $\varphi_0(x_0) = y_0$. Assume now that φ_i is defined; to define φ_{i+1}, we need to find some point y_{i+1} such that $d(y_{i+1}, \varphi_i(x_j)) = d(x_{i+1}, x_j)$ for all $j \leq i$. This is possible because P is finitely injective (that's precisely the extension property of P); setting $\varphi_{i+1}(x_{i+1}) = y_{i+1}$ defines a suitable extension of φ_i. This shows that one may embed isometrically any countable metric space in P; therefore, the theorem of extension of isometries, and the fact that P is complete, prove that any separable metric space may be embedded in P, so that P is universal.
Now, let $\varphi \colon A \to A'$ be an isometry between two finite subsets of P. To extend φ, one uses the so-called *back-and-forth method.* For this, begin by picking some countable dense subset $\{p_i\}_{i \geq 1}$ of P. Then, using the finite injectivity of P, one may build a sequence of finite subsets A_i of P, and isometric maps $\varphi_i \colon A_i \to P$ such that:
- $A_0 = A$, $\varphi_0 = \varphi$;

- $A_i \subset A_{i+1}$, and φ_{i+1} extends φ_i for all i;
- $\forall i$ $p_i \in A_{2i}$ ("forth");
- $\forall i$ $p_i \in \varphi_{2i+1}(A_{2i+1})$ ("back").

Assume that we have built A_i, φ_i for all $i \leq n$. If $n = 2k$, we first notice that the extension property of P ensures that there exists $z \in P$ such that $d(z, a) = (p_{n+1}, \varphi(a))$ for all $a \in A_n$. We then set $A_{n+1} = A_n \cup \{z\}$, and $\varphi(z) = p_{2n+1}$. A similar method works in the case when n is odd, so we assume that the sequence (A_i) is built. Let now $A = \cup A_i$; the maps φ_i induce an isometric map $\varphi_\infty \colon A \to P$. Since A is dense (this is what the "forth" step is for), φ_∞ extends to an isometric map from P into P; and the "back" step ensures that the image of φ_∞ is dense. Since φ_∞ is an isometry, and P is complete, $\varphi_\infty(P)$ must be closed; therefore, the back step ensures that $\varphi_\infty(P) = P$. Given that the first step ensured that φ_∞ extends φ, we are done.

Now, assume that P is both universal and ω-homogeneous, and let $A = \{a_1, \ldots, a_n\}$ be a finite subset of P, and $f \in E(A)$. Because of the universality of P, there exists an isometric copy of $A \cup \{f\}$ which is contained in P; call this copy $\{b_1, \ldots, b_n, z\}$, where the enumeration is such that $\varphi \colon a_i \mapsto b_i$ is an isometric map, and $d(z, b_i) = f(a_i)$. Then our assumption on P implies that φ extends to an isometry of P, which we still denote by φ. Let $y = \varphi^{-1}(z)$: we have $d(y, a_i) = d(\varphi(y), \varphi(a_i)) = d(z, b_i) = f(a_i)$, which proves that P is finitely injective . $\diamondsuit$

We gave the proof above in detail because it is a good illustration of how the back-and forth method works, and this method is the fundamental tool to study the geometry of the Urysohn space.

Theorem 3.3. *(Urysohn* [Ur2]*) Any two finitely injective Polish metric spaces are isometric.*

Exercise 4. Use the back-and-forth method to prove the theorem above.

We now have a nice characterization of the Urysohn space as being the only finitely injective Polish metric space; the problem, of course, is that we haven't proved that such a space exists. Before building a finitely injective metric space, we need to establish the following result.

Theorem 3.4. *(Urysohn* [Ur2]*) If X is complete and has the approximate extension property, then X actually has the extension property.*

It is obvious that the completion of a space with the approximate extension property also has the approximate extension property; therefore, the above theorem implies that the completion of a finitely injective metric space is also finitely injective.

Proof.
Let X satisfy the hypotheses of the theorem; pick $\{x_1, \dots, x_n\} \subset X$ and $f \in E(\{x_1, \dots, x_n\})$. Since X is complete, it is enough to build a sequence (z_p) such that $|d(z_p, x_i) - f(x_i)| \leq 2^{-p}$ for all i, and $d(z_p, z_{p+1}) \leq 2^{1-p}$.
The fact that X has the approximate extension property enables us to define z_0; assume now that we have defined $z_0, \dots, z_p$.
Let $f_p \in E(\{x_1, \dots, x_n\})$ be the map defined by $f_p(x_i) = d(z_p, x_i)$; by definition of z_p we have $d(f_p, f) \leq 2^{-p}$ (where the distance $d(f_p, f)$ is computed in $E(\{x_1, \dots, x_n\})$).
The map g_p defined on $\{x_1, \dots, x_n\} \cup \{z_p\}$ by $g_p(x_i) = f(x_i)$, $g_p(z_p) = d(f_p, f)$ is a Katětov map, since these distances are realized by a subset of $E(\{x_1, \dots, x_n\})$. Hence there exists $z \in X$ such that $|d(z, x_i) - g_p(x_i)| = |d(z, x_i) - f(x_i)| \leq 2^{-(p+1)}$ and $d(z, z_p) \leq d(f_p, f) + 2^{-(p+1)} \leq 2^{1-p}$.
We can now set $z_{p+1} = z$ and go on to the next step. $\Diamond$

Building a finitely injective Polish metric space is now rather straightforward: we only need to build a finitely injective separable metric space, and its completion will work. This is easier because such a space may be built inductively; Katětov was the first to notice this, and it is his construction which led to the current interest in the Urysohn space. Beginning with any separable metric space X, we build inductively an increasing sequence of separable metric spaces by setting $X_{i+1} = E(X_i, \omega)$ (at each step we identify X_i to a subspace of X_{i+1} via the Kuratowski map). Let now $Y = \cup X_i$; the construction ensures that Y is finitely injective. Indeed, any finite subset $\{y_1, \dots, y_n\}$ of Y is contained in X_m for some big enough m; then, the Katětov extension to X_m of any map $f \in E(\{y_1, \dots, y_n\})$ appears as an element of X_{m+1}, which shows that there is indeed a point $y \in Y$ such that $d(y, y_i) = f(y_i)$ for all $i = 1, \dots, n$. Hence, the completion of Y is a finitely injective Polish metric space, and we have finally proved the existence of the Urysohn space. For definiteness, we denote by $\mathbb{U}$ the space obtained by applying the above construction starting with $X_0 = \{0\}$. In particular, we always consider 0 as an element of the Urysohn space (this simplifies some statements).
It might be interesting to mention that the way Urysohn built his universal space was different, even though it was based on similar ideas. He began by building a countable metric space which is both universal for spaces with rational distances and ultrahomogenous; in modern terms, he built the Fraïssé limit of the finite metric spaces with rational distances (30 years before Fraïssé defined this notion in a general setting; this is perhaps the earliest

example of such a construction). Then, he proved that this space (which we denote by $\mathbb{QU}$, for "rational Urysohn space") has the approximate extension property (it actually has the *rational extension property*, meaning that one only considers Katětov maps with rational values on their support), and concluded that the completion of $\mathbb{QU}$ must have the extension property. This construction is quite remarkable, especially considering when it was done.
Notice that there are many possible variants of Katětov's construction: namely, one can build Urysohn spaces for spaces of diameter $\leq d$, for spaces with distances in $\mathbb{N}$, for spaces with distances in $\mathbb{Q}$ (obtaining $\mathbb{QU}$), in $\{q \in \mathbb{Q} \colon q \geq 1\}$, etc. By "Urysohn space for spaces with distances in $A \subset \mathbb{R}$", we mean a Polish metric space $\mathbb{U}_A$ with distances in A, which is ω-homogenous and universal for spaces with distances in A; equivalently, a space with the extension property for extensions with values in A. Such Urysohn spaces don't exist for all $A \subset \mathbb{R}$, but for simple A (as the ones above) one may simply mimic Katětov's construction to obtain $\mathbb{U}_A$.

Exercise 5. When $A = \{1, 2\}$, one obtains a corresponding countable Urysohn space $\mathbb{U}_{\{1,2\}}$; define a graph structure on $\mathbb{U}_{\{1,2\}}$ by saying that there is an edge between $x, y \in \mathbb{U}_{\{1,2\}}$ if and only if $d(x, y) = 1$. Prove that this graph is isomorphic to the random graph (see for instance [Bol] for a definition and characterizations of the random graph).

This statement explains why one may consider the Urysohn space as a "generalized random graph"; A. Vershik proved that the analogy goes further, showing that the Urysohn space is the generic Polish space, just as the random graph is the generic countable graph (see [Ve]). Recently A. Usvyastov proved (in the context of model theory for metric structures) that the analogy is even more far-reaching; see the paper [Usv] for detailed statements and explanations.

4 Simple geometric properties of $\mathbb{U}$.

In this section, we try to give the reader a feel for the geometry of $\mathbb{U}$; for this, we discuss a few examples, and propose some exercises which seem helpful for learning basic methods that are adapted to proving statements about $\mathbb{U}$.

4.1 Geodesic segments.

Say that a map $\gamma \colon I \to \mathbb{R}$, where I is an interval of $\mathbb{R}$, is a *geodesic* if one has $d(\gamma(t), \gamma(t')) = |t - t'|$ for all $t, t' \in I$. In other words, it is just an isometric map from I into $\mathbb{U}$. It is clear that any two points x, y in $\mathbb{U}$ are joined by a geodesic segment: since $\mathbb{U}$ is universal, there exists an isometric image of the segment $[0, d(x,y)]$ that is contained in $\mathbb{U}$; let a be the image of 0, and b be the image of $d(x,y)$. Then $\{a,b\}$ and $\{x,y\}$ are isometric, so there exists $\varphi \in \mathrm{Iso}(\mathbb{U})$ such that $\varphi(a) = x$ and $\varphi(b) = y$. Composing by φ, the geodesic segment between a and b becomes a geodesic segment between x and y. The existence of geodesics is nothing really surprising. Here, however, geodesics usually have nontrivial intersections: for instance, if $\gamma\colon [0,1] \to \mathbb{U}$ and $\gamma'\colon [1,2] \to \mathbb{U}$ are geodesics such that $\gamma(1) = \gamma'(1)$, then their union $\gamma''\colon [0,2] \to \mathbb{U}$ is a geodesic if, and only if, $d(\gamma(0),\gamma(1)) + d(\gamma'(1),\gamma'(2)) = d(\gamma(0),\gamma'(2))$ (this is a direct consequence of the triangle inequality). Given that $\mathbb{U}$ is finitely injective, it is therefore very easy to build geodesic segments which coincide on some segment, then are different, then coincide again, etc. Thus, we see that there are actually *uncountably many* different geodesic segments between any two distinct points $x, y \in \mathbb{U}$. The properties above were already mentioned in Urysohn's original article.

Exercise 6. Prove the results about geodesics stated above.

Exercise 7. Let B be a nonempty ball centered in 0, S its boundary, and x a point outside of B. Prove that
$d(x,0) = \frac{1}{2}(\inf_{z\in B} d(x,z) + \sup_{z \in B} d(x,z)) = \frac{1}{2}(\inf_{z\in S} d(x,z) + \sup_{z\in S} d(x,z))$.
Find a similar formula, assuming now that $x \in B$.

4.2 Subsets isometric to the whole space.

Since $\mathbb{U}$ is universal, it is reasonable to expect that it contains many isometric copies of itself; let us give some concrete examples.
Pick $x_1, \dots, x_n \in \mathbb{U}$, and consider the set

$$\mathrm{Med}(x_1,\dots,x_n) = \{z \in \mathbb{U}\colon \forall i,j\ d(z,x_i) = d(z,x_j)\} .$$

We claim that this set is isometric to $\mathbb{U}$.
The proof is typical of how one proves that a given set is isometric to $\mathbb{U}$, so we give it in full. The set $M = \mathrm{Med}(x_1,\dots,x_n)$ is closed in $\mathbb{U}$, so we simply need to prove that M is finitely injective. To that end, pick $a_1,\dots,a_p \in M$, and $f \in E(\{a_1,\dots,a_p\})$. We want to find some point $z \in M$ such that $d(z,a_i) = f(a_i)$ for all i; in other words, we want to find some point in $\mathbb{U}$ such that $d(z,a_i) = f(a_i)$ for all $i = 1,\dots,p$ and $d(z,x_j) = d(z,x_k)$ for all $j,k = 1,\dots,n$. We need to use the universal property of $\mathbb{U}$: let g denote the Katětov extension of f to $\{a_1,\dots,a_p\} \cup \{x_1,\dots,x_n\}$; then necessarily $g(x_j) = g(x_k)$ for all $j,k = 1,\dots,n$. By the finite injectivity of $\mathbb{U}$, there

exists a point $z \in \mathbb{U}$ such that $d(z, a_i) = g(a_i)$ and $d(z, x_j) = g(x_j)$ for all i, j. This z witnesses that $\mathrm{Med}(x_1, \ldots, x_n)$ is finitely injective.

By definition of the Urysohn space, isometries between finite subsets of $\mathbb{U}$ can always be extended to isometries of $\mathbb{U}$; the example above shows that this is not true for countable subsets. Indeed, if one lets A_1, A_2 be isometric countable subsets with A_1 dense in $\mathbb{U}$ and A_2 dense in $\mathrm{Med}(x_1, x_2)$ (where $x_1 \neq x_2$), then an isometry between A_1 and A_2 obviously cannot be extended to an isometry of $\mathbb{U}$. We'll say more about this later on.
Up to now, all the isometric copies of $\mathbb{U}$ we have encountered have empty interior. It is then natural to wonder whether this is always the case. The following proposition shows that there are actually many isometric copies of $\mathbb{U}$ which have nonempty interior; recall that a Polish metric space X has the *Heine-Borel* property if all closed bounded balls in X are compact (these spaces are also known as *proper* metric spaces).

Proposition 4.1. [Me3] *Let $X \subset U$ be a Polish metric space with the Heine-Borel property, and $M \in \mathbb{R}$. Then $\{z \in \mathbb{U} \colon d(z, X) \geq M\}$ is isometric to $\mathbb{U}$. In particular, $\mathbb{U}$ and $\mathbb{U} \setminus B(0, 1[$ are isometric.*

Proof.
Let $Y = \{z \in \mathbb{U} \colon d(z, X) \geq M\}$; once again, since Y is a closed subset of $\mathbb{U}$, we only need to prove that Y is finitely injective.
Let $y_1, \ldots, y_n \in Y$ and $f \in E(\{y_1, \ldots, y_n\})$. We want to find some $z \in Y$ such that $d(z, y_i) = f(y_i)$ for all $i = 1, \ldots, n$. We begin by doing this under the additional assumption that X is compact.
Define $\varepsilon = \min\{f(y_i) \colon 1 \leq i \leq n\}$. We may of course assume $\varepsilon > 0$.
Since X is compact, we may find $x_1, \ldots, x_p \in X$ with the property that for all $x \in X$ there exists j such that $d(x, x_j) \leq \varepsilon$.
Let then g be the Katětov extension of f to $\{y_1, \ldots, y_n\} \cup \{x_1, \ldots, x_p\}$.
There exists some $z \in \mathbb{U}$ such that $d(z, y_i) = g(y_i) (= f(y_i))$ for all $i \leq n$ and $d(z, x_j) = g(x_j) = d(x_j, y_{i_j}) + f(y_{i_j}) \geq M + \varepsilon$ for all $j \leq p$.
Since for all $x \in X$ there is $j \leq p$ such that $d(x, x_j) \leq \varepsilon$, the triangle inequality shows that $d(z, x) \geq d(z, x_j) - d(x_j, x) \geq M$, hence $z \in Y$. This proves that Y is finitely injective.
Suppose now that X is Heine-Borel but not compact; pick some $x \in X$ and let $m = f(y_1) + d(y_1, x)$. Since $B(x, M + m] \cap X$ is compact, there exists, by the above argument, a point $z \in \mathbb{U}$ such that $d(z, y_i) = f(y_i)$ for all $i \leq n$, and $d(z, B(x, M + m] \cap X) \geq M$. Then we claim that for all $x' \in X$ we have $d(z, x') \geq M$; indeed, if $d(x', x) \leq M + m$ then this is true by definition of z, and if $d(x', x) > M + m$ then one has $d(z, x') \geq d(x, x') - d(z, x) > M$ (because $d(z, x) \leq f(y_1) + d(y_1, x) = m$). $\Diamond$

4.3 Spheres and sets of uniqueness.

We saw earlier that there existed analogues of the Urysohn space for spaces of diameter bounded by some constant $\lambda \in \mathbb{R}$, i.e ultrahomogenous metric spaces which are universal for separable metric spaces of diameter less than λ. Equivalently, these spaces are characterized among Polish metric spaces of diameter $\leq \lambda$ by the analogue of the extension property where one asks for the extension to still be of diameter $\leq \lambda$. We then have the following fact (which was already mentioned in Urysohn's original article): spheres of diameter 2λ in $\mathbb{U}$ (i.e boundaries of balls of diameter λ) are isometric to the Urysohn space of the corresponding diameter. This is not true for balls, since they are not homogenous (any isometry of a ball with center x must have x as a fixed point).

To prove that the sphere $S = S(0,1)$ has the extension property for spaces of diameter 2, pick $x_1, \ldots, x_n \in S$ and some $f \in E(\{x_1, \ldots, x_n\})$ such that $\sup(f(x_i)) \leq 2$. Define a mapping $g \colon \{x_1, \ldots, x_n\} \cup \{0\} \to \mathbb{R}$ by setting $g(x_i) = f(x_i)$, and $g(0) = 1$. Then g is a Katětov map, so that there exists $z \in \mathbb{U}$ such that $d(z,0) = g(0) = 1$ (so z is in S) and $d(z,x_i) = f(x_i)$ for all $i = 1, \ldots, n$.

That universal property of spheres may be used to prove the following fact.

Exercise 8. Let $S_1, \ldots, S_n \subset \mathbb{U}$ be spheres. Prove that $S_1 \cap \ldots \cap S_n$, if nonempty, is isometric to the sphere of smallest diameter.

Definition 4.2. [Me3] We say that $A \subset \mathbb{U}$ is a *set of uniqueness* if it has the following property:

$$\forall x, y \in \mathbb{U} \ (\forall a \in A \ d(x,a) = d(y,a)) \Rightarrow x = y \ .$$

Then one has the following folklore result, which has been rediscovered several times, the first person to notice it being apparently Mati Rubin:

Proposition 4.3. *(Rubin) Nonempty spheres are sets of uniqueness.*

Proof. It is enough to prove the proposition in the case when $S = S(0,1]$. Let now x, y be two points in $\mathbb{U}$, and assume that $x \neq y$. Assume also, without loss of generality, that $d(y,0) \geq d(x,0)$. Now define, for $\varepsilon > 0$, a map g_ε on $\{0,x,y\}$ by setting $g_\varepsilon(0) = 1$, $g_\varepsilon(x) = 1 + d(x,0)$ and $g_\varepsilon(y) = 1 + d(y,0) - \varepsilon$. Then a simple verification shows that for ε small enough g_ε is a Katětov map, so that there exists $z \in \mathbb{U}$ with $d(z,0) = g_\varepsilon(0) = 1$, $d(z,x) = g_\varepsilon(x) = 1 + d(x,0)$ and $d(z,y) = g_\varepsilon(y) = 1 + d(y,0) - \varepsilon$. If ε is well-chosen then $g_\varepsilon(x) \neq g_\varepsilon(y)$, so that the above point z is an element of S such that $d(z,x) \neq d(z,y)$. $\Diamond$

Notice that, since obviously a set containing a set of uniqueness is also a set of uniqueness, this proves that balls, and more generally subsets of $\mathbb{U}$ with nonempty interior, are sets of uniqueness. In turn, this proves that an isometric map defined on $\mathbb{U}$ and which has a set of fixed points with nonempty interior must actually leave every point fixed; equivalently, an isometric map which coincides with an isometry on a nonempty open ball must coincide with it everywhere, so it has to be onto. To prove this, assume that $\varphi \in \mathrm{Iso}(\mathbb{U})$ is such that $\varphi(x) = x$ for all x in a nontrivial ball B. Then one has, for any $x \in X$, and any $z \in B$, that $d(\varphi(x), z) = d(\varphi(x), \varphi(z)) = d(x, z)$. But then one must have $\varphi(x) = x$.

There are many other examples of sets of uniqueness, as the following exercises show.

Exercise 9. Let $x_1, \ldots, x_n \in \mathbb{U}$ and $f\colon \{x_1, \ldots, x_n\} \to \mathbb{U}$ be a Katětov map such that

$$\forall i \neq j \ |f(x_i) - f(x_j)| < d(x_i, x_j) \text{ and } f(x_i) + f(x_j) > d(x_i, x_j) \ .$$

Show that $K = \{x_1, \ldots, x_n\} \cup \{z \in \mathbb{U}\colon \forall i \ d(z, x_i) = f(x_i)\}$ is a set of uniqueness (This is proposition 3.2 in [Me3]).

Exercise 10. Let $x_1, \ldots, x_n \in \mathbb{U}$. Prove that $\mathrm{Med}(x_1, \ldots, x_n) \cup \{x_1, \ldots, x_n\}$ is a set of uniqueness. Prove that $\mathrm{Med}(x_1, x_2) \cup \{x_1\}$ also is a set of uniqueness, whereas $\mathrm{Med}(x_1, x_2)$ obviously is not if $x_1 \neq x_2$!

We saw above that if an isometric map coincides with an isometry on a ball (or even just on a sphere), then both maps must coincide everywhere; it should be mentioned that this is not true for isometric maps (it is true only if the image of the ball is a set of uniqueness).

Exercise 11. Build two isometric maps $\varphi, \varphi'\colon \mathbb{U} \to \mathbb{U}$ such that $\varphi = \varphi'$ on $B(0, 1]$ but $\varphi(x) \neq \varphi'(x)$ everywhere else.

4.4 Extensions of isometries.

We saw that, given any finite metric space $A \subset \mathbb{U}$ and any isometry φ of A, φ extends to an isometry of $\mathbb{U}$. This property does not hold for general subsets of $\mathbb{U}$. Let us check this for balls in $\mathbb{U}$, for instance. We saw above that $\mathbb{U}$ and $\mathbb{U} \setminus B(0, 1[$ are isometric; let $\varphi\colon \mathbb{U} \to \mathbb{U} \setminus B(0, 1[$ witness this fact, and $x \in \mathbb{U}$ be such that $d(x, 0) \geq 2$. There exists, because of the ultrahomogeneity of $\mathbb{U} \setminus B(0, 1[$, an isometry ψ of $\mathbb{U} \setminus B(0, 1[$ such that $\psi(\varphi(x)) = x$. Thus, composing if necessary φ with ψ, we may suppose that x is a fixed point of φ.

But then φ must send the ball of center x and radius 1 (in $\mathbb{U}$) onto the ball of center x and radius 1 (in $\mathbb{U} \setminus B(0,1[)$. Since by choice of x both balls are the same, we see that $\varphi_{|B(x,1]}$ is an isometry of $B(x,1]$, yet it cannot coincide on this ball with an isometry of $\mathbb{U}$, since otherwise it would have to be onto because of the fact that balls are sets of uniqueness. Notice that the same fact holds for spheres.
There exists at least one other proof of this fact, which we sketch in the following exercise.

Exercise 12. Prove that there exists a sequence (φ_n) of isometries of $\mathbb{U}$ and $z \in \mathbb{U}$ such that $\varphi(0) = 0$, $\varphi_n(x) \to x$ for all $x \in B(0,1[$ but $\varphi_n(z)$ does not converge. Using automatic continuity of Baire-measurable morphisms between Polish groups (see [Ke1]), use this to prove that there exist isometries of $B(0,1]$ which do not extend to $\mathbb{U}$.

Question. Does there exist an isometry φ of $B(0,1]$ which cannot be extended to an isometric map $\tilde{\varphi}\colon Z \to \mathbb{U}$, where $Z \supsetneq B(0,1]$? Same question for spheres. Very little is known (at least to the author) on this question, so one may ask a similar question in the opposite direction: is it true that, given any isometry φ of a ball, there exists $Z \subset \mathbb{U}$ isometric to $\mathbb{U}$ such that φ extends to an isometric map from Z to $\mathbb{U}$? One could ask the same question replacing the ball by an arbitrary subset of $\mathbb{U}$.

4.5 Compact homogeneity.

We saw that $\mathbb{U}$ is characterized, among universal Polish metric spaces, by the fact that it is ultrahomogeneous, i.e any isometry between two finite metric subspaces extends to the whole space. It actually has a (apparently) stronger property, which is called *compact homogeneity*: any isometry between two compact subspaces of $\mathbb{U}$ extends to the space itself. This was first proved by Huhunaišvili [Hu] in 1955; this result seems to have been largely unnoticed, since it was proved again in the special case of countable compact metric subspaces of $\mathbb{U}$ by Joiner in 1973 [Jo], then it appears (without reference to the preceding articles) as an exercise in Gromov's book [Gro], and it was again independently proved (without reference to any of the aforementioned papers) by Bogatyi in 2002 ([Bog]).
As in the case of ω-homogeneity, compact homogeneity has an equivalent formulation (for universal Polish metric spaces), which we call *compact injectivity*: a space X is *compactly injective* if, and only if,

$$\forall K \text{compact} \subset \mathbb{U}\ \forall f \in E(K)\ \exists z \in X\ \forall k \in K\ d(z,k) = f(k)\ .$$

Exercise 13. Prove that compact injectivity and compact homogeneity are indeed equivalent for universal Polish metric spaces.

Let us now explain how to prove that $\mathbb{U}$ is compactly injective; pick some compact $K \subset \mathbb{U}$ and $f \in E(K)$. Fix also $\varepsilon > 0$. Since K is totally bounded, there exist $x_1, \ldots, x_n \in K$ such that for all $k \in K$ $d(k, x_i) \leq \varepsilon$ for some i. By the universal property of $\mathbb{U}$, there exists $z \in \mathbb{U}$ such that $d(z, x_i) = f(x_i)$ for all $i = 1, \ldots, n$. Then, the triangle inequality implies that $|d(z,x) - f(x)| \leq 2\varepsilon$ for all $x \in K$. We just proved that for any compact subset $K \subset \mathbb{U}$, any map $f \in E(K)$ and any $\varepsilon > 0$, there exists $z \in \mathbb{U}$ such that $|d(z,x) - f(x)| \leq \varepsilon$ for all $x \in K$. Now, we may conclude as in the proof of the fact that the approximate extension property and the extension property are equivalent for Polish metric spaces: what we saw above implies that we may define inductively a sequence (z_n) such that:
- $\forall n \geq 0$ $d(z_n, z_{n+1}) \leq 2^{1-n}$.
- $\forall x \in K$ $|d(z_n, x) - f(x)| \leq 2^{-n}$.
The sequence (z_n) is Cauchy, so it converges to some z, which must be such that $d(z,x) = f(x)$ for all $x \in K$.

4.6 Translations.

In [CV], Cameron and Vershik established the remarkable result that $\mathbb{U}$ could be endowed with a structure of *monothetic Polish group*, i.e a Polish group with an element generating a dense subgroup. In particular, this proves that one may define "translations" in $\mathbb{U}$, i.e continuous maps $(x,y) \mapsto \varphi_{x,y}$ from $\mathbb{U}^2$ to $\mathrm{Iso}(\mathbb{U})$ with the property that $\varphi_{x,y}(x) = y$, and $\varphi_{y,z} \circ \varphi_{x,y} = \varphi_{x,z}$ (cocycle identity).
The translation cocycle obtained as a corollary of Cameron and Vershik's construction is particularly simple, but not so easy to visualize geometrically. Here is another way to build one; though it is more complicated than Cameron and Vershik's, we think it is worth including here because the map built here is actually continuous with regard to a stronger topology on $\mathrm{Iso}(\mathbb{U})$, the so-called "uniform topology" (defined later in the article). It also gives a hint of why the situation is different when one tries to build finite-order isometries of $\mathbb{U}$, as opposed to arbitrary isometries: in the second case, one is obliged to ensure that the isometric map obtained at the end of the construction is onto, which leads to using some type of back-and-forth method. In the first case, however, it is enough to define $\varphi(x), \ldots, \varphi^{n-1}(x), \varphi^n(x) = x$, and then the map obtained is necessarily onto. In particular, building isometric involutions is very different from building general isometries.
Let us now go on to the construction; we first define a continuous map (rel-

ative to the uniform topology) $x \mapsto \varphi_x$ such that each φ_x is an isometric involution and $\varphi_x(x) = 0$. Then, setting $\varphi_{x,y} = \varphi_y \circ \varphi_x$ defines the desired translation operator.
Let $\{0 = x_0, x_1, \dots, x_n, \dots\}$ be a countable dense subset of $\mathbb{U}$ (we assume our enumeration to be injective).
We wish to define a sequence (φ_n) of isometries of $\mathbb{U}$ such that :

- $\varphi_0 = id_{\mathbb{U}}$;
- $\forall n\ \varphi_n(x_n) = 0$;
- $\forall n\ \varphi_n^2 = id_{\mathbb{U}}$;
- $\forall n, m\, \forall x \in \mathbb{U}\ d(\varphi_n(x), \varphi_m(x)) = d(x_n, x_m)$

If we manage to do this, then the map $x_n \mapsto \varphi_n$ extends to a map $x \mapsto \varphi_x$ from $\mathbb{U}$ into $\mathrm{Iso}(\mathbb{U})$, such that $\varphi_x^2 = id_{\mathbb{U}}$ for all x, $\varphi_x(x) = 0$, and $d(\varphi_x(z), \varphi_y(z)) = d(x, y)$ for all $x, y, z \in \mathbb{U}$.
In particular, this map is a continous right inverse to the orbit map (from $\mathrm{Iso}(\mathbb{U})$ to $\mathbb{U}$); notice that each of our translations was obtained as a product of two isometric involutions ("reflections"). The construction proceeds as follows: we first let $\varphi_0 = id_{\mathbb{U}}$. Now, assume that $\varphi_0, \dots, \varphi_n$ have been built; we need to explain how to obtain φ_{n+1}.
We use a variant of the back-and-forth method adapted to building involutions. To apply it, we first pick a countable set $\{y_i\}_{i\in\mathbb{N}}$ which is dense in $\mathbb{U}$; then we build by induction a sequence of finite sets F_i, and isometric involutions $\psi_i \colon F_i \to F_i$ such that :

- $F_0 = \{x_{n+1}, 0\}$ and $\psi_0(x_{n+1}) = 0$;
- $y_i \in F_i \subset F_{i+1}$, and ψ_{i+1} extends ψ_i;
- $\forall j \le n\ \forall i\ \forall x \in F_i\ d(\psi_i(x), \varphi_j(x)) = d(x_{n+1}, x_j)$.

First, we need to show that the third assertion is true when $i = 0$; in other words, we need to check that $d(0, \varphi_j(x_{n+1})) = d(x_{n+1}, \varphi_j(0)) = d(x_{n+1}, x_j)$. This is obvious, since by definition we have $0 = \varphi_j(x_j)$, and φ_j is an involution so we also have $x_j = \varphi_j(0)$.
We now need to explain how to build F_{i+1} and ψ_{i+1} from F_i, ψ_i.
If $y_{i+1} \in F_i$, we let $F_{i+1} = F_i$, and we are done. Otherwise, we define a map g on $F_i \cup \{\varphi_j(y_{i+1}) \colon j \le n\}$ by setting :
- $g(z) = d(y_{i+1}, \psi_i(z))$ for all $z \in F_i$;
- $g(\varphi_j(y_{i+1})) = d(x_{n+1}, x_j)$ for all $j \le n$.
(Notice that if some $\varphi_j(y_{i+1})$ belongs to F_i, then both lines give the same definition for $g(\varphi_j(y_{i+1}))$, since then one must have $d(\psi_i(\varphi_j(y_{i+1})), y_{i+1}) = d(\psi_i(\varphi_j(y_{i+1})), \varphi_j(\varphi_j(y_{i+1}))) = d(x_{n+1}, x_j)$ by definition of F_i).

We claim that this is a Katětov map. The only nonobvious inequalities are those involving $g(z) + g(\varphi_j(y_{i+1}))$ and $|g(z) - g(\varphi_j(y_{i+1}))|$ (where $z \in F_i$).
We have
$g(z)+g(\varphi_j(y_{i+1})) = d(y_{i+1}, \psi_i(z))+d(x_{n+1}, x_j) = d(y_{i+1}, \psi_i(z))+d(\psi_i(z), \varphi_j(z))$ (since $z \in F_i$), so $g(z) + g(\varphi_j(y_{i+1}) \geq d(y_{i+1}, \varphi_j(z)) = d(z, \varphi_j(y_{i+1}))$ (remember that φ_j is an involution). Similarly, we have
$|g(z) - g(\varphi_j(y_{i+1}))| = |d(y_{i+1}, \psi_i(z)) - d(\psi_i(z), \varphi_j(z))| \leq d(y_{i+1}, \varphi_j(z))$, and we are done.
Since g is a Katětov map and $F_i \cup \{\varphi_j(y_{i+1})\colon j \leq n\}$ is finite, there exists some b in $\mathbb{U}$ such that $d(b, .) = g$; we may now let $F_{i+1} = F_i \cup \{y_{i+1}, b\}$ and set $\psi_{i+1}(y_{i+1}) = b$, $\psi_{i+1}(b) = y_{i+1}$.
The fact that this is a suitable extension of ψ_i is a direct consequence of the definition of z, and the fact that ψ_i is an involution (so that $d(b, z) = d(y_{n+1}, \psi_i(z))$ is equivalent to $d(b, \psi_i(z)) = d(y_{n+1}, z)$ for $z \in F_i$).
This construction enables us to define φ_{n+1} by setting $\varphi_{n+1}(y_i) = \psi_i(y_i)$ for all i, and using the theorem of extension of isometries. $\diamondsuit$

5 Algebraic and topological properties of Iso($\mathbb{U}$).

Let us first emphasize a consequence of Katětov's construction: recall that, to build the Urysohn space, one may start with any separable metric space $X = X_0$, then let $X_{i+1} = E(X_i, \omega)$ (identifying X_i to a subset of X_{i+1} via the Kuratowski map). This yields an increasing sequence of metric spaces (X_i); if we let $Y = \cup X_i$, it is finitely injective by construction, so that its completion is a Urysohn space. Recall that we saw that any isometry of a separable metric space X extends uniquely to an isometry of $E(X, \omega)$, and that the extension morphism from $\mathrm{Iso}(X)$ to $\mathrm{Iso}(E(X, \omega))$ is continuous. Thus, we see that all isometries of X extend to isometries of $Y = \cup X_i$, and what we described above actually defines a continuous morphism from $\mathrm{Iso}(X)$ to $\mathrm{Iso}(Y)$. It is a classical result that all isometries of Y extend to isometries of its completion (which we identify with $\mathbb{U}$) and that once again the associated morphism between the isometry groups is continuous. This way, we see that there is a continuous morphism $\Psi\colon \mathrm{Iso}(X) \to \mathrm{Iso}(\mathbb{U})$ such that for all $\varphi \in \mathrm{Iso}(X)$ $\Psi(\varphi)$ is an extension of φ.

Definition 5.1. We follow [Pe2] and say that a space X is *g-embedded* in another space Y if it isometrically embeds in Y in such a way that all isometries of X extend to Y and the associated morphism is continuous.

What we saw above implies that any separable metric space may be g-embedded in $\mathbb{U}$; now, notice that any Polish group G admits a left-invariant

distance. Denote by X the completion of G endowed with this distance; then the left-translation action of G extends to an action by isometries of G on X, so we see that G is isomorphic to a (necessarily closed) subgroup of $\mathrm{Iso}(X)$. Hence, any Polish group is a subgroup of the isometry group of some Polish metric space. Actually, Gao and Kechris proved that any Polish group is isomorphic to $\mathrm{Iso}(X)$ for some suitable Polish metric space X (see [GK] for their original proof or [Me2] for a shorter one).
Going back to the Urysohn space, the discussion above established the following result.

Theorem 5.2. *(Uspenskij* [Usp2]*) Any separable metric space may be g-embedded in $\mathbb{U}$; consequently, any Polish group is isomorphic to a (necessarily closed) subgroup of Iso($\mathbb{U}$).*

It is common usage to state this by saying that $\mathrm{Iso}(\mathbb{U})$ is *universal* for Polish groups.
We will see in the next section that one can give a more accurate version of Theorem 5.2, which shows what the isomorphic image of G "looks like" in $\mathrm{Iso}(\mathbb{U})$.
Remark. The term "universal" is a bit misleading, since there is not a unique (up to isomorphism of topological groups) universal Polish group. For instance, the homeomorphism group of the Hilbert Cube is also universal (see [Usp1]) in the above sense, yet it is not isomorphic to $\mathrm{Iso}(\mathbb{U})$: indeed, the former group admits a transitive action on a compact space (the Hilbert Cube), while Pestov established in [Pe1] that the latter is *extremely amenable*, which means that any continuous action of $\mathrm{Iso}(\mathbb{U})$ on a compact space admits a (global) fixed point. Perhaps we should borrow terminology from algebraic geometers here and simply call such groups "versal Polish groups". I'm grateful to Mathieu Florence for pointing out to me this inconsistency in terminology, and how algebraic geometers deal with it.

5.1 Iso($\mathbb{U}$) is not divisible.

Let's turn to some algebraic properties of $\mathrm{Iso}(\mathbb{U})$. In [Pe3], Pestov asks whether it is a divisible group; in other words, given $\varphi \in \mathrm{Iso}(\mathbb{U})$ and $n \in \mathbb{N}^*$, does there always exist some isometry τ such that $\tau^n = \varphi$? It turns out that the answer is negative, as established by the following theorem.

Theorem 5.3. *There exists an isometry σ of $\mathbb{U}$ such that σ doesn't admit a n-th root for any $n > 1$.*

Proof.
The proof is based on a variant of Katětov's construction; the idea is to begin by finding a Polish metric space which has an isometry with no n-th root for any $n > 1$, and then build a suitable embedding of this space into $\mathbb{U}$. The following easy lemma takes care of the first step.

Lemma 5.4. *Let $\sigma\colon \mathbb{Z} \to \mathbb{Z}$ denote the shift, i.e $\sigma(k) = k+1$ for any $k \in \mathbb{Z}$. Then the only maps from $\mathbb{Z}$ to $\mathbb{Z}$ which commute with σ are its powers.*

Exercise 14. Prove this lemma.

Let now X_0 denote $\mathbb{Z}$ endowed with the discrete distance; σ may be seen as an isometry of X_0.
We build an embedding of X_0 into $\mathbb{U}$, and an extension $\tilde{\sigma} \in \mathrm{Iso}(\mathbb{U})$ of σ which has the following property:

$$\forall x, y \in \mathbb{U} \ \liminf_{|n| \to \infty} d(\tilde{\sigma}^n(x), y) \geq 1 + d(x, X_0) + d(y, X_0) \ .$$

Assume for now that such an embedding has been built, and that $\tau \in \mathrm{Iso}(\mathbb{U})$ is such that $\tau^m = \tilde{\sigma}$ for some $m \in \mathbb{N}$.
Then τ and $\tilde{\sigma}$ commute, so one has, for all $i \in X_0$ and all $n \in \mathbb{Z} \setminus \{0\}$, that

$$d(\tau(i), \tilde{\sigma}^n(\tau(i))) = d(\sigma(i), \tau^{m-1}\tilde{\sigma}^n \tau(i)) = d(\sigma(i), \sigma^{n+1}(i)) = 1 \ .$$

Hence $\tau(i)$ must belong to X_0 for all $i \in X_0$, and the same is true for τ^{-1}; this implies that $\tau_{|X_0}$ is an isometry of $X_0 = \mathbb{Z}$ which commutes with σ, so the lemma above tells us that $\tau_{|X_0} = \sigma^p$ for some $p \in \mathbb{N}$, and this combined with $\tau^m = \tilde{\sigma}$ eventually gives us $\sigma^{mp} = \sigma$, which is only possible if $mp = 1$, hence $m = 1$, and we are done.
One may notice that the proof also shows that $\mathrm{Iso}(\mathbb{QU})$ is not divisible, but this was already known: it is a direct consequence of the result, due to Cameron and Vershik ([CV]), stating that there exists a transitive isometry of $\mathbb{QU}$. Indeed, a transitive automorphism of a countable structure cannot have a root of any order $n \geq 2$, as shown by the lemma. It is not clear (at least to the author) whether one can use this result to find another proof of the fact that $\mathrm{Iso}(\mathbb{U})$ is not divisible.

Going back to the proof, we still need to explain how to obtain the desired embedding of X_0 in $\mathbb{U}$, and the isometry $\tilde{\sigma}$.
If X is a metric space, we let $E(X, \omega, \mathbb{Q})$ denote the set of Katětov maps on X which take rational values on some finite support.
The construction proceeds as follows: we define inductively a sequence X_i

of countable metric spaces with rational distances, such that $X_i \subset X_{i+1}$ and for all $f \in E(X_i, \omega, \mathbb{Q})$ there exists $z \in X_{i+1}$ such that $d(z,x) = f(x)$ for all $x \in X_i$. We also define inductively a sequence of isometries σ_i of X_i which are such that:

- $\liminf d(\sigma_i^n(x), y) \geq 1 + d(x, X_0) + d(y, X_0)$ for all $x, y \in X_i$;
- $\sigma_0 = \sigma$, and σ_{i+1} extends σ_i.

Then $\cup X_i$ is isometric to the rational Urysohn space $\mathbb{QU}$, so its completion is isometric to $\mathbb{U}$; also, the isometries σ_i induce an isometry of $\cup X_i$ which extends σ, and which may be extended by uniform continuity to an isometry $\tilde{\sigma}$ of $\widehat{\cup X_i}$ which extends σ and has the desired property.

Assume now that (X_i, σ_i) has been built.

If $f_i \in E(X, \omega, \mathbb{Q})$, then we define for all $j \in \mathbb{Z}$ a one-point metric extension $X_i^j = X_i \cup \{y_j^f\}$ of X_i by setting $d(y_j^f, x) = f(\sigma_i^{-j}(x))$. We let X_i^f denote the metric amalgam of the X_i^j over X_i.

Now, we define X_{i+1} as the metric amalgam of the X_i^f over X_i; σ_i extends to an isometry of X_{i+1} which maps each y_j^f to y_{j+1}^f, and which we denote by σ_{i+1}. For the proof to be complete, we only need to prove by induction that for all i and for all $x, y \in X_i$ one has

$$\lim \inf_{|n| \to \infty} \{d(\sigma_{i+1}^n(x), y)\} \geq 1 + d(x, X_0) + d(y, X_0) \ .$$

This is true when $i = 0$.

To prove that the property is hereditary, notice that is is enough to show that each (X_i^f, σ_{i+1}) has it whenever X_i has it .

One has $d(\sigma_{i+1}^n(y_p^f), y_q^f) = \inf\{d(y_{p+n}^f, x) + d(x, y_q^f) \colon x \in X_i\}$. by definition of σ_{i+1} and of a metric amalgam; let $\{x_1, \ldots, x_m\}$ denote a finite support for f, pick $\varepsilon > 0$, and assume that M is big enough that

$$\forall |n| \geq M \ \forall j, k \ d(\sigma_i^n(x_j), x_k) \geq 1 + d(x_j, X_0) + d(x_k, X_0) - \varepsilon \ .$$

By definition, for all n and all $x \in X_i$ we have $d(y_{p+n}^f, x) = f(x_k) + d(\sigma^{p+n}(x_k), x)$ and $d(y_q^f, x) = f(x_j) + d(\sigma_i^q(x_j), x)$ for some (j, k); hence

$d(y_{p+n}^f, x) + d(x, y_q^f) \geq f(x_k) + f(x_j) + d(\sigma_i^{p+n}(x_k), \sigma_i^q(x_j))$, so for $|n| \geq M + |p-q|$

$$d(y_{p+n}^f, x) + d(x, y_q^f) \geq 1 + f(x_k) + d(x_k, X_0) + f(x_j) + d(x_j, X_0) - \varepsilon \geq 1 + 2d(f, X_0) - \varepsilon \ ,$$

and $1 + 2d(f, X_0) = 1 + d(y_p^f, X_0) + d(y_q^f, X_0)$, so the above inequality is what we were looking for.

We also need to check that $d(\sigma_{i+1}^n(y_p^f), x) \geq 1 + d(f, X_0) + d(x, X_0)$ for $|n|$ big enough; let again $\{x_1, \ldots, x_m\}$ denote a finite support for f, pick $\varepsilon > 0$ and let M be big enough that $d(\sigma_i^n(x_j), x) \geq 1 + d(x_j, X_0) + d(x, X_0) - \varepsilon$ for all $|n| \geq M$ and all j,k. One has $d(\sigma_{i+1}^n(y_p^f),x) = f(x_j) + d(\sigma_i^{n+p}(x_j),x)$ for some j, so that for all $|n| \geq M + |p|$ one has

$$d(\sigma_{i+1}^n(y_p^f),x) \geq f(x_j)+1+d(x_j,X_0)+d(x,X_0)-\varepsilon \geq 1+d(y_p^f,X_0)+d(x,x_0)-\varepsilon\ ,$$

and the proof is complete . $\Diamond$

Remarks. 1. The proof can easily be adapted to show that the isometry groups of the *bounded* Urysohn spaces (i.e Urysohn spaces for spaces of diameter at most d) are not divisible either. To see it for instance for $d = 1$, it is enough to reproduce the above proof, except that one needs to replace the metric amalgam in the definition of X_i^f and X_{i+1} by "metric amalgams of diameter 1", i.e one needs to replace the metric d used in the proof by $\min(d,1)$.
2. C. Rosendal has proved that a generic element of Iso($\mathbb{U}$) does have roots of any order; so the behavior described above is pathological. Actually, it seems that one can prove that a generic element of Iso($\mathbb{U}$) embeds in a flow.

5.2 The uniform topology on Iso($\mathbb{U}$).

The topology of Iso($\mathbb{U}$) is now completely understood : it is homeomorphic to the Hilbert space [1] (notice that Uspenskij proved that the same is true of $\mathbb{U}$ itself, see [Usp4]). There is more than one "natural" topology on Iso($\mathbb{U}$), however: first, define $d'(x,y) = \min(d(x,y),1)$ for $x,y \in \mathbb{U}$ (beware: $(\mathbb{U},d')$ is *not* the Urysohn space for spaces of diameter 1, but for our purposes this does not matter). Then, define the *uniform distance* $d_\infty(\varphi,\psi)$ between two elements φ and ψ of Iso($\mathbb{U}$) by setting $d_\infty(\varphi,\psi) = \sup\{d'(\varphi(x),\psi(x))\colon x \in X\}$. Then $(\mathrm{Iso}(\mathbb{U}),d_\infty)$ is a topological group with a complete metric(it is perhaps more natural to consider the uniform topology on the isometry group Iso($\mathbb{U}_1$); the facts and questions below have obvious counterparts in that setting).
The following two exercises sum up all that the author knows about $(\mathrm{Iso}(\mathbb{U}),d_\infty)$.

Exercise 15. Prove that $(\mathrm{Iso}(\mathbb{U}),d_\infty)$ is not separable.

Exercise 16. Prove that, if $A \subset \mathbb{U}$ is finite and $\varphi\colon A \to \mathbb{U}$ is an isometric map such that $d(a,\varphi(a)) \leq \lambda$ for all $a \in A$, then φ extends to an isometry (still denoted by φ) of $\mathbb{U}$ such that $d(z,\varphi(z)) \leq \lambda$ for all $z \in Z$. Deduce from this that $(\mathrm{Iso}(\mathbb{U}),d_\infty)$ is not discrete (this is lemma 11 in [CV], and answers a question asked by Pestov in [Pe3]). Notice that the construction of the translation operator in Section 4 was already enough to prove this, since we saw that $(\mathbb{U},\min(d,1))$ isometrically embeds in $(\mathrm{Iso}(\mathbb{U}),d_\infty)$.

[1]This is an as yet unpublished result of the author, see the webpage http://www.math.uiuc.edu/~melleray for a draft of proof

Open problems about the uniform topology on Iso$(\mathbb{U})$.

- (Pestov [Pe3]) Does $\mathrm{Iso}(\mathbb{U})$ possess a uniform neighborhood of 0 covered by 1-parameter subgroups?
- (Pestov [Pe3]) Does $\mathrm{Iso}(\mathbb{U})$ have a uniform neighborhood of 0 containing non-trivial subgroups?
(The two questions above were asked of $\mathrm{Iso}(\mathbb{U}_1)$ instead of $\mathrm{Iso}(\mathbb{U})$)
- Linked to these questions, one may wonder whether $(\mathrm{Iso}(\mathbb{U}), d_\infty)$ is path-connected; the proof above does not adapt. It is possible to build a path of nonsurjective isometries which is continuous with regard to the above uniform distance (which is still well defined even it the isometries are not onto); the problem is that it turns out to be difficult in that case to find a back-and-forth argument that would ensure surjectivity of these maps.

6 Action of Iso$(\mathbb{U})$ on $\mathcal{F}(\mathbb{U})$.

A classical fact of descriptive set theory is that the set $\mathcal{F}(P)$ of closed sets of a given Polish metric space P may be endowed with a Borel structure, the *Effros Borel structure*, which is the σ-algebra generated by sets of the form $\{F \in \mathcal{F}(\mathbb{U}) \colon F \cap U = \emptyset\}$, where U varies over open subsets of P. Endowed with this structure, $\mathcal{F}(\mathbb{U})$ is a *standard Borel space*, i.e the σ-algebra above is isomorphic to the σ-algebra of Borel subsets of $[0,1]$ (or any other uncountable Polish space; see [Ke1] for detailed explanations). Then one may see the left-translation action of $\mathrm{Iso}(\mathbb{U})$, defined by $\varphi.F = \varphi(F)$, as a Borel action of $\mathrm{Iso}(\mathbb{U})$ on the standard Borel space $\mathcal{F}(\mathbb{U})$. The complexity of this action was computed by Gao and Kechris in [GK]: it is Borel bireducible to the universal relation for actions of Polish groups. Loosely speaking, this means that the induced relation is as complicated as a relation induced by a Borel action of a Polish group can be.

Theorem 6.1. [Me1] *Let G be a Polish group. Then there exists a closed set $F \subset \mathbb{U}$ such that G is isomorphic (as a topological group) to the stabilizer of F for the left-translation action; explicitly, this means that G is isomorphic to $\{\varphi \in Iso(\mathbb{U}) \colon \varphi(F) = F\}$.*

Actually, the proof gives slightly more: it produces a set $F \subset \mathbb{U}$ such that G is isomorphic to $\mathrm{Iso}(F)$, and any isometry of F extends uniquely to an isometry of $\mathbb{U}$.
This result answers a question asked by Gao and Kechris in [GK]. It is an illustration of the complexity of the action of $\mathrm{Iso}(\mathbb{U})$ on $\mathcal{F}(\mathbb{U})$: indeed, a result of Becker and Kechris [BK] states that, given a Borel action of a

Polish group H, the stabilizer of any point is necessarily a closed subgroup of H. In other words, stabilizers of points are always Polish groups; the above theorem states that the converse holds in that case, meaning that all the "theoretically possible" stabilizers are actually obtained. Notice though that, since the relation is not Borel, the map which to a closed set $F \subset \mathbb{U}$ associates its stabilizer, from $\mathcal{F}(\mathbb{U})$ to the set of closed subgroups of $\mathrm{Iso}(\mathbb{U})$ (which is a Borel subset of $\mathcal{F}(\mathrm{Iso}(\mathbb{U}))$), cannot be Borel either (see [Ke1]).
This result was published in [Me1]; the proof below is a simplified rendering of the original proof.

Proof of Theorem 6.1.
The starting point of this proof is the following result, due to Gao and Kechris (see [GK] or [Me2] for a proof): any Polish group is (isomorphic to) the isometry group of some Polish metric space. Let now G be a Polish group, and find X such that G is isomorphic to $\mathrm{Iso}(X)$; if one applies Katětov's construction with X as a starting point, then one obtains an increasing sequence of subsets (X_n) in $\mathbb{U}$ with dense union such that $X_1 = X$ and each isometry of X_i extends uniquely to an isometry of X_{i+1}, which proves that G is isomorphic to $\{\varphi \in \mathrm{Iso}(\mathbb{U})\colon \forall i \in \mathbb{N}\ \varphi(X_i) = X_i\}$. It is not very hard to ensure also that each X_i is closed, so that one obtains that G is isomorphic to the subgroup of isometries which stabilize each member of a countable sequence of closed subsets of $\mathbb{U}$; this was first proved by Gao and Kechris. Here, we want to show that it is possible to replace the sequence by a single closed set; for that, we use a variant of Katětov's construction, based on the following remark: to ensure that $\cup X_i$ is finitely injective, it is not necessary to have $X_{i+1} = E(X_i, \omega)$; it is sufficient that $X_{i+1} \supset E(X_i, \omega)$. Thus, one may add "control points" at each step, which enables us to construct the set F.
Before proceeding with the proof, we need to introduce some new notation: if Y is a bounded, nonempty subset of a metric space X, we set

$$E(X,Y) = \{f \in E(X)\colon \exists d \in \mathbb{R}^+\ \forall x \in X\ f(x) = d + d(x,Y)\} \ .$$

Notice that $E(X,Y)$ is isometric to $\mathbb{R}^+$; in particular, it is closed in $E(X)$.
For technical reasons, assume w.l.o.g that the space X such that $G = \mathrm{Iso}(X)$ that we have chosen is bounded, of diameter ≤ 1, and has more than two elements.
We begin by setting $X_0 = X$, and then define inductively a sequence of bounded Polish metric spaces X_i, of diameter d_i, by:

$$X_{i+1} = \Big\{f \in \overline{E(X_i,\omega)} \cup \bigcup_{j<i} E(X_i, X_j)\colon \forall x \in X_i\ f(x) \leq 2d_i\Big\} \ .$$

(We endow X_{i+1} with the distance induced by that of $E(X_i)$; the Kuratowski map isometrically embeds X_i in X_{i+1}, and we identify X_i with the corresponding subspace of X_{i+1}).
Then we see that $d_i \to +\infty$ with i, and each X_i is a Polish metric space. The construction ensures that $\cup X_i$ has the extension property, consequently its completion Y is isometric to $\mathbb{U}$. Notice also that any isometry $g \in G$ extends to an isometry g_i of X_i for all i.

Exercise 17. Prove that, for all i, g_i is the unique isometry of X_i whose restriction to X is g and such that $g_i(X_j) = X_j$ for all $j \le i$. Show also that each map $g \mapsto g_i$, from G into $\mathrm{Iso}(X_i)$, is continuous.

This enables us to associate to each $g \in G$ an isometry g^* of Y defined by ${g^*}_{|X_i} = g_i$, and this induces an embedding of topological groups from G into $\mathrm{Iso}(Y)$. Notice now that, if $f \in X_{i+1}$ is defined by $f(x) = d + d(x, X_j)$ for some d, some $j < i$, and all $x \in X_i$, then $g^*(f) = f$ for all $g \in G$; indeed, any element of X_{i+1} is uniquely determined by its distances to elements of X_i, and any g^* has to fix X_j.
The construction implies that an isometry $\varphi \in \mathrm{Iso}(Y)$ is equal to some g^* if, and only if, $\varphi(X_n) = X_n$ for all n. We now wish to build a closed set $F \subset Y$ such that for all $\varphi \in \mathrm{Iso}(Y)$ one has $\varphi(F) = F$ if and only if $\varphi(X_n) = X_n$ for all n.
Begin by fixing an enumeration $(k_i)_{i \ge 1}$ of the nonnegative integers, such that each integer appears an infinite number of times.
The definition of X_i ensures that one can choose inductively points $a_i \in \cup X_n$, positive reals e_i, and an increasing sequence of integers (j_i) such that:

- $e_1 \ge 4$; $\forall i \ge 1\ e_{i+1} > 4e_i$.
- $\forall i \ge 1\ j_i \ge k_i$, $a_i \in X_{j_i+1}$ and $\forall x \in X_{j_i}\ d(a_i, x) = e_i + d(x, X_{k_i-1})$.
- $\forall i \ge 1\ \forall g \in G\ g^*(a_i) = a_i$.

Exercise 18. Check that this is indeed possible.

We now let $F = X_0 \cup \{a_i\}_{i \ge 1}$; since X_0 is complete and $d(a_i, X_0) = e_i \to +\infty$, we see that F is a closed subset of Y. It is also clear that $\varphi \in G^* \Rightarrow \varphi(F) = F$ (since each a_i is fixed by G^*). All that remains to be done is to prove the converse; for that, we use the following lemma:

Lemma 6.2. *For all $\varphi \in Iso(F)$, one has $\varphi(X_0) = X_0$ and $\varphi(a_i) = a_i$ for all i. Furthermore, there exists some (necessarily unique) $g \in G$ such that $\varphi = g^*_{|F}$.*

Proof of lemma 6.2:
Notice that we only need to prove that $\varphi(X_0) = X_0$; then, each a_i has to be fixed since the mapping $i \mapsto d(a_i, X_0)$ is injective, and this proves that φ coincides with the restriction to F of $\varphi_{|X_0}$. The fact that $\varphi(X_0) = X_0$ is a consequence of the fact that each $e_i = d(a_i, X_0)$ is large: since X_0 has more than two elements and $\text{diam}(X_0) \leq 1$, the definition of F implies that

$$\forall x \in F \ (x \in X_0) \Leftrightarrow (\exists y \in F \ \ 0 < d(x,y) \leq 1)$$

The right-hand side of this equivalence is invariant under the action of isometries of F, which is enough to ensure that $\varphi(X_0) = X_0$ for all $\varphi \in \text{Iso}(F)$. $\Diamond$

The idea behind the definition of the a_i's is that, if $\varphi \in \text{Iso}(Y)$ maps some element of X_{j_i} into $X_{k_{i-1}}$, then one must have $\varphi(a_i) \neq a_i$; but the distances $d(a_i, a_j)$ are such that each isometry of F must leave each a_i fixed.
Lemma 6.2 implies that G is isomorphic to $\text{Iso}(F)$; furthermore, each isometry of F extends to Y (since they coincide with elements of G), and this induces a continuous morphism from $\text{Iso}(F)$ into $\text{Iso}(Y)$ is continuous.
To finish the proof of Theorem 6.1, it is therefore enough to show that each isometry of F admits a unique extension to Y. We saw above that it is enough to show that, if $\varphi \in \text{Iso}(Y)$ is such that $\varphi(F) = F$ then $\varphi(X_n) = X_n$ for all $n \geq 0$. Pick some $\varphi \in \text{Iso}(Y)$ such that $\varphi(F) = F$.
It is enough to show that $\varphi(X_n) \supseteq X_n$ for all $n \in \mathbb{N}$; assume that this is not true, i.e that there exists some $n \in \mathbb{N}$ and $x \notin X_n$ such that $\varphi(x) \in X_n$. Let $\delta = d(x, X_n) > 0$ (recall that X_n is complete); pick $y \in \cup X_m$ such that $d(x,y) \leq \frac{\delta}{4}$. Then $y \in X_m \setminus X_n$ for some $m > n$; one may find i such that $k_i = n+1$ and $j_i \geq m$.
Then one has $d(\varphi(y), \varphi(a_i)) = d(y, a_i) = e_i + d(y, X_n) \geq e_i + \frac{3\delta}{4}$, and $d(a_i, \varphi(y)) \leq d(a_i, \varphi(x)) + d(x,y) \leq e_i + \frac{\delta}{4}$, so $d(\varphi(a_i), a_i) \geq \frac{\delta}{2}$, and this contradicts Lemma 6.2. $\Diamond$

Now that we saw what the stabilizers look like for the left-translation action of $\text{Iso}(\mathbb{U})$ on $\mathcal{F}(\mathbb{U})$ (or rather now that we saw that the stabilizers look like nothing in particular, since any Polish group is the stabilizer of some closed set), it is natural to ask what the orbits under this action are. Of course, the orbit of F is contained in $\{F' \in \mathcal{F}(\mathbb{U})\colon F' \text{ is isometric to } F\}$. The universal property of $\mathbb{U}$ ensures that the converse is true if F is finite; we saw earlier in the paper a proof, originally due to Huhunaišvili, that it also holds when F is compact. In the original paper of Urysohn, the question of determining for which sets the converse holds is asked; he was already aware that it could not hold for all sets. We saw in the examples of Section 4 that there are many

proper subsets of $\mathbb{U}$ which are isometric to it, so in particular the converse does not hold for $F = \mathbb{U}$.
There is also a related question: which Polish metric spaces X are such that given any $X', X'' \subset \mathbb{U}$ isometric to X, any isometry $\varphi \colon X \to X'$ extends to an isometry of $\mathbb{U}$? This is obviously a stronger property than the one considered above, and we already saw that spheres do not have this property. It turns out that both properties are equivalent, as implied by the following theorem (published for the first time in [Me3]), which provides the answer to Urysohn's question.

Theorem 6.3. [Me3] *Let X be a Polish metric space. The following assertions are equivalent:*
(a) X is compact.
(b) If $X_1, X_2 \subseteq \mathbb{U}$ are isometric to X and $\varphi \colon X_1 \to X_2$ is an isometry, then there exists $\tilde{\varphi} \in Iso(\mathbb{U})$ which extends φ.
(c) If $X_1, X_2 \subseteq \mathbb{U}$ are isometric to X, then there exists $\varphi \in Iso(\mathbb{U})$ such that $\varphi(X_1) = X_2$.
(d) If $X_1 \subseteq \mathbb{U}$ is isometric to X and $f \in E(X_1)$, there exists $z \in \mathbb{U}$ such that $d(z,x) = f(x)$ for all $x \in X_1$.

As explained above, $(a) \Rightarrow (b)$ has been known for 50 years; $(b) \Rightarrow (c)$ is obvious. We have to note here that E. Ben Ami and C. Ward Henson independently obtained (different) proofs of the equivalence between (a), (b) and (c); to the knowledge of the author, there is as yet no preprint or article containing any of these two proofs, so the curious reader will have to look up references himself.

Exercise 19. Using the fact that there exists a copy of X which is g-embedded in $\mathbb{U}$, prove that $(c) \Rightarrow (d)$.

The proof of $(d) \Rightarrow (a)$ is much more intricate; we postpone it for the moment (it will be a consequence of Proposition 6.10 below). We begin by analyzing what it means for a Polish metric space to have property (d), and establish that it is necessary that X have the collinearity property. Then we will provide a construction that proves that a Polish metric space with the collinearity property can only have property (d) if it is compact, which will be enough to finish the proof of Theorem 6.3.
For a subset X of $\mathbb{U}$, the map $\Phi^X \colon \mathbb{U} \to E(X)$ defined by $z \mapsto (x \mapsto d(z,x))$ is continuous (it is 1-Lipschitz), so the image of $\mathbb{U}$ is separable. Property (d) is equivalent to $\Phi^{X'}$ being onto for *any* isometric copy X' of X contained in $\mathbb{U}$; it is possible that $\Phi^{X'}$ is onto for *some* isometric copy X' of X contained in $\mathbb{U}$ only if $E(X)$ is separable. Therefore, for X to have property (d), it is

necessary that $E(X)$ be separable. As a side remark, notice that if $E(X)$ is separable then there does exist some $X' \subset \mathbb{U}$ isometric to X and such that all $f \in E(X')$ are realized in $\mathbb{U}$: just begin Katětov's construction with $X_0 = E(X)$.

Recall that we provided in section 2 a characterization of Polish metric spaces X such that $E(X)$ is separable in terms of the collinearity property. In order to prove theorem 6.3, we need to show that, given any noncompact Polish metric space X with the collinearity property, there exists an isometric copy X' of X which is contained in $\mathbb{U}$ and is such that for some $f \in E(X')$ there is no $z \in \mathbb{U}$ satisfying $d(z, x') = f(x')$ for all $x' \in X'$.

We first need to introduce a new definition; to try to motivate it, we consider the case $X = \mathbb{N}$. We wish to build an embedding of $\mathbb{N}$ into $\mathbb{U}$ such that there is some $f \in E(\mathbb{N})$ which is not realized in $\mathbb{U}$. Turning the question on its head, we ask the following question: what kind of condition on $f \in E(\mathbb{N})$ ensures that, for any embedding of $\mathbb{N}$ into $\mathbb{U}$, f must be realized in $\mathbb{U}$? If f happens to be completely determined by its values on some finite subset of $\mathbb{N}$, then it must be realized, because of the finite injectivity of $\mathbb{U}$. Say now that $f \in E(\mathbb{N})$ is *strongly saturated* if there exist $n < m \in \mathbb{N}$ such that $f(n) + f(m) = m - n$. Then, for any $p \geq m$, we must have $f(p) \geq p - n - f(n) = p + f(m) - m$, and $f(p) \leq f(m) + p - m$, so that $f(p) = f(m) + p - m$ for all $p \geq m$; similarly, this holds for all $p \leq n$, so that f is completely determined by its values on $[n, m]$. Therefore, for any isometric embedding of $\mathbb{N}$ into $\mathbb{U}$ and any strongly saturated $f \in E(\mathbb{N})$, f must be realized in $\mathbb{U}$; this also has to be true for any f which is in the closure of the set of strongly saturated Katětov maps on $\mathbb{N}$. We call such an f a *saturated* Katětov map. It turns out that the converse is true, i.e f must be realized for any embedding of $\mathbb{N}$ if and only if it is saturated. Note that this definition may also be expressed in terms of model theory for metric structures (saturated maps are actually the same thing as d-isolated 1-types over X).

Definition 6.4. Let X be a Polish metric space. We say that $f \in E(X)$ is *ε-saturated* if there exists a compact subset K of X such that, for any $g \in E(X)$, one has $g_{|K} = f_{|K} \Rightarrow d(f, g) \leq \varepsilon$.

We say that f is *saturated* if it is ε-saturated for all $\varepsilon > 0$.

First, let us note the following.

Proposition 6.5. *Let X be a Polish metric space. Then the set of saturated maps on X is closed in $E(X)$.*

Proof. First, we need to point out the following fact: let $f \in E(X)$, $\varepsilon > 0$, and pick $Y \subset X$ and $g \in E(Y)$ such that $\sup\{|f(y) - g(y)| \colon y \in Y\} \leq \varepsilon$.

Then for any $x \in X \setminus Y$ one can extend g to a Katětov map (still denoted by g) on $Y \cup \{x\}$ that satisfies $|f(x) - g(x)| \leq \varepsilon$ (just look at the inequalities $g(x)$ must satisfy). Thus, using transfinite induction, one sees that actually g extends to a map $\tilde{g} \in E(X)$ such that $d(f,\tilde{g}) \leq \varepsilon$.
Now, let (f_n) be a sequence of saturated maps in $E(X)$ that converges to some f. Let $\varepsilon > 0$, and pick n such that $d(f_n, f) \leq \varepsilon$. Then pick a compact set K that witnesses that f_n is ε- saturated, and, let $g \in E(X)$ be any Katětov map such that f, g coincide on K. One has $\sup\{|g(x) - f_n(x)| \colon x \in K\} \leq \varepsilon$, so there exists a map $h \in E(X)$ that extends $f_{n_{|K}}$ and is such that $d(h, g) \leq \varepsilon$. Since we must have $d(h, f_n) \leq \varepsilon$ because of the choice of K, we obtain:

$$d(f, g) \leq d(f, f_n) + d(f_n, h) + d(h, g) \leq 3\varepsilon \ .$$

So the compact set K witnesses the fact that f is 3ε-saturated, and (since $\varepsilon > 0$ was arbitrary) we are done. $\Diamond$

Also, it is obvious that if X is compact then all Katětov maps on X are saturated; the converse is true.

Lemma 6.6. *If X is a noncompact Polish metric space, then there exists $f \in E(X)$ which is not saturated.*

Proof. We only prove this in the case when X is Heine-Borel, since this is the only case that we are concerned with while trying to prove Theorem 6.3. Since X is noncompact, there exists a sequence x_n such that $d(x_0, x_n) \to \infty$; we may assume that $d(x_{n+1}, x_0) \geq d(x_n, x_0) + 1$. Set then $f(x) = 1 + d(x_0, x)$. We claim that f is not saturated. Indeed, given a compact subset K of X, one may find $n \in \mathbb{N}$ such that $d(x_n, x_0) \geq d(x_0, k) + 1$ for all $k \in K$. Let then $\tilde{f}$ be the map on $K \cup \{x_n\}$ defined by $\tilde{g}(k) = f(k)$ for all $k \in K$, and $\tilde{f}(x_n) = f(x_n) - 1$. Then $\tilde{f}$ is 1-Lipschitz because of the choice of n, and one has, for all $k \in K$, that
$\tilde{f}(x_n) + \tilde{f}(k) = d(x_n, x_0) + d(x_0, k) + 1 \geq d(x_n, k) + 1$. This shows that $\tilde{f}$ is a Katětov map on $K \cup \{x_n\}$, so its Katětov extension to X witnesses the fact that f is not saturated. $\Diamond$

Exercise 20. Prove lemma 6.6 in the case when X is bounded.

In order to help the reader understand better what saturated Katětov maps are, we regroup a few of their properties in the following exercise; we will use these properties in the proof of Proposition 6.10.

Lemma 6.7. *Let X be a Polish metric space with the collinearity property. Then the following assertions hold:*

(1) If $\varepsilon > 0$ and $f \in E(X)$ is not ε-saturated, then for any compact $K \subseteq X$ there is some $x \in X$ such that $f(x) + f(k) > d(x,k) + \varepsilon$ for all $k \in K$.
(2) If $f \in E(X)$ is saturated, then for any $\varepsilon > 0$ there exists some compact $K \subseteq X$ such that

$$\exists M\, \forall x \in X\ d(x,K) \geq M \Rightarrow \exists z \in K\ f(z) + f(x) \leq d(z,x) + \varepsilon.$$

(3) Let $f_n \in E(X)$ be ε_n-saturated maps such that :
- For any n there exists a compact K_n which witnesses the fact that f_n is $2\varepsilon_n$-saturated, and such that $m \geq n \Rightarrow f_{m_{|K_n}} = f_{n_{|K_n}}$.
- $\varepsilon_n \to 0$.
- $\cup K_n = X$
Then f_n converges uniformly to a saturated Katětov map f.

Exercise 21. Prove Lemma 6.7.

Definition 6.8. If $Y \subset X$ are metric spaces, we let $E(X, Y, \omega)$ denote the set of maps $f \in E(X)$ which have a support contained in $Y \cup F$, where F is some finite subset of X. For instance, $E(X, \emptyset, \omega) = E(X, \omega)$ and $E(X, X, \omega) = E(X)$. The interest for us is that $E(X, Y, \omega)$ is separable if $E(Y)$ is.

We can now describe our construction: we begin by picking a Polish metric space with the collinearity property X, then we set $X_0 = X$ and define

$$X_{i+1} = \{f \in E(X_i, X_0, \omega)\colon f_{|X_0} \text{ is saturated }\}\ .$$

(We identify as usual X_i to a subspace of X_{i+1}).
For the remainder of this section, the notation X_i will denote one of the spaces defined above; we first establish a technical lemma.

Lemma 6.9. *Let $x_1, \ldots, x_n \in X_p$, $f \in E(\{x_1, \ldots, x_n\})$.*
Let also $f' \in E(X_p, X_0, \omega)$ and $\varepsilon > 0$ be such that $f'(x_i) = f(x_i)$ for all i, and $f'_{|X_0}$ is not ε-saturated.
Then, for any compact $K \subset X_0$, there exists $g \in E(X_p, X_0, \omega)$ such that

$$\forall i = 1, \ldots, n\ g(x_i) = f(x_i),\ g_{|K} = f'_{|K} \text{ and } \exists x \in X_0 \backslash K\ g(x) \leq f'(x) - \frac{\varepsilon}{2}. \qquad (*)$$

Proof of Lemma 6.9.
Begin by picking some $z_0 \in K$ (which we may assume to be nonempty).
Since $f'_{|X_0}$ is not ε-saturated, lemma 6.7(1) shows that we can find $y_1 \in X_0 \setminus K$ such that $f'(y_1) + f'(z) > d(y,z) + \varepsilon$ for all $z \in K \cap X_0$. Letting $K_1 = B(z_0, 2d(z_0, y_1))$ we can apply the same process and find y_2, and so on. One can indefinitely continue this process, and thus build a sequence (y_n) of

elements of X_0 such that $d(y_n, z_0) \to +\infty$, an increasing sequence of compact sets (K_i) such that $K_0 = K$, $\cup K_i = X_p$, and

$$\forall i \geq 1 \ \forall z \in K_{i-1} \cap X_0 \ f'(y_i) + f'(z) > d(y_i, z) + \varepsilon \ .$$

We first point out that, if for all $I \in \mathbb{N}$ there exists $i \geq I$ such that $f'(y_i) + f'(x_k) \geq d(x_k, y_i) + \frac{\varepsilon}{2}$ for all $k = 1, \ldots, n$, then we can find a map g as in (*). Indeed, choose I such that $d(y_I, z_0) \geq \max\{f'(z) \colon z \in K_0\} + \frac{\varepsilon}{2}$, $f'(y_i) \geq f'(z)$ for all $z \in K_0$ and $i \geq I$, $K_I \supseteq B(z_0, 2\mathrm{diam}(K_0)]$, then find $i \geq I$ as above. Define a map g on $\{x_k\}_{k=1,\ldots,n} \cup K \cup \{y_i\}$ by $g(y_i) = f'(y_i) - \frac{\varepsilon}{2}$, $g(x) = f'(x)$ elsewhere.
By choice of i and since $f'(y_i) + f'(z) \geq d(y, z) + \frac{\varepsilon}{2}$ for all $z \in K_0$, we see that g is a Katětov map, and that its Katětov extension $\hat{g}$ to X_p is such that $\hat{g}(x_i) = f(x_i)$, $\hat{g}_{|K} = f'_{|K}$ and $\hat{g}(y_i) \leq f'(y_i) - \frac{\varepsilon}{2}$.
So, we may as well assume that there exists $I \in \mathbb{N}$ such that

$$\forall i \geq I \ \exists k_i \ f'(y_i) + f(x_{k_i}) < d(x_{k_i}, y_i) + \frac{\varepsilon}{2} \ . \qquad (**)$$

We show that this is impossible (which is enough to prove the lemma): up to some extraction, we may also assume that $k_i = k$ for all $i \geq I$. By definition of X_p, we know that the restriction to X_0 of the map $d(x_k, .)$ is saturated, so lemma 6.7(2) shows that there exists J such that

$$\forall j > J \, \exists z \in K_J \cap X_0 \ \ d(x_k, z) + d(x_k, y_j) \leq d(z, y_j) + \frac{\varepsilon}{4} \ .$$

This, combined with $(**)$, shows that for all $j > \max(I, J)$ there exists $z \in K_J \cap X_0 \subseteq K_{j-1} \cap X_0$ such that $f'(y_j) + f(x_k) + d(x_k, z) \leq d(z, y_j) + \frac{\varepsilon}{2} + \frac{\varepsilon}{4}$. This in turn implies that $f'(y_j) + f'(z) < d(z, y_j) + \varepsilon$, and that contradicts the definition of the sequence (y_i). ◊

Proposition 6.10. [Me3] *Let X be a Polish metric space with the collinearity property. Then there exists an isometric copy $X' \subseteq \mathbb{U}$ of X such that $\Phi^{X'}(z)$ is saturated for all $z \in \mathbb{U}$.*

Proof of Proposition 6.10.
It is enough to prove that $\cup X_i$ is finitely injective (recall that the X_i's are the spaces that were defined before the statement of Lemma 6.9); this will yield an isometric embedding of X in $\mathbb{U}$ with the desired property (notice that then $\Phi^{X'}(z)$ is saturated for a dense subset of $\mathbb{U}$, and by continuity and the fact that the set of saturated maps is closed in $E(X)$ we obtain that

actually $\Phi^{X'}(z)$ is saturated for all $z \in \mathbb{U}$).
Pick $\{x_1, \dots, x_n\} \subseteq X_p$ (for some $p \geq 0$) and $f \in E(\{x_1, \dots, x_n\})$. We are looking for $g \in E(X_p, X_0, \omega)$ such that $g(x_i) = f(x_i)$ for all i, and $g_{|X_0}$ is saturated. Letting $\varepsilon_0 = \inf\{\varepsilon > 0 \colon k(f)_{|X_0}$ is $\varepsilon -$ saturated $\}$, we only need to deal with the case $\varepsilon_0 > 0$.
Let $L_0 \subset X_0$ be a compact set witnessing the fact that $k(f)_{|X_0}$ is $2\varepsilon_0$-saturated, and choose $z_0 \in L_0$; Lemma 6.9 shows that there exists $f_1 \in E(X_p, X_0, \omega)$ such that $f_{1|_{L_0}} = k(f)_{|_{L_0}}$, $f_1(x_i) = f(x_i)$ for $i = 1, \dots, n$ and $z_1 \in X_0 \setminus L_0$ such that $f_1(z_1) \leq \min\{k(f)(z) + d(z, z_1) \colon z \in L_0\} - \frac{\varepsilon_0}{2}$.
Again, let $\varepsilon_1 = \inf\{\varepsilon > 0 \colon f_{1|_{X_0}}$ is $\varepsilon -$ saturated $\}$: if $\varepsilon_1 = 0$ we are finished, so assume it is not, let $X_0 \supseteq L_1 \supseteq B(z_0, \mathrm{diam}(L_0) + d(z_0, z_1)] \cap X_0$ be a compact set witnessing the fact that $f_{1|_{X_0}}$ is $2\varepsilon_1$-saturated and apply the same process as above to $(f_1, L_1, \varepsilon_1)$.
Then we obtain $z_2 \notin L_1$ and $f_2 \in E(X_p, X_0, \omega)$ such that $f_2(x_i) = f(x_i)$ for $i = 1, \dots, n$, $f_{2|_{L_1}} = f_{1|_{L_1}}$ and $f_2(z_2) \leq \min\{f_1(z) + d(z, z_2) \colon z \in L_1\} - \frac{\varepsilon_1}{2}$.
We may iterate this process, thus producing a (finite or infinite) sequence $(f_m) \in E(X_p, X_0, \omega)$ who has (among others) the property that
$f_m(x_i) = f(x_i)$ for all m and $i = 1, \dots, n$; the process terminates in finite time only if some $f_{m|_{X_0}}$ is saturated, in which case we have won.
So we may focus on the case where the sequence is infinite: then the construction produces a sequence of Katětov maps (f_m) whose restriction to X_0 is ε_m- saturated, an increasing sequence of compact sets (L_m) such that $\cup L_m = X_0$ witnessing that $f_{m|_{X_0}}$ is $2\varepsilon_m$-saturated, f_{m+1} and f_m coincide on L_m for all m, and points $z_m \in L_m \setminus L_{m-1}$ such that

$$f_m(z_m) \leq \min\{f_{m-1}(z) + d(z, z_m) \colon z \in L_{m-1}\} - \frac{\varepsilon_{m-1}}{2} \ .$$

If 0 is a cluster point of (ε_m), passing to a subsequence if necessary, we may apply lemma 6.7(3) and thus obtain a map $h \in E(X_0 \cup \{x_1, \dots, x_n\})$ such that $h(x_i) = f(x_i)$ for all $i = 1, \dots, n$ and $h_{|X_0}$ is saturated; then its Katětov extension to X_p has the desired properties.
Therefore, we only need to deal with the case when there exists $\alpha > 0$ such that $\varepsilon_m \geq 2\alpha$ for all m; we will show by contradiction that this never happens. To simplify notation, let $A = \{x_1, \dots, x_n\} \cup X_0$. Since the sequence (L_m) exhausts X_0, the sequence $(f_{m|_A})$ converges pointwise to some $h \in E(A)$ such that $h(z_m) = f_m(z_m)$ for all m.
Up to some extraction, we may assume, since X has the collinearity property, that $d(z_0, z_m) + d(z_m, z_{m+1}) \leq d(z_0, z_{m+1}) + \frac{\alpha}{2}$ for all m.
Also we know that $h(z_{m+1}) \leq h(z_m) + d(z_m, z_{m+1}) - \alpha$.
The two inequalities combined show that $h(z_{m+1}) - d(z_{m+1}, z_0) \leq h(z_m) - d(z_m, z_0) - \frac{\alpha}{2}$. This is clearly absurd, since if it were true the sequence

$(h(z_m) - d(z_m, z_0))$ would have to be unbounded, whereas we have necessarily $h(z_m) - d(z_m, z_0) \geq -h(z_0)$.
This is (finally) enough to conclude the proof. $\Diamond$

Proof of Theorem 6.3.
Recall that we only needed to prove that (d) $\Rightarrow$ (a); for this, we will prove that $\neg$(a) $\Rightarrow$ $\neg$(d). Assume that X is a noncompact Polish metric space; then, if X doesn't have the collinearity property we know that X cannot have property (d) because $E(X)$ is not separable. If X has the collinearity property, then Proposition 6.10 tells us that there is an isometric copy $X' \subset \mathbb{U}$ of X such that only maps in the closure of the set of saturated maps on X' are realized in $\mathbb{U}$; but since X' is not compact, there is a Katětov map on X' that is not in the closure of the set of saturated maps on X', so X' witnesses the fact that X does not have property (d). $\Diamond$.

Remarks. 1. If one applies the construction above to $X_0 = (\mathbb{N}, |.|)$, one obtains a countable set $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathbb{U}$ such that $d(x_n, x_m) = |n - m|$ for all n, m and

$$\forall z \in \mathbb{U}\, \forall \varepsilon > 0\, \exists n, m \in \mathbb{N}\ d(x_n, z) + d(z, x_m) \leq |n - m| + \varepsilon.$$

In particular, $\{x_n\}$ is an isometric copy of $\mathbb{N}$ which is not contained in any isometric copy of $\mathbb{R}$.

2. In general, given a Polish metric space X, one may consider the set of all closed subsets of $\mathbb{U}$ which are isometric to X, and look at the equivalence relation on that set induced by the left-translation action of X. We proved that X is compact iff there is only one equivalence class; what is the situation when X is noncompact? What is the descriptive set-theoretic complexity of the associated relation? When $X = \mathbb{U}$ one can prove that it is Borel bireducible to the universal relation for Borel actions of Polish groups! On the other hand, is it true that if X is "simple" (say, X has the collinearity property, or X is locally compact) then the associated relation is simple too?Also, one may define a quasi-ordering on the isometric copies of X contained in $\mathbb{U}$ by setting $X \leq X'$ if there exists an isometric map $\varphi\colon \mathbb{U} \to \mathbb{U}$ such that $\varphi(X) = X'$. Then, if X has the collinearity property, this quasi-ordering has a minimal element (which we built in the construction above) and a maximal element (the one obtained by applying Katětov's construction with $X_0 = E(X)$). What else can be said about this ordering and its descriptive complexity?

7 Conjugacy in Iso($\mathbb{U}$) and fixed points of isometries.

From a descriptive set-theoretic point of view, understanding the relation of conjugacy in Iso($\mathbb{U}$) is an interesting problem (mentioned in [GK]); we prove below that it is the universal relation for Borel actions of Polish groups. For that, it turns out to be enough to study the nature (up to isometry) of the sets of fixed points of isometries; the link of this with conjugacy is that if two isometries are conjugate then their sets of fixed points must be isometric. J. Clemens had conjectured in 2005 that a set of fixed points, if nonempty, had to be isometric to $\mathbb{U}$. It this were true, it would tell us nothing about the conjugacy relation; it turns out that the opposite result is true, even though Clemens' conjecture does hold for isometries with "small" orbits (see below).

Theorem 7.1. [Me3] *Let X be a Polish metric space.*
There exists an isometry $\varphi \in Iso(\mathbb{U})$ such that its set of fixed points $Fix(\varphi)$ is isometric to X.

Proof of Theorem 7.1.
The proof is based on ideas similar to those that were used to prove that there exists an isometry of $\mathbb{U}$ without a square root (actually the proof for fixed points was obtained first, and then the ideas were used to show the other result): we begin with an isometry which is the identity on X, and then build an embedding of X into $\mathbb{U}$ and an extension of that isometry, while trying to ensure that this extension "moves all points not in X as much as possible". We propose below a way to formulate this naive idea in proper mathematical language.

Definition 7.2. Let $X_0 \subset X$ be two metric spaces, and φ be an isometry; we say that (X, X_0, φ) has property (*) if :
- $\forall x \in X_0 \ \varphi(x) = x$.
- $\forall x, y \in X \ \ \liminf_{|p| \to +\infty} d(x, \varphi^p(y)) \geq d(x, X_0) + d(y, X_0)$.

If we manage, given a Polish metric space X, to build an embedding of X into $\mathbb{U}$ and an isometry φ of $\mathbb{U}$ such that $(\mathbb{U}, X, \varphi)$ has property (*) then we will be done. We again use an inductive construction, based on the following lemma.

Lemma 7.3. *Let $X_0 \subset X$ be Polish metric spaces and $\varphi \in Iso(X)$ be such that (X, X_0, φ) has property (*). Then there exists a Polish metric space $X' \supset X$, and an isometry φ' of X' which extends φ, which are such that any $f \in E(X, \omega)$ is realized in X' and (X', X_0, φ') still has property (*).*

If one admits for the time being that Lemma 7.3 is true, then Theorem 7.1 is very easy to prove: begin with the triple (X, X, φ_0) with $\varphi_0 = id_{|X}$, then define inductively $X_0 = X$, $X_{n+1} = X'_n$, $\varphi_{n+1} = \varphi'_n$. Since any $f \in E(X_n, \omega)$ is realized in X_{n+1}, the completion of $\cup X_i$ is isometric to $\mathbb{U}$; letting φ denote the isometry of $\mathbb{U}$ obtained at the end of the construction, the lemma also ensures that $(\mathbb{U}, X, \varphi)$ has property (*), so we are done.

Proof of Lemma 7.3. Let (X, X_0, φ) be as in the statement of the lemma. Consider now a set of disjoint isometric copies Y_n of $E(X, \omega)$ $(n \in \mathbb{Z})$, and let Y denote the amalgamation of those over X. If $y \in Y$ is such that $y \in Y_n$ and $d(y, x) = f(x)$ for all $x \in X$, we denote y by $f^{(n)}$. Notice that if $x \in X$ then our notation gives $x = d(x, .)^{(n)}$ for all $n \in \mathbb{Z}$, which we simply write $x = x^{(n)}$ for all $n \in \mathbb{Z}$.
Explicitly, the distance on Y is defined by $d(f^{(n)}, g^{(n)}) = d(f, g)$ (where d is the distance on $E(X)$), and $d(f^{(n)}, g^{(m)}) = \inf\{f(x) + g(x) \colon x \in X\}$ for $n \neq m$ and $f, g \notin X$.
Now, recall that each isometry φ of X extends to a unique isometry of $E(X, \omega)$, denoted by φ^*; let φ' denote the isometry of Y defined by setting

$$\varphi'(f^{(n)}) = (\varphi^*(f))^{(n+1)} \ .$$

Then φ' is an extension of φ. Let also X' denote the completion of Y, and denote again by φ' the unique extension of φ' to X'. We claim that (X', X_0, φ') has property (*). To prove it, it is enough to show that (Y, X_0, φ') has this property. The first part is an obvious consequence of the definition of X', φ' and our assumption that (X, X_0, φ) has property (*).
Let now $x, y \in Y$; we let $x = f^{(n)}$ and $y = g^{(m)}$, and assume that $g \notin X$ (if both f and g are in X then our assumption that (X, X_0, φ) has property (*) is enough to obtain what we wish).
By definition, f is supported by some finite set $\{x_1, \ldots, x_r\}$ and g by another finite set $\{y_1, \ldots, y_s\}$. For $|p|$ big enough, $(\varphi')^p(g^{(m)})$ and $f^{(n)}$ do not belong to the same Y_n, so that one has

$$d(x, (\varphi')^p(y)) = d(f^{(n)}, (\varphi^p(g))^{m+p}) = \inf\{f(z) + \varphi^p(g)(z) \colon z \in X\} \ .$$

The triangle inequality and the definition of f, g imply that there exists some $i_p \leq r, j_p \leq s$ such that $d(x, (\varphi')^p(y)) = f(x_{i_p}) + g(y_{j_p}) + d(x_{i_p}, \varphi^p(y_{j_p}))$. For p big enough, our assumption on (X, X_0, φ) implies that $d(x_i, \varphi^p(y_j)) \geq d(x_i, X_0) + d(y_j, X_0)$ for all $i \leq r, j \leq s$. Therefore, there exists some P such that $p \geq P$ implies

$$d(x, (\varphi')^p(y)) \geq f(x_{i_P}) + d(x_{i_p}, X_0) + g(y_{j_p}) + d(y_{j_p}, X_0) \geq d(x, X_0) + d(y, X_0).$$

This ends the proof of the fact that (X', X_0, φ') has property (*). $\Diamond$

In the construction above, we associate to each Polish metric space X an isometry φ_X of $\mathbb{U}$ such that X is isometric to $Fix(\varphi_X)$. The construction also has the property that, if X and X' are isometric, then φ_X and $\varphi_{X'}$ are conjugate: any isometry $\rho\colon X \to X'$ extends to an isometry of $\mathbb{U}$ such that $\rho \circ \varphi_X = \varphi_{X'} \circ \rho$ (identifying X, X' with their images in $\mathbb{U}$ under the embedding we defined above). The construction may be done uniformly, meaning that the map $X \mapsto \varphi_X$ may be assumed to be Borel (we do not go into detail here, see [Me2] for a sketch of proof). Admitting this, we see that $X \mapsto \varphi_X$ is a reduction of the relation of isometry between Polish metric spaces to the relation of conjugacy in Iso($\mathbb{U}$). Since Gao and Kechris proved that the former is universal for relations induced by a Borel action of a Polish group, and the latter is induced by a continuous action of a Polish group, this completely determines the Borel complexity of conjugacy in Iso($\mathbb{U}$).

Corollary 7.4. *(Of the construction)* [Me3] *The relation of conjugacy in Iso($\mathbb{U}$) is Borel bi-reducible to the universal relation for actions of Polish groups.*

Surprisingly, the situation turns out to be very different when it comes to studying isometries of finite order or, more generally, isometries with totally bounded orbits.

Theorem 7.5. [Me3] *If $\varphi\colon \mathbb{U} \to \mathbb{U}$ is an isometry whose orbits are totally bounded, and $Fix(\varphi)$ is nonempty, then $Fix(\varphi)$ is isometric to $\mathbb{U}$.*

To prove this theorem, we have to prove that $Fix(\varphi)$, if nonempty, has the approximate extension property. We need two lemmas, from which we deduce Theorem 7.5; so we don't begin the proof of that theorem yet. Still, it seemed interesting to state it as soon as possible, since it contrasts strongly from what we saw above and is the reason why we are interested in the two lemmas below.
We manipulate points x such that the diameter of their orbit under the action of φ, which we denote by $\rho_\varphi(x)$, is smaller than a given $\varepsilon > 0$. The first question is then: assuming that $Fix(\varphi)$ is nonempty and that $\rho_\varphi(x)$ is small, does x have to be close to $Fix(\varphi)$? The answer is a consequence of the following lemma.

Lemma 7.6. *Let $\varphi \in Iso(\mathbb{U})$ have totally bounded orbits; assume that $Fix(\varphi)$ is nonempty, and that $x \in \mathbb{U}$ is such that $\rho_\varphi(x) \leq 2\varepsilon$. Then, for any $\delta > 0$, there exists $y \in \mathbb{U}$ such that :*
- $\forall n \in \mathbb{Z}\ d(y, \varphi^n(x)) = d(y, x) \leq \varepsilon + \delta$;
- $\rho_\varphi(y) \leq \varepsilon$.

A direct consequence of this lemma is that, if $Fix(\varphi)$ is nonempty and $\rho_\varphi(x) \leq \varepsilon$ for some $\varepsilon > 0$, then there exists a fixed point x' of φ such that $d(x, x') \leq 2\varepsilon$.

Remark. Actually, using model theory for metric structures, one can prove (under the additional assumption that there is a uniform bound on the entropy of the orbits of φ) that for any $x \in \mathbb{U}$ there must exist a fixed point x' of φ such that $d(x, x') = \frac{1}{2}\rho_\varphi(x)$.

Proof of Lemma 7.6.
Let x, φ be as above; let also

$$E = \{y \in \mathbb{U} \colon \forall n \in \mathbb{Z}\ d(y, \varphi^n(x)) = d(y, x) \text{ and } \rho_\varphi(y) \leq \varepsilon\}$$

Notice that E is nonempty, since any fixed point of φ belongs to E. We want to prove that $\alpha = d(x, E) \leq \varepsilon$. We proceed by contradiction, so assume $\alpha > \varepsilon$. For technical reasons, we need to split the proof in two cases here.
(1) For all $p \in \mathbb{N}^*$ there exists a fixed point y_p such that $\alpha \leq d(y_p, x) < \alpha + \frac{1}{p}$. If so, let p be such that $\frac{1}{p} < \frac{\varepsilon}{2}$ and $\alpha - \frac{1}{p} > \varepsilon$, then consider the map g defined by the following equations:
- $g(y_p) = \frac{1}{p}$
- $\forall n \in \mathbb{Z}\ g(\varphi^n(x)) = d(y_p, x) - \frac{1}{p}$.

Then g belongs to $E(\{\varphi^n(x)\} \cup \{y_p\})$, therefore there is $z \in \mathbb{U}$ with the desired distances; to conclude, notice that $z \in E$ and $d(z, x) < \alpha$, which is absurd.
(2) Assume we are not in case (1); we may pick a point $y \in E$ such that no fixed point is as close as y to x. Let now $\rho_\varphi(y) = \rho \leq \varepsilon$; a direct verification shows that the map g defined below belongs to $E(\{\varphi^n(x)\} \cup \{\varphi^n(y)\})$:
- $\forall n \in \mathbb{Z}\ g(\varphi^n(x)) = \max(\varepsilon, d(y, x) - \frac{\rho}{2})$.
- $\forall n \in \mathbb{Z}\ g(\varphi^n(y)) = \frac{\rho}{2}$.

Since the orbits of φ are totally bounded, there exists $z \in \mathbb{U}$ with the prescribed distances; consequently $z \in E$. Indeed, one has, for all $n \in \mathbb{Z}$, that

$$d(z, \varphi^m(z)) \leq d(z, y) + d(y, \varphi^m(z)) = \rho\ .$$

We may iterate this construction, which yields a sequence of points $y_i \in E$ such that $\rho_\varphi(y_{i+1}) \leq \rho_\varphi(y_i)$, and $d(y_{i+1}, y_i) = \frac{\rho_\varphi(y_i)}{2}$. If $\sum \rho_\varphi(y_i)$ converges, then the sequence y_i converges to a fixed point which is closer to x than y, and this is impossible by definition of y. Since $d(x, y_{i+1}) > \varepsilon \Rightarrow d(x, y_{i+1}) = d(x, y_i) - \frac{\rho_\varphi(y_i)}{2}$, we see that then there must be some i such that $d(x, y_i) = \varepsilon$. This concludes the proof of Lemma 7.6. $\Diamond$

Before we can move on to the next lemma, we need to establish a Claim (which mostly consists in checking some inequalities).

Claim.
Let $\varphi \in \mathrm{Iso}(\mathbb{U})$, $x_1, \ldots, x_m \in Fix(\varphi)$, $f \in E(\{x_1, \ldots, x_m\})$, and $z \in \mathbb{U}$. Assume that $\min\{f(x_i)\} \geq \rho_\varphi(z) > 0$. Partition $\{1, \ldots m\}$ in three subsets A, B, C by setting $i \in A$ iff $d(z, x_i) < f(x_i) - \frac{\rho_\varphi(z)}{2}$, $i \in B$ iff $d(z, x_i) > f(x_i) + \frac{\rho_\varphi(z)}{2}$, and $i \in C$ iff $|d(z, x_i) - f(x_i)| \leq \frac{\rho_\varphi(z)}{2}$.
Then the following equations define a Katětov map on $\{\varphi^n(z)\}_{n \in \mathbb{Z}} \cup \{x_i\}_{1 \leq i \leq m}$:
- $\forall n \in \mathbb{Z}$ $g(\varphi^n(z)) = \frac{\rho_\varphi(z)}{2}$,
- $\forall i \in A$ $g(x_i) = d(z, x_i) + \frac{\rho_\varphi(z)}{2}$,
- $\forall i \in B$ $g(x_i) = d(z, x_i) - \frac{\rho_\varphi(z)}{2}$,
- $\forall i \in C$ $g(x_i) = f(x_i)$.

Proof of the Claim.
To simplify notation, we let $\rho = \rho_\varphi(z)$. We begin by checking that g is 1-Lipschitz:
First, we have that $|g(x_i) - g(\varphi^n(z))| = |d(z, x_i) + \alpha - \frac{\rho}{2}|$, where $|\alpha| \leq \frac{\rho}{2}$. If $\alpha = \frac{\rho}{2}$ there is nothing to prove, otherwise it means that $d(z, x_i) \geq f(x_i) - \frac{\rho}{2}$, so that $d(z, x_i) \geq \frac{\rho}{2}$, which is enough to show that $|d(z, x_i) + \alpha - \frac{\rho}{2}| \leq d(z, x_i) = d(\varphi^n(z), x_i)$.
We now let $1 \leq i, j \leq m$ and assume w.l.o.g that $|g(x_i) - g(x_j)| = g(x_i) - g(x_j)$; there are three nontrivial cases.
(a) $g(x_i) = d(z, x_i) + \alpha$, $g(x_j) = d(z, x_j) + \beta$, with $\alpha > \beta \geq 0$.
Then one must have $g(x_j) = f(x_j)$, and also $g(x_i) \leq f(x_i)$, so that $g(x_i) - g(x_j) \leq f(x_i) - f(x_j) \leq d(x_i, x_j)$.
(b) $g(x_i) = d(z, x_i) + \alpha$, $g(x_j) = d(z, x_j) - \beta$, $0 \leq \alpha, \beta \leq \frac{\rho}{2}$. Then the definition of g ensures that $g(x_i) \leq f(x_i)$ and $g(x_j) \geq f(x_j)$, so that $g(x_i) - g(x_j) \leq f(x_i) - f(x_j) \leq d(x_i, x_j)$.
(c)$g(x_i) = d(z, x_i) - \alpha$, $g(x_j) = d(z, x_j) - \beta$, $0 \leq \alpha < \beta$.
Then we have $g(x_i) = f(x_i)$, and $g(x_j) \geq f(x_j)$, so $g(x_i) - g(x_j) \leq f(x_i) - f(x_j)$.
We proceed to check the remaining inequalities necessary for g to be a Katětov map:
- $g(\varphi^n(z)) + g(\varphi^p(z)) = \rho \geq d(\varphi^n(z), \varphi^p(z))$ by definition of ρ;
- $g(\varphi^n(z)) + g(x_i) = \frac{\rho}{2} + d(z, x_i) + \alpha$, where $|\alpha| \leq \frac{\rho}{2}$, so $g(\varphi^n(z)) + g(x_i) \geq d(z, x_i) = d(\varphi^n(z), x_i)$.
The last remaining inequalities to examine are that involving x_i, x_j; we again break the proof in subcases, of which only two are not trivial:
(a) $g(x_i) = d(z, x_i) + \alpha$ and $g(x_j) = d(z, x_j) - \beta$, where $0 \leq \alpha < \beta$. Then

$g(x_i) = f(x_i)$, and $g(x_j) \geq f(x_j)$, so that $g(x_i) + g(x_j) \geq d(x_i, x_j)$.
(b) $g(x_i) = d(z, x_i) - \alpha$, $g(x_j) = d(z, x_j) - \beta$: then we have that both $g(x_i) \geq f(x_i)$ and $g(x_j) \geq f(x_j)$, so again $g(x_i) + g(x_j) \geq d(x_i, x_j)$..
This concludes the proof of the Claim. $\Diamond$

Lemma 7.7. *Let φ be an isometry of $\mathbb{U}$ with totally bounded orbits, $x_1, \dots, x_m \in Fix(\varphi)$, $f \in E(\{x_1, \dots, x_m\})$, and $\varepsilon > 0$. Then one (or both) of the following assertions is true:*
- There exists $z \in \mathbb{U}$ such that $\rho_\varphi(z) \leq \varepsilon$ and $d(z, x_i) = f(x_i)$ for all i.
- There is $z \in Fix(\varphi)$ such that $|f(x_i) - d(z, x_i)| \leq \varepsilon$ for all i.

Proof of Lemma 7.7.
Once again, this proof is an inductive construction based on the compact injectivity of $\mathbb{U}$; the claim above provides us with the tool necessary to make that construction work.
We may assume that

$$\gamma = \inf \Big\{ \sum_{i=1}^{m} |f(x_i) - d(x, x_i)| \colon x \in Fix(\varphi) \Big\} > 0 \ .$$

Then, pick $x \in Fix(\varphi)$ such that $\sum_{i=1}^{m} |f(x_i) - d(x, x_i)| \leq \gamma + \frac{\varepsilon}{4}$.
With a proof similar to that of the Claim above, one can show that there exists $z \in \mathbb{U}$ such that
- $d(z, x) = \frac{\varepsilon}{2}$;
- $\forall i = 1, \dots, m \ |d(x, x_i) - f(x_i)| \leq \frac{\varepsilon}{2} \Rightarrow d(z, x_i) = f(x_i)$;
- $\forall i = 1, \dots, m \ d(x, x_i) > f(x_i) + \frac{\varepsilon}{2} \Rightarrow d(z, x_i) = d(x, x_i) - \frac{\varepsilon}{2}$;
- $\forall i = 1, \dots, m \ d(x, x_i) < f(x_i) - \frac{\varepsilon}{2} \Rightarrow d(z, x_i) = d(x, x_i) + \frac{\varepsilon}{2}$.

If this point is fixed, then either for all i one had $|d(z, x_i) - f(x_i)| \leq \dfrac{\varepsilon}{2}$ and thus $\gamma = 0$, which is against our initial assumption; or z is a fixed point such that $\displaystyle\sum_{i=1}^{m} |f(z_i) - d(x, x_i)| \leq \gamma - \frac{\varepsilon}{4}$, and this again contradicts the definition of γ. Thus, z cannot be fixed.
Using the Claim, we may then build a sequence (z_n) of elements of $\mathbb{U}$ such that $z_0 = z$ and

(1) $0 < \rho(z_n) \leq \varepsilon$;
(2) $\forall p \in \mathbb{Z} \ d(z_{n+1}, \varphi^p(z_n)) = \frac{\rho_\varphi(z_n)}{2}$;
(3) $\forall i \ \ d(z_n, x_i) < f(x_i) - \frac{\rho_\varphi(z_n)}{2} \Rightarrow d(z_{n+1}, x_i) = d(z_n, x_i) + \frac{\rho_\varphi(z_n)}{2}$;
(4) $\forall i \ d(z_n, x_i) > f(x_i) + \frac{\rho_\varphi(z_n)}{2} \Rightarrow \ d(z_{n+1}, x_i) = d(z_n, x_i) - \frac{\rho_\varphi(z_n)}{2}$;
(5) $\forall i \ |d(z_n, x_i) - f(x_i)| \leq \frac{\rho_\varphi(z_n)}{2} \Rightarrow \ d(z_{n+1}, x_i) = f(x_i)$.

(In (3), (4) and (5),"$\forall i$" means "for any integer i between 1 and m)

To see that such a sequence can be constructed, assume that its terms have been defined up to rank n. Then, the Claim above ensures that there exists $z = z_{n+1} \in \mathbb{U}$ which satisfies conditions (2) through (6). Condition (2) implies that $\rho_\varphi(z_{n+1}) \leq \rho_\varphi(z_n) \leq \varepsilon$; as above, z_{n+1} cannot be fixed because it would contradict the definition of γ.
If some z_n is such that $d(z_n, x_i) = f(x_i)$ for all $i = 1, \ldots, m$, then we are done. If it's not the case, then either A_n or B_n is nonempty for all n, thus conditions (4) and (5) ensure that $\sum \rho_\varphi(z_n)$ converges, and then condition (2) tells us that z_n converges to some z_∞, which must then be a fixed point of φ because of condition (2). But by construction $\sum_{i=1}^m |f(x_i) - d(z_\infty, x_i)| \leq \sum_{i=1}^m |f(x_i) - d(z_0, x_i)| < \gamma$, which again contradicts the definition of γ. $\Diamond$

We are now ready to finish the proof of Theorem 7.5.

Proof of Theorem 7.5.
Let φ be an isometry of the Urysohn space with totally bounded orbits, and assume that $Fix(\varphi)$ is nonempty. We need to prove that $Fix(\varphi)$ has the approximate extension property; for that, pick $x_1, \ldots, x_n \in Fix(\varphi)$ and $\varepsilon > 0$. Then Lemma 7.7 tells us that there exists a fixed point z of φ such that $|d(z, x_i) - f(x_i)| \leq \varepsilon$ for all $i = 1, \ldots, n$, or there exists z with $d(z, x_i) = f(x_i)$ for all $i = 1, \ldots n$ and $\rho_{\varphi(z)} \leq \varepsilon$. In the first case we are done; in the second case, Lemma 7.6 ensures that there is a fixed point z' such that $d(z', z) \leq 2\varepsilon$; and by the triangle inequality one must have $|d(z', x_i) - d(z, x_i)| \leq 2\varepsilon$ for all $i = 1, \ldots, n$, so that $|d(z', x_i) - f(x_i)| \leq 2\varepsilon$ and we are done. $\Diamond$

Looking at the proof, one sees that Theorem 7.5 may be generalized: indeed, the same arguments work to prove that if G is any group acting on $\mathbb{U}$ by isometries in such a way that all the orbits for this action are totally bounded (in particular, if G is a compact group acting continuously on $\mathbb{U}$ by isometries), then the set of global fixed points of G is either empty or isometric to $\mathbb{U}$.

Exercise 22. Prove this result.

Theorem 7.5 shows that there is a big difference between general isometries and isometries with totally bounded orbits. It may be worth mentioning quickly another difference: using similar methods to those used to prove Theorem 7.1 one can build an isometry of $\mathbb{U}$ such that $\inf\{\rho_\varphi(x) \colon x \in \mathbb{U}\} = 0$, yet $Fix(\varphi) = \emptyset$. However, using model theory for metric structures (see [BBHU]), one may prove that, if φ has totally bounded orbits (with uniformly

bounded entropy) and $\inf\{\rho_\varphi(x)\colon x \in \mathbb{U}\} = 0$, then $Fix(\varphi)$ is nonempty (and thus isometric to $\mathbb{U}$). To prove this, one can use Lemma 7.6 to show that the set of fixed points of φ is a so-called "definable set" and then use some model theory (going to a monster model; this is the step that requires the uniform bound on the entropy of the orbits).
This leads to a question: do all the "natural geometric invariants" for an isometry with totally bounded orbits satisfy the dichotomy of Theorem 7.5? Examples of such invariants are $\{x \in \mathbb{U}\colon d(x,\varphi(x)) = 1\}$ or (in the case where $\varphi^n = 1$ for some n) $\{x \in \mathbb{U}\colon d(x,\varphi(x)) = a_1,\ d(x,\varphi^2(x)) = a_2,\ d(x,\varphi^{n-1}(x)) = a_{n-1}\}$.
This is also the occasion to point out that model theory for metric structures seems to provide a natural setting to study (and solve) some questions about the geometry of $\mathbb{U}$; C.W. Henson established the equivalence of (a), (b) and (c) of Theorem 6.3 using this particular machinery.

8 Linear Rigidity of $\mathbb{U}$.

In [Hol1], M.R Holmes, following earlier work of Sierpinski [Si] on isometric embeddings of $\mathbb{U}$ in Banach spaces, proved a very surprising result, which we state below. Before this, we have to introduce some notation: if $(X,0)$ is a pointed metric space, then $(X,0)'$ denotes the set of 1-Lipschitz maps f on X such that $f(0)=0$.

Theorem 8.1. *(Holmes* [Hol1]*) If $\mathbb{U}$ is isometrically embedded in a Banach space B, and $0 \in \mathbb{U}$, then one has, for all $x_1,\dots,x_n \in \mathbb{U}$ and $\lambda_1,\dots,\lambda_n \in \mathbb{R}$:*

$$||\sum_{i=1}^{n} \lambda_i x_i|| = \sup\left\{\left|\sum_{i=1}^{n} \lambda_i f(x_i)\right|\colon f \in \{0,x_1,\dots,x_n\}'\right\} .$$

Proof. The proof below is a simplified rendering of the proof in Holmes' article [Hol1], which is quite long and intricate (Holmes was not directly interested in this result; he answered a question of Sierpinski about the embeddings of the Urysohn space in $\mathcal{C}([0,1]$, and the theorem above is a corollary of his proof); It is based on two simple lemmas.

Remark. After writing this article I was made aware of [Hol2], in which Holmes gives his own simplified version of his proof; it still seems worthwile to write down the proof here, since it can be generalized rather easily, which is not the case of Holmes's proof (because of the rather cumbersome use of $\mathcal{C}([0,1])$); all the ideas are already present in Holmes's paper.

Lemma 8.2. *Let $\mathbb{U}$ be embedded in a Banach space B in such a way that $0 \in \mathbb{U}$, and let $x_1, \dots, x_n \in \mathbb{U}$. Then there exists a continuous linear functional φ such that $||\varphi|| = 1$ and $\varphi(x_i) = \varphi(x_1) + d(x_1, x_i)$ for all $i = 1, \dots, n$.*

To prove this, let $x_1, \dots, x_n$ be as above, set $D = \text{diam}\{x_1, \dots, x_n\}$ and set $f(x_i) = 2D - d(x_1, x_i)$. Then $f \in E(\{x_1, \dots, x_n\})$, so there exists $z \in \mathbb{U}$ such that $d(z, x_i) = f(x_i)$ for all $i = 1, \dots, n$.
By the Hahn-Banach theorem, there exists a continuous linear map φ such that $||\varphi|| = 1$ and $\varphi(z) = \varphi(x_1) + d(x_1, z)$. Then one has, for all $i = 1, \dots, n$, that $\varphi(x_1) + d(x_1, x_i) \geq \varphi(x_i)$, and $\varphi(x_i) \geq \varphi(z) - d(z, x_i) = \varphi(x_1) + d(x_1, z) - d(z, x_i) = \varphi(x_1) + d(x_1, x_i)$. ◊

This ends the proof of Lemma 8.2; using it, one may prove the lemma below, which is enough to conclude the proof of Holmes's theorem.

Lemma 8.3. *Let $\mathbb{U}$ be embedded in B as above, let $x_1, \dots, x_n \in \mathbb{U}$, and let $f \in \{0, x_1, \dots, x_n\}'$. Then there exists a continuous linear functional φ such that $||\varphi|| = 1$ and $\varphi(x_i) = f(x_i)$ for all $i = 1, \dots, n$.*

Before proving Lemma 8.3, notice that it is sufficient to conclude the proof; indeed we have, for all $x_1, \dots, x_n \in \mathbb{U}$, and all $a_1, \dots, a_n \in \mathbb{R}$, that $||\sum a_i x_i|| = \sup\{|\sum a_i \varphi(x_i)| \colon \varphi \text{ is linear and } ||\varphi|| = 1\}$, so

$$||\sum a_i x_i|| \geq \sup\left\{\left|\sum_{i=1}^{n} \lambda_i f(x_i)\right| \colon f \in \{0, x_1, \dots, x_n\}'\right\} .$$

It is clear (again because of the Hahn-Banach theorem) that the converse inequality is always true, so we get that the equality holds. Thus, we only need to prove Lemma 8.3.

Proof of Lemma 8.3. Pick $f \in \{0, x_1, \dots, x_n\}'$, denote this time $D = \text{diam}(\{0, x_1, \dots, x_n\})$ and set $g(0) = 2D$, $g(x_i) = 2D + f(x_i)$. One checks easily that $g \in E(\{0, x_1, \dots, x_n\})$, so that there exists some $z \in \mathbb{U}$ such that $d(z, 0) = 2D$, and $d(z, x_i) = 2D + f(x_i)$ for all $i = 1, \dots, n$. Applying Lemma 8.2 to $z, 0, x_1, \dots, x_n$ (in that order), we obtain a linear functional φ such that $||\varphi|| = 1$, $\varphi(0) = \varphi(z) + d(z, 0) = \varphi(z) + 2D$ (so that $\varphi(z) = -2D$), and $\varphi(x_i) = \varphi(z) + d(z, x_i) = -2D + 2D + f(x_i) = f(x_i)$ for all i. ◊

This concludes the proof of Holmes's theorem, which has a remarkable consequence: assume that X, X' are isometric to $\mathbb{U}$, and that $0 \in X \subset B$, $0 \in X' \subset B'$, where B and B' are Banach spaces. Then any isometry $\varphi \colon X \to X'$ mapping 0 to 0 extends to a linear isometry $\tilde{\varphi}$ which maps the

closed linear span of X (in B) onto the closed linear span of X' (in B'): to see that, one simply has to check that the mapping $\tilde{\varphi}\colon \sum_{i=1}^n \lambda_i x_i \mapsto \sum_{i=1}^n \lambda_i \varphi(x_i)$ is an isometry from the linear span of X to that of X', and this is a direct consequence of the formula we obtained for the norm of a linear combination of elements of $\mathbb{U}$.
Also, if $P \subset \mathbb{U}$ is a Polish metric space containing 0, then Holmes' result shows that the closed linear span of P (in B) is isometric to the Lipschitz-free space over P (see [We] for information on these spaces).
In particular, if $0 \in \mathbb{U} \subset B$ for some Banach space B, then the closed linear span of $\mathbb{U}$ is isometric to the Lipschitz-free Banach space of $\mathbb{U}$; this might be stated as "the Urysohn space generates a unique Banach space". This leads to the introduction of a new property of metric spaces.

Definition 8.4. Let X be a metric space. We say that X is *linearly rigid* if, for any embedding of X in a Banach space B satisfying $0 \in X$, one has, for all $x_1, \dots, x_n \in X$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, that

$$||\sum_{i=1}^n \lambda_i x_i|| = \sup \{ \big| \sum_{i=1}^n \lambda_i f(x_i) \big| \colon f \in \{0, x_1, \dots, x_n\}' \} .$$

In other words, a space X is linearly rigid if and only there is a unique norm compatible with the metric on X.

We saw the Urysohn space is linearly rigid; in [MPV], it is established that linear rigidity is a "Urysohn-type" property, meaning that a space is linearly rigid if, and only if, some Katětov maps over finite subsets are (approximately) realized in the space.

Hints for the exercises.

(1) If $\hat{f}$, $\hat{g}$ denote the extensions of f and g from Y to X, and $x \in X$, then one has $\hat{f}(x) = f(y) + d(x,y)$ for some $y \in Y$; since $\hat{g}(y) \geq g(y) + d(y,x)$, one gets $f(x) - g(x) \leq f(y) - g(y) \leq d(f,g)$.
(2) Look at the proof of Proposition 2.2.
(3) This is proved by induction: saying that X has the extension property is exactly the statement "for any finite metric spaces $A \subset A' = A \cup \{z\}$, any isometric embedding of A into X extends to an isometric embedding of A' into X".
(4) Let X, Y be two finitely injective metric spaces; pick two countable sets $\{x_n\}$ and $\{y_n\}$ that are dense in X and Y respectively. Then use the back-and-forth method to build an isometry between two countable sets X' and

Y' such that $\{x_n\} \subset X'$ and $\{y_n\} \subset Y'$.
(5) Use the characterization of the random graph as being the unique universal countable graph G such that any isomorphism between finite subgraphs extends to an isomorphism of G (see [Bol]).
(6) Use the triangle inequality and the extension property of $\mathbb{U}$.
(7) Look at geodesics going through x and hitting the sphere in two points, one being as close to x as possible and the other being as far from x as possible. The second formula is

$$\forall x \in B \ d(x,0) = \frac{1}{2}(\sup_{x \in S} d(x,z) - \inf_{x \in S} d(x,z))$$

(8) Prove that the intersection, if nonempty, has the extension property for spaces of diameter d (where d is the smallest of the diameters of the spheres); use the extension property of $\mathbb{U}$ to prove this.
(9) Use the extension property to show that, if $x \neq y$, then there exists $k \in K$ such that $d(x,k) \neq d(x,y)$ (assume that $d(x,x_i) = d(x,y_i)$ for all i, and use an argument similar to the one used in the preceding exercise).
(10) For the beginning, use the same method as in Exercise 9. To prove that $\mathrm{Med}(\{x_1,x_2\}) \cup \{x_1\}$ is a set of uniqueness, pick $x, x' \in \mathbb{U}$ such that $d(x,x_1) = d(x',x_1)$. If $d(x,x_2) \neq d(x',x_2)$, prove that there exists z such that $d(z,x_1) = d(z,x_2)$ but $d(z,x) \neq d(z,x)$ (think of z as being very far from x, x', x_1, x_2). Conclude.
(11) Let X denote the amalgam of two copies of $\mathbb{U}$ over $B = B(0,1]$; consider an isometry ψ_0 of X which is the identity on B but has no other fixed points (draw a picture). Then embed X in $\mathbb{U}$, and pick any isometry φ which maps $\mathbb{U}$ to one of the two copies of $\mathbb{U}$ that were used to define X (sending B to the set the two copies of $\mathbb{U}$ are amalgamated on). Then φ and $\varphi' = \psi_0 \circ \varphi$ provide the example we're looking for.
(12) Show first that, if any isometry of B extends to an isometric map defined on $\mathbb{U}$, then any isometry of B actually has to extend to an isometry of $\mathbb{U}$. Also, if all isometries of B extend, then it defines a morphism from $\mathrm{Iso}(B)$ into $\mathrm{Iso}(\mathbb{U})$. Since this morphism has an inverse (the restriction to B), it is actually an isomorphism. The restriction morphism is clearly continuous, so the extension morphism has to be continuous too. Picking some $z \in \mathbb{U}$ such that $d(z,0) \geq 10$ (for instance), build a sequence of isometries (φ_n) of $\mathbb{U}$ such that $\varphi_n(x) \to x$ for all $x \in B$, but $d(z,\varphi_n(z)) = 1$ for all n. You may for instance ensure that φ_n fixes the first n points of a given dense subset of B, including its center, and $d(\varphi_n(z),z) = 1$. This shows that the extension morphism is not continuous, which is a contradiction.
(13) Use the back-and-forth method as in the proof of Theorem 3.2.
(14) Saying that τ commutes with σ is saying that $\tau(n+1) = \tau(n)+1$. But

then $\tau(n) = \tau(0) + n$ for all $n \in \mathbb{Z}$.
(15) It would be too long to detail the construction here; use the Gram-Schmidt orthonormalization process to build paths (you may even use finite-dimensional subspaces instead of compact subspaces).
(16) Notice that, for instance, the set of permutations on $\mathbb{N}$, endowed with the discrete distance, isometrically embeds in $\mathrm{Iso}(\mathbb{U})$ when it is equipped with the uniform metric. (There exists a copy of $\mathbb{N}$, with the discrete distance, which isometrically embeds in $\mathbb{U}$).
(17) Let $f \in E(A)$ be the map that corresponds to $\varphi^*(a)$ in $E(A)$; show that $d(a, f) \leq \lambda$ (where the distance is computed in $E(A)$). Use this to obtain a point $z \in \mathbb{U}$ such that setting $z = \varphi(a)$ is a suitable extension of φ.
(18) This proof is similar to the one that says that isometries of X extend uniquely to isometries of $E(X)$ (the idea is that a point in X_{i+1} is uniquely characterized (among points in X_{i+1}) by its distances to the points in X_i).
(19) Pick any number strictly larger than $4e_i$ and call it e_{i+1}. Then there exists j_i such that $d_{j_i} \geq 2e_{i+1}$; the definition of X_{j_i+1} ensures that setting $a_i(x) = e_{i+1} + d(x, X_{k_i-1})$ defines an element $a_i \in X_{j_i+1}$; this element has to be such that $g^*(a_i) = a_i$ for all $g \in G$. Indeed, $d(g^*(a_i), x) = e_{i+1} + d(x, X_{k_i-1})$ for all $x \in X_{j_i}$ (because X_{j_i} is fixed by g^*), and elements of X_{j_i+1} are uniquely characterized by their distances to elements of X_{j_i}.
(20) Prove that if (c) is satisfied then all copies of X are g-embedded in $\mathbb{U}$. To prove the implication, use this fact, and the fact that for any $f \in E(X)$ there exists an isometric copy of $X \cup \{f\}$ contained in $\mathbb{U}$.
(21) There exists $m, M > 0$ and a sequence (x_i) such that $m < d(x_i, x_j) < M$ for all $i \neq j$. Then check that the map defined by $f(x) = 2M + \inf(d(x, x_i))$ is a Katětov map and is not saturated.
(22) See [Me3], Lemma 4.8.
(23) The same proof works, replacing everywhere $\{\varphi^n(x) \colon n \in \mathbb{Z}\}$ by the orbit of x under the action of G (for instance, replace $\rho_\varphi(x)$ by the diameter of that orbit).

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