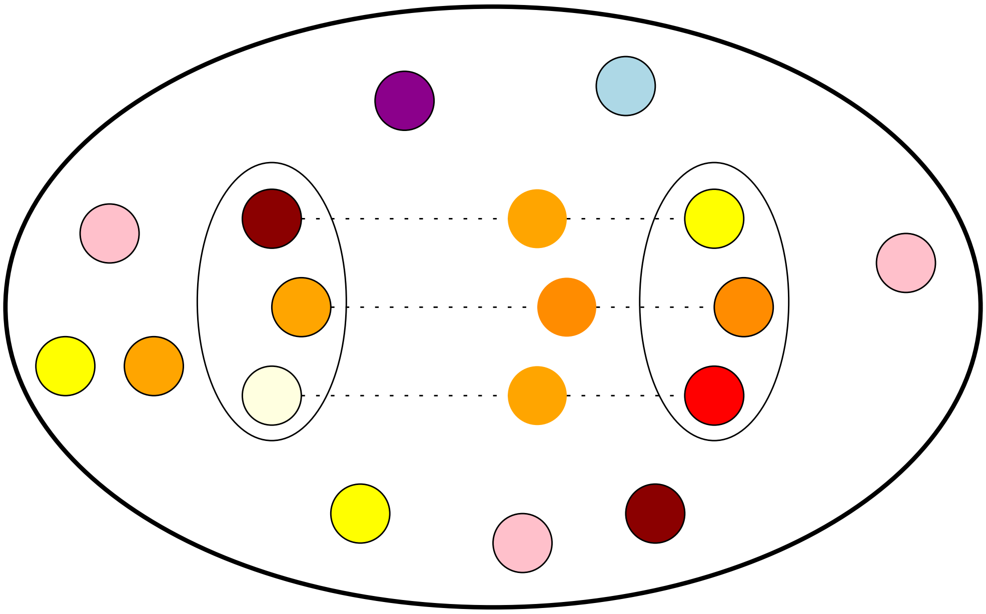
Institut Camille Jordan
UMR 5208 du CNRS

Structures métriques et leurs groupes d'automorphismes : reconstruction, homogénéité, moyennabilité et continuité automatique



Adriane Kaïchouh

Thèse de doctorat

Université Claude Bernard Lyon 1

Université Claude Bernard Lyon 1
École doctorale **InfoMath**, ED 512
Spécialité : **Mathématiques**
N. d'ordre 80-2015

Structures métriques et leurs groupes d'automorphismes : reconstruction, homogénéité, moyennabilité et continuité automatique

Thèse de doctorat

Soutenue publiquement le 26 juin 2015 par

Adriane Kaïchouh

devant le jury composé de:

Itaï Ben Yaacov	Université Lyon 1	Directeur de thèse
Krzysztof Krupiński	Uniwersytet Wrocławski	Rapporteur
Julien Melleray	Université Lyon 1	Directeur de thèse
Christian Rosendal	University of Illinois at Chicago	Examinateur
Marcin Sabok	McGill University	Examinateur
Todor Tsankov	Université Paris 7	Rapporteur

C'est véritablement utile parce que c'est joli.

Le Petit Prince

Contents

Remerciements 9

Introduction en français 13

Introduction 25

Trunk. Polish groups as automorphism groups 35

Chapter 1. Polish groups 37
1. Polish spaces 37
2. Polish groups 39
3. Uniformities on Polish groups 40

Chapter 2. The Urysohn space 45
1. Katětov spaces 45
2. Tower construction of the Urysohn space 48

Chapter 3. Model theory 51
1. Languages and structures 51
2. Isomorphisms, automorphism groups 54
3. Formulas 55
4. Theories and models 57
5. Spaces of types 58
6. Definability 60

Chapter 4. Fraïssé theory 63
1. Classical Fraïssé theory 63
2. Metric Fraïssé theory 69
3. Approximate maps 71
4. An approximate route to metric Fraïssé theory 77
5. Topology on the automorphism group 81

Chapter 5. Leitmotiv 83
1. Non-archimedean Polish groups 83
2. General Polish groups 84

Branch 1. Reconstruction 85

Chapter 6. Categoricity 87
1. Ryll-Nardzewski 88
2. Action of the automorphism group 90
3. Homogeneity 93

Chapter 7. Reconstruction of separably categorical structures 95
1. Reconstruction up to interdefinability 95
2. Reconstruction up to bi-interpretability 96

Branch 2. Automatic continuity 103

Chapter 8. Automatic continuity for infinite powers of Polish groups 105
1. Automatic continuity 106
2. The juxtaposed structure 109
3. The extension property 110
4. The existence property 111
5. Isolation 111
6. Connected groups with ample generics 115
7. Perspectives 117

Branch 3. Amenability and Ramsey theory 119

Chapter 9. Amenability and convex Ramsey theory 121
1. The metric convex Ramsey property 123
2. The metric convex Ramsey property for the automorphism group 126
3. A criterion for amenability 129
4. Structural consequences 131
5. Concluding remarks 134

Branch 4. Homogeneity 135

Chapter 10. Homogeneity 137
1. Countably categorical case 138
2. Somewhere realized types 138
3. When types are finitely determined 139
4. When they are not 142
5. Perspectives 147

Bibliography 149

Remerciements

Thank you very much.
Barack Obama[1]

Mes premiers remerciements vont à mes directeurs de thèse, Itaï Ben Yaacov et Julien Melleray, qui, s'amusant parfois à jouer good cop/bad cop, m'ont encadrée avec beaucoup d'humanité et d'humour et m'ont fait découvrir de très belles mathématiques dans la joie. Je ne saurais exprimer toute ma gratitude envers Julien d'avoir été toujours disponible pour m'aiguiller avec passion dans les maths et avec réalisme et bienveillance dans l'avenir. Au travers des mille-et-une questions fascinantes qu'il m'a posées, j'ai navigué avec plaisir entre les multiples facettes de la topo-logique et en ai découvert la richesse avec émerveillement. Au-delà des mathématiques, je remercie vivement Julien d'avoir égayé mes quatre années de thèse par de rocambolesques récits d'aventures et d'avoir affuté mon gout certain pour les blagues d'un gout moins certain ! Je suis aussi très reconnaissante à Itaï d'avoir patiemment toléré mes questions idiotes de théorie des modèles (même avec multiplicité) et d'avoir été, dans ses réponses, un véritable oracle. Je le remercie aussi de sa rafraichissante spontanéité et d'avoir été toujours un excellent public auxdites blagues de logiciennes !

Tout au long de ma thèse (et même un peu avant), Todor Tsankov m'a menée avec un grand enthousiasme de conférences en balades, de topologies en pièces de théâtre. Nos conversations, autant mathématiques que linguistiques, ont été très enrichissantes et il est pour moi un modèle. C'est un grand honneur qu'il m'a fait en acceptant de relire mon travail et je l'en remercie chaleureusement. I also thank Krzysztof Krupiński for accepting to referee my thesis. I am very grateful for his incredibly thorough proofreading : my thesis owes him a lot of clarity and the last chapter owes him correctness and a good fright !

I am also very honored that Christian Rosendal and Marcin Sabok took an interest in my work (which, for a substantial part, originates in theirs). Christian, de par ses excellents articles, ses excellents exposés, et les (trop rares) échanges que nous avons eus, m'a beaucoup appris et a été une grande source d'inspiration pour mon travail. Je le remercie d'avoir accepté de faire partie du jury. I also thank Marcin for accepting to be on the committee, as well as for his kindness and for our enlightening discussions about automatic continuity.

Je tiens également à remercier les personnes avec qui j'ai eu le plaisir de collaborer et à qui, bien entendu, cette thèse doit beaucoup. Itaï, merci encore ! I have had the chance to work with Isabel Müller on several projects and several occasions and it has been an immense pleasure every time. Feeling absolutely free to express the most stupid thoughts (possibly again and again) made for a nice back-and-forth understanding and a fruitful homogeneous collaboration. Together, we also have braved the triangle inequality if not with success, with great enthusiasm. I thank her for being an inexhaustible source of motivation and cheerfulness, and for helping me fix the last chapter at such short notice. It was also very pleasant and enriching to work with Aristotelis Panagiotopoulos on those questions of homogeneity. I thank him for his easygoingness and his constant mathematical optimism. Enfin, j'apprécie beaucoup de travailler (mais pas que) avec François Le Maître, surtout sur un vélo, et je le remercie pour notre premier

[1] *Nobel lecture*

bébé. Je suis très heureuse d'annoncer qu'un autre est en route (qui sera moins générique mais plus moyennable) !

J'ai bénéficié à l'ICJ d'un accueil très chaleureux et je voudrais remercier tout aussi chaleureusement toutes les personnes avec qui j'ai eu la chance d'interagir et qui ont rendu ces années de thèse bien agréables. Je remercie l'équipe de logique (sans toutefois oublier les côtés A et G de la force) pour l'ambiance si conviviale et stimulante qui règne en séminaire et en groupe de travail. Un grand merci en particulier à Amador pour son oreille attentive et son énergie débordante, à Thomas pour avoir patiemment joué l'interprète entre la langue de Morley et celle de Cantor-Bendixson, à Pierre pour ses gentils conseils (sur les maths comme sur le reste), à Nadja pour sa divertissante véhémence contre l'administration française et pour les pique-niques et à Frank pour les escargots. Mille mercis à mon petit frère Tomás pour son infinie gentillesse et pour m'avoir plus d'une fois illuminé la théorie des modèles. Ça a été un très grand plaisir de discuter et de réfléchir avec lui et je regrette de ne pas l'avoir fait plus souvent. Lionel, dont la gentillesse n'a d'égale que la générosité, m'a aussi été d'un grand soutien pendant ma thèse. Il a pris le temps de relire mon travail, de me transmettre son intuition de la théorie de Ramsey et de me donner de précieux conseils. Je lui suis très reconnaissante de ces conversations éclairantes et rassurantes.

M'éloignant un peu de l'ICJ, je remercie les divers Gilles de m'avoir réservé un accueil excellent dans leurs équipes d'analyse fonctionnelle respectives, ainsi que les organisateurs et les participants du semestre d'homogénéité à Bonn, d'en avoir fait un séjour à la fois productif et agréable. More generally, I would like to thank all the other people I shared a bit of maths, a coffee or a dive with at the bend of a conference. Especially Sławek, for our illuminating discussions, for his refreshing sense of humor and for our first Ferris wheel ride ; Ola, for teaching me projective Fraïssé theory and for her delighting French ; Dana and Anush, for their kind and constructive advice ; Phillip, for his impressive capacity for mathematical enthusiasm, for enhancing my (almost non-existant) culture about tdlc groups and for the Belgian beer ; Andy, for patiently explaining his beautiful results ; Eli, for very enriching discussions and/on a bike ride to Koblenz ; and Mike, for the math puzzles !

Quand petite thésarde, je suis arrivée au labo, j'ai été très bien accueillie par mes cobureaux en rédaction : Erwan, Julien, Marianne, Mohammed et Romain, qui m'ont prise sous leurs ailes. Grâce à eux, et à ceux qui ont partagé mon bureau plus longtemps : Cécilia, Colin, Évrad, François, Ivan, Raouf et un autre Romain, le quotidien de la thésarde a été grandement adouci, et les joies, les frustrations et les exercices de TD ont été partagés dans la bonne humeur ; merci beaucoup !

J'ai aussi été très bien accueillie, et nourrie[2], dans le bureau d'à côté, où nous avons discuté de varivariétés et de fibrés en droites, décomposé des fractions rationnelles en éléments simples et pleuré nos malheurs. Je n'ose imaginer ce qu'aurait été cette thèse sans ce joyeux bureau étendu, sans le groupe de 11h37, de 14h du samedi matin et 21h30 : Benoît, Bloup, Cécile (bis), Cécilia, Colin, Coline, Corentin, Élodie, François (ter, dont un Mini-Bloup), Fred, Hugo (bis), Jean-Cyrille, Mathias, Niccoló et Thomas. Outre des maths, nous avons partagé un joyeux séminaire de probas alternatives, beaucoup de thé et d'hostilités, de chouettes randonnées, des marteaux, des ressorts comiques qui grincent (oui !), des $n \times 100$ mètres crawl, de beaux palindromes, de la physique quantique de bureau, de jolies perles[3], de la grande et moins grande

[2]Merci à Blanche, Thomas, François et Coline, pour les biscuits et l'huile de palme.

[3]On peut même les consulter en se préparant une tasse de théorèmes, donc CQFD ! Je me suis d'ailleurs beaucoup épanouie dans l'enseignement ces trois dernières années (et j'espère continuer !) et j'en profite donc pour remercier les collègues avec qui j'ai enseigné (en particulier, merci Blanche !), les étudiants qui m'ont

musique et des violettes ; c'est avec des bombes-étoiles plein les yeux que les remercie !

Un peu plus à la frontière de la thèse, je voudrais remercier mes amis et ma famille, qui m'ont accompagnée durant la thèse et ont contribué grandement à ce que je m'y sente bien. Merci aux amis de Lyon que j'ai pu retrouver à l'occasion du séminaire doctoroïde, de randonnées chantantes et calées, de soirées ciné et d'après-midis jeux. Et, comme dans la chanson, merci maman, merci papa ! Merci à ma famille pour son grand soutien (et pas seulement pendant la thèse), ses versions d'anglais et son sens de l'à-propos sans pareil.

Enfin, il resterait bien peu de cette thèse si François n'avait pas été là (ou à Bruxelles) pour me soutenir jour après jour. Depuis presque sept ans, il illumine ma vie, et les mots me manquent pour l'en remercier. J'emprunterai donc au JSL[4] ces quelques mots : merci champion !

regardé calculer des développements limités (merci Jacques, Anouk, Tony, Julie) et les auteurs desdites perles pour les grands moments de divertissement (merci Timmy®).

[4]Journal ~~of Symbolic Logic~~ de Saône-et-Loire.

Introduction en français

Quel est le point commun entre un arbre régulier, l'ordre sur les rationnels, un espace de Hilbert et un corps algébriquement clos ? Ils admettent beaucoup de symétries : ils sont *ultrahomogènes*. Une structure est ultrahomogène si tout isomorphisme entre sous-structures finiment engendrées s'étend en un automorphisme de la structure toute entière. En d'autres termes, l'ultrahomogénéité garantit que les configurations finies se retrouvent partout dans la structure, donnant ainsi un groupe de symétries très riche.

À l'instar de leurs sous-structures finies, les structures ultrahomogènes sont omniprésentes. Le premier exemple d'une telle structure est un ensemble, sans structure supplémentaire, dans lequel les bijections finies s'étendent toujours. Plus intéressant, les rationnels, en tant qu'ensemble ordonné, sont ultrahomogènes. Cela peut se montrer par un argument de *va-et-vient*, qui s'avère être la technique la plus efficace pour prouver l'ultrahomogénéité de structures dénombrables. En effet, la méthode du va-et-vient consiste à écrire la structure dénombrable comme une union dénombrable d'ensembles finis puis de construire par récurrence un isomorphisme comme limite d'applications entre ces ensembles finis. Par la même méthode[5], les rationnels peuvent être caractérisés comme l'unique ordre total dénombrable sans extrémités, propriétés que l'on peut vérifier sur les sous-structures finies. S'inspirant de l'exemple des rationnels et de l'efficacité de la méthode du va-et-vient, Fraïssé a introduit dans [**F**] une approche unifiée des structures ultrahomogènes dénombrables, dans laquelle les structures finies jouent un rôle central.

Un autre exemple de structure ultrahomogène qui illustre bien la théorie de Fraïssé est le graphe aléatoire, bien qu'il soit apparu plus tard. Son nom lui vient de ce qu'Erdös et Rényi ([**ER**]) l'ont construit comme suit : partant du graphe complet sur les entiers, pour chaque arête, on décide de la garder ou de l'enlever en tirant à pile ou face. Il se trouve que ce procédé donne toujours le même graphe, à isomorphisme près : le graphe obtenu a presque surement la propriété que pour chaque paire de sous-graphes finis disjoints, on peut trouver un sommet relié à tous les sommets du premier sous-graphe mais à aucun du second. Une application du va-et-vient montre alors que deux graphes satisfaisant cette propriété sont isomorphes. Mais le graphe aléatoire doit aussi son nom à ce qu'il contient une copie isomorphe de tout graphe fini et même de tout graphe dénombrable. La construction du graphe aléatoire comme objet universel est due à Rado[6] ([**R1**]). L'idée sous-jacente à une telle construction est que les graphes finis se recollent bien ensemble pour donner le graphe aléatoire ; si bien que non seulement tout graphe fini se retrouve dans le graphe aléatoire, mais qu'en plus, on l'y retrouve partout !

La théorie de Fraïssé est justement une manière de construire des structures ultrahomogènes en recollant des structures finies les unes aux autres. Plus précisément, la classe de toutes les sous-structures finies d'une structure ultrahomogène, qu'on appelle son *âge*, jouit de bonnes propriétés d'amalgamation (voir figure 0.1).

Une *classe de Fraïssé* est une classe dénombrable (à isomorphisme près) qui satisfait le même type de propriétés. Le théorème de Fraïssé affirme que toute classe de Fraïssé est en fait l'âge d'une certaine structure ultrahomogène dénombrable. De plus, une telle structure est unique ; on l'appelle la *limite de Fraïssé* de la classe. Ainsi, les rationnels sont la limite de Fraïssé de la classe de tous les ensembles finis ordonnés et le graphe aléatoire celle de la classe de tous les

[5] Dans [**P6**], Plotkin soutient que Cantor n'a utilisé dans [**C**] qu'un argument de *va* et attribue l'introduction et la popularisation du *vient* à Huntington ([**H4**]) et Hausdorff ([**H1**]).

[6] Le graphe aléatoire est parfois appelé graphe de Rado.

$$
\begin{tikzcd}
 & \mathbf{D} & \\
\mathbf{B} \arrow[ur, dotted] & & \mathbf{C} \arrow[ul, dotted] \\
 & \mathbf{A} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

FIGURE 0.1. La propriété d'amalgamation : si une structure **A** se plonge dans deux structures différentes **B** and **C** de la classe, alors ces deux structures se plongent elles-mêmes dans une quatrième structure **D**, toujours dans la classe, de telle sorte que le diagramme commute.

graphes finis. Ce résultat puissant assure que les structures ultrahomogènes sont bien caractérisées par leurs sous-structures finies, comme le laissaient présager les exemples des rationnels et du graphe aléatoire. Par conséquent, la théorie de Fraïssé permet un traitement combinatoire des structures ultrahomogènes et en particulier des propriétés dynamiques de leurs groupes d'automorphismes, comme nous le verrons en détail plus tard.

Le groupe d'automorphismes d'une structure ultrahomogène dénombrable, muni de la topologie de convergence simple, est un sous-groupe fermé de S_∞, le groupe de permutations d'un ensemble dénombrable infini. C'est donc un *groupe polonais* : un groupe topologique séparable qui admet une distance compatible complète. La classe des groupes polonais est très riche : elle admet notamment des objets universels, comme le groupe d'homéomorphismes du cube de Hilbert ([**U2**]) et le groupe d'isométries de l'espace d'Urysohn ([**U4**]). De plus, elle est assez étendue, puisque tout groupe localement compact métrisable est polonais. Mais la classe des groupes polonais va bien au-delà : le groupe des unitaires de l'espace de Hilbert séparable ou le groupe des homéomorphismes croissants de l'intervalle sont des groupes polonais aussi, qui diffèrent grandement des groupes localement compacts. Par exemple, toutes leurs actions continues sur un espace compact admettent un point fixe global (on dit qu'ils sont *extrêmement moyennables*), tandis qu'un résultat de Veech ([**V2**]) exclut ce phénomène pour les groupes localement compacts non triviaux.

On ne dispose pas d'un outil aussi puissant que la mesure de Haar dans le contexte des groupes polonais plus gros. Cependant, la séparabilité nous donne une prise sur les espaces et autorise les arguments inductifs, tandis que la complétude permet d'utiliser des méthodes de Baire, ce qui rend la théorie des groupes polonais relativement robuste (voir [**K4**] et [**G1**]). En particulier, le théorème de Baire détermine une notion de complexité, et de généricité, qui concurrence celle d'ensemble de mesure pleine (voir [**O**] pour les mérites comparés de la mesure et de la catégorie). Cela fait des groupes polonais un terrain idéal pour la théorie descriptive des ensembles, et ils sont étudiés comme tels de manière systématique depuis les travaux des mathématiciens polonais (comme leur nom l'indique) du début du vingtième siècle.

Un résultat de Pettis ([**P5**]) dit que tout morphisme de groupes entre groupes polonais, pourvu qu'il soit Baire-mesurable (une hypothèse bénigne satisfaite par toutes les applications boréliennes), est automatiquement continu. Par ailleurs, Effros a caractérisé dans [**E1**] le fait pour une orbite sous l'action continue d'un groupe polonais d'être polonaise : cela équivaut à ce que l'application orbitale soit ouverte. Ces deux résultats forts ont mené à l'étude très prospère des actions définissables (principalement continues et boréliennes) de groupes polonais (voir [**BK2**]) et aux relations d'équivalence orbitale qu'elles induisent (voir [**G1**]). C'est une théorie très riche ; mentionnons, par exemple, l'existence d'actions universelles pour un groupe polonais ([**BK1**], [**H4**]). La structure de ces relations d'équivalence a en particulier été décrite par de beaux théorèmes de dichotomies ([**HKL**], [**KST**], [**M9**]) et par des résultats de (non-)classification ([**H3**], [**K5**]).

La théorie descriptive des ensembles et la théorie des modèles entretiennent des liens étroits. Le premier exemple notable d'un tel lien réside peut-être dans la reformulation de la conjecture de Vaught ([**V1**]) en termes d'une action spécifique d'un groupe polonais. La conjecture de Vaught stipule qu'une théorie dans un langage dénombrable admet soit une quantité dénombrable soit un continuum de modèles dénombrables. Elle a été généralisée en une conjecture de Vaught topologique, dont la formulation rappelle la dichotomie de Glimm-Effros : elle affirme que toute action continue d'un groupe polonais admet soit un nombre dénombrable soit un continuum d'orbites. Pour cette version topologique, des résultats partiels puissants ont été obtenus, voir [**B2**]. L'autre interaction notable entre les deux domaines, qui se trouve au cœur de cette thèse, est que l'on peut aborder les groupes polonais comme des groupes d'automorphismes. Plus précisément, on a mentionné que les groupes d'automorphismes de structures dénombrables (ultrahomogènes) sont des sous-groupes fermés de S_∞. Il s'avère que la réciproque est vraie aussi : tout sous-groupe de S_∞ est isomorphe au groupe d'automorphismes d'une structure dénombrable. Mieux, en nommant les orbites sous l'action du groupe, on peut rendre la structure ultrahomogène.

Cette correspondance éclairante ne se limite pas aux sous-groupes de S_∞ et s'étend en fait à tous les groupes polonais, via la *logique continue*. Cette dernière a été développée par Ben Yaacov et Usvyatsov dans [**BU2**], ainsi que Berenstein et Henson ([**BBHU**]), dans le but d'étudier les structures métriques. L'idée est de remplacer les points par les fonctions qui donnent la distance à ces points, ainsi que les valeurs de vérités habituelles vrai et faux par un continuum de valeurs de vérités, généralement l'intervalle $[0, 1]$. Une relation devient alors une fonction uniformément continue qui prend ses valeurs dans $[0, 1]$ au lieu de $\{0, 1\}$ et une *structure métrique* est un espace métrique complet, muni d'une famille de telles relations. Remarquons que les structures classiques rentrent dans ce cadre : munies de la distance discrète, elles deviennent des structures métriques. Dans ce contexte métrique, les automorphismes sont en particulier des isométries. De plus, l'homologue continu naturel de la dénombrabilité est la séparabilité, de telle sorte que les groupes d'automorphismes de structures métriques séparables sont polonais. L'observation cruciale que ceux-ci englobent tous les groupes polonais est due à Melleray ([**M5**]).

En outre, comme dans le cas discret, on peut imposer que la structure soit hautement symétrique. Pour ce faire, on relaxe l'hypothèse d'ultrahomogénéité. Une structure métrique est *approximativement ultrahomogène* si tout isomorphisme entre sous-structures finies s'étend, à une erreur arbitrairement petite près, en un automorphisme de la structure toute entière. Toujours en nommant les adhérences des orbites dans le langage, tout groupe polonais peut alors se voir comme le groupe d'automorphismes d'une structure métrique approximativement ultrahomogène.

De nombreuses structures métriques qui apparaissent naturellement sont approximativement ultrahomogènes : l'algèbre de mesure d'un intervalle, (la boule unité d')un espace de Hilbert séparable ou les espaces L^p. Tous ces exemples sont en fait *exactement* ultrahomogènes. Cependant, il n'est pas vrai en général que l'on peut se passer du mot "approximativement" : les treillis de Banach ou l'espace de Gurarij, qui émergent tout aussi naturellement, sont seulement approximativement ultrahomogènes (voir [**BBHU**] et [**M2**]). Melleray a demandé si tout groupe polonais était néanmoins le groupe d'automorphismes d'une certaine structure exactement ultrahomogène. Ce n'est pas le cas : Ben Yaacov a récemment fourni des exemples de groupes polonais qui ne peuvent pas agir transitivement continument par isométries sur un espace métrique complet ([**B7**]). La question de l'exacte ultrahomogénéité des structures métriques apparaitra tout de même comme une intrigue secondaire tout au long de la thèse.

Un exemple important et très illustratif de structure métrique ultrahomogène est l'*espace d'Urysohn*. L'espace d'Urysohn est, à isométrie près, l'unique espace métrique qui est à la fois ultrahomogène et universel pour la classe de tous les espaces métriques finis. En rapprochement avec le graphe aléatoire, cela suggère l'idée d'une version métrique de la théorie de Fraïssé. Une telle théorie, très analogue à la théorie classique (avec une propriété d'amalgamation plus faible), a en effet été développée par Schoretsanitis ([**S2**]) et par Ben Yaacov ([**B5**]). Comme attendu, l'espace d'Urysohn est la limite de Fraïssé de la classe des espaces métriques finis, l'espace de Hilbert celle de la classe des espaces de Hilbert de dimension finie, l'algèbre de mesure de l'intervalle celle de la classe des algèbres de mesure finies. Cet analogue de la théorie de Fraïssé fournit un bon cadre dans lequel appliquer des arguments combinatoires aussi aux groupes polonais généraux.

L'interaction entre la théorie descriptive des groupes polonais et la théorie des modèles des structures de Fraïssé s'est avérée très fructueuse. Dans cette thèse, nous verrons un échantillon de divers aspects de cette correspondance florissante.

La question se pose naturellement de savoir à quel point la correspondance est bonne : quelles propriétés de la structure son groupe d'automorphismes retient-il ? En d'autres termes, la structure peut-elle être reconstruite à partir du seul groupe d'automorphismes ? Ahlbrandt et Ziegler([**AZ**]) ont donné une réponse positive pour une classe spéciale de structures dénombrables homogènes, les structures *dénombrablement catégoriques*[7]. Une structure est dénombrablement catégorique si c'est l'unique modèle dénombrable de sa théorie : toute structure dénombrable satisfaisant les mêmes propriétés (du premier ordre) lui est isomorphe. Il y a beaucoup de structures dénombrablement catégoriques, en particulier parmi les limites de Fraïssé. Par exemple, les rationnels et le graphe aléatoire sont dénombrablement catégoriques, puisque non seulement ils sont caractérisés par leurs sous-structures finies mais la caractérisation s'exprime par des énoncés du premier ordre et appartient donc à la théorie.

D'autre part, la catégoricité dénombrable fournit suffisamment de rigidité pour permettre une correspondance plus riche entre la structure et son groupe d'automorphismes. En effet, est à notre disposition le très puissant théorème de Ryll-Nardzewski, qui décrit les types dans une structure dénombrablement catégorique. Cela a de nombreuses conséquences, pour le groupe d'automorphismes tout particulièrement. Le théorème affirme que le groupe doit agir de manière *oligomorphe*, c'est-à-dire avec un nombre fini d'orbites et que l'espace des types doit être fini. Cela produit en particulier une caractérisation de la définissabilité qui s'exprime purement en termes du groupe d'automorphismes : dans une structure dénombrablement catégorique, la définissabilité se résume à l'invariance sous l'action du groupe d'automorphismes. Cette caractérisation constitue un outil essentiel dans la reconstruction.

La reconstruction qu'Ahlbrandt et Ziegler proposent est la suivante. Ils montrent que deux structures dénombrablement catégoriques dont les groupes d'automorphismes sont isomorphes (en tant que groupes topologiques) sont *bi-interprétables*. Plus précisément, si deux structures sont bi-interprétables, chacune se plonge dans les imaginaires de l'autre, c'est-à-dire dans un quotient définissable d'une puissance finie de l'autre. Ainsi, ils retrouvent, sinon la structure originelle, du moins toutes ses propriétés modèle-théoriques.

Avec Itaï Ben Yaacov, nous étendons ce résultat de reconstruction au cadre continu, dans lequel la catégoricité dénombrable est remplacée par la *catégoricité séparable*. La classe des structures séparablement catégoriques est assez vaste aussi puisqu'elle comprend l'algèbre de mesure, (la boule unité d')un espace de Hilbert séparable et la sphère d'Urysohn. De plus, elle jouit de propriétés très semblables au cas dénombrable. Ici encore, la définissabilité se caractérise par l'invariance sous l'action du groupe d'automorphismes. En effet, le théorème de Ryll-Nardzewski se généralise naturellement aux structures métriques (voir [**BU1**] et [**BBHU**]) : l'espace des types dans une structure séparablement catégorique doit être compact (pour une

[7]La parenté de ce résultat est en fait assez floue. Ahlbrandt et Ziegler l'attribuent à Coquand (dans une note non publiée) et il semble que l'idée que ce résultat devait être vrai était déjà dans l'air depuis un certain temps.

distance naturelle) et le groupe d'automorphismes doit agir de manière *approximativement oligomorphe*, c'est-à-dire avec un nombre compact d'orbites. Ainsi, la structure topologique de l'espace des types admet une description sous forme d'arbre, que nous utilisons amplement dans la preuve.

En théorie des modèles métriques, la définition des imaginaires doit être légèrement modifiée : on a besoin d'autoriser les quotients définissables infinis pour prendre en compte les suites convergentes d'epsilons (voir la discussion dans [**BU2**]). Dans [**BK3**], nous définissons la notion d'interprétation entre structures métriques en conséquence. Alors nous montrons comme escompté que deux structures métriques séparablement catégoriques dont les groupes d'automorphismes sont topologiquement isomorphes sont bi-interprétables. Notre démonstration diffère toutefois de la preuve classique puisqu'elle est de nature intrinsèquement métrique. En effet, nous exploitons la preuve de Melleray du fait que tout groupe polonais est le groupe d'automorphismes d'une structure métrique ([**M5**]) : en se servant d'une construction standard, il produit une structure métrique canonique[8] associée à un groupe polonais, sa *structure chapeau*. Nous montrons en fait que toute structure séparablement catégorique est bi-interprétable avec la structure chapeau de son groupe d'automorphismes.

Ce résultat de reconstruction assure que toute propriété modèle-théorique des structures séparablement catégoriques est encodée dans leur groupe d'automorphismes, que *c'est* véritablement une propriété topologique. La construction du dictionnaire à proprement parler a été initiée par Ben Yaacov et Tsankov dans [**BT1**], avec une entrée sur la stabilité. La bonne traduction fait intervenir la compactification de Roelcke du groupe d'automorphismes. En effet, les groupes d'automorphismes de structures séparablement catégoriques sont *Roelcke-précompacts* (cela découle de l'oligomorphie approximative de leur action, voir [**R3**]). Réciproquement, Ben Yaacov et Tsankov, et, indépendamment, Rosendal ([**R4**]) ont montré que tout groupe polonais Roelcke-précompact se réalise comme tel. Forts de cette observation, Ben Yaacov et Tsankov montrent que la stabilité correspond à la faible presque-périodicité de la compactification de Roelcke du groupe d'automorphismes, puis étudient ces compactifications d'un point de vue modèle-théorique. Ibarlucía ([**I**]) a ensuite étendu le dictionnaire aux structures métriques dépendantes. Il est amusant de noter que le mot même d'"indépendant" était couramment utilisé du côté topologique (voir [**R5**] ou [**GM1**]), pour désigner la même notion, bien avant que la connexion avec la théorie des modèles n'ait été remarquée !

Ce type de reconstruction fait intervenir le groupe d'automorphismes, avec sa topologie. Il est donc légitime de se demander si le groupe abstrait suffit à retrouver la structure. Malheureusement, en général, il ne suffit pas : Evans et Hewitt ([**EH**]) ont construit un contre-exemple dans le cas dénombrablement catégorique, répondant ainsi à une question de Rubin ([**R6**]). Néanmoins, les théoriciens des modèles se sont tout de même intéressés (et s'intéressent) à la question de reconstruire la topologie d'un groupe à partir de sa structure algébrique. Dans ce but, la *propriété de petit indice* a été largement étudiée. Cette propriété dit que tous les sous-groupes d'indice dénombrable sont ouverts. La propriété de petit indice équivaut à ce que tout morphisme du groupe à valeurs dans S_∞ est continu. Ainsi, pour le groupe d'automorphismes d'une structure dénombrable, la propriété de petit indice garantit que la structure algébrique du groupe encode déjà sa topologie et que la reconstruction plus forte s'opère. C'est le cas pour un grand nombre de groupes d'automorphismes, comme S_∞ lui-même (Semmes, [**S3**]) ou le groupe d'automorphismes des rationnels (Truss, [**T2**]). Voir [**HHLS**] et [**L1**] pour plus de détails sur la propriété de petit indice et le problème de reconstruction.

Dans [**HHLS**], Hodges, Hodkinson, Lascar et Shelah ont introduit un outil puissant, les *amples génériques*, pour montrer la propriété de petit indice pour les groupes d'automorphismes de structures dénombrables. La notion a été affinée et étudiée plus avant par Kechris et Rosendal ([**KR**]). Un groupe polonais G a des amples génériques si pour tout n dans $\mathbb{N}$, l'action diagonale

[8] Techniquement, la structure chapeau est associée à une distance invariante à gauche sur le groupe polonais. Néanmoins, à bi-interprétabilité près, elle ne dépend pas du choix d'une telle distance invariante à gauche.

par conjugaison de G sur G^n admet une orbite comaigre. Cela implique en particulier que G a une classe de conjugaison comaigre, phénomène dont Wesolek a montré qu'il ne peut pas se produire dans les groupes polonais localement compacts ([**W**]). Les amples génériques sont particulièrement intéressants : outre la propriété de petit indice, ils impliquent une propriété très forte, la *propriété de continuité automatique*. À savoir, si G a des amples génériques, alors tout morphisme de groupe de G dans un groupe topologique séparable quelconque — pas seulement un sous-groupe fermé de S_∞ — est automatiquement continu. Cette jolie propriété permet de retrouver complètement la topologie du groupe à partir de sa structure algébrique, ce qui constitue un résultat de reconstruction très puissant.

Pour les sous-groupes fermés de S_∞, vus comme groupes d'automorphismes de structures de Fraïssé, Kechris et Rosendal ont exhibé des conditions combinatoires qui impliquent les amples génériques. En particulier, si une classe de Fraïssé satisfait la *propriété d'amalgamation libre* et la *propriété d'extension*, alors le groupe d'automorphisme de sa limite de Fraïssé a des amples génériques (voir [**M6**]). La propriété d'extension dit que tout isomorphisme partiel d'une structure finie s'étend en un automorphisme global d'une structure plus grande, mais toujours finie. Elle a été d'abord prouvée pour la classe des graphes finis par Hrushovski[9] ([**H2**]). Ce résultat a été généralisé par Herwig et Lascar ([**HL**], voir aussi [**S7**]) aux classes de structures finies qui omettent une certaine famille de configurations, tels que les graphes sans triangle. Leurs techniques ont été utilisées par Solecki ([**S6**]) pour montrer que la propriété d'extension est vraie pour les espaces métriques finis. Quant à la propriété d'amalgamation libre, elle assure que les unions d'isomorphismes partiels s'étendent de manière cohérente, en amalgamant leurs domaines le plus indépendamment possible. En appliquant ce critère combinatoire, on obtient que le groupe d'automorphisme du graphe aléatoire, celui du graphe sans triangle de Henson et S_∞ ont les amples génériques. Le groupe d'automorphismes des rationnels, au contraire, a une classe de conjugaison comaigre mais n'a pas d'amples génériques. Un certain nombre d'autres sous-groupes polonais de S_∞, comme le groupe des homéomorphismes de l'espace de Cantor (Kwiatkowska, [**K7**]), ont aussi des amples génériques et par conséquent, la propriété de continuité automatique.

Malheureusement, les groupes polonais plus gros sont souvent loin d'avoir des amples génériques : dans le groupe d'isométries de l'espace d'Urysohn ou dans le groupe d'automorphismes de l'algèbre de mesure de l'intervalle, dans le groupe unitaire d'un espace de Hilbert séparable, toutes les classes de conjugaison sont maigres ([**K6**]). Néanmoins, même parmi les groupes qui n'ont pas d'amples génériques, plusieurs groupes polonais satisfont la propriété de continuité automatique. Rosendal et Solecki ([**RS**]) ont contourné l'absence d'amples génériques et montré la propriété de continuité automatique pour le groupe d'automorphismes des rationnels, le groupes des homéomorphismes de la droite réelle et le groupe des homéomorphismes du cercle.

L'observation que certains gros groupes polonais n'ont pas d'amples génériques a amené Ben Yaacov, Berenstein et Melleray ([**BBM**]) à concevoir une version plus faible des amples génériques : les *amples génériques topométriques*, que plus de groupes polonais satisfont. Le fait que tout groupe polonais soit le groupe d'automorphismes d'une structure métrique est crucial dans leur travail. Un tel groupe d'automorphismes, en plus de son habituelle topologie de la convergence simple, est naturellement muni de la distance uniforme. Cette distance est bi-invariante et, au moins dans le cas où la structure est séparablement catégorique, définit une uniformité bi-invariante canonique qui raffine la topologie polonaise usuelle. Les amples génériques topométriques sont définis en entrelaçant la distance uniforme et la topologie. Suivant la preuve de Kechris et Rosendal, Ben Yaacov, Berenstein et Melleray montrent que si G a des amples génériques topométriques, il satisfait une version topométrique de la propriété de continuité automatique : tout morphisme de groupe de G dans un groupe séparable qui est continu pour la distance uniforme doit également être continu pour la topologie usuelle. Pour vérifier que des structures spécifiques ont des amples génériques topométriques, Ben Yaacov,

[9] Par conséquent, la propriété d'extension est souvent appelée propriété de Hrushovski.

Berenstein et Melleray font appel à des structures dénombrables denses[10] qui, elles, ont des amples génériques. En combinant ceci avec des résultats de Kittrell et Tsankov ([**KT**]), ils obtiennent la propriété de continuité automatique pour le groupe d'automorphismes de l'algèbre de mesure de l'intervalle. Avec des idées similaires, Tsankov ([**T3**]) a prouvé, que le groupe unitaire d'un espace de Hilbert satisfait également la propriété de continuité automatique.

Dans la même veine, Sabok ([**S1**]) a introduit une méthode pour prouver la propriété de continuité automatique, dont une étape consiste à extraire une structure dénombrable avec la propriété d'amalgamation libre et la propriété d'extension. Sa preuve se base sur celle de Rosendal et Solecki de la propriété de continuité automatique pour le groupe Homéo$(2^{\mathbb{N}})$ ([**RS**], qui a été écrit avant que Kwiatkowska ne montre qu'il a en fait des amples génériques). Plus précisément, Sabok isole un ensemble de propriétés combinatoires sur une structure métrique qui miment les amples génériques et impliquent la propriété de continuité automatique pour son groupe d'automorphismes. Les conditions comprennent la propriété d'extension, une version métrique de la propriété d'amalgamation libre, une condition ad hoc d'homogénéité locale appelée la *propriété d'isolement*, ainsi que la condition notable que la structure soit exactement ultrahomogène. Elles ont ensuite permis a Sabok d'obtenir que le groupe d'isométries de l'espace d'Urysohn a la propriété de continuité automatique.

Nous nous intéressons à une propriété de stabilité de la classe des groupes qui ont la propriété de continuité automatique : quand est-il vrai que si G satisfait la propriété de continuité automatique, alors $G^{\mathbb{N}}$ aussi ? Bien qu'il ne soit pas vrai en général que la puissance infinie d'un groupe avec la propriété de continuité automatique a encore la propriété de continuité automatique, les amples génériques, eux, passent aux puissances dénombrables ! Dans [**RS**], Rosendal et Solecki ont montré la propriété de continuité automatique non seulement pour Homéo$(2^{\mathbb{N}})$ mais aussi pour Homéo$(2^{\mathbb{N}})^{\mathbb{N}}$ (avant que l'on ne sache que Homéo$(2^{\mathbb{N}})$ a des amples génériques). Naturellement, cela soulève la question pour les conditions de Sabok : passent-elles aux puissances infinies ? La réponse n'est pas claire ; cependant, il se trouve qu'elles n'en sont pas loin. En effet, Malicki ([**M1**]) a proposé une version légèrement modifiée des conditions de Sabok, conçue pour mimer encore mieux les amples génériques, afin d'en retrouver plus de conséquences. D'une manière semblable à [**RS**], à partir d'une structure métrique $\mathbf{M}$, nous construisons la *structure juxtaposée* de $\mathbf{M}$, dont le groupe d'automorphismes est la puissance infinie de celui de $\mathbf{M}$. Nous prouvons alors que les conditions de Malicki passent aux structures juxtaposées. Ainsi, nous obtenons que les groupes $\mathrm{Iso}(\mathbb{U})^{\mathbb{N}}$, $\mathcal{U}(\ell^2)^{\mathbb{N}}$ et $\mathrm{Aut}(\mu)^{\mathbb{N}}$ satisfont la propriété de continuité automatique.

Comme on l'a mentionné plus haut, les gros groupes polonais ont tendance à ne pas avoir d'amples génériques. Kechris et Rosendal ont même demandé s'il existait un groupe polonais avec des amples génériques en dehors de S_∞. Un exemple étonnamment simple d'un tel groupe est ressorti d'une discussion avec François Le Maître ([**KLM**]) sur les généralisations possibles du résultat précédent sur la propriété de continuité automatique pour les puissances de groupes. En effet, nous avons remarqué que le groupe $G^{\mathbb{N}}$ n'est autre que le groupe $L^0(\mathbb{N}, G)$ des fonctions (mesurables) sur les entiers à valeurs dans G. Il est par conséquent naturel de considérer le groupe polonais $L^0([0,1], G)$ des variables aléatoires sur l'intervalle à valeurs dans G. La preuve du fait que si G a des amples génériques, alors $G^{\mathbb{N}}$ aussi, s'adapte à $L^0([0,1], G)$ en utilisant le théorème d'uniformisation de Jankov-van Neumann. De plus, le groupe $L^0([0,1], G)$ est toujours connexe (et même contractile), tandis que S_∞ est totalement discontinu, donc $L^0([0,1], G)$ ne peut pas être un sous-groupe de S_∞. En conséquence, comme G se plonge dans $L^0([0,1], G)$, nous prouvons même que tout groupe polonais qui a des amples génériques se plonge dans un groupe connexe qui a des amples génériques. Nous exhibons une autre classe d'exemples, les groupes pleins de relations d'équivalence ergodiques hyperfinies de type III, dont il se trouve

[10] En fait, de nombreuses limites de Fraïssé dénombrables admettent des analogues continus, et vice versa. Par exemple, l'espace d'Urysohn est l'homologue continu de l'espace d'Urysohn rationnel, qui est la limite de Fraïssé de tous les espaces métriques finis à distances rationnelles.

qu'ils se plongent également dans $L^0([0,1], S_\infty)$. Notons que simultanément, Malicki a produit encore une autre classe de groupes polonais avec des amples génériques qui ne sont pas des sous-groupes fermés de S_∞ ([**M1**]).

Quant à étendre les techniques de Malicki au groupe $L^0([0,1], G)$, malheureusement, nous ne sommes pas allés très loin. Si $\mathbf{M}$ est une structure métrique, alors $L^0([0,1], \mathrm{Aut}(\mathbf{M}))$ est le groupe d'automorphismes d'une randomisée de $\mathbf{M}$, qui reste ultrahomogène quand $\mathbf{M}$ l'est. Néanmoins, pour faire passer les autres conditions à la randomisée, il semble que l'on a besoin qu'elles soient, en un certain sens, uniformes, ce qui n'a pas l'air d'être le cas dans nos exemples. Par exemple, on ne sait toujours pas si les groupes $L^0([0,1], \mathrm{Iso}(\mathbb{U}))$ ou $L^0([0,1], \mathrm{Aut}(\mu))$ satisfont la propriété de continuité automatique.

Passons maintenant à un aspect différent de la correspondance entre groupes et structures : une approche combinatoire de la dynamique topologique. On a mentionné qu'un certain nombre de gros groupes polonais sont extrêmement moyennables, c'est-à-dire que toutes leurs actions continues sur un espace compact admettent un point fixe. L'étude de la *propriété de point fixe sur les compacts*[11] a commencé dans le contexte des semi-groupes (voir [**M10**] et [**G4**]). Les premiers exemples de groupes topologiques extrêmement moyennables ont été construits par Herer et Christensen ([**HC**]) et appelés exotiques[12]. Plus tard, des exemples plus naturels de groupes extrêmement moyennables sont apparus : le groupe unitaire d'un espace de Hilbert séparable (Gromov-Milman, [**GM2**]), le groupe des homéomorphismes croissants de l'intervalle (Pestov, [**P4**]), $L^0([0,1], S^1)$ (Glasner, [**G2**]). Les preuves utilisaient la concentration de la mesure (voir par exemple [**L2**]). Pestov a ensuite mis en lumière une relation entre l'extrême moyennabilité et la théorie de Ramsey structurale. En utilisant le théorème de Ramsey fini, il a démontré que le groupe d'isométries de l'espace d'Urysohn est extrêmement moyennable ([**P2**]).

Le cadre général qui sous-tend cette connexion a été mis en lumière par Kechris, Pestov et Todorčević dans [**KPT**] pour les sous-groupes fermés de S_∞. Ils ont prouvé que le groupe d'automorphismes d'une limite de Fraïssé est extrêmement moyennable si et seulement si la classe de Fraïssé satisfait la *propriété de Ramsey* (sous l'hypothèse que les objets soient rigides ; typiquement, c'est le cas des structures ordonnées). Une classe $\mathcal{K}$ a la propriété de Ramsey si pour toute palette de k couleurs, pour toute petite structure $\mathbf{A}$ et toute moyenne structure $\mathbf{B}$ dans $\mathcal{K}$, il existe une grosse structure $\mathbf{C}$ toujours dans $\mathcal{K}$ telle que pour tout coloriage de l'ensemble des copies de la petite structure $\mathbf{A}$ dans $\mathbf{C}$ en des couleurs de la palette, il existe une copie de la moyenne structure $\mathbf{B}$ dans $\mathbf{C}$ à l'intérieur de laquelle toutes les copies de $\mathbf{A}$ ont la même couleur (voir figure 0.2). En d'autres termes, la propriété de Ramsey assure l'existence de grosses sous-structures monochromatiques.

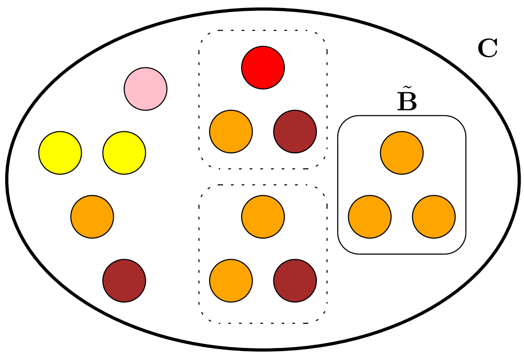


FIGURE 0.2. La propriété de Ramsey

[11]Le nom "extrêmement moyennable" est né d'un parallèle avec (l'une des multiples caractérisations de) la moyennabilité, dont on parlera plus tard. Il est probablement dû à Granirer.

[12]Comme ce phénomène ne peut pas avoir lieu dans le contexte plus communément étudié des groupes localement compacts, un tel résultat est effectivement très surprenant.

À partir de cette puissante caractérisation, des théorèmes de Ramsey déjà connus ont fourni de nouveaux exemples de groupes extrêmement moyennables. Par exemple, Nešetřil et Rödl ([**NR**]) ont montré que la classe de Fraïssé des graphes finis ordonnés a la propriété de Ramsey, le groupe d'automorphismes du graphe aléatoire ordonné est donc extrêmement moyennable. Remarquablement, l'équivalence a aussi été utilisée dans l'autre direction : Pestov a prouvé que le groupe d'isométries de l'espace d'Urysohn est extrêmement moyennable, ce qui a poussé Nešetřil à vérifier que la classe des espaces métriques finis à distances rationnelles a la propriété de Ramsey (voir [**NVT**]). De plus, la méthode de Kechris, Pestov et Todorčević a été appliquée pour calculer le flot universel minimal de plusieurs groupes polonais ; pour plus de détails sur les développements de la correspondance de Kechris-Pestov-Todorčević, voir la récente étude de Nguyen van Thé ([**T1**]).

Cette correspondance a été étendue au cadre continu par Melleray et Tsankov ([**MT1**]). Ils introduisent une version approchée de la propriété de Ramsey et montrent que l'extrême moyennabilité du groupe d'automorphismes d'une limite de Fraïssé métrique équivaut à ce que la classe satisfasse cette propriété. Cela capture un résultat similaire obtenu par Pestov ([**P2**]) sur les groupes d'isométries d'espaces métriques ultrahomogènes. Pour cette propriété de Ramsey approchée, un changement notable est le fait qu'ils imposent une condition de régularité sur les coloriages, afin de contrôler les divers epsilons qu'engendre l'ultrahomogénéité approchée. Par ailleurs, ils fournissent un critère pour vérifier la propriété de Ramsey approchée, dont l'intuition provient de l'observation de Kechris et Rosendal que si une classe de Fraïssé a la propriété d'extension, alors le groupe d'automorphismes de sa limite contient une union dense de sous-groupes compacts, ce qui implique qu'il est moyennable ([**KR**]). Plus précisément, le critère fait intervenir un affaiblissement de la propriété d'extension, ainsi qu'une condition qui permet l'utilisation de la concentration de la mesure. Bien que cela leur permette de retrouver l'extrême moyennabilité de plusieurs groupes polonais, le critère n'a pas encore été appliqué pour trouver de nouveaux groupes extrêmement moyennables. Néanmoins, le caractère finitaire de la propriété de Ramsey approchée permet à Melleray et Tsankov ([**MT2**]) de calculer sa complexité : l'extrême moyennabilité est une notion G_δ, ce qui est intéressant du point de vue de la catégorie de Baire.

Par ailleurs, la correspondance de Kechris-Pestov-Todorčević a été transférée par Moore ([**M12**]) à la *moyennabilité* pour les sous-groupes fermés de S_∞. La moyennabilité a été introduite par von Neumann ([**N**]) à la fin des années vingt pour mieux comprendre le paradoxe de Banach-Tarski[13]. C'est une notion bien mieux connue que sa petite sœur, puisqu'elle a été étudiée en profondeur pour les groupes discrets. Dans ce cadre, la moyennabilité est définie comme l'existence d'une moyenne invariante sur le groupe, et admet de nombreuses définitions équivalentes. Nous n'évoquerons pas celles-ci mais nous concentrerons sur les groupes topologiques. Pour ces derniers, elle est définie comme suit : un groupe topologique est moyennable si chaque fois qu'il agit continument sur un espace compact, l'action admet une mesure de probabilité borélienne invariante. Cette définition coïncide bien sûr avec la notion précédente dans le cas discret. Les exemples de groupes polonais moyennables abondent, d'autant plus que la moyennabilité n'exclut pas les groupes localement compacts : les groupes compacts, les groupes abéliens, les groupes résolubles, S_∞, et plus généralement tous les groupes d'automorphismes de structures de Fraïssé avec la propriété d'extension sont moyennables, pour ne nommer qu'eux.

La moyennabilité admet aussi une description en termes de combinatoire finie, comme l'ont montré Tsankov dans une note non publiée et Moore ([**M12**]). L'idée sous-jacente à cette description est que l'extrême moyennabilité, qui fournit des points fixes aux actions, correspond au fait pour les coloriages d'avoir de gros ensembles monochromatiques (donc, en un sens, fixes). De même, la moyennabilité donne des mesures de probabilités fixes, que l'on peut envisager comme des barycentres de mesures de Dirac. Ainsi, le pendant de la propriété de Ramsey adapté à ce contexte devrait garantir l'existence de grosses "combinaisons convexes monochromatiques d'ensembles". C'est précisément le contenu combinatoire de la moyennabilité. En effet, Moore

[13] Connaissez-vous un bon anagramme de "Banach-Tarski" ?

a introduit la *propriété de Ramsey convexe* et a montré qu'une classe de Fraïssé satisfait cette propriété si et seulement si le groupe d'automorphismes de sa limite est moyennable.

En combinant ceci avec les idées de Melleray et Tsankov, nous étendons la caractérisation de Moore au cadre continu. Exactement comme dans [**MT1**], nous remplaçons la notion classique de coloriage par des coloriages lipschitziens pour introduire la *propriété de Ramsey convexe métrique*, et, refermant le diagramme, nous montrons l'exact analogue du résultat de Moore pour les groupes d'automorphismes de limites de Fraïssé métriques. Ces descriptions combinatoires de la moyennabilité n'ont pas encore mené à de nouveaux exemples de groupes moyennables. En fait, il n'y a pas de technique connue pour montrer la propriété de Ramsey convexe directement, encore moins pour la version métrique. Néanmoins, notre caractérisation a conduit à de belles propriétés structurales sur la moyennabilité. En particulier, nous montrons que la moyennabilité est elle aussi une condition G_δ. Cela signifie que la moyennabilité revient essentiellement à une condition $\forall\exists$, alors qu'elle était définie dans l'autre sens (de manière simplifiée, elle consiste à dire qu'"*il existe* une mesure qui donne la même mesure aux translatés de *tous* les ensembles"). Moore avait prouvé que pour les groupes discrets, on pouvait effectivement échanger les quantificateurs dans la définition de la moyennabilité. Nous montrons que c'est aussi le cas pour les groupes polonais dans la définition de la moyennabilité. Remarquons qu'il découle des travaux de Melleray et Tsankov qu'il en est de même pour l'extrême moyennabilité.

Enfin, nous étudions une dernière facette de notre correspondance directrice : à quel point le groupe d'automorphismes d'une limite de Fraïssé permet-il de décrire les isomorphismes à l'intérieur de la structure. Plus précisément, les limites de Fraïssé sont ultrahomogènes, ce qui veut dire que les classes d'isomorphisme de structures finies sont exactement les orbites sous l'action du groupe d'automorphismes. Qu'en est-il des classes d'isomorphisme de structures plus grandes ? Panagiotopoulos ([**P1**]) donne des conditions sur une structure de Fraïssé pour que les isomorphismes entre structures génériques s'étendent. Nous abordons la question sous un angle différent. Ayant fixé une structure de Fraïssé précise, on voudrait trouver toutes les sous-structures qui ont la *propriété d'homogénéité*, c'est-à-dire trouver les sous-structures telles que tout isomorphisme entre deux copies de la sous-structure s'étende en un automorphisme de la structure toute entière. Cette question a été posée à l'autrice par Julien Melleray pendant un programme semestriel à Bonn.

Les premiers résultats allant dans cette direction se trouvent dans les travaux de Huhunaišvili ([**H3**]) sur l'espace de Urysohn. En effet, Huhunaišvili a prouvé que non seulement l'espace d'Urysohn est ultrahomogène, mais que plus encore, toute isométrie entre sous-espaces compacts s'étend en une isométrie globale de l'espace d'Urysohn. Melleray ([**M3**], voir [**M4**]) a ensuite montré que les sous-espaces (relativement) compacts sont les seuls sous-ensembles satisfaisant cette propriété. Pour cela, il caractérise les espaces compacts comme étant les seuls espaces dont chaque extension métrique par un point est déterminée par un ensemble compact. Avec Isabel Müller et Aristotelis Panagiotopoulos, nous introduisons un analogue de cette propriété pour les classes de Fraïssé classiques : la *finitude typique*, qui affirme que tous les types au-dessus de notre sous-structure sont déterminés par un ensemble fini. Nous montrons alors qu'une sous-structure a la propriété d'homogénéité si et seulement si elle est typiquement finie.

Nous étudions plus avant la finitude typique dans des exemples concrets ; dans beaucoup d'exemples de structures de Fraïssé, nous montrons que les sous-structures typiquement finies doivent en fait être finies. C'est le cas de toutes les structures dénombrablement catégoriques ainsi que de l'espace d'Urysohn rationnel. On espère obtenir une description plus explicite des structures typiquement finies. De plus, notre résultat ouvre plusieurs perspectives de généralisation. Par exemple, nous essayons actuellement de voir ce qui se passe dans le cadre métrique, en dehors du cas particulier de l'espace d'Urysohn. Dans la mesure où les limites de Fraïssé ne sont pas toujours ultrahomogènes, il est probablement nécessaire de se restreindre à celles dont on sait déjà qu'elles sont ultrahomogènes. La caractérisation attendue devrait alors ressembler à la détermination compacte des types ; la signification précise de détermination compacte est cependant très loin d'être claire.

Par ailleurs, on s'intéresse à une version plus faible de la propriété d'homogénéité : on peut ne demander seulement que le groupe d'automorphismes de toute la structure de Fraïssé agisse transitivement sur les copies de la sous-structure. Dans l'espace d'Urysohn, Melleray a montré que c'est en fait équivalent à la propriété d'homogénéité. Nous montrons que c'est aussi le cas dans les structures de Fraïssé classiques qui admettent une *relation d'indépendance stationnaire* (au sens de [**TZ2**]), dont un cas particulier sont les structures avec la propriété d'amalgamation libre. Nous aimerions trouver un exemple de structure dans laquelle les deux propriétés diffèrent.

Nous avons opté pour une organisation arborée de la thèse. La partie tronc contient essentiellement des prérequis sur les mots du titre. Elle se dirige vers la correspondance entre groupes polonais et groupes d'automorphismes de structures approximativement ultrahomogènes. Plus spécifiquement, le chapitre 1 survole les propriétés de base dont on aura besoin sur les groupes polonais. En particulier, on passe en revue les différentes structures uniformes dont on peut munir les groupes polonais. Dans le chapitre 2, on présente l'espace d'Urysohn et sa construction moderne. Bien que l'espace d'Urysohn ne soit pas à strictement parler essentiel pour établir la correspondance, nous estimons qu'il mérite, en tant que bel objet qui capture très bien les idées de la logique continue, d'être mis en valeur. Il sera notre espace compagnon tout au long de cette thèse, comme il l'a été durant ces quatre dernières années. Le chapitre 3 pose ensuite en détail les bases de la théorie des modèles afin de fixer précisément nos conventions. De plus, nous profitons de l'occasion pour présenter les deux cadres en parallèle, de manière à souligner leurs similarités et leurs différences, et ce dans une présentation, on l'espère, auto-contenue. De même, nous exposons à la fois les théories de Fraïssé classiques et métriques dans le chapitre 4. Toutefois, les outils élégants utilisés du côté métrique n'étant pas bien connus, nous leur consacrons une partie conséquente du chapitre. Ainsi équipés, nous donnons finalement au chapitre 5 la preuve du fait que tout groupe polonais est le groupe d'automorphismes d'une limite de Fraïssé métrique.

Chaque branche porte sur un aspect de cette correspondance, les branches apparaissant dans le même ordre que dans l'introduction. La branche 1 commence par un chapitre de prérequis sur la catégoricité dénombrable et séparable, que l'on présente côte à côte également. Nous procédons ensuite à la version métrique de la reconstruction d'Ahlbrandt et Ziegler dans le chapitre 6, collaboration avec Itaï Ben Yaacov qui est paru dans [**BK3**]. La deuxième branche traite des questions de continuité automatique pour les puissances infinies de groupes polonais. Elle contient également une partie d'un travail fait en collaboration avec François Le Maître dans lequel nous exhibons des exemples de groupes polonais connexes qui ont des amples génériques. La branche 3, qui est parue dans [**K1**], décrit notre caractérisation combinatoire de la moyennabilité pour les groupes polonais généraux. Enfin, la branche 4 contient un travail en cours et en collaboration avec Isabel Müller et Aristotelis Panagiotopoulos sur la propriété d'homogénéité dans les structures de Fraïssé.

Introduction

What do a regular tree, the order on the rationals, a Hilbert space and an algebraically closed field have in common? They admit plenty of symmetries: they are *ultrahomogeneous*. A structure is ultrahomogeneous if every isomorphism between finitely generated substructures can be extended to an automorphism of the whole structure. In other words, ultrahomogeneity guarantees that finite configurations can be found everywhere in the structure, thus yielding a very rich group of symmetries.

Just like their finite substructures, ultrahomogeneous structures are ubiquitous. The first example of such a structure is a set, with no further structure, where finite bijections always extend. More interestingly, the rationals as an ordered set are ultrahomogeneous. This can be proven by a *back-and-forth* argument, which turns out to be the most fruitful technique in proving ultrahomogeneity for countable structures. Indeed, the back-and-forth method consists in exhausting the countable structure as a union of finite sets and then in constructing an isomorphism inductively as the limit of maps between those finite sets. With the same method[14], the rationals can be characterized as the only countable dense order without endpoints (Cantor, [**C**]), properties that one may check on finite substructures. Building on the example of the rationals and the efficiency of the back-and-forth technique, Fraïssé introduced in [**F**] a unified approach to countable ultrahomogeneous structures, where finite structures play a central role.

Another example of an ultrahomogeneous structure that provides insight into Fraïssé theory is the random graph, although it appeared later. It is so called because Erdös and Rényi ([**ER**]) built it as follows: starting from the complete graph on the integers, for each edge, decide whether to keep this edge or to take it out by flipping a coin. This process happens to give almost surely the same graph: the resulting graph almost surely has the property that for any two disjoint finite subgraphs, one can find a vertex that is related to every vertex in the first graph but no vertex in the second one. A stroke of back-and-forth then shows that any two graphs with that property are isomorphic. But the random graph deserves its name for yet another reason: it contains an isomorphic copy of every finite graph, and actually, an isomorphic copy of every countable graph. The construction of the random graph as a universal object for the class of finite graphs is due to Rado[15] ([**R1**]). The idea behind such a construction is that finite graphs can be glued together nicely so as to yield the random graph; so nicely that not only every finite graph can be found inside the random graph, but everywhere at that!

Fraïssé theory is exactly a way of building ultrahomogeneous structures by gluing finite structures together. More precisely, the class of all finite substructures of an ultrahomogeneous structure, which is called its *age*, enjoys good amalgamation properties (see figure). A *Fraïssé class* is a countable (up to isomorphism) class that satisfies the same amalgamation properties. Fraïssé's theorem states that every Fraïssé class is in fact the age of some countable ultrahomogeneous structure. Moreover, such a structure is unique; we call it the *Fraïssé limit* of the class. The rationals are thus the Fraïssé limit of the class of all finite ordered sets and the random graph that of the class of all finite graphs. This powerful result ensures that ultrahomogeneous structures are indeed characterized by their substructures, as was expected from the examples of the rationals and the random graph. As a consequence, Fraïssé theory allows for a combinatorial treatment of ultrahomogeneous structures, and especially of the dynamical properties

[14] In [**P6**], Plotkin argues that Cantor only used a *forth* argument in [**C**], and credits Huntington ([**H4**]) and Hausdorff ([**H1**]) for introducing and popularizing the *back* step alongside.

[15] The random graph is sometimes called the Rado graph.

$$\begin{tikzcd} & \mathbf{D} & \\ \mathbf{B} \arrow[ur, dotted] & & \mathbf{C} \arrow[ul, dotted] \\ & \mathbf{A} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

FIGURE 0.3. The amalgamation property: if a structure **A** embeds in two different structures **B** and **C** of the class, then both those structures themselves embed in a fourth structure **D**, still in the class, so that the diagram commutes.

of their automorphism groups, as we will see in more detail later on.

The automorphism group of a countable ultrahomogeneous structure, endowed with the topology of pointwise convergence, is a closed subgroup of S_∞, the permutation group of an infinite countable set. Hence, it is a *Polish group*: a topological group which is separable and admits a complete compatible metric. The class of Polish groups is very rich: notably, it admits universal objects, such as the homeomorphism group of the Hilbert cube ([**U2**]) and the isometry group of the Urysohn space ([**U4**]). Moreover, it is quite wide, as every metrizable locally compact group is Polish. But the class of Polish groups reaches far beyond that: the unitary group of the separable Hilbert space or the group of increasing homeomorphisms of the interval are Polish groups too, which differ greatly from locally compact ones. For instance, any of their continuous actions on a compact space must admit a global fixed point (they are called *extremely amenable*), whereas a result of Veech ([**V2**]) excludes this phenomenon for non-trivial locally compact groups.

We do not have so powerful a tool as Haar measure in the context of larger Polish groups. However, separability gives us a hold over the spaces and allows for inductive arguments, while completeness enables the use of Baire category methods, yielding a quite robust theory for Polish groups (see [**K4**] and [**G1**]). In particular, Baire category determines a notion of complexity, and of genericity, competing with that of full measure subsets (see [**O**] for the comparative merits of measure and category). This makes Polish groups an ideal ground for descriptive set theory, and they are systematically studied as such since the work of Polish mathematicians (as the name suggests) in the early twentieth century.

A result of Pettis ([**P5**]) states that any group homomorphism between Polish groups, provided it is Baire-measurable (a mild assumption that all Borel maps satisfy), is automatically continuous. Besides, Effros characterized in [**E1**] the Polishness of orbits under continuous actions of Polish spaces: this is equivalent to the orbit maps being open. These two strong results have led to the booming study of definable (mostly continuous and Borel) actions of Polish groups (see [**BK2**]) and the orbit equivalence relations they induce (see [**G1**]). This is a very rich theory; let us mention, for instance, the existence of universal actions for a Polish group ([**BK1**], [**H4**]). The structure of those equivalence relations were in particular described by beautiful dichotomy theorems ([**HKL**], [**KST**], [**M9**]) and (non-)classification results ([**H3**], [**K5**]).

Descriptive set theory and model theory maintain close relations. Maybe the first notable example of such a relation resides in the reformulation of the Vaught conjecture ([**V1**]) in terms of a specific action of a Polish group. The Vaught conjecture states that a theory in a countable language admits either countably many or continuously many countable models. It was generalized to a topological Vaught conjecture ([**M8**]), whose phrasing is reminiscent of the Glimm-Effros dichotomy: it asserts that every continuous action of a Polish group admits either countably many or continuously many orbits. For this topological version, some strong partial results have been obtained, see [**B2**]. The other notable interaction between the two fields, which is at the heart of this thesis, is that Polish groups can be approached as automorphism groups. More precisely, we have mentioned that automorphism groups of countable (ultrahomogeneous) structures are closed subgroups of S_∞. It turns out that the converse also holds: every closed subgroup of S_∞ is isomorphic to the automorphism group of some countable structure! Even better, by naming the orbits under the action of the group, we can make the structure ultrahomogeneous.

This enlightening correspondence is not limited to subgroups of S_∞, and can actually be extended to all Polish groups, via *continuous logic*. The latter was developed by Ben Yaacov and Usvyatsov in [**BU2**], along with Berenstein and Henson ([**BBHU**]), in order to study metric structures. The idea is to replace points with the distance functions to the points, as well as the usual truth values true and false with a continuum of truth values, generally the interval $[0,1]$. A relation then becomes a uniformly continuous function that takes its values in $[0,1]$ instead of $\{0,1\}$ and a *metric structure* is a complete metric space, equipped with a family of such relations. Note that classical structures fit into that framework: when endowed with the discrete metric, they become metric structures. In this metric context, automorphisms are in particular isometries. Besides, the natural continuous counterpart of countability is separability, so that the automorphism groups of separable metric structures are Polish. The crucial observation that those encompass all Polish groups is due to Melleray ([**M5**]).

Furthermore, as in the discrete setting, we may require the structure to be highly symmetric. To that purpose, the ultrahomogeneity assumption is relaxed. A metric structure is *approximately ultrahomogeneous* if every isomorphism between finite substructures can be extended, up to an arbitrarily small distance error, to an automorphism of the whole structure. Then, again by naming the closures of orbits in the language, every Polish group can be made into the automorphism group of an approximately ultrahomogeneous structure.

Many metric structures that arise naturally are approximately ultrahomogeneous: the measure algebra of an interval, (the unit ball of) a separable Hilbert space or L^p spaces. All of those examples are in fact *exactly* ultrahomogeneous. This is however not the case in general that the word "approximately" can be dropped: Banach lattices or the Gurarij space, which surface as naturally, are only approximately ultrahomogeneous (see [**BBHU**] and [**M2**]). Melleray asked whether every Polish group was nonetheless the automorphism of some exactly ultrahomogeneous structure. It it not the case: Ben Yaacov recently provided examples of Polish groups that cannot act transitively continuously by isometries on any complete metric space ([**B7**]). The question of the exact ultrahomogeneity of metric structures will still come up as a subplot throughout the thesis.

An important example of an ultrahomogeneous metric structure, which conveys much intuition, is the *Urysohn space*. The Urysohn space is the unique metric space, up to isometry, that is both ultrahomogeneous and universal for the class of all finite metric spaces. In analogy with the random graph, this whispers the idea of a metric version of Fraïssé theory. Such a theory, much analogous to the classical one (with a relaxed amalgamation property), was indeed developed by Schoretsanitis ([**S2**]) and Ben Yaacov ([**B5**]). As expected, the Urysohn space is the Fraïssé limit of the class of all finite metric spaces, the Hilbert space that of the class of finite-dimensional Hilbert spaces, the measure algebra of the interval that of the class of all finite measure algebras. This analogue of Fraïssé theory provides a nice setting in which we can apply combinatorial arguments to general Polish groups as well.

The interplay between descriptive set theory of Polish groups and model theory of Fraïssé structures has proven very fruitful. In this thesis, we shall see a sample of various aspects of this flourishing correspondence.

The question naturally arises of how good the correspondence is: what does the automorphism group remember about the structure? In other words, can the structure be reconstructed from the automorphism group alone? Ahlbrandt and Ziegler ([**AZ**]) gave a positive answer for a special class of homogeneous countable structures, *countably categorical* ones[16]. A structure is countably categorical if it is the only countable model of its theory: any countable structure that satisfies the same (first-order) properties is isomorphic to it. There are many countably categorical structures especially among Fraïssé limits. For example, the rationals and the random graph are countably categorical, since not only are they characterized by their finite substructures but the characterization is expressed by first-order sentences, and hence belongs to the theory.

On the other hand, countable categoricity provides enough rigidity to enable a richer correspondence between the structure and its automorphism group. Indeed, at our disposal is the very powerful Ryll-Nardzewski theorem, which describes types in a countably categorical structure. This has many consequences, especially for the action of the automorphism group. The theorem asserts that the group must act *oligomorphically*, that is, with finitely many orbits and that the space of types must be finite. In particular, this yields a characterization of definability that is expressed purely in terms of the automorphism group: in a countably categorical structure, definability simply boils down to invariance under the action of the automorphism group. This characterization constitutes an essential tool in the reconstruction.

The reconstruction Ahlbrandt and Ziegler propose is the following. They show that two countably categorical structures whose automorphism groups are isomorphic (as topological groups) are *bi-interpretable*. More precisely, if two structures are bi-interpretable, each of them embeds in the imaginaries of the other, that is, in a definable quotient of a finite power of the other. Thus, they recover, if not the original structure, at least all its model-theoretic properties.

With Itaï Ben Yaacov, we extend this reconstruction result to the continuous setting, where countable categoricity is replaced by *separable categoricity*. The class of separably categorical structures is quite large too, as it includes the measure algebra, (the unit ball of) a separable Hilbert space, and the Urysohn sphere. Besides, it enjoys very similar properties to the countable case. Again, definability can be characterized as invariance under the action of the automorphism group. Indeed, the Ryll-Nardzewski theorem generalizes naturally to metric structures (see [**BU1**] and [**BBHU**]): the space of types in a separably categorical structure must be compact (for a natural metric) and the automorphism group must act *approximately oligomorphically*, that is, with compactly many orbits. Thus, the space of types has a tree-like topological structure, a description that we use extensively in the proof.

In metric model theory, the definition of imaginaries has to be slightly modified: one needs to allow infinite definable quotients to take converging sequences of epsilons into account (see the discussion in [**BU2**]). In [**BK3**], we define the notion of an interpretation between metric structures accordingly. Then, as anticipated, we prove that two separably categorical metric structures whose automorphism groups are topologically isomorphic are bi-interpretable. Our proof, however, differs from the classical one and is intrinsically metric in nature. Indeed, we exploit Melleray's proof ([**M5**]) that every Polish group is the automorphism group of some metric structure: using a standard construction, he produces a canonical[17] metric structure associated to a Polish group, its *hat structure*. We actually show that every separably categorical structure is bi-interpretable with the hat structure of its automorphism group.

This reconstruction result ensures that every model-theoretic property of separably categorical structures is encoded in their automorphism group, that it *is* really a topological property. The construction of the actual dictionary was initiated by Ben Yaacov and Tsankov in [**BT1**], with an entry on stability. The right translation involves the Roelcke compactification

[16]The parentality of this result is a bit unclear, actually. Ahlbrandt and Ziegler attribute it to Coquand (in an unpublished note) and it appears that the idea was out there for quite some time that this result should be true.

[17]Technically, the hat structure is associated to a left-invariant metric on the Polish group. However, up to bi-interpretability, it does not depend on the choice of such a left-invariant metric.

of the automorphism group. Indeed, automorphism groups of separably categorical structures are *Roelcke-precompact* (this follows from the approximate oligomorphicity of their action, see [**R3**]). Conversely, Ben Yaacov and Tsankov, and independently Rosendal ([**R4**]), showed that every Roelcke-precompact Polish group arises as such. With this observation in hand, Ben Yaacov and Tsankov prove that stability corresponds to the weak almost periodicity of the Roelcke compactification of the automorphism group, and they study those compactifications from a model-theoretic viewpoint. Ibarlucía ([**I**]) then extended the dictionary to dependent metric structures. Amusingly enough, the very word "independent" was commonly used on the topological side (see [**R5**] or [**GM1**]), to designate the same notion, long before the connection with model theory was noticed!

This type of reconstruction involves the automorphism group together with its topology. It is therefore legitimate to ask whether the abstract group alone suffices to recover the structure. Unfortunately, in general, it does not: Evans and Hewitt ([**EH**]) constructed a counterexample in the countably categorical case, answering a question of Rubin ([**R6**]). However, model theorists were (and are) still interested in the question of recovering the topology of a group from its algebraic structure. To that aim, the *small index property* was extensively studied. This property states that all subgroups of countable index are open. The small index property is equivalent to every homomorphism from the group to S_∞ being continuous. Thus, for the automorphism group of a countable structure, the small index property guarantees that the algebraic structure of the group already encodes its topology, and that the stronger reconstruction goes through. This is the case for a great many automorphism groups, such as S_∞ itself (Semmes, [**S3**]) or the automorphism group of the rationals (Truss, [**T2**]). See [**HHLS**] and [**L1**] for more details on the small index property and the reconstruction problem.

In [**HHLS**], Hodges, Hodkinson, Lascar and Shelah introduced a powerful tool, *ample generics*, to prove the small index property for automorphism groups of countable structures. The notion was refined and further investigated by Kechris and Rosendal ([**KR**]). A Polish group G has ample generics if for every n in $\mathbb{N}$, the diagonal conjugacy action of G on G^n admits a comeager orbit. This implies in particular that G has a comeager conjugacy class, a phenomenon Wesolek ruled out for locally compact Polish groups ([**W**]). Ample generics are particularly interesting, for beyond the small index property, they imply a very strong property: the *automatic continuity property*. Namely, if G has ample generics, then every group homomorphism from G to any separable topological group — not just a closed subgroup of S_∞ — is automatically continuous. This beautiful property guarantees that the topology on the group is the unique Polish topology compatible with its algebraic structure, a powerful reconstruction result.

For closed subgroups of S_∞, seen as automorphism groups of Fraïssé structures, Kechris and Rosendal exhibited combinatorial conditions that imply ample generics. In particular, if a Fraïssé class satisfies the *free amalgamation property* and the *extension property*, then the automorphism group of its Fraïssé limit has ample generics (see [**M6**]). The extension property states that any partial isomorphism of a finite structure extends to a global isomorphism of a bigger, but still finite, structure. It was first proved for the class of finite graphs by Hrushovski[18] ([**H2**]). This result was generalized by Herwig and Lascar ([**HL**], see also [**S7**]) to classes of finite structures that omit a certain set of configurations, such as triangle-free graphs. Their techniques were used by Solecki ([**S6**]) to prove that the extension property holds for finite metric spaces. As for the free amalgamation property, it guarantees that unions of partial isomorphisms extend coherently, by amalgamating their domains as independently as possible. Applying this combinatorial criterion, we get that the automorphism group of the random graph, that of Henson's triangle-free graph, and S_∞ have ample generics. The automorphism group of the rationals, on the other hand, has a comeager conjugacy class but does not have ample generics. Quite a number of other Polish subgroups of S_∞, such as the homeomorphism

[18]Consequently, the extension property is often named the Hrushovski property.

group of the Cantor space (Kwiatkowska, [**K7**]), also have ample generics and therefore the automatic continuity property.

Unfortunately, for larger Polish groups, ample generics often fail badly: in the isometry group of the Urysohn space, in the automorphism group of the measure algebra of the interval, in the unitary group of a separable Hilbert space, all conjugacy classes are meager ([**K6**]). Nevertheless, even among groups that do not have ample generics, several Polish groups satisfy the automatic continuity property. Rosendal and Solecki ([**RS**]) circumvented the absence of ample generics and proved the automatic continuity property for the automorphism group of the rationals, the homeomorphism group of the reals and the homeomorphism group of the circle.

The observation that some large Polish groups fail to have ample generics led Ben Yaacov, Berenstein and Melleray ([**BBM**]) to devise a weaker version of ample generics, *ample topometric generics*, that more Polish groups satisfy. Crucial to their work is the fact that every Polish group is the automorphism group of some metric structure. Such an automorphism group, apart from its usual pointwise convergence topology, then naturally comes equipped with the metric of uniform convergence. This metric is bi-invariant and in most cases, defines a canonical bi-invariant uniformity that refines the usual Polish topology. Ample topometric generics are defined by intertwining the uniform metric and the topology. Following Kechris and Rosendal, Ben Yaacov, Berenstein and Melleray prove that if G has ample topometric generics, it enjoys a topometric version of the automatic continuity property: any group homomorphism from G to a separable group which is continuous for the uniform topology must be continuous for the usual topology as well. In order to check ample topometric generics for specific metric structures, Ben Yaacov, Berenstein and Melleray call on dense countable structures[19] that do have ample generics. Combining this with results of Kittrell and Tsankov ([**KT**]), they obtain the automatic continuity property for the automorphism group of the measure algebra of the interval. With the same ideas, Tsankov ([**T3**]) proved that the unitary group of a Hilbert space also satisfies the automatic continuity property.

Along the same line, Sabok ([**S1**]) introduced a method of proving the automatic continuity property, a step of which is to extract a countable structure with the free amalgamation property and the extension property. His proof originates in Rosendal and Solecki's that the group $\mathrm{Homeo}(2^{\mathbb{N}})$ has the automatic continuity property ([**RS**], which was written before Kwiatkowska showed that in fact, it had ample generics). More precisely, Sabok isolates a set of combinatorial properties on a metric structure which mimic ample generics and imply the automatic continuity property for its automorphism group. The conditions consist of the extension property, a metric version of the free amalgamation property, an ad hoc condition of local homogeneity called the *isolation property*, as well as the notable condition that the structure be exactly ultrahomogeneous[20]. From this, Sabok obtained that the isometry group of the Urysohn space has the automatic continuity property.

We take an interest in a stability property of the class of groups that have the automatic continuity property: when is it true that if G satisfy the automatic continuity property, then so does $G^{\mathbb{N}}$? Although it is not true in general that the infinite power of a group with the automatic continuity property still has the automatic continuity property, ample generics do carry to countably infinite powers! In their aforequoted paper, Rosendal and Solecki proved the automatic continuity property not only for $\mathrm{Homeo}(2^{\mathbb{N}})$ but also for $\mathrm{Homeo}(2^{\mathbb{N}})^{\mathbb{N}}$ (before $\mathrm{Homeo}(2^{\mathbb{N}})$ was known to have ample generics). Naturally, it brings the question for Sabok's conditions: do they go through to infinite powers? The answer is not clear; however, it turns

[19] In fact, many countable Fraïssé limits admit continuous analogues, and vice versa. For instance, the Urysohn space is the continuous counterpart of the rational Urysohn space, which is the Fraïssé limit of the class of all finite metric spaces with rational distances.

[20] As we pointed out before, it would be very nice to have a combinatorial characterization of exact ultrahomogeneity as well!

out they almost do. Indeed, Malicki ([**M1**]) proposed a slightly modified version of Sabok's conditions, designed to mimic ample generics better still, so as to recover the small index property at the same time as the automatic continuity property. In a similar way to [**RS**], from a metric structure $\mathbf{M}$, we build the *juxtaposed structure* of $\mathbf{M}$, whose automorphism group is the infinite power of that of $\mathbf{M}$. Then, we prove that Malicki's conditions carry to juxtaposed structures. Hence, we obtain that the groups $\mathrm{Iso}(\mathbb{U})^{\mathbb{N}}$, $\mathcal{U}(\ell^2)^{\mathbb{N}}$ and $\mathrm{Aut}(\mu)^{\mathbb{N}}$ satisfy the automatic continuity property.

As mentioned before, large Polish groups tend to fail having ample generics. Kechris and Rosendal have actually asked whether there existed a Polish group outside S_∞ with ample generics. A surprisingly simple example of such a group came out of a discussion with François Le Maître ([**KLM**]) on possible generalizations of the previous result on automatic continuity for group powers. Indeed, we remarked that the group $G^{\mathbb{N}}$ is no other than the group $L^0(\mathbb{N}, G)$ of all G-valued (measurable) functions on the integers. It is therefore natural to consider the Polish group $L^0([0,1], G)$ of G-valued random variables on the interval. The proof that if G has ample generics, then so does $G^{\mathbb{N}}$, readily adapts to $L^0([0,1], G)$ by using the Jankov-van Neumann uniformization theorem. Moreover, the group $L^0([0,1], G)$ is always connected (even contractible) while S_∞ is totally disconnected, so $L^0([0,1], G)$ cannot be a subgroup of S_∞. Hence, since G embeds in $L^0([0,1], G)$, we even prove that every Polish group with ample generics embeds into a connected one with ample generics. Another class of examples that we exhibit consists of full groups of type III hyperfinite ergodic equivalence relations, which happen to embed into $L^0([0,1], S_\infty)$. Note that simultaneously, Malicki provided yet another class of Polish groups with ample generics that are not closed subgroups of S_∞ ([**M1**]).

As for extending Malicki's techniques to the group $L^0([0,1], G)$, sadly, we did not go very far. If $\mathbf{M}$ is a metric structure, then $L^0([0,1], \mathrm{Aut}(\mathbf{M}))$ is the automorphism group of a randomization of $\mathbf{M}$, which stays exactly ultrahomogeneous when $\mathbf{M}$ is. However, to carry the other conditions over to the randomization, it appears that one would need them to be uniform in some sense, which they do not seem to be in our examples. For instance, we still do not know whether the groups $L^0([0,1], \mathrm{Iso}(\mathbb{U}))$ or $L^0([0,1], \mathrm{Aut}(\mu))$ satisfy the automatic continuity property.

Let us now go over a different aspect of the correspondence between groups and structures: a combinatorial approach to topological dynamics. We have mentioned that a number of large Polish groups are extremely amenable, that is, any of their continuous actions on a compact space admits a fixed point. The study of the *fixed point on compacta property*[21] started in the context of semi-groups (see [**M10**] and [**G4**]). The first examples of extremely amenable topological groups were built by Herer and Christensen ([**HC**]) and were called exotic[22]. Later, more natural examples of extremely amenable groups turned up: the unitary group of a separable Hilbert space (Gromov-Milman, [**GM2**]), the group of increasing homeomorphisms of the interval (Pestov, [**P4**]), $L^0([0,1], S^1)$ (Glasner, [**G2**]). The proofs used concentration of measure, see for instance [**L2**]. Pestov then revealed a relationship between the phenomenon of concentration of measure and structural Ramsey theory. Using the finite Ramsey theorem, he proved that the automorphism group of the rationals is extremely amenable ([**P4**]). With similar ideas, he also proved that the isometry group of the Urysohn space is extremely amenable ([**P2**]).

The general setting underlying this connection was uncovered by Kechris, Pestov and Todorčević in [**KPT**] for closed subgroups of S_∞. They proved that the automorphism group of a

[21]The name "extremely amenable" emerged from a parallel with (one of the multiple characterizations of) amenability, which we shall discuss later on. It is probably due to Granirer.

[22]Since this phenomenon cannot appear in the more commonly studied context of locally compact groups, such a result is indeed very surprising.

Fraïssé limit is extremely amenable if and only if the Fraïssé class satisfies the *Ramsey property* (under the mild assumption that the objects are rigid; typically, it is the case for ordered structures). A class $\mathcal{K}$ has the Ramsey property if for every palette of k colors, for every small structure $\mathbf{A}$ and medium structure $\mathbf{B}$ in $\mathcal{K}$, there is a big structure $\mathbf{C}$ still in $\mathcal{K}$ such that for every coloring of the set of copies of the small structure $\mathbf{A}$ in $\mathbf{C}$ using colors from the palette, there exists a copy of the medium structure $\mathbf{B}$ in $\mathbf{C}$ within which all copies of $\mathbf{A}$ have the same color (see figure). In other words, the Ramsey property guarantees the existence of large monochromatic substructures.

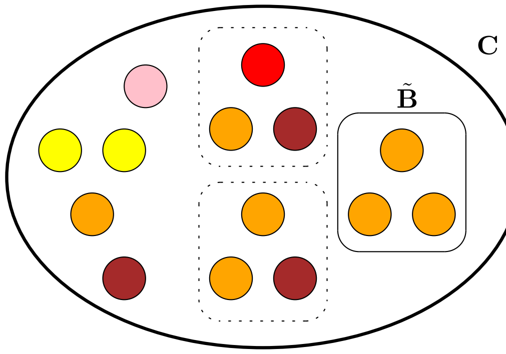


FIGURE 0.4. The Ramsey property.

From this powerful characterization, known Ramsey theorems provided new extremely amenable groups. For instance, Nešetřil and Rödl ([**NR**]) proved that the Fraïssé class of finite ordered graphs has the Ramsey property, the automorphism group of the random ordered graph is then extremely amenable. Remarkably, the equivalence was also used in the reverse direction: Pestov proved that the isometry group of the Urysohn space is extremely amenable, which prompted Nešetřil to check that the class of finite ordered metric spaces with rational distances has the Ramsey property (see [**NVT**]). Moreover, Kechris, Pestov and Todorčević's method was applied to compute the universal minimal flow of several Polish groups; for more details on the developments of the Kechris-Pestov-Todorčević correspondence, see the recent survey by Nguyen van Thé ([**T1**]).

This correspondence was extended to the continuous setting by Melleray and Tsankov ([**MT1**]). They introduce an approximate version of the Ramsey property and show that extreme amenability of the automorphism group of a metric Fraïssé limit is equivalent to the class satisfying this property. This captures a similar result obtained by Pestov ([**P2**]) for isometry groups of ultrahomogeneous metric spaces. For this approximate Ramsey property, a notable change is that they impose a regularity condition on colorings, in order to control the various epsilons generated by the approximate ultrahomogeneity. Besides, they provide a criterion to verify the approximate Ramsey property, the intuition for which stems from Kechris and Rosendal's observation that if a Fraïssé class has the extension property, then the automorphism group of its limit admits a dense union of compact subgroups, and is hence amenable ([**KR**]). More precisely, their criterion involves a weakening of the extension property, as well as a condition that enables the use of concentration of measure. Although this allows them to recover the extreme amenability of several Polish groups, the criterion has not yet been applied to find new examples of extreme amenable groups. Nevertheless, the finitary character of the approximate Ramsey property allows Melleray and Tsankov ([**MT2**]) to compute its complexity: extreme amenability is a G_δ notion, which is very nice from a Baire category perspective.

In another direction, the Kechris-Pestov-Todorčević correspondence was transferred by Moore ([**M12**]) to *amenability* for closed subgroups of S_∞. Amenability was introduced by von Neumann ([**N**]) in the late twenties to get a better understanding of the Banach-Tarski[23]

[23]Do you know a good anagram of "Banach-Tarski"?

paradox. It is far better-known a notion than its little sister, as it has been extensively studied for discrete groups. In that setting, it is defined as the existence of an invariant mean on the group, and admits equivalent definitions aplenty. We will not touch upon these, though, and concentrate on topological groups. For the latter, a right definition (which of course coincides with the previous notion in the discrete case) turns out to be the following. A topological group is amenable if every time it acts continuously on a compact space, the action admits an invariant Borel probability measure. Examples of amenable Polish groups abound, all the more so as amenability does not exclude locally compact groups: compact groups, abelian groups, solvable groups, S_∞, and more generally all automorphism groups of Fraïssé structures with the extension property are amenable, to name a few.

Amenability also admits a description in terms of finite combinatorics, as was shown by Tsankov in an unpublished note and by Moore ([**M12**]). The idea behind this is that extreme amenability, which provides fixed points for actions, corresponds to colorings admitting large monochromatic (hence fixed, in some sense) sets. Similarly, amenability provides fixed probability measures, which can be thought of as barycenters of Dirac measures. Thus, the appropriate counterpart of the Ramsey property in that context should guarantee the existence of large "monochromatic convex combinations of sets". That is precisely the combinatorial content of amenability. Indeed, Moore introduced the *convex Ramsey property* and proved that a Fraïssé class satisfies this property if and only if the automorphism group of its limit is amenable.

Combining this with the ideas of Melleray and Tsankov, we extend Moore's characterization to the continuous setting. Exactly as in [**MT1**], we replace the classical notion of coloring with Lipschitz ones to introduce the *metric convex Ramsey property*, and, closing the diagram, we prove the exact analogue of Moore's result for automorphism groups of metric Fraïssé limits. Those combinatorial descriptions of amenability have not yet led to any new examples of amenable groups. In fact, there is no known technique to prove the convex Ramsey property directly, let alone the metric one. Nevertheless, our characterization leads to nice structural properties on amenability. In particular, we obtain that amenability also is a G_δ condition. This means that amenability essentially boils down to a $\forall\exists$ condition, when it was defined the other way around (basically, "*there exists* a measure that gives the same measure to the translates of *all* sets"). Moore had showed that for discrete groups, one may indeed swap the quantifiers in the actual definition of amenability. We prove that the same holds for Polish groups in the definition of amenability. Note that it follows from the work of Melleray and Tsankov that the same is true of extreme amenability as well.

Finally, we investigate a last facet of our guiding correspondence: how well does the automorphism group of a Fraïssé limit describe isomorphisms inside the structure? More precisely, Fraïssé limits are ultrahomogeneous, which means that isomorphism classes of finite structures are exactly the orbits under the action of the automorphism group. What about isomorphism classes of larger substructures? Panagiotopoulos ([**P1**]) provides conditions on a Fraïssé structure for the isomorphisms between generic structures to extend. We approach the question from a different angle. Fixing a particular Fraïssé structure, we would like to find out which substructures have the *homogeneity property*, that is, to find substructures every isomorphism between two copies of which extends to an automorphism of the whole structure. This question was asked to the author by Julien Melleray during a semester program in Bonn.

The first results in that direction can be found in work of Huhunaišvili ([**H3**]) on the Urysohn space. Indeed, Huhunaišvili proved that not only is the Urysohn space ultrahomogeneous, but actually, every isometry between compact subspaces extend to a global isometry of the Urysohn space. Melleray ([**M3**], see [**M4**]) then showed that (relatively) compact subspaces are the only subsets with this property. To this aim, he characterizes compact spaces as the only spaces whose one-point metric extensions are compactly determined. With Isabel Müller and Aristotelis Panagiotopoulos, we introduce an analogue of this property for classical Fraïssé classes: *typical finiteness*, which asserts that all types over our substructure are determined by a finite set. Then, we prove that a substructure has the homogeneity property if and only if is typically finite.

We further study typical finiteness in concrete examples; in many examples of Fraïssé structures, we show that typically finite substructures must in fact be finite. It is the case of all countably categorical structures, as well as the rational Urysohn space. We hope to obtain a more explicit description of typically finite substructures. Moreover, our result opens several perspectives of generalization. For instance, we are currently trying to see what happens in the metric setting, outside the particular case of the Urysohn space. Inasmuch as metric Fraïssé limits are not always ultrahomogeneous, it is probably necessary to restrict to those we already know are ultrahomogeneous. Then, the expected characterizing property should resemble compact determination of types; the exact meaning of compact determination is however not at all clear.

Besides, we are interested in a weaker version of the homogeneity property: we may only ask that the automorphism group of the whole Fraïssé structure act transitively on the copies of the substructure. In the Urysohn space, Melleray showed that this is actually equivalent to the homogeneity property. We prove that this also holds in classical Fraïssé structures that admit a *stationary independence relation* (in the sense of [**TZ2**]), an important particular case of which is structures with the free amalgamation property. In general, however, we do not know whether the two properties are equivalent.

We opted for a tree-like organization of the thesis. The trunk part essentially contains prerequisites about the words of the title. It aims at the correspondence between Polish groups and automorphism groups of approximately ultrahomogeneous structures. More specifically, chapter 1 skims through basic facts we will need about Polish groups. In particular, we briefly go over the different uniform structures one can equip Polish groups with. In chapter 2, we present the Urysohn space and its modern construction. Although the Urysohn space is not strictly speaking essential in the establishing of the correspondence, we felt it deserved to be given special prominence as a nice object that captures the ideas of continuous logic quite well. It will be our companion space throughout the thesis as it has been during the last four years. Chapter 3 then lays the basics of model theory in detail in order to set our conventions precisely. Besides, we took the opportunity to present both frameworks in parallel, so as to highlight their similarities and differences in a (hopefully) self-contained exposition. Similarly, we expose both classical and metric Fraïssé theories in chapter 4. However, the elegant tools involved on the metric side not being quite well-known, we devote a substantial part of the chapter to a thorough presentation of those tools. Thus armed, we finally give the proof in chapter 5 that every Polish group is the automorphism group of a metric Fraïssé limit.

Each branch is then concerned with one aspect of this correspondence, the branches appearing in the same order as in the introduction. Branch 1 begins by a chapter of prerequisites on countable and separable categoricity, presented side by side as well. Then we carry out the metric version of Ahlbrandt and Ziegler's reconstruction in chapter 6, which is joint work with Itaï Ben Yaacov and appeared in [**BK3**]. The second branch discusses questions of automatic continuity for infinite powers of Polish groups. It also contains part of a joint work with François Le Maître, in which we exhibit examples of connected Polish groups with ample generics. Branch 3, which appeared in [**K1**], describes our combinatorial characterization of amenability for general Polish groups. Finally, Branch 4 contains joint work in progress with Isabel Müller and Aristotelis Panagiotopoulos on the homogeneity property in Fraïssé structures.

Trunk

Polish groups as automorphism groups

CHAPTER 1

Polish groups

À gauche, à droite!
Yannick[1]

1. Polish spaces **37**
2. Polish groups **39**
3. Uniformities on Polish groups **40**
3.1. Uniform spaces 40
3.2. Metrizable uniformities 41
3.3. Group completion 43
3.4. Precompactness 43

A great many structures that we encounter in mathematics come equipped with a natural Polish topology: a separable topology that admits a compatible complete metric. Those two properties combine perfectly: completeness provides the Baire category theorem, while separability allows to construct many countable intersections of open sets to apply it to. Furthermore, Polish spaces also enjoy many stability properties, which we present and illustrate in the first section of this chapter.

Polish groups, more particularly, constitute the cynosure of this thesis. Those groups abound as well, often surfacing as isometry groups. Besides, they have strong rigidity properties, which we glimpse at in the second section. In the third section of this chapter, we go over several compatible metrics a Polish group can be endowed with. Actually, we describe these metrics in the more intrinsic framework of uniform structures, and then compare the properties of Polish groups with respect to these different uniformities.

1. Polish spaces

Definition 1.1. A **Polish space** is a separable topological space that admits a compatible complete metric.

Note that the metric is not part of the data: a Polish space is just a topological space. We illustrate this fact with the following simple example. However, we will call a metric space Polish if it is separable and complete.

Example 1.2. The space $\mathbb{R}$ is Polish, for its usual metric is complete. This metric does not induce a complete distance on the open interval $]0,1[$. Yet, the space $]0,1[$ is homeomorphic to $\mathbb{R}$, hence it is Polish.

Proposition 1.3. Let $(X_n)_{n\in\mathbb{N}}$ be a family of Polish spaces. Then the space $\displaystyle\prod_{n\in\mathbb{N}} X_n$, endowed with the product topology, is Polish. In particular, countable powers of Polish spaces are Polish.

[1] *Ces soirées-là*

PROOF. If d_n is a complete compatible metric on X_n, then the metric defined by

$$d^\omega(x,x') = \sum_{n\in\mathbb{N}} \frac{1}{2^n} \min(1, d_n(x_n, x'_n))$$

is complete and compatible with the product topology on $\prod_{n\in\mathbb{N}} X_n$. Moreover, the product is separable: if $(U_m^{(n)})_{m\in\mathbb{N}}$ is a countable basis of open subsets of X_n, then a countable basis of open subsets of $\prod_{n\in\mathbb{N}} X_n$ is given by all sets of the form

$$U_{m_1}^{(1)} \times \dots \times U_{m_p}^{(p)} \times \prod_{n>p} X_n,$$

for p and $m_1, \dots, m_p$ in $\mathbb{N}$. □

It is clear that closed subsets of Polish spaces are Polish. Moreover, we saw in example 1.2 an example of an open subset that is Polish. In fact, all G_δ subsets of Polish spaces are Polish themselves. Recall that a subset is called G_δ if it is a countable intersection of open subsets.

THEOREM 1.4. (Alexandroff) Let X be a Polish space and let A be a G_δ subset of X. Then the space A (endowed with the induced topology) is Polish.

PROOF. First, assume that A is open and let d be a complete compatible metric on X. We claim that the following defines a complete compatible metric on A:

$$d_A(x,y) = d(x,y) + \left| \frac{1}{d(x, X\setminus A)} - \frac{1}{d(y, X\setminus A)} \right|.$$

Because A is open, this is well-defined on A.

The metric d_A induces the same topology as d on A. Indeed, if a sequence (x_n) converges to x in A for d, then it does for d_A too, as the map $d(\cdot, X\setminus A)$ is continuous. The other direction is trivial.

We now prove that d_A is complete. Let (x_n) be a d_A-Cauchy sequence in A. It is in particular Cauchy for d, so, by completeness of d, it converges to some point x in $\overline{A}$. It follows that $d(x_n, X\setminus A)$ converges to $d(x, X\setminus A)$. Moreover, the sequence $\left(\frac{1}{d(x_n, X\setminus A)}\right)$ is Cauchy in $\mathbb{R}$, so it converges. Thus, $(d(x_n, X\setminus A))$ is bounded away from 0, hence $d(x, X\setminus A)$ is non-zero and x belongs to A.

For the general case, write A as the intersection of open sets U_i. The first case yields complete compatible metrics d_{U_i} on the U_i's. Now A embeds continuously as a closed subset of the product of the U_i's. By proposition 1.3, this product is Polish, hence A is Polish too. □

Actually, the converse also holds, providing a very useful characterization of Polish subspaces, a proof of which can be found in [**K4**, theorems 3.8 and 3.9].

THEOREM 1.5. (Kuratowski, Lavrentiev) Let X be a complete metric space and let A be a subset of X. Then, A is completely metrizable if and only if A is G_δ in X.

One of the most important tools in studying Polish spaces is the Baire category theorem (see e.g. [**K4**, theorem 8.4]).

THEOREM 1.6. (Baire category) Let (X,d) be a complete metric space. Then every countable intersection of dense open subsets of X is also dense. Equivalently, every countable union of closed sets with empty interiors also has empty interior.

An application is for instance the following non-example.

NON-EXAMPLE 1.7. The space $\mathbb{Q}$, with its usual topology, is not Polish. Indeed, if it were, then the Baire category theorem would imply that every countable union of closed sets with empty interior would have empty interior in $\mathbb{Q}$. But the space $\mathbb{Q}$ itself is of this form: $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$, a contradiction.

A comprehensive exposition of the theory of Polish spaces can be found in [**K4**, chapter I].

2. Polish groups

The focus of this thesis is on *Polish structures* (metric structures on a Polish space) and on their *automorphism groups*, which will be Polish too. Even better, their topology is compatible with their group structure.

DEFINITION 1.8. A **Polish group** is a topological group whose topology is Polish.

We present the two main examples of Polish groups we will encounter throughout the thesis.

EXAMPLE 1.9. The group S_∞ of all permutations of $\mathbb{N}$ (all of them, not just those with finite support). We equip S_∞ with the topology induced by the product topology on $\mathbb{N}^\mathbb{N}$, that is, the topology of pointwise convergence. A basis of neighborhoods of a permutation σ is given by sets of the form

$$\{\tau \in S_\infty : \forall i \in F, \tau(i) = \sigma(i)\},$$

for a finite set F of integers. With this topology, S_∞ is a topological group. Moreover, it is G_δ in the Polish space $\mathbb{N}^\mathbb{N}$. Indeed, elements of S_∞ are characterized as follows:

$$\sigma \in S_\infty \Leftrightarrow [\forall i \neq j, \sigma(i) \neq \sigma(j) \text{ and } \forall j, \exists i, \sigma(i) = j].$$

For fixed i and j, the conditions $\sigma(i) \neq \sigma(j)$ and $\sigma(i) = j$ are clopen, so the whole condition inside brackets is G_δ. Thus, S_∞ is a Polish group.

EXAMPLE 1.10. The isometry group $\mathrm{Iso}(X, d)$ of a Polish metric space (X, d). Again, we endow $\mathrm{Iso}(X, d)$ with the topology of pointwise convergence. Basic open sets are the sets of all isometries that extend a given partial isometry between finite subsets up to a small error: a basis of neighborhoods of an isometry g is given by the sets of the form

$$\{f \in \mathrm{Iso}(X, d) : \forall x \in F, d(f(x), g(x)) < \epsilon\},$$

for a finite subset F of X and a positive ϵ.

Enumerate a countable dense subset of X: $\{x_i : i \in \mathbb{N}\}$. Consider the map $g \mapsto (g(x_i))_{i \in \mathbb{N}}$ from $\mathrm{Iso}(X, d)$ to $X^\mathbb{N}$. It is a homeomorphism onto its image. Indeed, isometries extend uniquely from the dense set $\{x_i : i \in \mathbb{N}\}$ to the complete set X, because they are uniformly continuous. Therefore, the map is injective and, since pointwise convergence in $\mathrm{Iso}(X, d)$ amounts to pointwise convergence on the x_i's, it is also a homeomorphism on its image.

Moreover, the space $\mathrm{Iso}(X, d)$ is characterized by the following conditions:

$$g \in \mathrm{Iso}(X, d) \Leftrightarrow [\forall i, j \in \mathbb{N}, d(g(x_i), g(x_j)) = d(x_i, x_j) \text{ and } \forall n \geqslant 1, \forall j \in \mathbb{N}, \exists i \in \mathbb{N}, d(g(x_i), x_j) < \frac{1}{n}].$$

Thus, $\mathrm{Iso}(X, d)$ is homeomorphic to a G_δ subset of the Polish space $X^\mathbb{N}$, hence $\mathrm{Iso}(X, d)$ is a Polish group.

Note that the metric on S_∞ we described in example 1.9 coincides with this one, when S_∞ is viewed as the isometry group of the space $\mathbb{N}$ for the discrete metric.

REMARK 1.11. A beautiful result of Gao and Kechris ([**GK**]) states that every Polish group is isomorphic to such an isometry group. As a first step towards seeing Polish groups as automorphism groups, we will use the weaker result that every Polish group is isomorphic to a subgroup the isometry group of some Polish space (theorem 5.2).

Here is a very useful application of the Baire category theorem to Polish groups.

THEOREM 1.12. Let G be a Polish group and let H be a subgroup of G. If H is Polish (with respect to the induced topology), then H is closed in G.

PROOF. We place ourselves in $\overline{H}$, which is a closed subgroup of G, hence a Polish group. Since H is Polish, theorem 1.5 implies that H is G_δ in $\overline{H}$. In other words, H is a dense G_δ subset of $\overline{H}$.

Now, if g is any element of $\overline{H}$, the coset gH is also a dense G_δ of $\overline{H}$ because multiplication by g is a homeomorphism. Thus, by the Baire category theorem in the space $\overline{H}$, these two dense G_δ sets H and gH have a dense intersection. In particular, the two cosets must intersect. It follows that they actually coincide, so g belongs to H. Finally, this means that $H = \overline{H}$, hence H is closed. $\square$

Apart from here, the Baire category theorem only barely appears in this thesis. We refer the reader to [**M7**] for a deeper overview of Baire category methods. Also, see [**G1**] for more details on the very rich theory of Polish groups.

3. Uniformities on Polish groups

In this section, we present several uniformities and compatible metrics one can endow a Polish group with and we compare their properties.

3.1. Uniform spaces. Uniformities are designed to mimic the behavior of a metric, and in particular to provide a notion of uniform continuity, on pure topological spaces. Here, we review some of the basic definitions and facts we will need later on; for a more thorough introduction to uniformities, see for example [**P3**, chapter 1] or [**E2**, chapter 8].

DEFINITION 1.13. A **uniform space** is a pair $(X, \mathcal{E})$, where X is a set and $\mathcal{E}$ is a family of subsets of $X \times X$, called **entourages of the diagonal**, such that

- every entourage V in $\mathcal{E}$ contains the diagonal $\Delta_X = \{(x,x) : x \in X\}$ of $X \times X$;
- if V is in $\mathcal{E}$ and $V \subseteq U$, then U belongs to $\mathcal{E}$ too;
- if U and V are in $\mathcal{E}$, then so is $U \cap V$;
- if V is in $\mathcal{E}$, then the set $V^{-1} = \{(y,x) \in X \times X : (x,y) \in V\}$ is in $\mathcal{E}$;
- for every V in $\mathcal{E}$, there exists U in $\mathcal{E}$ so that $U^2 = \{(x,y) \in X \times X : \exists z \in X, (x,z) \in U \text{ and } (z,y) \in U\}$ is contained in V;
- the intersection of all the entourages in $\mathcal{E}$ is the diagonal[2].

Such a family $\mathcal{E}$ is called a **uniformity** on X.

We say that a family $\mathcal{B}$ of subsets of $X \times X$ is a **base** for the uniformity $\mathcal{E}$ if for every entourage V in $\mathcal{E}$, there exists U in $\mathcal{B}$ such that $U \subseteq V$. If $\mathcal{B}$ is a base for a uniformity, then it satisfies the following properties.

- If V and W are in $\mathcal{B}$, then there exists U in $\mathcal{B}$ such that $U \subseteq V \cap W$.
- For every V in $\mathcal{B}$, there exists U in $\mathcal{B}$ such that $U^2 \subseteq V$.
- The intersection of all sets in $\mathcal{B}$ is the diagonal.

Conversely, if $\mathcal{B}$ is any family of subsets of $X \times X$ with those properties, then it is the base of a unique uniformity $\mathcal{E}_\mathcal{B}$ on X, defined by

$$V \in \mathcal{E}_\mathcal{B} \Leftrightarrow \exists U \in \mathcal{B}, U \subseteq V.$$

We say that $\mathcal{E}_\mathcal{B}$ is the uniformity **generated** by $\mathcal{B}$.

Intuitively, if (x,y) belongs to an entourage V, it means that x and y are V-close.

EXAMPLE 1.14. The main example is, as expected, given by metric spaces. If (X,d) is a metric space, then the corresponding uniformity $\mathcal{E}_d$ is generated by all sets of the form $\{(x,y) \in X \times X : d(x,y) < \epsilon\}$, where ϵ is a positive real.

[2]We only consider Hausdorff uniform spaces, so we include this condition in the definition.

Conversely, a uniformity naturally induces a topology: if $(X, \mathcal{E})$ is a uniform space and x is an element in X, then a basis of neighborhoods of x is given by the sets $V[x] = \{y : (x, y) \in V\}$, for V in $\mathcal{E}$ (see [**E2**, theorem 8.1.1]).

Proposition 1.15. (See [**E2**, theorem 8.3.13]) Let X be a compact space. Then there exists a unique uniformity on X that induces the topology of X.

Definition 1.16. Let $(X, \mathcal{E})$ and $(Y, \mathcal{F})$ be uniform spaces. Let $f : X \to Y$ be a function. We say that f is **uniformly continuous** if for all V in $\mathcal{F}$, there exists U in $\mathcal{E}$ such that for all x, x' in X, if $(x, x') \in U$, then $(f(x), f(x')) \in V$.

If the uniformities are generated by metrics, then this notion coincides with usual uniform continuity with respect to the metrics.

We now turn to uniform structures on groups. We can equip a topological group with four natural uniformities that induce its topology. Let G be a topological group.

- The **left uniformity** $\mathcal{E}_L$ on G is generated by all entourages of the form
$$\{(g, h) \in G \times G : g^{-1}h \in V\},$$
where V is a neighborhood of the identity in G. Note that all these entourages are invariant under left translation and thus, the group acts uniformly continuously on itself by left translation.
- The **right uniformity** $\mathcal{E}_R$ on G is generated by all entourages of the form
$$\{(g, h) \in G \times G : gh^{-1} \in V\},$$
where V is a neighborhood of the identity in G. These entourages are invariant under right translation. Note also that inversion swaps the left and right uniformities.
- The **two-sided uniformity** $\mathcal{E}_{ts}$ on G is the coarsest common refinement of the left and right uniformities. It is generated by all entourages of the form
$$\{(g, h) \in G \times G : g^{-1}h \in V \text{ and } gh^{-1} \in V\},$$
where V is a neighborhood of the identity in G.
- The **Roelcke unifomity** $\mathcal{E}_{\text{Roelcke}}$ on G is the finest uniformity that is coarser than both the left and right uniformities. It is generated by all entourages of the form
$$\{(g, h) \in G \times G : h \in VgV\},$$
where V is a neighborhood of the identity in G.

The reader will find an extensive study of uniformities on groups in [**R4**].

Left and right uniformly continuous functions on a group will be important in the study of the amenability of a topological group. Indeed, we will make use of the compactification with respect to the right uniformity: the *Samuel compactification* (see chapter 9).

3.2. Metrizable uniformities. If G is a Polish group, then the four uniformities on G defined above are countably generated. This yields that they are induced by a metric.

Theorem 1.17. (Birkhoff [**B**], Kakutani [**K2**]) Let G be group acting on a uniform space $(X, \mathcal{E})$. Assume that the uniformity $\mathcal{E}$ admits a countable generating family of entourages, each of which is invariant under the action of G. Then there exists a metric on X that induces the uniformity $\mathcal{E}$ and that makes the action of G on X isometric.

Proof. The theorem follows from the same construction as in [**E2**, theorem 8.1.21] and [**G1**, theorem 2.1.1]. It is easy to check that it will indeed yield an isometric action. □

In particular, applying the Birkhoff-Kakutani theorem to the action of a Polish group G on itself by left (right) translation, we obtain a left-invariant (right-invariant) metric on G which generates the left (right) uniformity.

If d_L is a left-invariant metric, note that we can also produce a right-invariant metric directly by putting $d_R(g, h) = d_L(g^{-1}, h^{-1})$.

EXAMPLES 1.18.
- If σ and τ are distinct permutations in S_∞, put
$$n(\sigma,\tau) = \min\{n \in \mathbb{N} : \sigma(n) \neq \tau(n)\}.$$
Then
$$d_L(\sigma,\tau) = \begin{cases} 0 & \text{if } \sigma = \tau \\ 2^{-n(\sigma,\tau)} & \text{if } \sigma \neq \tau \end{cases}$$
defines a left-invariant compatible metric on S_∞.
- Let (X,d) be a Polish metric space and let $\{x_n : n \in \mathbb{N}\}$ be a countable dense subset of X. Then the following defines a left-invariant compatible metric on $\mathrm{Iso}(X,d)$:
$$d_L(g,h) = \sum_{n\in\mathbb{N}} \frac{1}{2^n} \min(1, d(g(x_n), h(x_n))).$$

Conversely, every left-invariant compatible metric generates the left uniformity (and similarly on the right).

PROPOSITION 1.19. Let G be group and let d and ρ be two left-invariant metrics that generate the same topology on G. Then they generate the same uniformity on G.

PROOF. Let $V = \{(g,h) \in G \times G : d(g,h) < \epsilon\}$ be a basic entourage of the uniformity induced by d. By left-invariance, we have $V = \{(g,h) \in G \times G : d(1, g^{-1}h) < \epsilon\}$. Now, since the metrics d and ρ induce the same topology on G, there exists a positive δ such that $B_\rho(1,\delta)$ is contained in $B_d(1,\epsilon)$. Thus, the set $U = \{(g,h) \in G \times G : \rho(1, g^{-1}h) < \delta\}$, which by left-invariance of ρ, is equal to the entourage $\{(g,h) \in G \times G : \rho(g,h) < \delta\}$ is contained in V. This completes the proof. $\square$

With Polish groups admitting a left-invariant metric and a complete metric, it is very tempting to believe that they might admit a metric that is *both* complete and left-invariant. However, that is not the case. To see this, we need the following definition.

DEFINITION 1.20. Let $(X,\mathcal{E})$ be a uniform space, with the uniformity $\mathcal{E}$ being countably generated.
- A sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X is **Cauchy** in $(X,\mathcal{E})$ if for every entourage V in $\mathcal{E}$, there exists an integer N such that for all $n,m \geqslant N$, we have $(x_n,x_m) \in V$.
- We say that $(X,\mathcal{E})$ is **complete** if every Cauchy sequence in $(X,\mathcal{E})$ converges to some element of X (for the induced topology).
- The **completion** of the uniform space $(X,\mathcal{E})$ is the quotient space of Cauchy sequences of $(X,\mathcal{E})$ under the equivalence relation
$$(x_n)_{n\in N} \sim (y_n)_{n\in\mathbb{N}} \Leftrightarrow \forall V \in \mathcal{E}, \exists N \in \mathbb{N}, \forall n \geqslant N, (x_n,y_n) \in V.$$

When the uniformity is generated by a metric, it is complete if and only if the metric is complete. In other words, completeness only depends on the generated uniformity. Also, a sequence is Cauchy for a metrizable uniformity if and only if it is Cauchy for one (or equivalently every) metric that generates the uniformity. Thus, as a consequence of proposition 1.19, a Polish group admits a compatible metric that is both complete and left-invariant (we call the group **cli**) if and only if *all* its compatible left-invariant metrics are complete.

NON-EXAMPLE 1.21. The infinite permutation group is not cli, for the left-invariant metric from example 1.18 is not complete. Indeed, for each i, consider the permutation σ_i defined by:
$$\sigma_i(n) = \begin{cases} n+1 & \text{if } n < i \\ 0 & \text{if } n = i \\ n & \text{if } n > i. \end{cases}$$
As $n(\sigma_i,\sigma_j) = \min(i,j)$, the sequence (σ_i) is d_L-Cauchy. In $\mathbb{N}^{\mathbb{N}}$, it converges to the shift, which is not in S_∞.

Actually, the completion of $(S_\infty, \mathcal{E}_L)$ is the set of all injective maps from $\mathbb{N}$ to $\mathbb{N}$.

The previous non-example thus shows that the left uniformity on a Polish group is not in general complete[3]. On the other hand, in the next subsection, we prove that Polish groups are *Rajkov-complete*: complete for the two-sided uniformity.

3.3. Group completion. Let G be a separable topological group and consider the completion $\overline{G}$ with respect to the two-sided uniformity. If d_L is any left-invariant metric on G, the two-sided uniformity on the group G is generated by the following metric:

$$D(g,h) = d_L(g,h) + d_L(g^{-1},h^{-1}).$$

Then $\overline{G}$ is the metric completion of (G,D). Note that $\overline{G}$ does not depend on the choice of the left-invariant compatible metric, by proposition 1.19.

PROPOSITION 1.22. The completion $\overline{G}$ is a topological group: the **group completion**.

PROOF. Inversion is an isometry of (G,D) so it extends to an isometry of the completion $\overline{G}$. It remains to show that if (g_n) and (h_n) are D-Cauchy sequences in G, then so is (g_nh_n). Again, since inversion is an isometry, it suffices to prove that (g_nh_n) is d_L-Cauchy.

Let ϵ be positive. For every n,m,N, we have

$$\begin{aligned} d_L(g_nh_n, g_mh_m) &\leqslant d_L(g_nh_n, g_nh_N) + d_L(g_nh_N, g_mh_N) + d_L(g_mh_N, g_mh_m) \\ &= d(h_n,h_N) + d(g_nh_N, g_mh_N) + d(h_N,h_m). \end{aligned}$$

Since right multiplication by h_N is continuous (and d_L induces the group topology on G), this is smaller than ϵ for large enough n,m,N, so the product of two elements of $\overline{G}$ is well-defined.

Moreover, multiplication in $\overline{G}$ is continuous. To see this, take $\bar{g}$ and $\bar{h}$ in $\overline{G}$, and V an open neighborhood of the identity in G. Let then W be an open neighborhood of the identity such that $W^2 \subseteq V$. Since $\bar{h}$ is in $\overline{G}$, there exists h in G such that $\bar{h} \in hW$. Moreover, since right multiplication by h is continuous in G, there exists an open neighborhood of 1 such that $Uh \subseteq hW$. Finally, let g be an element of G such that $\bar{g} \in gU$. Then, we have

$$\bar{g}\bar{h} \in gUhW \subseteq g(hW)W = ghW^2 \subseteq ghV.$$

Thus, if $\bar{k}$ is in gU and $\bar{l}$ is in hW, then the product $\bar{k}\bar{l}$ will be in the open neighborhood ghV of $\bar{g}\bar{h}$. It follows that multiplication is continuous in $\overline{G}$. Since inversion is an isometry, it is also continuous, hence $\overline{G}$ is a topological group. ☐

COROLLARY 1.23. If G is a Polish group, then $G = \overline{G}$ and thus G is Rajkov-complete.

PROOF. Since G is separable, its group completion $\overline{G}$ is a Polish group. Now, G is a Polish subgroup of $\overline{G}$ so by theorem 1.12, G is closed in $\overline{G}$. But G is dense in its completion, so G is in fact the whole of $\overline{G}$. ☐

3.4. Precompactness.

DEFINITION 1.24. Let $(X,\mathcal{E})$ be a uniform space. We say that $(X,\mathcal{E})$ is **precompact** if for every V in $\mathcal{E}$, there exists a finite subset F of X such that $X = V[F] = \bigcup_{x\in F} V[x]$.

A special class of Polish groups is that of *Roelcke-precompact* ones. A topological group G is said to be **Roelcke-precompact** if the Roelcke uniformity on G is precompact: if for every neighborhood V of 1, there exists a finite subset F of G such that $G = VFV$.

We will see in chapter 6 that Roelcke precompactness is tightly connected to a very important model-theoretic property: *categoricity*. Bridges between model theory and topological group theory in the context of Roelcke-precompact groups are therefore being established (see for instance [**BT1**] and [**I**]).

Note that precompactness with respect to any of the other three uniformities is not relevant a notion for Polish groups, as illustrated by the following proposition.

[3]Since inversion exchanges the left and right uniformities, a topological group is cli if and only if its right uniformity is complete.

PROPOSITION 1.25. Let G be a Polish group. Then the following are equivalent.

(1) The left uniformity on G is precompact: for every neighborhood V of 1, there exists a finite subset F of G such that $G = FV$.

(2) The right uniformity on G is precompact: for every neighborhood V of 1, there exists a finite subset F of G such that $G = VF$.

(3) The two-sided uniformity on G is precompact: for every neighborhood V of 1, there exists a finite subset F of G such that $G = \bigcup_{h \in F} (hV \cap Vh)$.

(4) The group G is compact.

PROOF. (4) $\Rightarrow$ (1),(2),(3)] A compact space admits a unique compatible uniformity (see proposition 1.15). Since the left, right, and two-sided uniformities on G generate the topology on G, this implies that they all coincide and are all precompact.

(3) $\Rightarrow$ (4)] By corollary 1.23, the two-sided uniformity on G is complete. If it is further precompact, then G is compact (by [**E2**, theorem 8.3.16]).

(1),(2) $\Rightarrow$ (4)] See [**S5**, lemma 1.2] or [**BT2**, proposition 4.3]. $\square$

CHAPTER 2

The Urysohn space

À mon avis personnel, c'est un triangle.
Corinne Bouchard[1]

1. Katětov spaces **45**
1.1. One-point metric extensions 45
1.2. Separability 47
2. Tower construction of the Urysohn space **48**

This section is devoted to the flagship space of this thesis. The Urysohn space $\mathbb{U}$ is a **universal** Polish space: it is a complete separable metric space that contains an isometric copy of every (complete) separable metric space. The space $\mathbb{U}$ was built by Urysohn in the early twenties ([**U1**]), but was almost forgotten after that. Indeed, another universal and much more famous Polish space, $\mathcal{C}([0,1],\mathbb{R})$ (Banach-Mazur, see [**B1**] and [**S4**]), put the Urysohn space in the shade for sixty years.

As it turns out, the Urysohn space is remarkable not only for its universality but mainly for its strong homogeneity properties: up to isometry, it is the unique Polish space that is both universal and *ultrahomogeneous.*

Definition 2.1. A metric space X is **ultrahomogeneous** if every isometry between finite subsets of X extends to a global isometry of X.

Ultrahomogeneity is a central theme in our work: we will meet this notion again in chapter 4 in a broader context, study it in more detail in chapter 10, and we will use a number of variations of ultrahomogeneity throughout the thesis.

The Urysohn space attracted renewed interest in the eighties when Katětov ([**K3**]) provided a new construction for it. From this construction, Uspenskij ([**U4**]) proved that not only is $\mathbb{U}$ universal but also its isometry group is a universal Polish group (every Polish group embeds in $\mathrm{Iso}(\mathbb{U})$ as a topological subgroup). In this chapter, we present Katětov's construction of the Urysohn space and explain how it yields the universality of its isometry group.

1. Katětov spaces

1.1. One-point metric extensions. Let X be a metric space.

Definition 2.2. A **Katětov map** on X is a map $f : X \to \mathbb{R}^+$ such that for all x and x' in X, one has

$$|f(x) - f(x')| \leqslant d(x,x') \leqslant f(x) + f(x').$$

A Katětov map corresponds to a one-point metric extension of X: if f is a Katětov map on X, then we can define a metric on $X \cup \{f\}$ that extends the metric on X by putting, for all x in X,

$$d(f,x) = f(x).$$

This will indeed be a metric because Katětov maps are exactly those which satisfy the triangle inequality.

[1] *La vie des charançons est assez monotone*

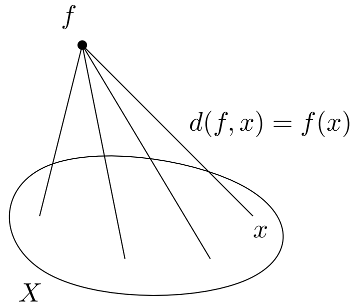


FIGURE 2.1. One-point metric extension of X

EXAMPLE 2.3. If x is a point in X, then the map $\delta_x : X \to \mathbb{R}^+$ defined by $\delta_x(x') = d(x,x')$ is a Katětov map on X. It correspond to a trivial extension of X: we are adding the point x to X.

REMARK 2.4. The condition of being a Katětov map can be rewritten as follows: for all x, x' in X,

$$|f(x) - d(x,x')| \leqslant f(x').$$

We denote by $E(X)$ the space of all Katětov maps on X. We equip the space $E(X)$ with the supremum metric:

$$d(f,g) = \sup_{x \in X} |f(x) - g(x)|.$$

This supremum is always finite: indeed, if f and g are two Katětov maps and x and x_0 are two points in X, we have $|f(x) - d(x,x_0)| \leqslant f(x_0)$ and $|g(x) - d(x,x_0)| \leqslant g(x_0)$ so $|f(x) - g(x)| \leqslant f(x_0) + g(x_0)$, hence the difference between f and g is bounded.

Geometrically, this metric represents the smallest possible distance between the two extension points.

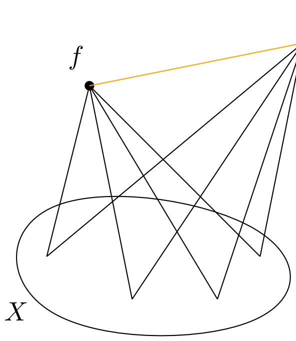


FIGURE 2.2. Distance between two Katětov maps

The maps δ_x of example 2.3 define an isometric embedding of the space X into $E(X)$. We therefore identify X with its image in $E(X)$ via this embedding.

REMARK 2.5. This identification is consistent with the metric we put on $X \cup \{f\}$ before: $d(f, \delta_x) = \sup_{x' \in X} |f(x') - d(x,x')| \leqslant f(x)$, with equality if $x = x'$, so $d(f, \delta_x) = f(x)$.

This observation will allow us to build towers of extensions in the next section. The essential property of those towers is the following.

PROPOSITION 2.6. Every isometry of X extends uniquely to an isometry of $E(X)$.

In particular, the uniqueness implies that the extension defines a group homomorphism from $\mathrm{Iso}(X)$ to $\mathrm{Iso}(E(X))$.

PROOF. Let φ be an isometry of X. If ψ extends φ, we must have $d(\psi(f), \delta_x) = d(f, \delta_{\varphi^{-1}(x)}) = f(\varphi^{-1}(x))$ for all x in X and f in $E(X)$, hence the uniqueness.

Thus, we extend φ to the space $E(X)$ by putting $\psi(f) = f \circ \varphi^{-1}$ for all f in $E(X)$. It is easy to check that the map ψ is an isometry of $E(X)$ that extends φ. □

1.2. Separability. In general, the space $E(X)$ is not separable even if X is, which is unfortunate since we are interested in building Polish spaces. The separability of the Katětov space was extensively studied by Melleray in [**M3**] (see also [**M4**]): he characterized the Polish spaces X for which $E(X)$ is separable as those satisfying the *collinearity property*. From this, he proved that the only Polish spaces on which the isometry group of the Urysohn space acts transitively are compact. We will discuss this more thoroughly in chapter 10.

To circumvent the problem of separability, Katětov considers only Katětov maps with finite support.

DEFINITION 2.7. Let S be a subset of X and let f be a Katětov map on X. We say that S is a **support** for f if for all x in X, we have

$$f(x) = \inf_{y \in S} f(y) + d(x, y).$$

In other words, S is a support for f if the map f is the largest 1-Lipschitz map on X that coincides with f on S. In this case, we also say that f is the **Katětov extension** of $f_{\restriction S}$.

REMARK 2.8. If f and g have a common support S, then the distance between f and g can be expressed in the following way:

$$d(f, g) = \sup_{x \in S} |f(x) - g(x)|.$$

We denote by $E(X, \omega)$ the space of all Katětov maps that admit a finite support[2]. If the metric space X is separable, then $E(X, \omega)$ remains separable. Luckily, in restricting to this separable subspace, we keep the essential properties of the Katětov space.

- The space $E(X, \omega)$ still embeds X isometrically. Indeed, the map δ_x has $\{x\}$ as its (finite!) support.
- Isometries of X still extend uniquely to isometries of $E(X, \omega)$. To see this, note that if φ is an isometry of X and $f \in E(X)$ has support S, then the Katětov map $f \circ \varphi^{-1}$ has support $\varphi(S)$. Thus, the isometry $f \mapsto f \circ \varphi^{-1}$ defined in proposition 2.6 induces an isometry of $E(X, \omega)$.

Furthermore, the extension homomorphism is continuous.

PROPOSITION 2.9. (Uspenskij) If X is separable, then the extension homomorphism from $\mathrm{Iso}(X)$ to $\mathrm{Iso}(E(X, \omega))$ defined above is continuous.

PROOF. Let (φ_n) be a sequence of isometries of X converging to an isometry φ and fix f in $E(X, \omega)$. We show that $f \circ \varphi_n^{-1}$ converges to $f \circ \varphi^{-1}$ in $E(X, \omega)$.

Let ϵ be a positive real and let S be a finite support for f. Since $\varphi_n(S) \cup \varphi(S)$ is a common support for $f \circ \varphi_n^{-1}$ and $f \circ \varphi^{-1}$, remark 2.8 gives that

$$d(f \circ \varphi_n^{-1}, f \circ \varphi^{-1}) = \sup_{x \in \varphi_n(S) \cup \varphi(S)} |f \circ \varphi_n^{-1}(x) - f \circ \varphi^{-1}(x)| \leqslant d(\varphi_n(x), \varphi(x)).$$

But for n large enough, we have that $d(\varphi_n(x), \varphi(x)) < \epsilon$ for all x in S (because S is finite), hence the desired convergence. □

We now take on the construction of the Urysohn space. For a more detailed description of the Katětov spaces, we refer the reader to Melleray's survey [**M4**] on the Urysohn space.

[2]The letter ω is the set-theoretic name for $\mathbb{N}$.

2. Tower construction of the Urysohn space

The construction of the Urysohn space we present highlights its universality: we start with an arbitrary Polish space and we build a copy of the Urysohn space around it. Besides, the construction keeps track of the isometries of the original Polish space, which points to the universality of its isometry group as well.

Let X be our starting Polish space. We build an increasing sequence (X_n) of metric spaces inductively, by setting

- $X_0 = X$;
- $X_{n+1} = E(X_n, \omega)$.

The discussion above guarantees that isometries extend continuously at each step: every isometry of X_n extends to an isometry of X_{n+1} and the extension homomorphism from $\mathrm{Iso}(X_n)$ to $\mathrm{Iso}(X_{n+1})$ is continuous. Thus, if we write $X_\infty = \bigcup_{n\in\mathbb{N}} X_n$, we obtain a continuous extension homomorphism from $\mathrm{Iso}(X)$ to $\mathrm{Iso}(X_\infty)$.

Now, consider the completion $\widehat{X_\infty}$ of X_∞. Since all the X_n are separable, the space $\widehat{X_\infty}$ is Polish. Moreover, isometries of X_∞ extend to isometries of $\widehat{X_\infty}$ by uniform continuity, so we get a continuous extension homomorphism from $\mathrm{Iso}(X)$ to $\mathrm{Iso}(\widehat{X_\infty})$.

It remains to explain why the space $\widehat{X_\infty}$ is the promised ultrahomogeneous and unique Urysohn space. The key defining property of X_∞ is that every one-point metric extension of a finite subset of X_∞ is realized in X_∞ over this finite set.

Definition 2.10. A metric space X is said to have the **Urysohn property** if for every finite subset A of X and every Katětov map $f \in E(A)$, there exists x in X such that for all a in A, we have $d(x, a) = f(a)$.

Let us show that the space $\widehat{X_\infty}$ also has the Urysohn property. We first show that it satisfies a relaxed Urysohn property and then we use completeness to get the full Urysohn property.

Lemma 2.11. Let X be a separable metric space with the Urysohn property. Then the completion $\widehat{X}$ of X satisfies the **approximate Urysohn property**: for every $\epsilon > 0$, every finite subset A of $\widehat{X}$ and every Katětov map f in $E(A)$, there exists y in $\widehat{X}$ such that for all a in A, we have $|d(y, a) - f(a)| < \epsilon$.

Proof. Let ϵ be positive, let $A = \{a_1, ..., a_p\}$ be a finite subset of $\widehat{X}$ and let f be a Katětov map on A. Assume that the elements of A are ordered so that $f(a_1) \geqslant ... \geqslant f(a_p)$ and put $\delta = \min\{d(a_i, a_j) : i \neq j\}$. Pick a positive η such that

$$\begin{cases} (2p+2)\eta & < & \delta \\ (2p+1)\eta & < & \epsilon. \end{cases}$$

By density of X in $\widehat{X}$, there exist $x_1, ..., x_p$ in X such that $d(x_i, a_i) < \eta$ for all i.

Consider the map g defined on $\{x_1, ..., x_p\}$ by $g(x_i) = f(a_i) + 2i\,\eta$. We prove that g is a Katětov map. For $i < j$, we have

$$\begin{aligned} g(x_i) - g(x_j) &= f(a_i) - f(a_j) - 2(j-i)\eta \\ &\leqslant d(a_i, a_j) - 2\eta \\ &\leqslant d(a_i, a_j) - d(x_i, a_i) - d(x_j, a_j) \\ &\leqslant d(x_i, x_j) \end{aligned}$$

and also

$$\begin{aligned} g(x_j) - g(x_i) &= f(a_j) - f(a_i) + 2(j-i)\eta \\ &\leqslant 2p\eta \\ &\leqslant (2p+2)\eta - 2\eta \\ &\leqslant \delta - d(x_i, a_i) - d(x_j, a_j) \\ &\leqslant d(x_i, x_j). \end{aligned}$$

For the other direction, we have

$$\begin{aligned} d(x_i, x_j) &\leqslant d(x_i, a_i) + d(a_i, a_j) + d(a_j, x_j) \\ &\leqslant \eta + (f(a_i) + f(a_j)) + \eta \\ &\leqslant g(x_i) + g(x_j). \end{aligned}$$

Thus, we can apply the Urysohn property to g and obtain a point x in X such that $d(x, x_i) = g(x_i)$ for all i. It follows that

$$\begin{aligned} |d(x, a_i) - f(a_i)| &\leqslant |d(x, a_i) - d(x, x_i)| + |d(x, x_i) - f(a_i)| \\ &\leqslant \eta + |g(x_i) - f(a_i)| \\ &\leqslant \eta + 2i\eta \\ &< \epsilon, \end{aligned}$$

showing that x is as desired. □

Proposition 2.12. Let X be a separable metric space with the Urysohn property. Then the completion $\widehat{X}$ of X also satisfies the Urysohn property.

Proof. By lemma 2.11, the space $\widehat{X}$ satisfies the approximate Urysohn property. Let us prove that it has the full Urysohn property: let A be a finite subset of $\widehat{X}$ and f be a Katětov map on A. Inductively, we build a sequence (x_n) of points of $\widehat{X}$ such that

- for all a in A, we have $|d(x_n, a) - f(a)| < 2^{-n}$;
- $d(x_n, x_{n+1}) < 2^{-(n-1)}$.

The resulting sequence will thus be Cauchy in $\widehat{X}$ and converge, by completeness, to a point x witnessing the Urysohn property for f.

The approximate Urysohn property applied to f directly gives x_0. Assume that x_n has been built. Then the restriction f_n of the Katětov map δ_{x_n} to A satisfies that $d(f, f_n) < 2^{-n}$. Now consider the Katětov map g defined on $A \cup \{x_n\}$ by $g(a) = f(a)$ and $g(x_n) = d(f, f_n)$. We apply the approximate Urysohn property to find a point x_{n+1} in $\widehat{X}$ such that $|d(x_{n+1}, a) - g(a)| = |d(x_{n+1}, a) - f(a)| < 2^{-(n+1)}$ for all a in A and $|d(x_{n+1}, x_n) - g(x_n)| = |d(x_{n+1}, x_n) - d(f, f_n)| < 2^{-(n+1)}$ so $d(x_{n+1}, x_n) < 2^{-n} + 2^{-(n+1)} < 2^{-(n-1)}$. □

We now prove that the Urysohn property yields ultrahomogeneity.

Theorem 2.13. (Urysohn) Let X be a complete separable metric space. If X has the Urysohn property, then X is ultrahomogeneous.

Proof. We carry out a *back-and-forth* argument (we will see plenty of those later on, especially in chapter 4). Let $i : A \to B$ an isometry between two finite subsets of X. Enumerate a dense subset $\{x_n : n \geqslant 1\}$ of X. Recursively, we build finite subsets A_n and B_n of X and isometries $i_n : A_n \to B_n$ such that

- $A_0 = A$ and $B_0 = B$;
- $i_0 = i$;
- $A_n \subseteq A_{n+1}$ and $B_n \subseteq B_{n+1}$;
- $x_n \in A_n \cap B_n$;
- i_{n+1} extends i_n.

To this aim, assume A_n and B_n have been built. Consider the metric extension of A_n by x_{n+1}: the corresponding Katětov map is $\delta_{x_{n+1}}$. We push it forward to a Katětov map on B_n via the isometry i_n. Now, since the space X satisfies the Urysohn property, we can find an element y_{n+1} that realizes it; we add it to B_n and extend i_n by setting $i'_{n+1}(x_{n+1}) = y_{n+1}$. This constitutes the *forth* step.

For the *back* step, we apply the same argument to the inverse of the isometry i'_{n+1} to find a preimage to x_{n+1} (see figure 2).

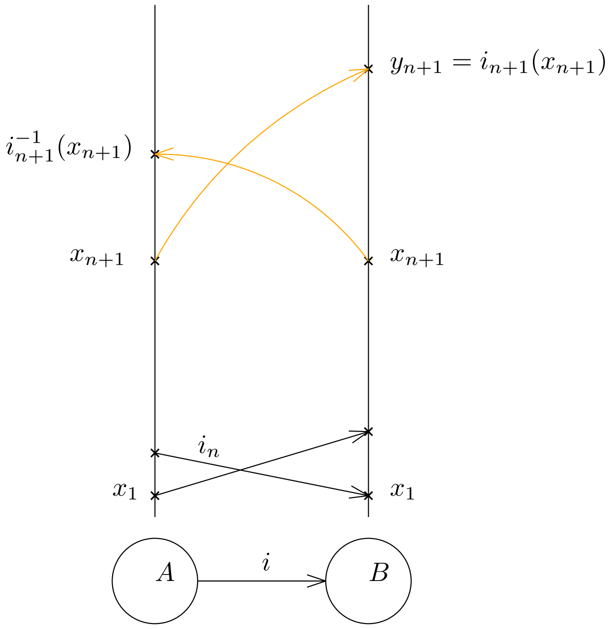


FIGURE 2.3. The back-and-forth argument

In the end, the union of all the isometries i_n defines an isometry of a dense subset of X, so it extends to an isometry of the whole space X (because X is complete). This is the desired extension of i. $\square$

Finally, the same back-and-forth argument shows that any two complete separable metric spaces with the Urysohn property are isometric.

Thus, we may define the Urysohn space to be the space obtained from any Polish space by applying the tower construction above.

DEFINITION 2.14. The Urysohn space $\mathbb{U}$ is the completion of X_∞, with $X = \{0\}$.

This uniqueness result guarantees that $\mathbb{U}$ indeed embeds every Polish space isometrically. Moreover, the construction also yields that its isometry group $\mathrm{Iso}(\mathbb{U})$ embeds all isometry groups of Polish spaces, hence all subgroups thereof. The aforementioned theorem 5.2 states that those actually encompass all Polish groups, so we conclude that $\mathrm{Iso}(\mathbb{U})$ is a universal Polish group.

CHAPTER 3

Model theory

The smaller the bone, the truer the make of the beast.
Robert Bakewell[1]

1.	**Languages and structures**	**51**
1.1.	Classical setting	51
1.2.	Continuous setting	52
1.3.	Parameters	54
2.	**Isomorphisms, automorphism groups**	**54**
3.	**Formulas**	**55**
3.1.	Classical connectives	55
3.2.	Continuous connectives	56
3.3.	Sentences and conditions	57
4.	**Theories and models**	**57**
5.	**Spaces of types**	**58**
5.1.	Types	58
5.2.	The logic topology	58
5.3.	The metric topology	59
6.	**Definability**	**60**

Model theory singles out the concept of an abstract structure and chooses to study these structures from the set of all properties they satisfy. Thereby, from a model-theoretic point of view, more important than actual elements will be their *types*: two elements have the same type if they satisfy the same properties. In order to define the very notion of *property*, we need to establish a precise framework for structures and formulas. When it comes to describing discrete algebraic structures, the framework is well established, and references abound (see for instance [**H1**] and [**TZ1**]). More recently, a model theory more suited for the study of metric structures was developed (see [**BBHU**] and [**BU2**]), generalizing the classical one. Here, we present both frameworks side by side.

1. Languages and structures

1.1. Classical setting.

Definition 3.1. A **(classical) language** is a family $\{R_i, n_i\}_{i\in I} \cup \{F_j, m_j\}_{j\in J}$, where, for all $i \in I$ and $j \in J$,

- n_i and m_j are non-negative integers;
- R_i is a relation symbol of arity n_i;
- F_j is a function symbol of arity m_j.

Function symbols of arity 0 will be called **constant symbols**.

Throughout the text, all languages will be countable.

Definition 3.2. Let $\mathcal{L} = \{R_i, n_i\}_{i\in I} \cup \{F_j, m_j\}_{j\in J}$ be a language. An **$\mathcal{L}$-structure M** is a set M (the **universe** of **M**) endowed with

[1]as quoted in [**D**].

- a subset $R_i^{\mathbf{M}}$ of M^{n_i} for every i in I;
- a map $F_j^{\mathbf{M}} : M^{m_j} \to M$ for every j in J.

CONVENTION 3.3. We always assume that languages contain a symbol $=$ that is interpreted as the equality in every structure. Thus, the **empty language** consists only of equality.

EXAMPLES 3.4. Here are a few examples of languages and structures.

- Structures in the empty language are pure sets, with no additional structure. For instance, the set $(\mathbb{N}, =)$ is an $\emptyset$-structure.
- The rationals can be endowed with many structures:
  - a group structure, in the language $\mathcal{L}_G = \{0, +, -\}$ of groups;
  - a ring structure, in the language $\mathcal{L}_R = \{0, 1, +, -, \times\}$ of rings;
  - an order structure, in the language $\mathcal{L}_{Or} = \{<\}$ of ordered sets.

  From the model-theoretic viewpoint, $(\mathbb{Q}, 0, +, -)$, $(\mathbb{Q}, 0, 1, +, -, \times)$ and $(\mathbb{Q}, <)$ really are different structures.
- Let $\mathcal{L}_P = \{P\}$ be a language consisting of a single unary relation. Define $P^{\mathbb{Q}}$ by $x \in P^{\mathbb{Q}} \Leftrightarrow x > 0$. Then $(\mathbb{Q}, P^{\mathbb{Q}})$ is an $\mathcal{L}_P$-structure.
- Graphs can be described with several languages:
  - the language $\mathcal{L}_{Gr} = \{E\}$ consisting of one binary symbol for the edge relation;
  - the language $\mathcal{L}_{GrMet} = \{d_n : n \in N\}$, where d_n is a binary relation for being at distance n in the graph metric.

DEFINITION 3.5. Let $\mathcal{L} = \{R_i, n_i\}_{i \in I} \cup \{F_j, m_j\}_{j \in J}$ be a language and let $\mathbf{M}$ be an $\mathcal{L}$-structure. A **substructure** of $\mathbf{M}$ is an $\mathcal{L}$-structure $\mathbf{N}$ such that

- the universe N of $\mathbf{N}$ is a subset of the universe of $\mathbf{M}$;
- $R_i^{\mathbf{N}} = R_i^{\mathbf{M}} \cap N^{n_i}$, for every i in I;
- $F_j^{\mathbf{N}}(\bar{x}) = F_j^{\mathbf{M}}(\bar{x})$, for every j in I and $\bar{x}$ in N^{m_j}.

DEFINITION 3.6. Let $\mathcal{L}$ be a classical language and let $\mathbf{M}$ be an $\mathcal{L}$-structure. Let A be a subset of M. The substructure of $\mathbf{M}$ **generated** by A is the smallest substructure of $\mathbf{M}$ that contains A.

DEFINITION 3.7. Let $\mathcal{L}$ be a language and let R be a relation symbol in $\mathcal{L}$ of arity n. Let $\mathbf{M}$ be an $\mathcal{L}$-structure and let $\bar{a}$ be a tuple in M^n. We say that $\mathbf{M}$ **satisfies** $R(\bar{a})$, and we write $\mathbf{M} \vDash R(\bar{a})$, if $\bar{a} \in R^{\mathbf{M}}$. We also say that $\bar{a}$ **satisfies** R or that $R(\bar{a})$ is **true** in $\mathbf{M}$.

1.2. Continuous setting. It seems rather tempting to say that in the structure $(\mathbb{Q}, Pos^{\mathbb{Q}})$ of example 3.4, $P(-2)$ is *less false* than $P(-1000)$, for instance. Yet, in a classical structure, there is no indication of how badly an element can fail to satisfy a relation, no measure of *how far* an element is from lying in a relation. This suggests the idea of distance and incites us to replace equality by a metric and the usual true and false by a continuum of *truth values*, a whole spectrum of maybes. Pursuing that line of thought, Ben Yaacov, Berenstein, Henson and Usvyatsov ([**BBHU**]) introduced a model theory for metric structures.

DEFINITION 3.8. A **continuous language** is a family $\{P_i, n_i, \kappa_i\}_{i \in I} \cup \{F_j, m_j, \Lambda_j\}_{j \in J}$, where, for all $i \in I$ and $j \in J$,

- n_i and m_j are non-negative integers;
- κ_i and Λ_j are non-negative reals;
- P_i is a predicate of arity n_i;
- F_j is a function symbol of arity m_j.

Sometimes we call predicates relation symbols by analogy with the classical setting.

As was the case for classical languages, all our continuous languages will be countable.

DEFINITION 3.9. Let $\mathcal{L} = \{P_i, n_i, \kappa_i\}_{i \in I} \cup \{F_j, m_j, \Lambda_j\}_{j \in J}$ be a continuous language. An **$\mathcal{L}$-structure $\mathbf{M}$** is a complete metric space (M, d) of diameter bounded by 1 endowed with

- a κ_i-Lipschitz predicate $P_i^{\mathbf{M}} : M^{n_i} \to [0,1]$ for every i in I;
- a Λ_j-Lipschitz map $F_j^{\mathbf{M}} : M^{m_j} \to M$ for every j in J.

Structures in a continuous language are called **metric structures**.

Convention 3.10. Like for classical languages, we always assume that continuous languages contain a symbol d for the metric.

Remarks 3.11.

- One may only ask that the predicates and functions are uniformly continuous, in which case the language must specify a modulus of uniform continuity for each symbol. This would not change the associated model theory too much though, for definable predicates would remain unchanged. Indeed, definable predicates will be defined as uniform limits of formulas (see section 6) and Lipschitz functions are uniformly dense in uniformly continuous ones (see the proof of proposition 9.17).
- The requirement that metric spaces must be bounded allows one to carry arguments of compactness. In particular, it guarantees that the space of types will be compact in the logic topology (theorem 3.51). However, we will conveniently drop this assumption whenever those compactness considerations are irrelevant.
- Note that unbounded metric spaces can be made bounded by replacing the metric with the following equivalent one
$$d'(x,y) = \frac{d(x,y)}{1+d(x,y)},$$
which is bounded by 1. We will especially apply this to the Urysohn space in chapters 6 and 8, where the boundedness requirement applies.

Example 3.12. Any classical structure becomes a metric structure when endowed with the discrete metric. Relations are identified with their indicator functions (that take their values in $\{0,1\}$). Note that this identification is somehow unusual, as *true* corresponds to 0 and *false* corresponds to 1.

From now on, we will state our definitions only for metric structures, with classical structures being an important particular case (where the motivation comes from). We shall still emphasize the different behaviors of the two settings whenever such differences occur.

Examples 3.13. Here are a few examples of metric structures, both bounded and unbounded.

- Complete metric spaces, with no additional structure, are metric structures in the empty continuous language.
- The measure algebra of the unit interval with its Lebesgue measure μ, denoted by MALG(μ), is a metric structure in the language $\mathcal{L}_{Bool} = \{0, 1, \cup, \cap, \mu\}$. The metric is given by $d(A,B) = \mu(A \triangle B)$.
- The Hilbert space ℓ^2 is a metric structure in the language $\mathcal{L}_{Hilb} = \{0, +, -, \langle\cdot\rangle, (m_\lambda)_{\lambda\in\mathbb{R}}\}$. Since it is not bounded, the structure we often consider instead is its unit ball, which captures all the relevant information about the Hilbert space.

Definition 3.14. Let $\mathcal{L} = \{P_i, n_i, \kappa_i\}_{i\in I} \cup \{F_j, m_j, \Lambda_j\}_{j\in J}$ be a continuous language and let $\mathbf{M}$ be a metric $\mathcal{L}$-structure. A **substructure** of $\mathbf{M}$ is an $\mathcal{L}$-structure $\mathbf{N}$ such that

- the universe (N, d_N) of $\mathbf{N}$ is a (necessarily closed) metric subspace of the universe (M, d_M) of $\mathbf{M}$;
- $P_i^{\mathbf{N}}(\bar{x}) = P_i^{\mathbf{M}}(\bar{x})$, for every i in I and $\bar{x}$ in N^{n_i};
- $F_j^{\mathbf{N}}(\bar{x}) = F_j^{\mathbf{M}}(\bar{x})$, for every j in I and $\bar{x}$ in N^{m_j}.

Definition 3.15. Let $\mathcal{L}$ be a continuous language and let $\mathbf{M}$ be an $\mathcal{L}$-structure. Let A be a subset of M. The substructure **generated** by A is the smallest substructure of $\mathbf{M}$ that contains A.

1.3. Parameters. Very often, we want to include some elements of a given structure in the language, treating them as parameters (for instance, to talk about types over a set).

If $\mathbf{M}$ is an $\mathcal{L}$-structure and X is a countable subset of M, then we define the language $\mathcal{L}_X$ as follows. For each element x of X, we add a constant symbol c_x to the language $\mathcal{L}$, so that $\mathcal{L}_X = \mathcal{L} \cup \{c_x : x \in X\}$.

This way, $\mathbf{M}$ naturally becomes an $\mathcal{L}_X$-structure: we interpret the symbol c_x as x. We denote this new structure by $(\mathbf{M}, X)$.

2. Isomorphisms, automorphism groups

DEFINITION 3.16. Let $\mathcal{L}$ be a continuous language. Let $\mathbf{M}$ and $\mathbf{N}$ be two metric $\mathcal{L}$-structures. An **embedding** of $\mathbf{N}$ into $\mathbf{M}$ is a map $f : \mathbf{N} \to \mathbf{M}$ such that

- for every predicate P in $\mathcal{L}$ of arity n and every tuple $(x_1, ..., x_n)$ in N^n, we have
$$P^{\mathbf{N}}(x_1, ..., x_n) = P^{\mathbf{M}}(f(x_1), ..., f(x_n));$$
- for every function symbol F in $\mathcal{L}$ of arity m and every tuple $(x_1, ..., x_m)$ in N^m, we have
$$f(F^{\mathbf{N}}(x_1, ..., x_n)) = F^{\mathbf{M}}(f(x_1), ..., f(x_m)).$$

REMARKS 3.17.
- Since the language contains the metric, embeddings are isometric. In particular, they are injective.
- If $f : \mathbf{N} \to \mathbf{M}$ is an embedding, then the image of $\mathbf{N}$ by f is a substructure of $\mathbf{M}$.

DEFINITION 3.18. An **isomorphism** is a surjective embedding.

DEFINITION 3.19. Let $\mathbf{M}$ be a metric structure. The **automorphism group** of $\mathbf{M}$, denoted by $\mathrm{Aut}(\mathbf{M})$, is the group of all self-isomorphisms of $\mathbf{M}$. We endow it with the topology of pointwise convergence.

EXAMPLE 3.20. The automorphism group of the pure set $\mathbb{N}$ is S_∞, as presented in example 1.9.

The topology on $\mathrm{Aut}(\mathbf{M})$ is generated by all sets of the form
$$\{g \in \mathrm{Aut}(\mathbf{M}) : d(g(\bar{a}), \bar{b}) < \epsilon\},$$
where $\bar{a}$, $\bar{b}$ are tuples in M and $\epsilon > 0$.

REMARK 3.21. If $\mathbf{M}$ is a classical structure, the topology on $\mathrm{Aut}(\mathbf{M})$ is the *permutation group topology*, generated by all sets of the form
$$\{g \in \mathrm{Aut}(\mathbf{M}) : g(\bar{a}) = \bar{b}\},$$
where $\bar{a}$, $\bar{b}$ are tuples in M.

Equivalently, the topology on $\mathrm{Aut}(\mathbf{M})$ is generated by all sets of the form
$$\{g \in \mathrm{Aut}(\mathbf{M}) : g \text{ extends } f\},$$
where f is an isomorphism between two finitely generated substructures of $\mathbf{M}$.

PROPOSITION 3.22. Let $\mathbf{M}$ be a metric structure. Then the automorphism group of $\mathbf{M}$ is a closed subgroup of $\mathrm{Iso}(M, d)$. In particular, if $\mathbf{M}$ is a countable classical structure, then $\mathrm{Aut}(\mathbf{M})$ is a closed subgroup of S_∞.

PROOF. Preserving the values of continuous predicates is a closed condition for the pointwise topology, so $\mathrm{Aut}(\mathbf{M})$ is closed in $\mathrm{Iso}(M, d)$. □

COROLLARY 3.23. If $\mathbf{M}$ is a separable metric structure, then $\mathrm{Aut}(\mathbf{M})$ is a Polish group.

Our leitmotiv (chapter 5) is that the converse of these two results also holds.

3. Formulas

Formulas are expressions, depending on variables, that can be written within the language. They are built inductively as follows. We fix a countable set $\mathcal{V}$ of variables beforehand.

Definition 3.24. Let $\mathcal{L}$ be a language. The class of **$\mathcal{L}$-terms** is the smallest class of expressions such that

- constant symbols in $\mathcal{L}$ are $\mathcal{L}$-terms;
- variables in $\mathcal{V}$ are $\mathcal{L}$-terms;
- if F is an m-ary function symbol in $\mathcal{L}$ and if $t_1, ..., t_m$ are $\mathcal{L}$-terms, then the expression $F(t_1, ..., t_m)$ is an $\mathcal{L}$-term.

Thus, each term depends on a finite number of variables. In any structure, terms can be interpreted by elements of the structure, by substituting a finite number of elements to the variables.

Definition 3.25. Let $\mathcal{L}$ be a language and let $\mathbf{M}$ be an $\mathcal{L}$-structure. Let $t(\bar{x})$ be an $\mathcal{L}$-term that depends on a finite tuple $\bar{x} = (x_1, ..., x_n)$ of variables. Let $\bar{a} = (a_1, ..., a_n)$ be a tuple of elements of M. By induction on the complexity of the term t, we define the element $t(\bar{a})$ of M.

- If $t(\bar{x})$ is a constant symbol c, then $t(\bar{a}) = c^{\mathbf{M}}$.
- If $t(\bar{x})$ is a variable x_i, then $t(\bar{a}) = a_i$.
- If $t(\bar{x}) = F(t_1(\bar{x}), ..., t_m(\bar{x}))$, where F is an m-ary function in $\mathcal{L}$ and $t_1(\bar{x}), ..., t_m(\bar{x})$ are $\mathcal{L}$-terms depending on $\bar{x}$, then
$$t(\bar{a}) = F^{\mathbf{M}}(t_1(\bar{a}), ..., t_m(\bar{a})).$$

Definition 3.26. Let $\mathcal{L}$ be a language. An **atomic $\mathcal{L}$-formula** is an expression of the form $P(t_1, ..., t_n)$, where P is an n-ary relation or predicate symbol in $\mathcal{L}$ and $t_1, ..., t_n$ are $\mathcal{L}$-terms.

Remark 3.27. If $\mathcal{L}$ is a classical language, it contains a relation symbol for equality, so in particular, if t_1 and t_2 are $\mathcal{L}$-terms, then the expression $t_1 = t_2$ is an atomic $\mathcal{L}$-formula.

Similarly, if $\mathcal{L}$ is a continuous language, there is a predicate for the metric, so if t_1 and t_2 are $\mathcal{L}$-terms, then the expression $d(t_1, t_2)$ is an atomic $\mathcal{L}$-formula.

In order to get the class of all formulas, we need to connect atomic formulas together. Connectives differ significantly from the classical to the metric case, so we will deal with the two settings separately.

3.1. Classical connectives. The connectives we use in the classical setting are the usual logical operations: conjunction ($\wedge$), disjunction ($\vee$) and negation ($\neg$), as well as the universal and existential quantifiers: $\forall$ and $\exists$.

Definition 3.28. Let $\mathcal{L}$ be a classical language. The class of $\mathcal{L}$-formulas is the smallest class such that

- atomic $\mathcal{L}$-formulas are $\mathcal{L}$-formulas;
- if φ and ψ are two $\mathcal{L}$-formulas, then the expressions $\varphi \wedge \psi$ and $\varphi \vee \psi$ are $\mathcal{L}$-formulas;
- if φ is an $\mathcal{L}$-formula, then the expression $\neg\varphi$ is an $\mathcal{L}$-formula;
- if φ is an $\mathcal{L}$-formula and x is a variable in $\mathcal{V}$, then the expressions $\forall x, \varphi$ and $\exists x, \varphi$ are $\mathcal{L}$-formulas.

Definition 3.29. Let φ be an $\mathcal{L}$-formula. Let x be a variable in $\mathcal{V}$ that appears in the expression of φ. We say that x is **bound** in φ if it lies in a subformula of φ of the form $\forall x, \psi$ or $\exists x, \psi$. Otherwise, we say that x is a **free variable** of φ.

We will say that a formula **depends** on its free variables.

Each formula also contains a finite number of variables. Formulas are then interpreted in any structure in the obvious way, by substituting elements of the structure to their free variables. The formulas, with parameters, that are obtained this way can be either true or false.

DEFINITION 3.30. Let $\mathcal{L}$ be a classical language and $\mathbf{M}$ be an $\mathcal{L}$-structure. Let $\varphi(\bar{x})$ be an $\mathcal{L}$-formula depending on the tuple of variables $\bar{x} = (x_1, ..., x_k)$. Let $\bar{a} = (a_1, ..., a_k)$ be a tuple of elements of M. By induction on the complexity of the formula φ, we define what it means that $\mathbf{M}$ **satisfies** $\varphi(\bar{a})$, denoted $\mathbf{M} \vDash \varphi(\bar{a})$.

- If $\varphi(\bar{x}) = R(t_1(\bar{x}), ..., t_n(\bar{x}))$, where R is an n-ary relation in $\mathcal{L}$ and $t_1(\bar{x}), ..., t_n(\bar{x})$ are $\mathcal{L}$-terms depending on $\bar{x}$, then
$$\mathbf{M} \vDash \varphi(\bar{a}) \Leftrightarrow \mathbf{M} \vDash R(t_1(\bar{a}), ..., t_n(\bar{a})).$$
- If $\varphi(\bar{x}) = \varphi_1(\bar{x}) \wedge \varphi_2(\bar{x})$, then
$$\mathbf{M} \vDash \varphi(\bar{a}) \Leftrightarrow [\mathbf{M} \vDash \varphi_1(\bar{a}) \text{ and } \mathbf{M} \vDash \varphi_2(\bar{a})].$$
- If $\varphi(\bar{x}) = \varphi_1(\bar{x}) \vee \varphi_2(\bar{x})$, then
$$\mathbf{M} \vDash \varphi(\bar{a}) \Leftrightarrow [\mathbf{M} \vDash \varphi_1(\bar{a}) \text{ or } \mathbf{M} \vDash \varphi_2(\bar{a})].$$
- If $\varphi(\bar{x}) = \neg\psi(\bar{x})$, then
$$\mathbf{M} \vDash \varphi(\bar{a}) \Leftrightarrow \mathbf{M} \nvDash \psi(\bar{a}).$$
- If $\varphi(\bar{x}) = \forall y, \psi(x_1, ..., x_k, y)$, then
$$\mathbf{M} \vDash \varphi(\bar{a}) \Leftrightarrow [\forall b \in \mathbf{M}, \mathbf{M} \vDash \psi(a_1, ..., a_k, b)].$$
- If $\varphi(\bar{x}) = \exists y, \psi(x_1, ..., x_k, y)$, then
$$\mathbf{M} \vDash \varphi(\bar{a}) \Leftrightarrow [\exists b \in \mathbf{M}, \mathbf{M} \vDash \psi(a_1, ..., a_k, b)].$$

3.2. Continuous connectives. In the continuous setting, we use a much broader class of connectives: all (uniformly) continuous functions.

DEFINITION 3.31. Let $\mathcal{L}$ be a continuous language. The class of $\mathcal{L}$-formulas is the smallest class such that

- atomic $\mathcal{L}$-formulas are $\mathcal{L}$-formulas;
- if $u : [0,1]^n \to [0,1]$ is a continuous function and $\varphi_1, ..., \varphi_n$ are $\mathcal{L}$-formulas, then the expression $u(\varphi_1, ..., \varphi_n)$ is an $\mathcal{L}$-formula;
- if φ is an $\mathcal{L}$-formula and x is a variable in $\mathcal{V}$, then the expressions $\sup_x \varphi$ and $\inf_x \varphi$ are $\mathcal{L}$-formulas.

The notions of bound and free variables are defined as in the classical case.

DEFINITION 3.32. Let φ be an $\mathcal{L}$-formula. Let x be a variable in $\mathcal{V}$ that appears in the expression of φ. We say that x is **bound** in φ if it lies in a subformula of φ of the form $\sup_x \psi$ or $\inf_x \psi$. Otherwise, we say that x is a **free variable** of φ.

We will say that a formula **depends** on its free variables.

Again in the continuous setting, each formula contains a finite number of variables and a formula is interpreted in any structure by substituting elements of the structures to its free variables. However, this time, the interpreted formulas take their values in $[0,1]$.

DEFINITION 3.33. Let $\mathcal{L}$ be a continuous language and $\mathbf{M}$ be an $\mathcal{L}$-structure. Let $\varphi(\bar{x})$ be an $\mathcal{L}$-formula depending on the tuple of variables $\bar{x} = (x_1, ..., x_k)$. Let $\bar{a} = (a_1, ..., a_k)$ be a tuple of elements of M. By induction on the complexity of the formula φ, we define the **value** $\varphi^{\mathbf{M}}(\bar{a})$ of $\varphi(\bar{a})$ in the structure $\mathbf{M}$.

- If $\varphi(\bar{x}) = P(t_1(\bar{x}), ..., t_n(\bar{x}))$, where P is an n-ary predicate in $\mathcal{L}$ and $t_1(\bar{x}), ..., t_n(\bar{x})$ are $\mathcal{L}$-terms depending on $\bar{x}$, then
$$\varphi^{\mathbf{M}}(\bar{a}) = P^{\mathbf{M}}(t_1(\bar{a}), ..., t_n(\bar{a})).$$
- If $\varphi(\bar{x}) = u(\varphi_1(\bar{x}), ..., \varphi_n(\bar{x}))$, where $u : [0,1]^n \to [0,1]$ is a continuous function, then
$$\varphi^{\mathbf{M}}(\bar{a}) = u(\varphi_1^{\mathbf{M}}(\bar{a}), ..., \varphi_n^{\mathbf{M}}(\bar{a})).$$

- If $\varphi(\bar{x}) = \sup_y \psi(x_1, ..., x_n, y)$, then
$$\varphi^{\mathbf{M}}(\bar{a}) = \sup_{b \in \mathbf{M}} \psi^{\mathbf{M}}(a_1, ..., a_n, b).$$
- If $\varphi(\bar{x}) = \inf_y \psi(x_1, ..., x_n, y)$, then
$$\varphi^{\mathbf{M}}(\bar{a}) = \inf_{b \in \mathbf{M}} \psi^{\mathbf{M}}(a_1, ..., a_n, b).$$

If $\varphi^{\mathbf{M}}(\bar{a}) = r$, we say that **M satisfies** that $\varphi(\bar{a}) = r$, and we write $\mathbf{M} \vDash \varphi(\bar{a}) = r$.

REMARK 3.34. The interpretation $\varphi^{\mathbf{M}} : M^k \to [0, 1]$ is uniformly continuous.

As hinted at in subsection 1.2, the intuition as for the truth of a continuous formula is: the smaller, the truer. Therefore, the continuous quantifiers sup and inf can be viewed as analogues of the classical ones $\forall$ and $\exists$. However, note that these analogues are in no way canonical: this interprets conjunction as a maximum, but it could as well be represented as a sum, for instance. As for negation, it does not —and should not — have any satisfying continuous analogue. See **[BM1]** for a more extensive discussion about the continuous analogues of Boolean operations (in the context of grey sets).

REMARK 3.35. The automorphism group preserves the interpretations of formulas: if $\varphi^{\mathbf{M}}(\bar{a}) = r$ and g is an automorphism of $\mathbf{M}$, then $\varphi^{\mathbf{M}}(g(\bar{a})) = r$ too.

3.3. Sentences and conditions.

DEFINITION 3.36. Let $\mathcal{L}$ be a language. An **$\mathcal{L}$-sentence** is an $\mathcal{L}$-formula with no free variable.

REMARK 3.37. If $\mathbf{M}$ is an $\mathcal{L}$-structure and φ is an $\mathcal{L}$-sentence, then the interpretation $\varphi^{\mathbf{M}}$ of φ in $\mathbf{M}$ is constant. In the classical setting, a sentence is either true in $\mathbf{M}$ or false in $\mathbf{M}$.

DEFINITION 3.38. Let $\mathcal{L}$ be a continuous language. An **$\mathcal{L}$-condition** is an expression of the form $\varphi(\bar{x}) = r$, where $\varphi(\bar{x})$ is an $\mathcal{L}$-formula and r is a real in $[0, 1]$.

DEFINITION 3.39. Let $\mathcal{L}$ be a continuous language. A closed **$\mathcal{L}$-condition** is an expression of the form $\varphi = r$, where φ is an $\mathcal{L}$-sentence and r is a real in $[0, 1]$.

REMARKS 3.40.
- In the classical case, the only relevant conditions are of the form "$\varphi(\bar{x})$ is true" and "$\varphi(\bar{x})$ is false". So we will identify formulas and conditions.
- Note that an expression of the form $\varphi \leqslant r$ can also be seen as condition, for it is a rewriting of the condition $\max(0, \varphi - r) = 0$.

The semantics of conditions is defined the natural way, as in definition 3.33.

4. Theories and models

DEFINITION 3.41. Let $\mathcal{L}$ be a language. An **$\mathcal{L}$-theory** is a set of closed $\mathcal{L}$-conditions.

An important example of theory is the theory of a structure: given a structure, look at all the closed conditions it satisfies.

DEFINITION 3.42. Let $\mathcal{L}$ be a language and let $\mathbf{M}$ be an $\mathcal{L}$-structure. The **theory** of $\mathbf{M}$, denoted $\mathrm{Th}(\mathbf{M})$, is the set of all closed $\mathcal{L}$-conditions satisfied in $\mathbf{M}$. Any theory of that form is called a **complete** theory.

Dually, given a theory, we can consider all structures which satisfy this theory.

DEFINITION 3.43. Let $\mathcal{L}$ be a language and let T be an $\mathcal{L}$-theory. A **model** of T is an $\mathcal{L}$-structure which satisfies every closed $\mathcal{L}$-condition in T. In other words, an $\mathcal{L}$-structure $\mathbf{M}$ is a model of T if and only if $T \subseteq \mathrm{Th}(\mathbf{M})$.

Example 3.44. The structure $(\mathbb{Q}, <)$ is a linear order, so its theory contains the closed condition $\forall x, \forall y, [(x < y) \vee (y < x) \vee (x = y)]$. Hence, all models of the theory of $(\mathbb{Q}, <)$ are linear orders. Similarly, all models of $\mathrm{Th}(\mathbb{Q}, <)$ are dense orders without endpoints. In fact, $(\mathbb{Q}, <)$ is the only countable dense linear order without endpoints, up to isomorphism: it is the only countable model of its theory (see chapter 6).

5. Spaces of types

Let $\mathcal{L}$ be a continuous language and let T be a complete $\mathcal{L}$-theory. Let $\mathbf{M}$ be a model of T and let X be a countable subset of M. Moreover, let T_X be the $\mathcal{L}_X$-theory of $(\mathbf{M}, X)$.

5.1. Types. The type of an element over the set X is everything the language can say, using parameters from X, to describe this element.

Definition 3.45. Let $\mathbf{N}_X$ be a model of T_X and let $\bar{a} = (a_1, ..., a_n)$ be a tuple in N_X. The **type** of $\bar{a}$ over X in $\mathbf{N}_X$ is the set of all $\mathcal{L}_X$-conditions of the form $\varphi(x_1, ..., x_n) = r$, where $\varphi(x_1, ..., x_n)$ is an $\mathcal{L}_X$-formula with free variables among $x_1, ..., x_n$, such that $\mathbf{N}_X \vDash \varphi(a_1, ..., a_n) = r$. We denote it $\mathrm{tp}(\bar{a}/X)$.

Remark 3.46. Let $\mathbf{N}_X$ be a model of T_X. Let g be an automorphism of $\mathbf{N}_X$ (g corresponds to an automorphism of $\mathbf{N}$ that fixes X pointwise). Then for every tuple $\bar{a}$ in N_X, the tuples $\bar{a}$ and $g(\bar{a})$ have the same type over X in $\mathbf{N}_X$. We will see in chapter 4 classes of structures in which the converse is true.

Types are all sets of conditions of the above form.

Definition 3.47. Let p be a set of $\mathcal{L}_X$-conditions with free variables among $x_1, ..., x_n$. The set p is an n**-type** over X if there exists a model $\mathbf{N}_X$ of T_X and a tuple $\bar{a}$ in N_X^n such that $p = \mathrm{tp}(\bar{a}/X)$.

We then say that the tuple $\bar{a}$ is a **realization** of p in the model $\mathbf{N}_X$.

The space of all n-types over X is denoted by $S_n(X)$, the Stone space of X.

Remark 3.48. Similarly, we can define **quantifier-free types**, as restrictions of types to formulas that do not contain any quantifier. When two tuples have the same quantifier-free type, it means that there exists an isomorphism between the structures they generate.

Example 3.49. In the Urysohn space, the Katětov space $E(X)$ is the space of quantifier-free 1-types over X. Actually, since the theory of $\mathbb{U}$ eliminates quantifiers, $E(X)$ is the space of all 1-types over X.

The space of types can be endowed with two different topologies, each of which enjoys nice properties. In the next two subsections, we go over those two topologies and explain how they intertwine.

5.2. The logic topology. The first topology, called the *logic topology*, is designed to deal with questions of *satisfiability* of sets of conditions. The powerful compactness theorem ([**BBHU**, corollary 5.12]) says that it suffices for every finite subset of conditions to be satisfiable (in the continuous setting, up to an arbitrarily small error) for all the conditions in the set to be satisfiable at the same time. As the name suggests, a restatement of this result is that the space of types, endowed with this topology, is compact.

Definition 3.50. Assume that $\mathcal{L}$ is a classical language. The **logic topology** on $S_n(X)$ is defined as follows. If p is an n-type over X, a basis of neighborhoods of p is given by all sets of the form

$$[\varphi] = \{q \in S_n(X) : \varphi \in q\},$$

where φ is an $\mathcal{L}_X$-condition contained in p.

Theorem 3.51. (See [**BBHU**, proposition 8.6]) Assume that $\mathcal{L}$ is a classical language. Then the space $S_n(X)$ is compact Hausdorff for the logic topology.

The logic topology has a natural continuous analogue. Assume now $\mathcal{L}$ is a continuous language.

DEFINITION 3.52. Let φ be a $\mathcal{L}_X$-formula with free variables among $x_1, ..., x_n$. Let also ϵ be a positive real. We define the set $[\varphi < \epsilon]$ as follows:

$$[\varphi < \epsilon] = \{q \in S_n(X) : \exists \delta \in [0, \epsilon[, \text{ the } \mathcal{L}_X\text{-condition } \varphi \leqslant \delta \text{ is in } q\}.$$

DEFINITION 3.53. The **logic topology** on $S_n(X)$ is defined as follows. If p is an n-type over X, a basis of neighborhoods of p is given by all sets of the form $[\varphi < \epsilon]$, where $\varphi = 0$ is an $\mathcal{L}_X$-condition contained in p and ϵ is a positive real.

THEOREM 3.54. (See [**BBHU**, proposition 8.6]) The space $S_n(X)$ is compact Hausdorff for the logic topology.

5.3. The metric topology. In the continuous setting, there is another natural topology to consider on the space of types, accounting for the distance between realizations of types.

Fix a model $\mathbf{N}_X$ of T_X in which every type over X is realized. Such a model exists (see [**BBHU**, proposition 7.6]) and is called **sufficiently saturated**. We are going to define a metric in this fixed model. Several results in the rest of this chapter will use more sophisticated model-theoretic tools, such as saturation, that are not needed to present the work of this thesis. Whenever such is the case, we will give a reference for the proof, together with some of the ideas involved.

DEFINITION 3.55. Let p and q be two types in $S_n(X)$. We define

$$d(p, q) = \inf\{\max_{1 \leqslant i \leqslant n} d(a_i, b_i) : \bar{a} \text{ and } \bar{b} \text{ are realizations of } p \text{ and } q \text{ in } \mathbf{N}_X \text{ respectively }\}.$$

It can be shown that this number does not depend on our choice of a sufficiently saturated model.

PROPOSITION 3.56. The map d defines a metric on the space $S_n(X)$.

PROOF. It is easy to see that d is a pseudometric on $S_n(X)$. To prove that d is a metric, we use the compactness theorem and the sufficient saturation of $\mathbf{N}_X$: see [**BBHU**, page 44]. □

REMARK 3.57. In the case when the language is classical, this metric is discrete.

THEOREM 3.58. The metric space $(S_n(X), d)$ is complete.

PROOF. We choose a sufficiently saturated model that has homogeneity properties to build limits of Cauchy sequences. See [**BBHU**, proposition 8.8]. □

PROPOSITION 3.59. The metric topology on $S_n(X)$ is finer than the logic topology.

PROOF. That follows from the uniform continuity of interpretations of formulas (remark 3.34). Let $[\varphi < \epsilon]$ be a basic open set of $S_n(X)$ in the logic topology and let p be a type in this open set. There exists $0 \leqslant \delta < \epsilon$ such that the $\mathcal{L}_X$-condition $\varphi \leqslant \delta$ is in p. Put now $r = \frac{\epsilon - \delta}{2}$. The interpretation $\varphi^{\mathbf{N}} : N^n \to [0, 1]$ of the formula φ is uniformly continuous so there exists a positive η such that for all $\bar{a}$, $\bar{b}$ in N^n with $d(\bar{a}, \bar{b}) < \eta$, we have $|\varphi^{\mathbf{N}}(\bar{a}) - \varphi^{\mathbf{N}}(\bar{b})| \leqslant r$.

We claim that the ball $B(p, \eta)$ around p in the metric topology is contained in the set $[\varphi < \epsilon]$. Indeed, let q be type in $B(p, \eta)$. There exist realizations $\bar{a}$ of p and $\bar{b}$ of q in N^n such that $d(\bar{a}, \bar{b}) < \eta$. This implies that $|\varphi^{\mathbf{N}}(\bar{a}) - \varphi^{\mathbf{N}}(\bar{b})| \leqslant r$. Besides, since the condition $\varphi \leqslant \delta$ is in p, we have $\varphi^{\mathbf{N}}(\bar{a}) \leqslant \delta$, so $\varphi^{\mathbf{N}}(\bar{b}) \leqslant r + \delta < \epsilon$, which completes the proof. □

The space of types is the archetype of a *topometric* space, in the sense of Ben Yaacov ([**B3**]): a space where we can juggle between a topology and a metric that refines it nicely. We will see in chapter 6 an illustration of those interactions and a characterization of when the logic and metric topologies coincide.

Another archetypal topometric space is the automorphism group of a metric structure, with its usual topology of pointwise convergence together with the distance of uniform convergence.

In topometric groups such as this one, this juggling between topologies has proved particularly successful: Ben Yaacov, Berenstein and Melleray ([**BBM**]) obtained a beautiful result of automatic continuity.

6. Definability

Definable objects include everything that the language can express. In the classical setting, definable predicates are exactly formulas and definable sets are the sets of tuples that verify a formula. In the continuous setting, however, the notion of definability is more delicate to delineate.

Definition 3.60. Let $\mathcal{L}$ be a continuous language. Let $\mathbf{M}$ be an $\mathcal{L}$-structure and let X be a countable subset of M. Let $P : M^n \to [0,1]$ be a function. We say that P is **definable** in $\mathbf{M}$ over X if it is the uniform limit of $\mathcal{L}_X$-formulas: if there exists a sequence $(\varphi_k)_{k\in\mathbb{N}}$ of $\mathcal{L}_X$-formulas in n free variables such that $(\varphi_k^{\mathbf{M}})$ converges uniformly to P.

Remarks 3.61.
- Definable predicates are uniformly continuous.
- Each formula only contains a finite number of elements of X in its expression. But unlike the classical case, the number of parameters from X needed to define a predicate can still be infinite.

We would still like to have a more concrete description of definable predicates, to mirror the classical case. To this aim, we widen the class of connectives: we allow infinitary connectives.

Endow $[0,1]^{\mathbb{N}}$ with the metric defined by

$$\rho(x,y) = \sum_{k\in\mathbb{N}} \frac{1}{2^k}|x_k - y_k|.$$

Proposition 3.62. Let $\mathcal{L}$ be a continuous language. Let $\mathbf{M}$ be an $\mathcal{L}$-structure and let X be a countable subset of M. Let $P : M^n \to [0,1]$ be a uniformly continuous function. Then P is definable is $\mathbf{M}$ over X if and only if there exists a (uniformly) continuous function $u : [0,1]^{\mathbb{N}} \to [0,1]$ and a sequence $(\varphi_k)_{k\in\mathbb{N}}$ of $\mathcal{L}_X$-formulas in n free variables such that, for all $\bar{a}$ in M^n,

$$P(\bar{a}) = u(\varphi_k^{\mathbf{M}}(\bar{a})|k\in\mathbb{N}).$$

Proof. $\Leftarrow$] Assume that P has the specified form and let ϵ be a positive real. To show that P is definable in $\mathbf{M}$ over X, we find an $\mathcal{L}_X$-formula whose interpretation in $\mathbf{M}$ uniformly approaches P up to ϵ. Since u is uniformly continuous, there exists a rank K in $\mathbb{N}$ such that for all sequences (r_k) and (s_k) in $[0,1]^{\mathbb{N}}$ that coincide up to the K-th coordinate, $|u(r_k|k\in\mathbb{N}) - u(s_k|k\in\mathbb{N})| \leqslant \epsilon$. Let now $u_K : [0,1]^{K+1} \to [0,1]$ be the associated truncated function: $u_K(r_0,...,r_K) = u(r_0,...,r_K,0,...0,...)$. This map u_K is (uniformly) continuous so the expression $\varphi(\bar{x}) = u_K(\varphi_0(\bar{x}),...,\varphi_K(\bar{x}))$ defines an $\mathcal{L}_X$-formula, which is as desired. Indeed, for every tuple $\bar{a}$ in M^n, we have

$$|P(\bar{a}) - \varphi^{\mathbf{M}}(\bar{a})| = |u(\varphi_k^{\mathbf{M}}(\bar{a})|k\in\mathbb{N}) - u_K(\varphi_0^{\mathbf{M}}(\bar{a}),...,\varphi_K^{\mathbf{M}}(\bar{a}))| \leqslant \epsilon.$$

$\Rightarrow$] Conversely, assume that the predicate P is definable in $\mathbf{M}$ over X: for every k in $\mathbb{N}$, there is an $\mathcal{L}_X$-formula φ_k such that for every $\bar{a}$ in M^n, we have $|P(\bar{a}) - \varphi_k^{\mathbf{M}}(\bar{a})| \leqslant \frac{1}{2^k}$. Note that P is the limit of the sequence $(\varphi_k^{\mathbf{M}})$.

We extend the limit map to a continuous map on $[0,1]^{\mathbb{N}}$. To this aim, consider the set $\mathcal{C}$ of all sequences (r_k) in $[0,1]^{\mathbb{N}}$ such that for every K in $\mathbb{N}$ and every two indices k and l greater than K, we have $|r_k - r_l| \leqslant \frac{1}{2^K}$. All sequences in $\mathcal{C}$ are Cauchy and thus converge. Moreover, the set $\mathcal{C}$ is closed in $[0,1]^{\mathbb{N}}$ and the limit map from $\mathcal{C}$ to $[0,1]$ is continuous, so the Tietze extension theorem yields a continuous map $u : [0,1]^{\mathbb{N}} \to [0,1]$ that extends the limit map.

Now our choice of formulas guarantees that for every $\bar{a}$ in M^n, the sequence $(\varphi_k^{\mathbf{M}}(\bar{a}))_{k\in\mathbb{N}}$ is in $\mathcal{C}$. Consequently, for every $\bar{a}$ in M^n, we have

$$P(\bar{a}) = \lim \varphi_k^{\mathbf{M}}(\bar{a}) = u(\varphi_k^{\mathbf{M}}(\bar{a})|k\in\mathbb{N}).$$

□

The following theorem characterizes definable predicates as continuous functions on the space of types.

THEOREM 3.63. Let $\mathcal{L}$ be a continuous language. Let $\mathbf{M}$ be an $\mathcal{L}$-structure and let X be a countable subset of M. Let $P : M^n \to [0,1]$ be a function. Then P is definable in $\mathbf{M}$ over X if and only if there exists a map $\Phi : S_n(X) \to [0,1]$ that is continuous with respect to the logic topology on $S_n(X)$ such that for all $\bar{a}$ in M^n,

$$P(\bar{a}) = \Phi(\mathrm{tp}(\bar{a}/X)).$$

PROOF. See [**BBHU**, theorem 9.9]. □

From the notion of a definable predicate, we can also define what it means for a set and for a function to be definable.

DEFINITION 3.64. Let $\mathcal{L}$ be a continuous language. Let $\mathbf{M}$ be an $\mathcal{L}$-structure and let X be a countable subset of M.

- Let D be a closed subset of M^n. We say that the subset D is **definable** in $\mathbf{M}$ over X if the predicate $d(\bar{x}, D) : M^n \to [0,1]$ is.
- Let F be a function from M^m to M. We say that the function F is **definable** in $\mathbf{M}$ over X if its graph is (as a subset of M^{m+1}).

We stress that in the continuous setting, being definable means much more than being the zeroset of a definable predicate. The following proposition illustrates this distinction and provides a more graspable characterization of definable sets. We shall use it in chapter 6 to describe principal types.

PROPOSITION 3.65. Let $\mathcal{L}$ be a continuous language. Let $\mathbf{M}$ be an $\mathcal{L}$-structure and let X be a countable subset of M. Let D be a closed subset of M^n. Then the following are equivalent.

- The set D is definable in $\mathbf{M}$ over X.
- There is a predicate $P : M^n \to [0,1]$, definable over X, such that for all $\bar{x}$ in D, we have $P(\bar{x}) = 0$, and for every positive ϵ, there exists $\delta > 0$ such that for all $\bar{x}$ in M^n, we have
$$P(\bar{x}) \leqslant \delta \Rightarrow d(\bar{x}, D) \leqslant \epsilon.$$
- There is a sequence (φ_m) of $\mathcal{L}_X$-formulas and a sequence (δ_m) of positive reals such that for all m and all $\bar{x}$ in D, we have $\varphi_m^{\mathbf{M}}(\bar{x}) = 0$, and for all m and all $\bar{x}$ in M^n, we have
$$\varphi_m^{\mathbf{M}}(\bar{x}) \leqslant \delta_m \Rightarrow d(\bar{x}, D) \leqslant 2^{-m}.$$

PROOF. See [**BBHU**, proposition 9.19]. □

CHAPTER 4

Fraïssé theory

C'est fait de tout petits riens.
Claude François[1]

1.	**Classical Fraïssé theory**	**63**
2.	**Metric Fraïssé theory**	**69**
3.	**Approximate maps**	**71**
3.1.	Operations on approximate maps	71
3.2.	Approximetries	72
3.3.	Partial isometries, extensions	73
3.4.	Measure of the totality of an approximetry	73
3.5.	Refinement	75
4.	**An approximate route to metric Fraïssé theory**	**77**
4.1.	Is-almost-phisms	77
4.2.	Near amalgamation	78
4.3.	Fraïssé limits	79
5.	**Topology on the automorphism group**	**81**
5.1.	An approximate basis of open sets	81
5.2.	The Roelcke uniformity	81

In [**F**], Fraïssé gave a way of building *random* structures as limits of finite objects that are glued together. A wide class of random structures is that of *ultrahomogeneous* structures, for which randomness means that every finite configuration can be found everywhere in the structure. Fraïssé theory permits, conversely, to study such structures combinatorially through their finitely generated substructures.

1. Classical Fraïssé theory

DEFINITION 4.1. A classical countable structure $\mathbf{M}$ is **ultrahomogeneous** if every isomorphism between finitely generated substructures of $\mathbf{M}$ extends to an automorphism of the whole structure $\mathbf{M}$. In other words, $\mathbf{M}$ is ultrahomogeneous if and only if any two tuples with the same quantifier-free type in $\mathbf{M}$ can be sent each to the other by an automorphism of $\mathbf{M}$.

Ultrahomogeneous structures thus have rich automorphism groups: automorphisms have to account for all local behaviors. Moreover, isomorphisms between two finitely generated substructures define non-empty open sets in the automorphism group (see definition 3.19).

REMARK 4.2. We shall see a notion of ultrahomogeneity more suited for metric structures in section 2.

EXAMPLES 4.3. Here a few examples of ultrahomogeneous structures.
(1) Pure sets: bijections between finite sets extend to bijections of the whole set.
(2) The rationals, with their order. If f is a finite increasing bijection, we extend f with piecewise linear maps. Another proof is by a back-and-forth argument, as we have done in chapter 2 for the Urysohn space.

[1] *Chanson populaire*

(3) The infinite k-regular tree T_k, with the graph distance. Let $\mathbf{A}$ and $\mathbf{B}$ be two subgraphs and let f be an isomorphism between $\mathbf{A}$ and $\mathbf{B}$. First, f extends uniquely to an isomorphism between the induced subtrees (see figure 4.1). Indeed, consider a connected component of $\mathbf{A}$ and one of the nearest other connected component. In the tree T_k, there exists a unique shortest path between them, say $(a, x_1, ..., x_n, a')$, with a and a' in $\mathbf{A}$, and the x_i's outside $\mathbf{A}$. Now, since $d(a, a') = n + 1$, we also have that $d(f(a), f(a')) = n + 1$. Thus, there exists a path $(f(a), y_1, ..., y_n, f(a'))$ between them in T_k. The choice of a and a' guarantees that the y_i's are not in $\mathbf{B}$. Thus, we may extend f by putting $f(x_i) = y_i$.

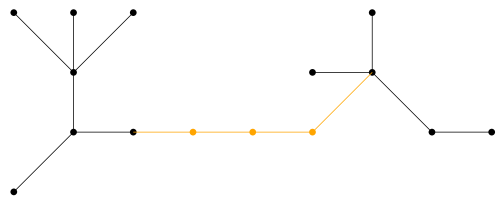


FIGURE 4.1. The induced subtree

Assume now that $\mathbf{A}$ and $\mathbf{B}$ are subtrees and consider a vertex a in $\mathbf{A}$. Since T_k is regular, the sets $\{x \in T_k \setminus \mathbf{A} : d(x, a) = 1\}$ and $\{y \in T_k \setminus \mathbf{B} : d(y, f(a)) = 1\}$ have the same size. Thus, we may extend f to these sets. Moreover, if x and y are neighbor of a and $f(a)$ respectively, the subtrees emerging from x and y are isomorphic, so f extends to those emerging subtrees as well.

We apply this process to every vertex in a. This is consistent, since no vertex outside $\mathbf{A}$ can be a neighbor of two distinct vertices in $\mathbf{A}$ (otherwise, since $\mathbf{A}$ is connected, it would create a cycle in T_k). Thus, the isomorphism f extends to an automorphism of the whole structure.

(4) The infinite infinitely splitting tree. The proof works the same way.

We now describe Fraïssé's construction of countable ultrahomogeneous structures as limits of their finitely generated substructures.

A *Fraïssé class* is a class of finitely generated structures that enjoys good amalgamation properties.

DEFINITION 4.4. Let $\mathcal{L}$ be a classical language. A **Fraïssé class** is a class $\mathcal{K}$ of finitely generated $\mathcal{L}$-structures with the following properties.

- **Hereditary property (HP):** If $\mathbf{B}$ is in $\mathcal{K}$ and $\mathbf{A}$ embeds in $\mathbf{B}$, then $\mathbf{A}$ is also in $\mathcal{K}$.
- **Joint embedding property (JEP):** If $\mathbf{A}$ and $\mathbf{B}$ are in $\mathcal{K}$, then there exists a structure $\mathbf{C}$ in $\mathcal{K}$ in which both $\mathbf{A}$ and $\mathbf{B}$ embed.

$$\begin{tikzcd} & \mathbf{C} & \\ \mathbf{A} \arrow[ur, dotted, hook] & & \mathbf{B} \arrow[ul, dotted, hook'] \end{tikzcd}$$

- **Amalgamation property (AP):** If $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are in $\mathcal{K}$, and $f_{\mathbf{B}} : \mathbf{A} \to \mathbf{B}$ and $f_{\mathbf{C}} : \mathbf{A} \to \mathbf{C}$ are embeddings, then there exists a structure $\mathbf{D}$ in $\mathcal{K}$ and embeddings

$g_{\mathbf{B}} : \mathbf{B} \to \mathbf{D}$ and $g_{\mathbf{C}} : \mathbf{C} \to \mathbf{D}$ such that $g_{\mathbf{B}} \circ f_{\mathbf{B}} = g_{\mathbf{C}} \circ f_{\mathbf{C}}$.

$$\begin{tikzcd}
& \mathbf{D} & \\
\mathbf{B} \arrow[ur, dotted, "g_{\mathbf{B}}"] & & \mathbf{C} \arrow[ul, dotted, "g_{\mathbf{C}}"'] \\
& \mathbf{A} \arrow[ul, "f_{\mathbf{B}}"] \arrow[ur, "f_{\mathbf{C}}"'] &
\end{tikzcd}$$

- **Denumerability:** The class $\mathcal{K}$ is countable, up to isomorphism.

REMARK 4.5. The joint embedding property is equivalent, modulo AP, to there being a unique structure generated by the empty set in $\mathcal{K}$.

The most essential property, and the trickiest to check, is the amalgamation property. In the examples, we will therefore often content ourselves with proving this one.

EXAMPLES 4.6. The following form Fraïssé classes.

(1) Finite sets. If $\mathbf{A}$ is included in both $\mathbf{B}$ and $\mathbf{C}$, a natural amalgam of $\mathbf{B}$ and $\mathbf{C}$ over $\mathbf{A}$ is given by $\mathbf{B} \cup (\mathbf{C} \setminus \mathbf{A})$.

(2) Finite ordered sets. To amalgamate $\mathbf{B}$ and $\mathbf{C}$ over $\mathbf{A}$, start with $\mathbf{A}$ and insert points of $\mathbf{B}$ and $\mathbf{C}$ at the right places by declaring that if $b \in \mathbf{B}$ and $c \in \mathbf{C}$ fall in the same interval, b is smaller than c (see figure 4.2). Note that this construction is not symmetric in $\mathbf{B}$ and $\mathbf{C}$.

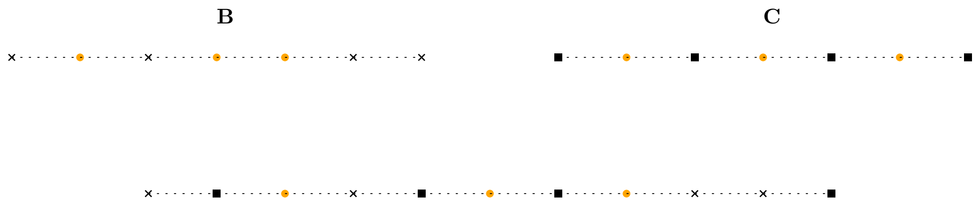


FIGURE 4.2. The amalgam of linear orders $\mathbf{B}$ and $\mathbf{C}$ over the orange suborder.

(3) Finite graphs. If $\mathbf{B} \cap \mathbf{C} = \mathbf{A}$, an amalgam of $\mathbf{B}$ and $\mathbf{C}$ over $\mathbf{A}$ is given by the union of $\mathbf{B}$ and $\mathbf{C}$, without any additional edge, so that there is no edge between $\mathbf{B} \setminus \mathbf{A}$ and $\mathbf{C} \setminus \mathbf{A}$.

(4) Finite triangle-free graphs. Since we added no superfluous edge, the above amalgam will stay triangle-free if both $\mathbf{B}$ and $\mathbf{C}$ were.

Note, though, that the class of bowtie-free graphs is not a Fraïssé class (see figure 4.3).

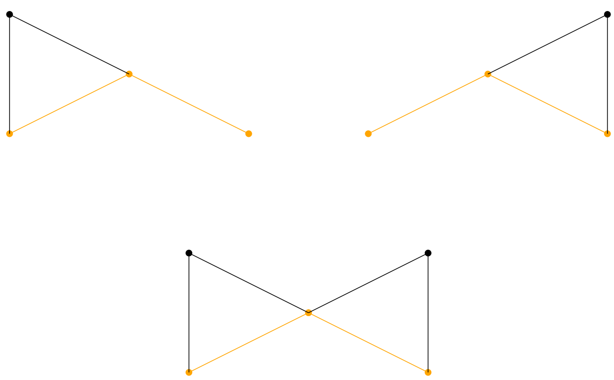


FIGURE 4.3. Any amalgam of those two graphs over the orange subgraph contains a bowtie.

(5) Finite metric spaces with rational distances, in the language consisting of predicates d_q, for all q in $\mathbb{Q}$, defined by $d_q(x, y) \Leftrightarrow d(x, y) = q$. If $\mathbf{A}$ is a non-empty metric subspace of both $\mathbf{B}$ and $\mathbf{C}$, then we can define the **maximal distance amalgam** of $\mathbf{B}$ and $\mathbf{C}$ over $\mathbf{A}$ as follows. In the amalgam, the distance between an element b of $\mathbf{B}$ and an element c of $\mathbf{C}$ is given by

$$d(b, c) = \min_{a \in \mathbf{A}} d(b, a) + d(a, c).$$

In other words, we put $\mathbf{B}$ and $\mathbf{C}$ as far away as the triangle inequality allows (see figure 4.4).

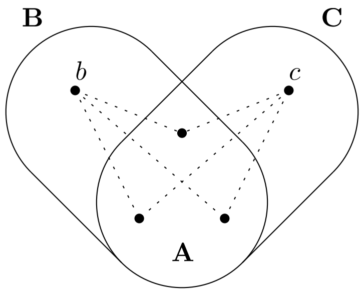


FIGURE 4.4. The maximal distance amalgam.

If $\mathbf{A}$ is empty, then we amalgamate $\mathbf{B}$ and $\mathbf{C}$ over an arbitrary extra point.

(6) Finitely generated vector spaces over $\mathbb{Q}$. Assume that $\mathbf{A}$ is the intersection of vector spaces $\mathbf{B}$ and $\mathbf{C}$. Let $\mathcal{A}$ be a basis for $\mathbf{A}$ and complete $\mathcal{A}$ into a basis $\mathcal{B}$ of $\mathbf{B}$ and a basis $\mathcal{C}$ of $\mathbf{C}$. Then an amalgam of $\mathbf{B}$ and $\mathbf{C}$ over $\mathbf{A}$ is given by the span of $\mathcal{B} \cup (\mathcal{C} \setminus \mathcal{A})$.

REMARK 4.7. In examples (1), (3) and (4), the amalgam that we described is the **free amalgam**: there is no relation between elements of $\mathbf{B} \setminus \mathbf{A}$ and $\mathbf{C} \setminus \mathbf{A}$. If such an amalgam can be found inside the class $\mathcal{K}$ for all triplets of structures, then we say that the class $\mathcal{K}$ has the **free amalgamation property**.

In the metric setting, it does not make any sense to say that there is no relation, for there is always the metric. However, the maximal distance amalgam is the freest amalgam. This falls within the more general framework of structures with a stationary independence relation, which was developed by Tent and Ziegler in [**TZ2**].

Ultrahomogeneous structures naturally provide Fraïssé classes, through their *ages*.

DEFINITION 4.8. Let $\mathbf{M}$ be an $\mathcal{L}$-structure. The **age** of $\mathbf{M}$, denoted by $\mathrm{Age}(\mathbf{M})$ is the class of all finitely generated $\mathcal{L}$-structures that embed in $\mathbf{M}$.

The age of a structure consists of all its finitely generated substructures (and of all the structures isomorphic to them).

PROPOSITION 4.9. Let $\mathbf{M}$ be a countable ultrahomogeneous $\mathcal{L}$-structure. Then the age of $\mathbf{M}$ is a Fraïssé class.

We will see in theorem 4.11 that all Fraïssé classes are actually obtained that way.

PROOF. The hereditary property is clear.

If $\mathbf{A}$ and $\mathbf{B}$ embed in $\mathbf{M}$ via f and g, then both $\mathbf{A}$ and $\mathbf{B}$ embed in $f(\mathbf{A}) \cup g(\mathbf{B})$, which proves the joint embedding property.

As for the amalgamation, the key assumption is ultrahomogeneity. Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be in the age of $\mathbf{M}$ and $f_{\mathbf{B}} : \mathbf{A} \to \mathbf{B}$, $f_{\mathbf{C}} : \mathbf{A} \to \mathbf{C}$ be embeddings. We may assume that $\mathbf{B}$ and $\mathbf{C}$ are substructures of $\mathbf{M}$. Now the map $f_{\mathbf{C}} \circ f_{\mathbf{B}}^{-1}$ is an isomorphism between the two copies $f_{\mathbf{B}}(\mathbf{A})$ and $f_{\mathbf{C}}(\mathbf{A})$ of $\mathbf{A}$. Since $\mathbf{M}$ is ultrahomogeneous, this isomorphism extends to an automorphism g of $\mathbf{M}$. Then $g \circ f_{\mathbf{B}} = f_{\mathbf{C}}$, hence the isomorphisms $g_{\mathbf{B}} = g_{\restriction \mathbf{B}}$ and $g_{\mathbf{C}} = \mathrm{id}_{\mathbf{C}}$ are as desired.

Moreover, it is easy to see that the age of a countable structure is countable, up to isomorphism. $\square$

If $\mathcal{K}$ is a Fraïssé class, we say that a countable structure $\mathbf{M}$ is a **Fraïssé limit** of $\mathcal{K}$ if $\mathbf{M}$ is ultrahomogeneous and the class $\mathcal{K}$ is the age of $\mathbf{M}$. Ultrahomogeneous structures are sometimes called **Fraïssé structures**.

Fraïssé limits of a class are actually characterized by a seemingly weaker property. We say that a structure $\mathbf{M}$ is $\mathcal{K}$**-rich** if for any two structures $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{K}$ and any two embeddings $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{M}$, there exists an embedding $h : \mathbf{B} \to \mathbf{M}$ such that $h \circ f = g$.

$$\begin{tikzcd} \mathbf{B} \arrow[r, dotted, "h"] & \mathbf{M} \\ & \mathbf{A} \arrow[ul, "f"] \arrow[u, "g"'] \end{tikzcd}$$

It is easily checked that a Fraïssé limit of $\mathcal{K}$ is $\mathcal{K}$-rich. The following theorem states the uniqueness of $\mathcal{K}$-rich structures of age $\mathcal{K}$. It yields that the Fraïssé limit is unique.

THEOREM 4.10. (Fraïssé) Let $\mathcal{K}$ be a Fraïssé class and let $\mathbf{M}$ and $\mathbf{N}$ be two countable structures of age $\mathcal{K}$. Assume that $\mathbf{M}$ and $\mathbf{N}$ are $\mathcal{K}$-rich. Then, for every isomorphism $f : \mathbf{A} \to \mathbf{B}$ between finitely generated substructures of $\mathbf{M}$ and $\mathbf{N}$ respectively, there exists an isomorphism between $\mathbf{M}$ and $\mathbf{N}$ that extends f.

When applied to $\mathbf{M} = \mathbf{N}$, the previous theorem yields that $\mathcal{K}$-rich structures of age $\mathcal{K}$ are in fact ultrahomogeneous, hence Fraïssé limits of $\mathcal{K}$.

PROOF. We proceed by back-and-forth. Exhaust $\mathbf{M}$ and $\mathbf{N}$ as increasing unions of finitely generated substructures (that is, structures in $\mathcal{K}$): $\mathbf{M} = \bigcup_{n \in \mathbb{N}} \mathbf{A}_n$ and $\mathbf{N} = \bigcup_{n \in \mathbb{N}} \mathbf{B}_n$. Since $\mathcal{K}$ is the age of the two structures, we may as well assume that $\mathbf{A}_0 = \mathbf{A}$ and $\mathbf{B}_0 = \mathbf{B}$. By induction, we build a chain $(f_n)_{n \in \mathbb{N}}$ of isomorphisms between finitely generated substructures of $\mathbf{M}$ and $\mathbf{N}$ such that

- $f_0 = f$;
- the domain of f_n contains $\mathbf{A}_n$;
- the range of f_n contains $\mathbf{B}_n$.

Assume that f_n has been built. For the *forth* step, consider the substructure $\mathbf{C}_{n+1}$ of $\mathbf{M}$ generated by $\mathrm{dom}(f_n) \cup \mathbf{A}_{n+1}$. This a structure in $\mathcal{K}$, thus in the age of $\mathbf{N}$: there exists a substructure $\mathbf{D}_{n+1}$ of $\mathbf{N}$ isomorphic to $\mathbf{C}_{n+1}$ via g. Thus, $g(\mathrm{dom}(f_n))$ embeds into $\mathbf{N}$ via $f_n \circ g^{-1}$, as well as into $g(\mathbf{C}_{n+1}) = \mathbf{D}_{n+1}$ via the inclusion map. This implies, by $\mathcal{K}$-richness of the structure $\mathbf{N}$, the existence of an embedding h of $\mathbf{D}_{n+1}$ into $\mathbf{N}$ that extends $f_n \circ g^{-1}$. Now set $f_{n+1} = h \circ g$. The map f_{n+1} is an extension of f_n to $\mathbf{C}_{n+1}$.

For the *back* step, we apply the same argument to the inverse of the map f_{n+1} we just obtained.

Finally, the union of all f_n's is an isomorphism between $\mathbf{M}$ and $\mathbf{N}$ which extends f, as desired. □

THEOREM 4.11. (Fraïssé) Every Fraïssé class admits a Fraïssé limit.

PROOF. Let $\mathcal{K}$ be a Fraïssé class. As per theorem 4.10, it suffices to build a $\mathcal{K}$-rich countable structure whose age is the class $\mathcal{K}$. We do so by induction: we build a chain $(\mathbf{C}_n)_{n \in \mathbb{N}}$ of structures in $\mathcal{K}$ such that for all $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{K}$ with $\mathbf{A}$ included in $\mathbf{B}$ and for every embedding $f : \mathbf{A} \to \mathbf{C}_n$, there exists $m > n$ and an embedding $g : \mathbf{B} \to \mathbf{C}_m$ that extends f.

Assume the chain has been built and put $\mathbf{M} = \bigcup_{n \in \mathbb{N}} \mathbf{C}_n$. Let us prove that $\mathbf{M}$ is the desired Fraïssé limit. The age of $\mathbf{M}$ is $\mathcal{K}$. Indeed, it is clear that the age of $\mathbf{M}$ is included in $\mathcal{K}$. Conversely, let $\mathbf{A}$ be a structure in $\mathcal{K}$. The joint embedding property gives a structure $\mathbf{B}$ into which both $\mathbf{A}$ and $\mathbf{C}_0$ embed. But now, the defining property of the chain ensures that $\mathbf{B}$ will embed into one of the $\mathbf{C}_n$'s, hence into $\mathbf{M}$. Thus, $\mathbf{A}$ belongs to the age of $\mathbf{M}$. Moreover, our assumption on the $\mathbf{C}_n$'s yields $\mathcal{K}$-richness directly.

It remains to construct the $\mathbf{C}_n$'s. To that aim, choose a countable set $\mathcal{P}$ of pairs $(\mathbf{A}, \mathbf{B})$ of structures in $\mathcal{K}$ with $\mathbf{A} \subseteq \mathbf{B}$ so that $\mathcal{P}$ contains a representative of each isomorphism type of such pairs. Also fix a bijection $\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all n and m, the integer $\pi(n, m)$ is greater than n. Now, start the construction with any structure $\mathbf{C}_0$ in $\mathcal{K}$. If the structure $\mathbf{C}_n$ has been built, enumerate all triples of the form $(f, \mathbf{A}, \mathbf{B})$ where $(\mathbf{A}, \mathbf{B})$ is in $\mathcal{P}$ and f is an embedding of $\mathbf{A}$ into $\mathbf{C}_n$ as $\{(f_{n,m}, \mathbf{A}_{n,m}, \mathbf{B}_{n,m}) : m \in \mathbb{N}\}$.

The idea is to take care of $f_{n,m}$ at step $\pi(n, m)$. The amalgamation property gives a structure $\mathbf{D}$ in $\mathcal{K}$ and embeddings such that the following diagram commutes.

$$\begin{tikzcd}
 & & \mathbf{D} & \\
 & & & \mathbf{C}_{\pi(n,m)} \arrow[ul, dotted] \\
\mathbf{B}_{n,m} \arrow[uurr, dotted] & & \mathbf{C}_n \arrow[ur] & \\
 & \mathbf{A}_{n,m} \arrow[ul] \arrow[ur, "f_{n,m}"'] & &
\end{tikzcd}$$

Then, we add $\mathbf{D}$ to the structure $\mathbf{C}_{\pi(n,m)}$ to make $\mathbf{C}_{\pi(n,m)+1}$. This process guarantees that every isomorphism $f_{n,m}$ will indeed extend to some $\mathbf{C}_i$, which completes the proof. $\square$

We now go over the Fraïssé classes in 4.6 and give their Fraïssé limits. We will recover the ultrahomogeneous structures from example 4.3 this way.

EXAMPLES 4.12. (1) Finite sets: the associated Fraïssé limit is the countably infinite set ($\mathbb{N}$ for instance).

(2) Finite ordered sets: the Fraïssé limit is $(\mathbb{Q}, <)$.

(3) The Fraïssé limit of the class of finite graphs is called the **random graph**, and is often denoted by $\mathcal{R}$.

There are several other ways to build the random graph. One possible construction is *percolation*: start with countably many vertices, and for every pair of vertices, flip a coin to decide whether to put an edge between them. Almost surely, the result is isomorphic to the random graph.

The random graph is characterized by the following property: for any two disjoint finite subgraphs $\mathbf{A}$ and $\mathbf{B}$, there exists a vertex in $\mathcal{R}$ that is related to every point in $\mathbf{A}$ but to no point in $\mathbf{B}$ (see figure 4.5).

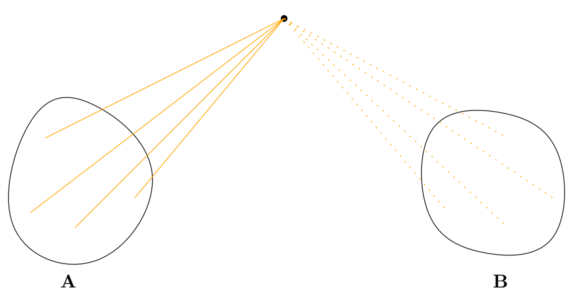


FIGURE 4.5. The defining property of the random graph.

(4) The Fraïssé limit of the class of finite triangle-free graphs was built by Henson in [**H2**], and is therefore called the **Henson triangle-free graph**.

(5) The Fraïssé limit of the class of finite metric spaces with rational distances is called the **rational Urysohn space**, and denoted by $\mathbb{QU}$.

The original construction of $\mathbb{U}$ by Urysohn — thirty years before Fraïssé! — resembled this one very much. The Urysohn metric space was actually constructed as the metric completion of the rational Urysohn space.

Note that, although $\mathrm{Aut}(\mathbb{QU})$ is the group of isometries of the rational Urysohn space, the topology on $\mathrm{Aut}(\mathbb{QU})$ does not correspond to pointwise convergence, it is still the permutation group topology.

(6) Finitely generated vector spaces over $\mathbb{Q}$: the Fraïssé limit is the countably infinite-dimensional $\mathbb{Q}$-vector space.

2. Metric Fraïssé theory

We described the Urysohn space as the completion of a classical Fraïssé limit, but it would be natural to see it as a Fraïssé limit in its own right: the Fraïssé limit of finite metric spaces. In this section, we present Ben Yaacov's generalization of Fraïssé's construction to separable metric structures (**[B5]**), which constitute the metric counterpart of countable structures.

To do so, we need to allow arbitrarily small errors and thus to relax the amalgamation properties of a Fraïssé class.

Definition 4.13. Let $\mathcal{L}$ be a metric language and let $\mathcal{K}$ be a class of finitely generated $\mathcal{L}$-structures. We say that $\mathcal{K}$ has the **near amalgamation property (NAP)** if for all $\mathbf{A} = \langle \bar{a} \rangle$, $\mathbf{B}$ and $\mathbf{C}$ in $\mathcal{K}$, for all embeddings $f_{\mathbf{B}} : \mathbf{A} \to \mathbf{B}$ and $f_{\mathbf{C}} : \mathbf{A} \to \mathbf{C}$, and for every $\epsilon > 0$, there exists a structure $\mathbf{D}$ in $\mathcal{K}$ and embeddings $g_{\mathbf{B}} : \mathbf{B} \to \mathbf{D}$ and $g_{\mathbf{C}} : \mathbf{C} \to \mathbf{D}$ such that $d(g_{\mathbf{B}} \circ f_{\mathbf{B}}(\bar{a}), g_{\mathbf{C}} \circ f_{\mathbf{C}}(\bar{a})) < \epsilon$.

$$\begin{tikzcd}
& \mathbf{D} & \\
\mathbf{B} \arrow[ur, dotted, "g_{\mathbf{B}}"] \arrow[rr, phantom, "\epsilon\text{-commutes}"] & & \mathbf{C} \arrow[ul, dotted, "g_{\mathbf{C}}"'] \\
& \bar{a} \arrow[ul, "f_{\mathbf{B}}"] \arrow[ur, "f_{\mathbf{C}}"'] &
\end{tikzcd}$$

Furthermore, the denumerability of the class needs to be replaced by a condition of separability together with a condition of completeness.

If $\mathcal{K}$ is a class of finitely generated structures that satisfies JEP and NAP, then, for every integer n, we denote by $\mathcal{K}_n$ the class consisting of all pairs $(\bar{a}, \mathbf{A})$, where $\mathbf{A}$ is in $\mathcal{K}$ and $\bar{a}$ is an n-tuple that generates $\mathbf{A}$. We equip $\mathcal{K}_n$ with the following pseudometric:

$$d_n((\bar{a}, \mathbf{A}), (\bar{b}, \mathbf{B})) = \inf\{d_{\mathbf{C}}(i(\bar{a}), j(\bar{b})) : \mathbf{C} \in \mathcal{K}, i : \mathbf{A} \to \mathbf{C} \text{ and } j : \mathbf{B} \to \mathbf{C} \text{ embeddings}\}.$$

Definition 4.14. Let $\mathcal{L}$ be a metric language. A **metric Fraïssé class** is a class $\mathcal{K}$ of finitely generated $\mathcal{L}$-structures that satisfies properties HP, JEP, NAP and such that for all n, the space $(\mathcal{K}_n, d_n)$ is separable and complete.

Examples 4.15. The following are metric Fraïssé classes.

(1) Finite metric spaces. A possible amalgam is again the maximal distance amalgam.

(2) Finite measure algebras. The amalgam of $\mathbf{B}$ and $\mathbf{C}$ over their intersection is the coarsest common refinement of $\mathbf{B}$ and $\mathbf{C}$.

(3) Finite-dimensional euclidean spaces. Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be finite-dimensional euclidean spaces, with $\mathbf{A} = \mathbf{B} \cap \mathbf{C}$. Let $\mathcal{A}$ be an orthonormal basis for $\mathbf{A}$ and complete $\mathcal{A}$ into orthonormal bases $\mathcal{B}$ and $\mathcal{C}$ of $\mathbf{B}$ and $\mathbf{C}$ respectively. Then define the amalgam of $\mathbf{B}$ and $\mathbf{C}$ over $\mathbf{A}$ to have $\mathcal{B} \cup \mathcal{C}$ as an orthonormal basis.

(4) Finitely generated Banach spaces. Define the amalgam of $\mathbf{B}$ and $\mathbf{C}$ over $\mathbf{A}$ as the space $\mathbf{B} \oplus \mathrm{Vect}(\mathbf{C} \setminus \mathbf{A})$ and set $\|b + c\| = \|b\| + \|c\|$ for b in $\mathbf{B} \setminus \mathbf{A}$ and c in $\mathbf{C} \setminus \mathbf{A}$.

(5) Finite metric spaces with a binary 1-Lipschitz predicate. Here, 1-Lipschitz refers to the ℓ^1 metric on X^2 (otherwise, the argument does not go through): a binary predicate P on X is 1-Lipschitz if for all x, x', y and y' in X, we have $|P(x,x') - P(y,y')| \leqslant d(x,x') + d(y,y')$. If $(\mathbf{B}, P)$ and $(\mathbf{C}, P)$ both contain $(\mathbf{A}, P)$, then it is easy to check that $P : \mathbf{B}^2 \cup C^2 \to \mathbb{R}$ is still 1-Lipschitz with respect to the metric in the maximal distance amalgam. Thus, it extends to a 1-Lipschitz map on the metric amalgam of $\mathbf{B}$ and $\mathbf{C}$ over $\mathbf{A}$.

The ultrahomogeneity of the limit also need relaxing.

Definition 4.16. A structure $\mathbf{M}$ is **approximately ultrahomogeneous** if for every finitely generated substructures $\mathbf{A} = \langle \bar{a} \rangle$ and $\mathbf{B}$ of $\mathbf{M}$, every isomorphism $f : \mathbf{A} \to \mathbf{B}$ and every $\epsilon > 0$, there exists an automorphism g of $\mathbf{M}$ such that one has $d(g(\bar{a}), f(\bar{a})) < \epsilon$.

Exactly as in proposition 4.9, the age of a separable approximately ultrahomogeneous structure is a metric Fraïssé class. Conversely, an approximately ultrahomogeneous structure $\mathbf{M}$ is a **metric Fraïssé limit** of a metric Fraïssé class $\mathcal{K}$ if $\mathcal{K}$ is the age of $\mathbf{M}$. We also call approximately ultrahomogeneous structures **metric Fraïssé structures**.

The analogue of Fraïssé's theorem holds. It was originally proven by Schoretsanitis in [**S2**], although in a somewhat different formalism, but Ben Yaacov ([**B5**]) proposed a different and elegant proof, which we will present in details in the next section.

Theorem 4.17. Every metric Fraïssé class admits a metric Fraïssé limit. Moreover, the metric Fraïssé limit is unique, up to isomorphism.

Our favorite examples of metric Fraïssé limits will be the following.

Examples 4.18. (1) As expected, the Urysohn space is the metric Fraïssé limit of the class of finite metric spaces.

(2) The measure algebra of the unit interval is the metric Fraïssé limit of the class of finite measure algebras.

To see that it is ultrahomogeneous, let $\mathbf{A}$ and $\mathbf{B}$ be two finite measure subalgebras and let f be an isomorphism between them. The atoms of $\mathbf{A}$ and $\mathbf{B}$ induce finite partitions $(A_1, ..., A_n)$ and $(B_1, ..., B_n)$ of $[0, 1]$, with $f(A_i) = B_i$. Since A_i and B_i have the same measure, the measure algebras of $(A_i, \mu_{\restriction A_i})$ and of $(B_i, \mu_{\restriction B_i})$ are isomorphic, say via f_i. Gluing the f_i's together, we obtain the desired extension of f.

(3) The separable Hilbert space (the space ℓ^2) is the metric Fraïssé limit of the class of finitely generated euclidean spaces.

(4) The metric Fraïssé limit of the class of finitely generated Banach spaces is called the **Gurarij space** and is denoted by $\mathbb{G}$. It was constructed by Gurarij in [**G6**] and its uniqueness was proved ten years later by Lusky ([**L3**]).

It is interesting to note that in the three first examples above, the metric Fraïssé limit is not only approximately ultrahomogeneous but *exactly* ultrahomogeneous. Exact homogeneity will be needed in chapter 8 when considering questions of automatic continuity. It would be very nice to have a characterization of metric Fraïssé classes whose limit is exactly ultrahomogeneous, but unfortunately, no such characterization is known. An obvious requirement is for the amalgamation property to be exact, but this does not suffice. Indeed, although the class of finitely generated Banach spaces satisfies AP, the Gurarij space is not exactly ultrahomogeneous (smooth points cannot be mapped to non-smooth points).

Question 4.19. Can exact ultrahomogeneity of the metric Fraïssé limit be read on the class?

Actually, when the limit is separably categorical (see chapter 6), exact ultrahomogeneity is equivalent to all the finite tuples in the class being *d-finite*. Ben Yaacov and Usvyatsov observed in [**BU1**] that in continuous structures, finite tuples tend to behave like infinite tuples do in the classical case. Thus, they introduce the notion of d-finiteness to discriminate those nice tuples that behave as we expect of finite tuples. However, d-finiteness can be quite a tricky condition to check, and even in very simple examples, the answer is unclear.

QUESTION 4.20. Is the metric Fraïssé limit of the class of finite metric spaces equipped with a binary 1-Lipschitz predicate exactly ultrahomogeneous?

3. Approximate maps

This section is devoted to the proof of theorem 4.17 and the tools this metric Fraïssé construction involves. Instead of considering embeddings that almost satisfy the amalgamation property and of keeping track of all the epsilons, we rather consider approximate maps: *approximetries* and *is-almost-phisms*[2]. This way, instead of dealing with maps that almost extend one another, we will have almost-maps that do extend one another. This elegant theory is due to Ben Yaacov ([**B5**]) and was inspired by Uspenskij, who introduced approximetries in [**U5**] to compute the Roelcke compactification of the isometry group of the Urysohn sphere.

Throughout this section, X, X', Y, Y', Z and Z' will denote metric spaces.

The idea is the same as in continuous logic: just like equality was replaced by a metric, and a set by the distance function to this set, we replace any isometry $f : X \to Y$ by the map $\psi_f : X \times Y \to \mathbb{R}^+$ defined by

$$\psi_f(x, y) = d(f(x), y).$$

3.1. Operations on approximate maps. Given an isometry $f : X \to Y$, let us see how this new function ψ_f behaves with regards to operations on f.

- Inversion: for all x in X and y in Y, we have
$$\psi_{f^{-1}}(y, x) = d(f^{-1}(y), x) = d(y, f(x)) = \psi_f(x, y).$$
- Composition: if $g : Y \to Z$ is another isometry, then, for all x in X and z in Z, for every $y \in Y$, we have
$$\begin{aligned}\psi_{g\circ f}(x, z) &= d(g \circ f(x), z)\\ &= d(f(x), g^{-1}(z))\\ &\leqslant d(f(x), y) + d(y, g^{-1}(z))\\ &= \psi_f(x, y) + \psi_g(y, z).\end{aligned}$$
Moreover, for $y = f(x)$, this is an equality, so
$$\psi_{g\circ f}(x, z) = \inf_{y\in Y} \psi_f(x, y) + \psi_g(x, z).$$

These observations provide us with the intuition for the following definitions.

DEFINITION 4.21. Let $\psi : X \times Y \to [0, \infty]$ and $\varphi : Y \times Z \to [0, \infty]$ be two functions. We define a **composition** $\varphi\psi : X \times Z \to [0, \infty]$ and an **inverse** $\psi^* : Y \times X \to [0, \infty]$ by:

$$\begin{aligned}\varphi\psi(x, z) &= \inf_{y\in Y} \psi(x, y) + \varphi(y, z),\\ \psi^*(y, x) &= \psi(x, y).\end{aligned}$$

These operations of composition and inversion behave as we expect them to: composition is associative, inversion is an involution and for all approximate maps ψ and φ (with compatible domain and range), we have $(\varphi\psi)^* = \psi^*\varphi^*$.

[2]Writing the thesis in English, my only regret is to not be able to use the beautiful words of *presqu'isométrie* and *presqu'isomorphisme*. To make up for this, instead of using Itaï's *approximate isometry* and *approximate isomorphism*, I opted for highly questionable puns. By the way, I thank François D. for coming up with *is-almost-phism*!

3.2. Approximetries. An **approximetry** must respect the metric structure. The right meaning for this is to preserve the triangle inequality.

DEFINITION 4.22. Let $\psi : X \times Y \to [0, \infty]$ be a function. We say that ψ is an **approximetry**, and we write $\psi : X \rightsquigarrow Y$, if ψ is **bi-Katětov**, that is, Katětov in each variable: for all x, x' in X, y, y' in Y,

$$\psi(x, y) \leqslant d_X(x, x') + \psi(x', y) \text{ and } d_X(x, x') \leqslant \psi(x, y) + \psi(x', y),$$
$$\psi(x, y) \leqslant d_Y(y, y') + \psi(x, y') \text{ and } d_Y(y, y') \leqslant \psi(x, y) + \psi(x, y').$$

The definition is consistent with our intuition: the approximate map associated to an isometry is indeed an approximetry.

EXAMPLES 4.23. There are two distinguished approximetries, which we will turn up quite often.

- The metric d_X on $X \times X$, which corresponds to the identity of X. Note that $d_X^* = d_X$.
- The constant map equal to ∞. It is the **empty approximetry**: it corresponds to the empty isometry.

REMARKS 4.24. The property of being Katětov translates into more global conditions on ψ involving the metric.

- Note that the inequality $\psi d_X \leqslant \psi$ always holds. Indeed, we have
$$\begin{aligned} \psi d_X(x, y) &= \inf_{x' \in X} d_X(x, x') + \psi(x', y) \\ &\leqslant \psi(x, y) \text{ with } x' = x. \end{aligned}$$
- The reverse inequality holds if and only the map ψ is 1-Lipschitz in the first variable.
- The map ψ is Katětov in the first variable (respectively in the second variable) if and only if $\psi = \psi d_X$ and $d_X \leqslant \psi^* \psi$ (respectively $\psi = d_Y \psi$ and $d_Y \leqslant \psi \psi^*$).
- Therefore, ψ is an approximetry if and only if it satisfies all those conditions:
$$\begin{aligned} \psi = \psi d_X &= d_Y \psi, \\ d_X &\leqslant \psi^* \psi, \\ d_Y &\leqslant \psi \psi^*. \end{aligned}$$
Thus, in the world of approximetries, the identification $d_X = \mathrm{id}_X$ is consistent with multiplication.

PROPOSITION 4.25. Let $\psi : X \rightsquigarrow Y$ and $\varphi : Y \rightsquigarrow Z$ be two approximetries. Then
(1) the inverse ψ^* is an approximetry;
(2) the composition $\varphi\psi$ is an approximetry.

PROOF. (1) We apply inversion to the defining properties of an approximetry, remarking that inversion preserves the order and recalling that the metric corresponds to the identity.
(2) First, $\varphi\psi = \varphi(\psi d_X) = (\varphi\psi) d_X$ and $\varphi\psi = (d_Z \varphi)\psi = d_Z(\varphi\psi)$ so the first item in the definition is satisfied. Second, we have
$$\begin{aligned} (\varphi\psi)^*(\varphi\psi) &= \psi^*(\varphi^*\varphi)\psi \\ &\geqslant \psi^*(d_Y \psi) \\ &= \psi^* \psi \\ &\geqslant d_X, \end{aligned}$$
and similarly with d_Z, hence $\varphi\psi$ is an approximetry.

□

EXAMPLE 4.26. If $f : X \to Y$ is an isometry and $\varphi : Y \rightsquigarrow Z$ is an approximetry, then the composition $\varphi\psi_f$ is defined by $\varphi\psi_f(x, z) = \varphi(f(x), z)$.

Approximetries were introduced by Uspenskij with the following geometric interpretation in mind.

PROPOSITION 4.27. Let $\psi : X \times Y \to [0, \infty[$. Let Z be the disjoint union of X and Y. Equip Z with the map d_Z that extends both d_X and d_Y defined by

$$d_Z(x, y) = d_Z(y, x) = \psi(x, y),$$

for all x in X and y in Y. Then d_Z is a pseudometric on Z if and only if the map ψ is an approximetry.

In other words, approximetries define amalgams of metric spaces: the Katětov conditions precisely say that the new pseudometric satisfies the triangle inequality.

3.3. Partial isometries, extensions. Approximetries present the advantages of encoding not only isometries but also partial isometries, as well as their compositions without needing to check whether their domains and ranges are compatible.

DEFINITION 4.28. Let $f : X \dashrightarrow Y$ be a partial isometry, let $f' : \mathrm{dom}(f) \to Y$ be the function f seen as a total map on its domain, and let $i : \mathrm{dom}(f) \to X$ the inclusion map. We define the approximetry $\psi_f : X \rightsquigarrow Y$ by $\psi_f = \psi_{f'}\psi_i^*$:

$$\psi_f(x, y) = \inf_{x_0 \in \mathrm{dom}(f)} d(x, x_0) + d(f(x_0), y).$$

We will identify partial isometries with their associated approximetries. Thus, extending partial isometries (at least formally) becomes possible, while it is not at all easy when staying in the range of ordinary functions.

DEFINITION 4.29. Let $\psi : X \rightsquigarrow Y$ be an approximetry and let $i : X \to X'$ et $j : Y \to Y'$ be isometric embeddings. Then the **trivial extension** of ψ to X' and Y' is the approximetry $j\psi i^* : X' \rightsquigarrow Y'$. Its value at x' and y' is given by $\inf_{y \in Y} \inf_{x \in X} d(x, x') + \psi(x, y) + d(y, y')$.

The trivial extension is a two-sided counterpart to the Katětov extension defined in chapter 2.

It carries to the composition and the inverse.

PROPOSITION 4.30. Let $\psi : X \rightsquigarrow Y$ and $\varphi : Y \rightsquigarrow Z$ be two approximetries. Let also $i : X \to X'$, $j : Y \to Y'$ and $k : Z \to Z'$ be isometric embeddings. Then

- The trivial extension of ψ^* to Y' and X' is the inverse of the trivial extension of ψ to X' and Y'.
- The trivial extension of $\varphi\psi$ to X' and Z' is the composition of the trivial extensions of φ to Y' and Z' and of ψ to X' and Y'.

PROOF. We have that $(j\psi i^*)^* = i\psi^* j^*$. Moreover, since j is an actual isometry, we have $j^* j = \mathrm{id}_Y$, so we have $(k\varphi j^*)(j\psi i^*) = k\varphi(\mathrm{id}_Y \psi)i^* = k(\varphi\psi)i^*$. □

Therefore, in all that follows, we will also identify approximetries with their trivial extensions.

3.4. Measure of the totality of an approximetry. The aim of this section is to quantize how far an approximetry is from being an actual map.

DEFINITION 4.31. Let $\psi : X \rightsquigarrow Y$ be an approximetry and let r be a positive real.

- We say that ψ is r-**total** if for all $x \in X$ and all $s > r$, there exists $y \in Y$ such that $\psi(x, y) < s$, that is, $\inf_{y \in Y} \psi(x, y) \leqslant r$.
- We say that ψ is r-**surjective** if for all $y \in Y$ and all $s > r$, there exists $x \in X$ such that $\psi(x, y) < s$, that is, $\inf_{x \in X} \psi(x, y) \leqslant r$.
- If ψ is both r-total and r-surjective, we say it is r-**bijective**.

REMARK 4.32. If $f : X \dashrightarrow Y$ is a partial map, then ψ_f is r-total if and only if the domain of f is r-dense in X. Similarly, ψ_f is r-surjective if and only if the range of f is r-dense in Y.

The notions of r-totality and r-surjectivity admit more intrinsic reformulations.

PROPOSITION 4.33. Let $\psi : X \rightsquigarrow Y$ be an approximetry and let r be a positive real.

- The approximetry ψ is r-total if and only if $\psi^*\psi \leqslant \mathrm{id}_X + 2r$.
- The approximetry ψ is r-surjective if and only if $\psi\psi^* \leqslant \mathrm{id}_Y + 2r$.

PROOF. We only give the argument for the first item; the second one is similar.

$\Rightarrow$] Assume that ψ is r-total and let x and x' be two elements of X. Since ψ is Katětov in the first variable, we have

$$\begin{aligned}\psi^*\psi(x,x') &= \inf_{y\in Y} \psi(x,y) + \psi(x',y)\\ &\leqslant \inf_{y\in Y} \psi(x,y) + d_X(x',x) + \psi(x,y)\\ &\leqslant d_X(x,x') + 2r.\end{aligned}$$

$\Leftarrow$] Conversely, assume that $\psi^*\psi \leqslant \mathrm{id}_X + 2r$ and let x be an element of X. We know that for all y in Y, we have $\psi(x,y) = \inf_{x'\in X} d_X(x,x') + \psi(x',y)$. Thus, we have

$$\begin{aligned}2\inf_{y\in Y} \psi(x,y) &= \inf_{y\in Y}\inf_{x'\in X} d_X(x,x') + \psi(x',y) + \psi(x,y)\\ &= \inf_{x'\in X} d_X(x,x') + \psi^*\psi(x,x')\\ &\leqslant \inf_{x'\in X} d_X(x,x') + d_X(x,x') + 2r\\ &= 2r,\end{aligned}$$

so ψ is r-total. □

The following proposition says that totality indeed corresponds to being an actual function.

PROPOSITION 4.34. Assume Y is complete and let $\psi : X \rightsquigarrow Y$ be r-total for all $r > 0$. Then there exists an isometric map $f : X \to Y$ such that $\psi = \psi_f$.

Note that since ψ is an approximetry, if such a function f exists, it is necessarily isometric. Moreover, it is unique.

PROOF. Let x be a point in X. For every n in $\mathbb{N}$, the 2^{-n}-totality gives a point y_n in Y such that $\psi(x,y_n) < 2^{-n}$. Then, since ψ is Katětov in the second variable, (y_n) is a Cauchy sequence. The completeness of Y ensures that (y_n) converges to some y in Y.

Note that $\psi(x,y) = 0$. Indeed, $|\psi(x,y) - \psi(x,y_n)| \leqslant d(y,y_n)$ so $\psi(x,y) \leqslant d(y,y_n) + 2^{-n} \longrightarrow 0$. It follows in particular that y does not depend on the choice of y_n's : if $y' \in Y$ satisfies $\psi(x,y') = 0$, then $d(y,y') \leqslant \psi(x,y) + \psi(x,y') = 0$. Thus we set $f(x) = y$.

We now show that the obtained map f is as desired. Let x be in X and y be in Y. Then we have $|\psi(x,y) - \psi(x,f(x))| \leqslant d(y,f(x)) = \psi_f(x,y)$ and since $\psi(x,f(x)) = 0$, this gives $\psi(x,y) \leqslant \psi_f(x,y)$. Conversely, we have $\psi_f(x,y) = d(f(x),y) \leqslant \psi(x,y) + \psi(x,f(x)) = \psi(x,y)$. □

REMARK 4.35. If ψ is moreover r-surjective for all $r > 0$, the corresponding map f will be surjective.

Therefore, in order to build real isometries from approximetries, we will construct finer and finer approximations of it while adding more totality at every step (see theorem 4.52). To do so, we first need to know what *finer* means.

3.5. Refinement. The notions of refinement we present mimic extension of isometric maps in the broader context of approximetries.

DEFINITION 4.36. Let $\psi, \varphi : X \rightsquigarrow Y$ be two approximetries. We say that ψ **approximates** φ, or that φ **refines** ψ, if $\varphi \leqslant \psi$ pointwise.

As in continuous logic, since the refinement is smaller, it is more precise (in particular, it vanishes more often).

EXAMPLE 4.37.
- If f and g are two isometries such that g extends f, then ψ_g refines ψ_f. Indeed, write $f : X \to Y$ and $g : X' \to Y$ with $X \subseteq X'$. Then (after identifying ψ_f with its trivial extension), we have
$$\begin{aligned}\psi_f(x', y) &= \inf_{x \in X} d(x', x) + d(f(x), y)\\ &= \inf_{x \in X} d(g(x'), g(x)) + d(g(x), y)\\ &\geqslant d(g(x'), y) = \psi_g(x', y).\end{aligned}$$
- The definition of an approximetry ensures that an approximetry refines any of its restrictions.

We would like approximate ultrahomogeneity to be witnessed by approximetries, so we introduce a notion of *strict refinement* that will define an open condition on approximetries in the product topology, and will be used later on to define open sets in the automorphism group (see section 5).

DEFINITION 4.38. Let $\psi, \varphi : X \rightsquigarrow Y$ be two approximetries. We say that ψ **strictly approximates** φ, or that φ **strictly refines** ψ, and we write $\varphi < \psi$, if there exist finite sets $X_0 \subseteq X$ and $Y_0 \subseteq Y$, a positive ϵ and an approximetry $\chi : X_0 \rightsquigarrow Y_0$ such that $\varphi + \varepsilon \leqslant \chi \leqslant \psi$.

The notation $\varphi < \psi$ can be misleading: it is stronger than having the inequality $\varphi(x, y) < \psi(x, y)$ everywhere. In particular, as startling as it can be, ψ does not necessarily strictly refine $\psi + r$. Actually, when ψ is the identity, this characterizes total boundedness, as the following proposition shows.

PROPOSITION 4.39. The following assertions are equivalent.
(1) The space X is totally bounded.
(2) For all $r > 0$, one has $\mathrm{id}_X < \mathrm{id}_X + r$.

PROOF. (1) $\Rightarrow$ (2)] Let $X_0 \subseteq X$ be a finite set such that $X \subseteq B(X_0, r)$. Then $\mathrm{id}_X < \mathrm{id}_X + 5r$. Indeed, we always have $\mathrm{id}_X < \mathrm{id}_{X_0} + r$ because $\mathrm{id}_X + r \leqslant \mathrm{id}_{X_0} + r$. For the other inequality, let x and x' be two elements of X and pick x_0 and x_0' in X_0 such that $d(x, x_0) < r$ and $d(x', x_0') < r$. Then
$$\begin{aligned}\mathop{\mathrm{id}}_{X_0}(x, x') + r &\leqslant d(x, x_0) + \mathop{\mathrm{id}}_{X_0}(x_0, x_0') + d(x_0', x') + r\\ &< d(x_0, x_0') + 3r\\ &\leqslant d(x, x') + 5r.\end{aligned}$$
(2) $\Rightarrow$ (1)] Assume, towards a contradiction, that X is not totally bounded. Then there exists a positive r such that for every finite subset X_0 of X, there is a point of X which is not in $B(X_0, r)$. In particular, for every χ is an approximetry between finite subsets X_0 and Y_0 of X, there is a point x such that $x \notin B(X_0 \cup Y_0, r)$. But then, for every such χ, we have
$$\inf_{x_0 \in X_0, y_0 \in Y_0} d(x, x_0) + \chi(x_0, y_0) + d(y_0, x) \geqslant 2r,$$
which prevents the inequality $\mathrm{id}_X + \epsilon \leqslant \chi \leqslant \mathrm{id}_X + r$. □

Nevertheless, when the domain and range are finite (as it is the case for χ in the definition), we fall back on our intuition.

PROPOSITION 4.40. Let $X_0 \subseteq X$ and $Y_0 \subseteq Y$ be two finite subsets. Let $\varphi : X \rightsquigarrow Y$ and $\chi : X_0 \rightsquigarrow Y_0$ be two approximetries. Then $\varphi < \chi$ if and only if for all x_0 in X_0, y_0 in Y_0, one has $\varphi(x_0, y_0) < \chi(x_0, y_0)$.

PROOF. By definition, the condition $\varphi < \chi$ is equivalent to the following: for all $x \in X$, $y \in Y$, $\varphi(x, y) < \inf_{x_0 \in X_0, y_0 \in Y_0} d(x, x_0) + \chi(x_0, y_0) + d(y_0, y)$.

$\Rightarrow$] If $\varphi < \chi$, then in particular, we have $\varphi(x_0, y_0) < \chi(x_0, y_0)$ for every x_0 in X_0 and y_0 in Y_0.

$\Leftarrow$] Conversely, if for every x_0 in X_0 and y_0 in Y_0, $\varphi(x_0, y_0) < \chi(x_0, y_0)$, then we have

$$\inf_{x_0 \in X_0, y_0 \in Y_0} d(x, x_0) + \varphi(x_0, y_0) + d(y_0, y) < \inf_{x_0 \in X_0, y_0 \in Y_0} d(x, x_0) + \chi(x_0, y_0) + d(y_0, y).$$

But since φ is bi-Katětov, it refines its restriction to $X_0 \times Y_0$, so for all x in X and y in Y, we have

$$\varphi(x, y) \leqslant \inf_{x_0 \in X_0, y_0 \in Y_0} d(x, x_0) + \varphi(x_0, y_0) + d(y_0, y),$$

which completes the proof. □

REMARK 4.41. In particular, if f is a partial isometry between finite sets and ϵ is positive, then $f + \epsilon$ strictly approximates f.

EXAMPLE 4.42. The empty approximetry strictly approximates every approximetry, including itself.

In the next proposition, we go over the stability properties of strict refinement.

PROPOSITION 4.43.
(1) If $\varphi < \psi$, then $\varphi^* < \psi^*$.
(2) Let $\psi, \varphi : Y \rightsquigarrow Z$ be two approximetries such that $\varphi < \psi$. Let $\rho : X \rightsquigarrow Y$ be an approximetry on X finite. Then $\varphi\rho < \psi\rho$.
(3) Let $\psi, \varphi : Y \rightsquigarrow Z$ and $\psi', \varphi' : X \rightsquigarrow Y$ be approximetries such that $\varphi < \psi$ and $\varphi' < \psi'$. Then $\varphi\varphi' < \psi\psi'$.
(4) Conversely, if $\rho > \varphi\psi$, then there exist φ' and ψ' such that $\varphi < \varphi'$, $\psi < \psi'$ and $\rho \geqslant \varphi'\psi'$.
(5) Let $\varphi, \varphi', \psi, \psi', \chi, \chi'$ be approximetries such that $\varphi < \psi$, $\varphi' < \psi'$ and $\chi \leqslant \chi'$. Then $\varphi\chi\varphi' < \psi\chi'\psi'$.

PROOF.
(1) If $\varphi + \epsilon \leqslant \chi \leqslant \psi$, then $\varphi^* + \epsilon \leqslant \chi^* \leqslant \psi$.
(2) Since $\varphi < \psi$, there are $\epsilon > 0$, finite sets $Y_0 \subseteq Y$, $Z_0 \subseteq Z$ and an approximetry $\chi : Y_0 \rightsquigarrow Z_0$ such that $\varphi + \epsilon \leqslant \chi \leqslant \psi$. Then the map $\chi i^* \rho$, where $i : Y_0 \to Y$ is the inclusion map, is an approximetry between the finite sets X and Z_0, and we have $\varphi\rho + \epsilon \leqslant \chi i^* \rho \leqslant \psi\rho$.

   Indeed,

$$\begin{aligned}
\varphi\rho(x, z) + \epsilon &= \inf_{y \in Y} \rho(x, y) + \varphi(y, z) + \epsilon \\
&\leqslant \inf_{y \in Y} \rho(x, y) + \chi(y, z) \\
&= \inf_{y \in Y, y_0 \in Y_0, z_0 \in Z_0} \rho(x, y) + d(y, y_0) + \chi(y_0, z_0) + d(z_0, z) \\
&= \inf_{z_0 \in Z_0} \chi i^* \rho(x, z_0) + d(z_0, z) \\
&\leqslant \inf_{y \in Y} \rho(x, y) + \psi(y, z) \\
&= \psi\rho(x, z).
\end{aligned}$$

(3) There exist finite sets $X_0 \subseteq X$, $Y_0, Y_0' \subseteq Y$ and $Z_0 \subseteq Z$, a positive ϵ and approximetries $\chi : Y_0 \rightsquigarrow Z_0$ and $\chi' : X_0 \rightsquigarrow Y_0'$ such that $\varphi + \epsilon \leqslant \chi \leqslant \psi$ and $\varphi' + \epsilon \leqslant \chi' \leqslant \psi'$.

   Set $\chi'' = \chi\chi'$. We show that $\varphi\varphi' + 2\epsilon \leqslant \chi'' \leqslant \psi\psi'$.

$$
\begin{aligned}
\varphi\varphi'(x,z) + 2\epsilon &= \inf_{y\in Y} (\varphi'(x,y) + \epsilon) + (\varphi(y,z) + \epsilon) \\
&\leqslant \inf_{y\in Y} \chi'(x,y) + \chi(y,z) = \chi\chi'(x,z) \\
&\leqslant \inf_{y\in Y} \psi'(x,y) + \psi(y,z) \\
&= \psi\psi'(x,z)
\end{aligned}
$$

(4) There exist a positive ϵ and finite sets $X_0 \subseteq X$, $Z_0 \subseteq Z$ such that $\rho \geqslant (\varphi\psi)_{\restriction X_0\times Z_0} + \epsilon$. Finiteness ensures that there exists a finite subset Y_0 of Y such that $(\varphi\psi)_{\restriction X_0\times Z_0} = (\varphi_{\restriction Y_0\times Z_0})(\psi_{\restriction X_0\times Y_0})$. Thus, we have that $\rho \geqslant (\varphi_{\restriction Y_0\times Z_0})(\psi_{\restriction X_0\times Y_0}) + \epsilon$, which is equal to $(\varphi_{\restriction Y_0\times Z_0} + \frac{\epsilon}{2})(\psi_{\restriction X_0\times Y_0} + \frac{\epsilon}{2})$.

(5) The proof works like for the third point.

□

It will be important, once we relate approximetries with Fraïssé theory, to know when an approximetry strictly approximates the identity (for instance, when defining strict is-almost-phisms, see definition 4.45).

PROPOSITION 4.44. Let $\psi : X \rightsquigarrow Y$ be an approximetry, with $X, Y \subseteq Z$ and X, Y finite. Then the following conditions are equivalent:

- ψ strictly approximates id_X.
- ψ strictly approximates id_Y.
- ψ strictly approximates id_Z.

We will thus say that ψ strictly approximates the identity without further precision.

PROOF. The triangle inequality gives that $\mathrm{id}_X \geqslant \mathrm{id}_Z$. Therefore, if $\psi > \mathrm{id}_X$, then $\psi > \mathrm{id}_Z$. Conversely, if $\psi > \mathrm{id}_Z$, then $\psi = \psi\,\mathrm{id}_X > \mathrm{id}_Z\,\mathrm{id}_X = \mathrm{id}_X$ by proposition 4.43. Likewise for Y. □

4. An approximate route to metric Fraïssé theory

4.1. Is-almost-phisms. We now add some structure to the theory of approximetries in order to approximate isomorphisms between structures of a Fraïssé class. Let $\mathcal{K}$ be class of finitely generated metric structures.

Intuitively, *is-almost-phisms* should be approximetries which approximate an isomorphic embedding. More precisely, it would be tempting to define an is-almost-phism as an approximetry all perturbations of which *strictly* approximate an isomorphic embedding. However, as we have seen in proposition 4.39, the identity does not satisfy this condition in general, so we have to be a bit more careful in the definition and speak only of finite generating sets for our structures.

DEFINITION 4.45.

- Let $\bar{a}$ and $\bar{b}$ be two finite subsets that generate structures in $\mathcal{K}$. Let $\psi : \bar{a} \rightsquigarrow \bar{b}$ be an approximetry. We say that ψ is a **strict is-almost-phism** from $\bar{a}$ to $\bar{b}$ if there exists $\mathbf{C}$ in $\mathcal{K}$ and embeddings $i : \langle\bar{a}\rangle \to \mathbf{C}$ and $j : \langle\bar{b}\rangle \to \mathbf{C}$ such that $j\psi i^* > \mathrm{id}$, or equivalently $\psi > j^* i$.

$$
\begin{tikzcd}
 & \mathbf{C} & \\
\bar{a} \arrow[ur, dotted, "i"] \arrow[rr, rightsquigarrow, "\psi"] & & \bar{b} \arrow[ul, dotted, "j"']
\end{tikzcd}
$$

- Let $\mathbf{A}$ and $\mathbf{B}$ be two structures in $\mathcal{K}$. An approximetry $\psi : \mathbf{A} \rightsquigarrow \mathbf{B}$ is a **strict is-almost-phism** if ψ approximates a strict is-almost-phism $\psi' : \bar{a} \rightsquigarrow \bar{b}$ between finite tuples $\bar{a}$ of $\mathbf{A}$ and $\bar{b}$ of $\mathbf{B}$.

- An approximetry from $\mathbf{A}$ to $\mathbf{B}$ is an **is-almost-phism** if all its strict approximations are strict is-almost-phisms.

We denote by $\mathrm{Alm}(\mathbf{A}, \mathbf{B})$ the set of is-almost-phisms from $\mathbf{A}$ to $\mathbf{B}$ and by $\mathrm{Str}(\mathbf{A}, \mathbf{B})$ the set of strict is-almost-phisms.

From the topological viewpoint, strict is-almost-phisms play the role of finite isomorphisms: they strictly approximate automorphisms and thus define open subsets in the automorphism group (see section 5). Note that strict is-almost-phisms are preserved under approximation.

EXAMPLE 4.46. As expected, actual isomorphisms are is-almost-phisms.

If ψ is a strict is-almost-phism, then so is ψ^*. Moreover, we see that an approximetry between structures of $\mathcal{K}$ is a strict is-almost-phism if and only if it strictly approximates an is-almost-phism, hence the terminology.

REMARK 4.47. If ψ is in $\mathrm{Str}(\bar{a}, \bar{b})$, then there exists $\delta > 0$ such that $\psi - \delta$ stays in $\mathrm{Str}(\bar{a}, \bar{b})$. Define $\Delta(\psi)$ to be the supremum of all such δ's.

In the remainder of this section, we endeavor to express properties of metric Fraïssé classes in terms of is-almost-phisms. The use of is-almost-phisms will allow us to build some kind of limit, which we will then recognize as the desired Fraïssé limit.

4.2. Near amalgamation. Let us first reformulate the **near-amalgamation property**: the class $\mathcal{K}$ has NAP if for all $\mathbf{A} = \langle \bar{a} \rangle$, $\mathbf{B}$ and $\mathbf{C}$ in $\mathcal{K}$, for all embeddings $f_{\mathbf{B}} : \mathbf{A} \to \mathbf{B}$ and $f_{\mathbf{C}} : \mathbf{A} \to \mathbf{C}$, and for every $\epsilon > 0$, there exists a structure $\mathbf{D}$ in $\mathcal{K}$ and embeddings $g_{\mathbf{B}} : \mathbf{B} \to \mathbf{D}$ and $g_{\mathbf{C}} : \mathbf{C} \to \mathbf{D}$ such that $f_{\mathbf{B}}^* g_{\mathbf{B}}^* g_{\mathbf{C}} f_{\mathbf{C}} \leqslant \mathrm{id}_{\bar{a}} + \epsilon$.

$$\begin{tikzcd}
& \mathbf{D} & \\
\mathbf{B} \arrow[ur, dotted, "g_{\mathbf{B}}"] & \text{refines } \mathrm{id}_{\mathbf{A}} + \epsilon & \mathbf{C} \arrow[ul, dotted, "g_{\mathbf{C}}"'] \\
& \bar{a} \arrow[ul, "f_{\mathbf{B}}"] \arrow[ur, "f_{\mathbf{C}}"'] &
\end{tikzcd}$$

The near amalgamation property guarantees that is-almost-phisms compose.

LEMMA 4.48. The following conditions are equivalent.

(1) The class $\mathcal{K}$ satisfies NAP.
(2) The composition of any two strict is-almost-phisms is again a strict is-almost-phism.
(3) The composition of any two is-almost-phisms is again an is-almost-phism.
(4) Every partial isomorphism between elements of $\mathcal{K}$ is an is-almost-phism.

PROOF. (1) $\Rightarrow$ (2)] Let $\psi : \mathbf{A} \rightsquigarrow \mathbf{B}$ et $\varphi : \mathbf{B} \rightsquigarrow \mathbf{C}$ be strict is-almost-phisms. Then ψ and φ are approximations of strict is-almost-phisms $\psi' : \bar{a} \rightsquigarrow \bar{b}$ and $\varphi' : \bar{b} \rightsquigarrow \bar{c}$ (modulo trivial extension, we may assume the domain and range tuples in $\mathbf{B}$ are the same). There exist structures $\mathbf{D}$ and $\mathbf{E}$ in $\mathcal{K}$ which embed $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{B}, \mathbf{C}$ respectively such that $\psi' > j_{\mathbf{D}}^* i_{\mathbf{D}}$ and $\varphi' > j_{\mathbf{E}}^* i_{\mathbf{E}}$.

$$\begin{tikzcd}
& & & \mathbf{F} & & & \\
& \mathbf{D} \arrow[urr, dotted, "i_{\mathbf{F}}"] & & & & \mathbf{E} \arrow[ull, dotted, "j_{\mathbf{F}}"'] & \\
\bar{a} \arrow[ur, "i_{\mathbf{D}}"] \arrow[rrr, rightsquigarrow, "\psi'"] & & & \bar{b} \arrow[ull, "j_{\mathbf{D}}"'] \arrow[urr, "i_{\mathbf{E}}"] \arrow[rrr, rightsquigarrow, "\varphi'"] & & & \bar{c} \arrow[ul, "j_{\mathbf{E}}"']
\end{tikzcd}$$

Pick $\epsilon < \Delta(\psi')$ so that $\psi' - \epsilon > j_{\mathbf{D}}^* i_{\mathbf{D}}$. We apply NAP to $\mathbf{B}$, $\mathbf{D}$, $\mathbf{E}$ (with embeddings $j_{\mathbf{D}}$ and $i_{\mathbf{E}}$) and ϵ to get a structure $\mathbf{F}$ and embeddings $i_{\mathbf{F}}$ and $j_{\mathbf{F}}$ such that $j_{\mathbf{D}}^* i_{\mathbf{F}}^* j_{\mathbf{F}} i_{\mathbf{E}} \leqslant \mathrm{id}_{\mathbf{B}} + \epsilon$. Then, we have

$$\begin{aligned}\varphi\psi &\geqslant \varphi'\psi' \\ &= \varphi'(\psi' - \epsilon) + \epsilon \\ &> j_{\mathbf{E}}^* i_{\mathbf{E}} j_{\mathbf{D}}^* i_{\mathbf{D}} + \epsilon \\ &= j_{\mathbf{E}}^* i_{\mathbf{E}} (\operatorname*{id}_{\mathbf{B}} + \epsilon)^* j_{\mathbf{D}}^* i_{\mathbf{D}} \\ &\geqslant j_{\mathbf{E}}^* i_{\mathbf{E}} (i_{\mathbf{E}}^* j_{\mathbf{F}}^* i_{\mathbf{F}} j_{\mathbf{D}}) j_{\mathbf{D}}^* i_{\mathbf{D}} \\ &= j_{\mathbf{E}}^* j_{\mathbf{F}}^* i_{\mathbf{F}} i_{\mathbf{D}},\end{aligned}$$

hence $\varphi\psi$ is a strict is-almost-phism.

(2) $\Rightarrow$ (3)] Let ψ and φ be two is-almost-phisms and let χ be a strict approximation of $\varphi\psi$. Then, by proposition 4.43, there exist φ' and ψ' such that $\varphi < \varphi'$, $\psi < \psi'$ and $\chi \geqslant \varphi'\psi'$. Since ψ and φ are is-almost-phisms, their strict approximations ψ' and φ' are strict is-almost-phisms. Now, by (2), the composition $\varphi'\psi'$ is a strict is-almost-phism, and so is χ. Thus, $\varphi\psi$ is an is-almost-phism.

(3) $\Rightarrow$ (4)] Let $f : \mathbf{A} \dashrightarrow \mathbf{B}$ be a partial isomorphism and let $f' : \mathrm{dom}(f) \to \mathrm{rng}(f)$ be the underlying isomorphism. We identify f with its trivial extension $j\psi_{f'}i^*$. Since $\psi_{f'}$ and the embeddings j and i are is-almost-phisms, item (3) implies that f is an is-almost-phism too.

(4) $\Rightarrow$ (1)] Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be structures in $\mathcal{K}$ and let $f_{\mathbf{B}} : \mathbf{A} \to \mathbf{B}$ and $f_{\mathbf{C}} : \mathbf{A} \to \mathbf{C}$ be embeddings and let ϵ be a positive real. Then the composition $\psi = f_{\mathbf{C}} \circ f_{\mathbf{B}}^*$ is again a partial isomorphism, hence an is-almost-phism. Moreover, since its domain is finite, $\psi + \epsilon$ is a strict approximation of ψ, hence a strict is-almost-phism. Thus, there exists a structure $\mathbf{D}$ in $\mathcal{K}$ and embeddings $g_{\mathbf{B}}$ and $g_{\mathbf{C}}$ of $\mathbf{B}$ and $\mathbf{C}$ in $\mathbf{D}$ such that $\psi + \epsilon > g_{\mathbf{C}}^* g_{\mathbf{B}}$, which precisely means that $\mathrm{id}_{\bar{a}} + \epsilon > f_{\mathbf{C}}^* g_{\mathbf{C}}^* g_{\mathbf{B}} f_{\mathbf{B}}$. ☐

4.3. Fraïssé limits. In a metric Fraïssé class, finite partial isomorphisms are is-almost-phisms, as per proposition 4.48. Moreover, all of their perturbations are strict is-almost-phisms (see remark 4.41). Thus, as approximate ultrahomogeneity (of a Fraïssé limit) states that every such finite partial isomorphism can be extended to an automorphism, its is-almost-phism counterpart will imply that every strict is-almost-phism can be refined by an automorphism. Actually, the definition of a Fraïssé limit we propose in this context is more finitary: for the back-and-forth argument of theorem 4.52, we will use it in the following guise.

DEFINITION 4.49. Let $\mathcal{K}$ be a Fraïssé class. A **limit** of $\mathcal{K}$ is a separable structure $\mathbf{M}$ whose age is contained in $\mathcal{K}$ that satisfies the following property: for every $(\bar{a}, \mathbf{A})$ in $\mathcal{K}_n$, every ψ in $\mathrm{Str}(\bar{a}, \mathbf{M})$ and every $\epsilon > 0$, there exists a strict is-almost-phism from $\bar{a}$ to $\mathbf{M}$ which is ϵ-total and strictly refines ψ.

All our classes and structures are separable; the following lemma says that it suffices to check the above condition for dense countable objects.

LEMMA 4.50. Let $\mathcal{K}$ be a Fraïssé class and let $\mathbf{M}$ be a separable structure whose age is contained in $\mathcal{K}$. Let $D \subseteq \mathbf{M}$ be a dense countable subset and, for all n, let $\mathcal{K}_{n,0}$ be d_n-dense in $\mathcal{K}_n$. Assume that the criterion for being a Fraïssé limit holds whenever ϵ is rational, $\bar{a} \in K_{n,0}$, $\bar{b} \in D^n$ for some n and $\psi_{\restriction \bar{a} \times \bar{b}}$ only takes rational values. Then $\mathbf{M}$ is a Fraïssé limit of $\mathcal{K}$.

PROOF. Let $\bar{a}$ be an element of $\mathcal{K}_n$ and ψ in $\mathrm{Str}(\bar{a}, \mathbf{M})$, which approximates some ψ' in $\mathrm{Str}(\bar{a}, \bar{b})$ and let ϵ be a positive real. Possibly increasing n and extending $\bar{a}$ et $\bar{b}$ arbitrarily, we may assume that the tuples $\bar{a}$ and $\bar{b}$ have the same length. We may also assume that ϵ is rational. Put $\delta = \frac{1}{4}\min(\epsilon, \Delta(\psi))$. Pick $\bar{a}'$ in $\mathcal{K}_{n,0}$ and $\bar{b}'$ in D^n such that $d(\bar{a}, \bar{a}') < \delta$ and $d(\bar{b}, \bar{b}') < \delta$.

Consider the is-almost-phisms $\chi = d_{\bar{b}\times\bar{b}'}$ and $\rho = d_{\bar{a}'\times\bar{a}}$ which witness the above inequalities and we put $\psi'' = \chi\psi\rho - 3\delta$. Then, by proposition 4.48 and by our choice of δ, the approximetry ψ'' is a strict is-almost-phism from $\bar{a}'$ to $\bar{b}'$. Pick a strict is-almost-phism $\tilde{\psi}$ with rational values such that $\tilde{\psi} < \psi''$. Then our assumption gives an $(\epsilon - \delta)$-total strict is-almost-phism $\tilde{\varphi}$ in $\mathrm{Str}(\bar{a}', \mathbf{M})$ such that $\tilde{\varphi} < \tilde{\psi}$. Then the strict is-almost-phism $\varphi = \tilde{\varphi}\rho^* \in \mathrm{Str}(\bar{a}, \mathbf{M})$ is ϵ-total and $\varphi < \psi''\rho^* < \psi'\rho^* < \chi\psi\rho\rho^* - 3\delta < \psi$. □

THEOREM 4.51. Every metric Fraïssé class admits a limit.

PROOF. Let $\mathcal{K}$ be a Fraïssé class. Fix a dense countable subclass $\mathcal{K}_{n,0}$ of $\mathbf{K}_n$, for each n. We build an increasing chain of structures $\mathbf{B}_n$ from $\mathcal{K}$, together with a dense subset $B_{n,0}$ of $\mathbf{B}_n$ such that

- $\mathbf{B}_0$ is the unique structure in $\mathcal{K}$ generated by the empty set;
- $B_{n,0}$ is contained in $B_{n+1,0}$;
- For every $\bar{a}$ in $\mathcal{K}_{n,0}$, every $\bar{b}$ in $B_{n,0}^n$ and every ψ in $\mathrm{Str}(\bar{a}, \bar{b})$ with rational values, there exists an m and an embedding $\varphi : \langle\bar{a}\rangle \to \mathbf{A}_m$ such that $\varphi < \psi$.

This is done inductively, as in the classical proof (see theorem 4.11), using proposition 4.48. Let $\mathbf{M}$ be the completion of the union of all $\mathbf{A}_n$'s. The structure $\mathbf{M}$ is separable, since $M_0 = \bigcup_{n\in\mathbb{N}} A_{n,0}$ is dense in $\mathbf{M}$. Lemma 4.50 ensures that our conditions suffice to make $\mathbf{M}$ a limit of $\mathcal{K}$. □

Any two limits of a metric Fraïssé class are isomorphic. Even better, we can impose the isomorphism to strictly refine any given strict is-almost-phism. From this, it follows in particular that a limit is approximately ultrahomogeneous.

THEOREM 4.52. Let $\mathcal{K}$ be a metric Fraïssé class. Let $\mathbf{M}$ and $\mathbf{N}$ be two limits of $\mathcal{K}$ and let $\psi : \mathbf{M} \rightsquigarrow \mathbf{N}$ be a strict is-almost-phism. Then there exists an isomorphism from $\mathbf{M}$ to $\mathbf{N}$ that strictly refines ψ.

PROOF. As in the classical case, we carry a back-and-forth argument. Let $\{a_n\}$ and $\{b_n\}$ be enumerations of dense subsets of $\mathbf{M}$ and $\mathbf{N}$ respectively. We will build two increasing sequences of finite tuples $\bar{c}_n$ in $\mathbf{M}$ et $\bar{d}_n$ in $\mathbf{N}$, as well as a decreasing sequence $\theta_n \in \mathrm{Str}(\bar{c}_n, \bar{d}_n)$ of strict is-almost-phisms such that

- $a_n \in \bar{c}_{n+1}$;
- $b_n \in \bar{d}_{n+1}$;
- θ_n is 2^{-n}-total if n is odd and 2^{-n}-surjective if n is even (and positive).

We start with $c_0 \in M$ and $d_0 \in N$ such that ψ approximates some θ_0 in $\mathrm{Str}(c_0, d_0)$ (for instance, we can restrict ψ itself). Given θ_n with n even, put $\bar{c}_{n+1} = (\bar{c}_n, a_n)$. Then, since $\mathbf{N}$ is a limit of $\mathcal{K}$, there exists a strict is-almost-phism θ_{n+1} in $\mathrm{Str}(\bar{c}_{n+1}, \bar{d}_{n+1})$, for some $\bar{d}_{n+1}$ in $\mathbf{N}$, that is 2^{-n}-total such that $\theta_n > \theta_{n+1}$. Trivially extending, we may assume that $\bar{d}_{n+1}$ extends $(\bar{d}_n, b_n)$. We deal with the odd case analogously, keeping in mind that an approximetry is r-total if and only if its inverse is r-surjective.

Now let θ be the pointwise limit of the decreasing sequence (θ_n). We show that θ induces the desired isomorphism. The map θ is the limit of 1-Lipschitz maps, so it is 1-Lipschitz. Thus, by density of $\{a_n\}$ and $\{b_n\}$, it extends to $\mathbf{M} \times \mathbf{N}$.

Moreover, θ is an actual isomorphism: indeed, proposition 4.34 ensures that there is an isometry $f : \mathbf{M} \to \mathbf{N}$ such that $\theta = \psi_f$. It remains to show that f preserves the structure. Let P be a predicate (for the sake of simplicity, assume that P is unary) and let a be an element of $\mathbf{M}$. We show that $P(f(\bar{a})) = P(\bar{a})$.

Consider a subsequence $(a_{\varphi(n)})$ converging to a. Since the predicates are continuous, we also have that $(P(a_{\varphi(n)})$ converges to $P(a)$. Now $\theta_{\varphi(n)}$ is an is-almost-phism, so there exist a structure $\mathbf{C}$ dans $\mathcal{K}$ and embeddings i and j into $\mathbf{C}$ such that :

$$\theta_{\varphi(n)}(a_{\varphi(n)}, f(a_{\varphi(n)})) + 2^{-\varphi(n)} \geqslant d(i(a_{\varphi(n)}), j(f(a_{\varphi(n)}))).$$

Taking the limit, we obtain that $d(i(a_{\varphi(n)}), j(f(a_{\varphi(n)})))$ goes to 0. Since P is continuous, it follows that $P(j(f(a_{\varphi(n)})))$ has the same limit as $P(i(a_{\varphi(n)}))$. But i and j are embeddings so $P(i(a_{\varphi(n)})) = P(a_{\varphi(n)}) \to P(a)$ and $P(j(f(a_{\varphi(n)}))) = P(f(a_{\varphi(n)})) \to P(f(a))$, so $P(f(a)) = P(a)$, which completes the proof. □

5. Topology on the automorphism group

5.1. An approximate basis of open sets. If $\mathcal{K}$ is a Fraïssé class and $\mathbf{M}$ is its Fraïssé limit, recall that the topology on $\mathrm{Aut}(\mathbf{M})$ is given by specifying the behavior of automorphisms on a finite set: the basic open sets are all those sets of the form

$$\{g \in \mathrm{Aut}(\mathbf{M}) : d(g(\bar{a}), f(\bar{a})) < \epsilon\},$$

where $f : \langle\bar{a}\rangle \to \langle\bar{b}\rangle$ is a finite partial isomorphism between elements of $\mathcal{K}$ and ϵ is a positive real. Ultrahomogeneity guarantees that such open sets always are non-empty.

In the context of approximate maps, we can rewrite this topology once again by replacing extension up to ϵ of an isomorphism by strict refinement of a strict is-almost-phism. If ψ is a strict is-almost-phism, we define

$$[\psi] = \{g \in \mathrm{Aut}(\mathbf{M}) : g < \psi\}.$$

Theorem 4.52 ensures that those sets are non-empty as well. Moreover, we defined strict refinement so as to make them open. The following proposition states that those sets too form a basis of the topology on the automorphism group.

PROPOSITION 4.53. The sets $[\psi]$, where ψ is a strict is-almost-phism of $\mathbf{M}$, form a basis of open sets for $\mathrm{Aut}(\mathbf{M})$.

PROOF. Let U be an open set in $\mathrm{Aut}(\mathbf{M})$. We may assume that $U = \{g \in \mathrm{Aut}(\mathbf{M}) : d(g(\bar{a}), f(\bar{a})) < \epsilon\}$, where $f : \bar{a} \to \bar{b}$ is a finite partial isomorphism and ϵ is a positive real.

Put $\psi = \psi_f + \varepsilon$. By lemma 4.48 and remark 4.41, ψ is a strict is-almost-phism. We show that U contains $[\psi]$. If g is in $[\psi]$, we have

$$\begin{aligned} d(g(\bar{a}), f(\bar{a})) &< \psi(\bar{a}, f(\bar{a})) \text{ by proposition 4.40} \\ &= d(f(\bar{a}), f(\bar{a})) + \epsilon \\ &= \epsilon, \end{aligned}$$

so $g \in U$, which completes the proof. □

5.2. The Roelcke uniformity. As mentioned before, the original motivation for introducing approximetries was to give a concrete description of the Roelcke compactification of the isometry group of the Urysohn sphere. Indeed, since the Urysohn sphere is a separably categorical structure (see chapter 6), its isometry group is Roelcke-precompact. Uspenskij proved in **[U5]** that its Roelcke compactification is the space of $[0,1]$-valued approximetries (which is indeed compact), endowed with its unique uniformity. He then studied a semi-group structure of this compact space to show that $\mathrm{Iso}(\mathbb{U}_1)$ is topologically simple and minimal.

In the general case of a metric Fraïssé limit, the Roelcke uniformity can also be expressed in terms of is-almost-phisms.

PROPOSITION 4.54. Let $\mathcal{K}$ be a metric Fraïssé class and let $\mathbf{M}$ be the Fraïssé limit of $\mathcal{K}$. Let G be the automorphism group of $\mathbf{M}$. The Roelcke uniformity on G is generated by the sets

$$E_\psi = \{(f, g) \in G^2 : f < \psi g \psi\},$$

where ψ ranges over strict is-almost-phisms such that $\mathrm{id} < \psi$.

PROOF. We first check that the E_ψ's belong to the Roelcke uniformity. Let ψ be a strict approximation of id and let $V = [\psi]$ be the associated open neighborhood of 1. Then the Roelcke entourage $\{(f, g) \in G^2 : f \in VgV\}$ is contained in E_ψ, by proposition 4.43.

Conversely, let V be an open neighborhood of 1 in G. By proposition 4.53, we may assume that $V = [\psi]$, with ψ in $\mathrm{Str}(\mathbf{M},\mathbf{M})$. Shrinking V, we may also assume that ψ is finite: that ψ is in $\mathrm{Str}(\bar{a},\bar{b})$, with finite tuples $\bar{a}$ and $\bar{b}$. Since ψ is a strict is-almost-phism, there exists a positive ϵ such that $\psi - 3\epsilon$ is still strict.

As $\mathbf{M}$ is a limit of $\mathcal{K}$, there exists an ϵ-total is-almost-phism φ which strictly refines $\psi - 3\epsilon$. Since ψ is finite, it follows that $\varphi + 2\epsilon < \psi$ (proposition 4.40).

We now show that if (f,g) is in E_φ, then $f \in VgV$. Pick an automorphism h in $[\varphi]$. Since $\varphi < \psi$, we have in particular that h is in V. Put $h' = g^{-1}h^{-1}f$ so that $f = hgh'$. It remains to show that h' belongs to V. We have $h < \varphi$, so $h^{-1} < \varphi^*$ (proposition 4.43). Combined with $f < \varphi g \varphi$, we get that

$$\begin{aligned} h' &= g^{-1}h^{-1}f \\ &\leqslant g^{-1}(\varphi^*\varphi)g\varphi \\ &\leqslant g^{-1}(\mathrm{id}+2\epsilon)g\varphi \\ &\leqslant \varphi + 2\varepsilon \\ &< \psi, \end{aligned}$$

hence h' is indeed in V. □

We may now wonder whether Uspenskij's result still holds for any separably categorical structure, with is-almost-phisms in place of approximetries. Let $\mathbf{M}$ be a separably categorical ultrahomogeneous structure and let G be its automorphism group. Now is a good time to assume that our structure is bounded so that G is Roelcke-precompact. Let also Θ be the compact space of $[0,1]$-valued is-almost-phisms of $\mathbf{M}$, which embeds the automorphism group G. Although Θ seems like a good candidate to be the Roelcke compactification of G, it is not always the case : the uniformity on G induced by Θ does not coincide with the Roelcke uniformity. Indeed, when the structure is discrete, the uniformity induced by Θ on the automorphism group is discrete, hence, it cannot be precompact (unless the group is finite).

CHAPTER 5

Leitmotiv

Qu'est-ce qu'il y a sous ton grand chapeau?
Annie Cordy[1]

1.	**Non-archimedean Polish groups**	**83**
2.	**General Polish groups**	**84**

We have seen that automorphism groups of separable metric structures are Polish groups. Actually, those encompass *all* Polish groups. From this observation arises a very fertile correspondence between topological groups and model-theoretic structures, which constitutes the guiding line of our work. We shall indeed explore several facets of this correspondence over the next chapters.

Further still, not only is every Polish group the automorphism group of some metric structure, but the structure can also be chosen to be approximately ultrahomogeneous. From here, Fraïssé theory opens a fruitful combinatorial approach to the study of Polish groups

1. Non-archimedean Polish groups

Let us first present the proof for closed subgroups of S_∞, which we know include automorphism groups of countable classical structures. Those groups admit a basis at the identity that consists of open subgroups, we say they are **non-archimedean**. Actually, every non-archimedean Polish group is isomorphic to a subgroup of S_∞ (see [**G1**, theorem 2.4.1]).

THEOREM 5.1. (Folklore) Let G be a Polish group. Then the following are equivalent.

(1) G is a closed subgroup of S_∞.
(2) There exists a countable ultrahomogeneous structure in a classical language whose automorphism group is G.

PROOF. (2) $\Rightarrow$ (1)] This is proposition 3.22 (and we do not need that the structure is ultrahomogeneous).

(1) $\Rightarrow$ (2)] Since G is a subgroup of S_∞, it acts on $\mathbb{N}$; we thus define a structure on $\mathbb{N}$. For every n, we consider on $\mathbb{N}^n$ the orbit equivalence relation induced by the diagonal action of G:

$$\bar{a} \sim_n \bar{b} \Leftrightarrow \exists g \in G, g(\bar{a}) = \bar{b}.$$

Call $\mathcal{C}_n$ the collection of all $\sim_n$-classes, and let $\mathcal{C}$ be the union of all the $\mathcal{C}_n$.

From this, we define a language by naming each orbit: let $\mathcal{L}$ consist of an n-ary relation symbol R_c, for each class c in $\mathcal{C}$. Note the $\mathcal{L}$ is countable. Now we build an $\mathcal{L}$-structure $\mathbf{M}$, with universe $\mathbb{N}$, the natural way: for every c in $\mathcal{C}$, put $R_c^{\mathbf{M}} = c$. We prove that $G = \mathrm{Aut}(\mathbf{M})$.

First, it is clear that G is contained in $\mathrm{Aut}(\mathbf{M})$. Conversely, let f be an automorphism of $\mathbf{M}$. Since G is closed, it suffices, for every n, to find an element g_n in G which coincides with f on $\{0, ..., n\}$. Now, since f is an automorphism, we have that $R_c^{\mathbf{M}}(0, ..., n) \Leftrightarrow R_c^{\mathbf{M}}(f(0), ..., f(n-1))$ for every c in $\mathcal{C}_{n+1}$. By definition of the relation R_c, this means there exists an element g_n in G such that for all $i \leqslant n$, we have $g_n(i) = f(i)$, as desired.

Moreover, the construction ensures that the structure $\mathbf{M}$ is ultrahomogeneous. $\square$

[1] *Tata Yoyo*

2. General Polish groups

As for general Polish groups, the argument goes similarly. Instead of naming orbits in the language, we will name closed orbits (or rather the associated distance predicate). But in order to speak about orbits, we need to view our Polish group as a closed subgroup of some isometry group. It turns out that there is a natural way to do so.

Let G be a Polish group. Equip G with a compatible left-invariant metric d_L (theorem 1.17), and consider the completion (X, d) of (G, d_L). This is a Polish space which fits our purposes.

THEOREM 5.2. The group G is isomorphic to a closed subgroup of $\mathrm{Iso}(X, d)$.

PROOF. Consider the action of G on itself by left translation. By continuity of the group operations, the action is continuous. Moreover, the action is isometric when G is endowed with d_L. Thus, it extends to an action of G on X by isometries.

We now show that this action is continuous. To this aim, let (g_m) be a sequence that converges to g in G and let (h_n) be a Cauchy sequence in (G, d_L) that converges to x in X. Then we have

$$d(g \cdot x, g_m \cdot x) \leqslant d(g \cdot x, gh_n) + d(gh_n, g_m h_n) + d(g_m h_n, g_m \cdot x).$$

For a large enough n, the lefthand and righthand terms are smaller than ϵ, and, by continuity of right multiplication in G, for large enough m's, the middle term is smaller than ϵ too. Thus, the action is continuous. Hence, we have a continuous injective homomorphism f from G to $\mathrm{Iso}(X, d)$.

Finally, let us show that the inverse of f is continuous. If $f(g_n)$ tends to id_X (with respect to the pointwise convergence topology), then $f(g_n)(1_G)$ tends to 1_G in X, that is, g_n tends to 1_G, so the inverse of f is continuous.

As a consequence, G is isomorphic to a subgroup of $\mathrm{Iso}(X, d)$. Since G is Polish, it must be closed in $\mathrm{Iso}(X, d)$ (theorem 1.12). □

This yields a canonical structure $\widehat{G}$, whose automorphism group is G and which we call the **hat structure** of G (see [**M5**, theorem 6] for its construction).

The universe of $\widehat{G}$ is the left completion (X, d) of G. Analogously to the classical case, we consider the following equivalence relation on X^n:

$$\bar{a} \sim_n \bar{b} \Leftrightarrow \bar{a} \in \overline{G \cdot \bar{b}}.$$

Let $\mathcal{C}_n$ be the collection of all $\sim_n$-classes and $\mathcal{C}$ the union of all $\mathcal{C}_n$. The language then consists of an n-ary predicate P_c for each c in $\mathcal{C}$. The predicates are naturally interpreted as follows: $P_c^{\widehat{G}}(\bar{x}) = d(\bar{x}, c)$.

As desired, we have the following.

THEOREM 5.3. (Melleray) The automorphism group of the structure $\widehat{G}$ is G.

PROOF. Again, it is clear that G is contained in $\mathrm{Aut}(\widehat{G})$. For the converse direction, let f be an automorphism of $\widehat{G}$. Let $\bar{a}$ be a tuple in X^n. Then, since f is an automorphism, we have $\bar{a} \sim_n f(\bar{a})$. Thus, for every positive ϵ, there exists a group element g in G such that $d(g(\bar{a}), f(\bar{a})) < \epsilon$. It follows that f is a pointwise limit of elements of G, hence in G, because G is closed in $\mathrm{Iso}(X, d)$. □

Moreover, the construction again ensures that $\widehat{G}$ is approximately ultrahomogeneous, thus proving that Polish groups are exactly automorphism groups of separable approximately ultrahomogeneous structures.

REMARK 5.4. Melleray asked whether every Polish group is the automorphism group of an *exactly* ultrahomogeneous structure. That is not the case, as was proved by Ben Yaacov in [**B7**]. He provided examples of Roelcke-precompact groups that cannot admit any transitive continuous action by isometries on a non-trivial complete metric space.

Branch 1

Reconstruction

CHAPTER 6

Categoricity

Jongle avec tout ce que tu as.
Maxime Le Forestier[1]

1. Ryll-Nardzewski **88**
1.1. Principal types 88
1.2. Atomic models 89
2. Action of the automorphism group **90**
2.1. Definability 91
2.2. Oligomorphicity 91
3. Homogeneity **93**

In this chapter, we present a distinguished class of structures, rigid enough to allow for a rich correspondence between them and their automorphism groups.

Let $\mathcal{L}$ be a language and let T be a complete $\mathcal{L}$-theory.

DEFINITION 6.1.
- If $\mathcal{L}$ is classical, we say that T is **countably categorical** or **$\aleph_0$-categorical** if any two countable models of T are isomorphic.
- If $\mathcal{L}$ is continuous, we say that T is **separably categorical** if any two separable models of T are isomorphic.
- An $\mathcal{L}$-structure is called **countably** or **separably categorical** if its theory is.

REMARK 6.2. Throughout the chapter, the requirement that metric structures be bounded is in order, as the compactness theorem will be used fully.

The previous conditions may seem too restrictive, but in fact, categorical structures are plentiful.

EXAMPLES 6.3.
- The countably infinite pure set is $\aleph_0$-categorical. The theory says exactly that the structure is infinite: for all n, the formula

$$\exists x_1, ..., x_n, \left(\bigwedge_{i \neq j} x_i \neq x_j \right)$$

  is in the theory. Now, any two countable infinite sets are in bijection.
- The rationals with their order are $\aleph_0$-categorical. The structure $(\mathbb{Q}, <)$ is a dense linear order without endpoints, and these properties can be expressed by formulas:
  - transitivity: $\forall x, \forall y, \forall z, (x < z) \vee \neg[(x < y) \wedge (y < z)]$;
  - antisymmetry: $\forall x, \forall y, \neg[(x < y) \wedge (y < x)]$;
  - linearity: $\forall x, \forall y, [(x < y) \vee (y < x) \vee (x = y)]$;
  - density: $\forall x, \forall z, [\neg(x < z) \vee (\exists y, x < y < z)]$;
  - no endpoints: $\forall x, \exists y, (x < y)$ and $\forall x, \exists y, (y < x)$,

  so this is part of the theory. Now an easy back-and-forth argument shows that any two countable dense linear orders without endpoints are isomorphic.

[1] *Saltimbanque*

- The random graph is countably categorical. As mentioned in example 4.12, the random graph is characterized by the following property: for any two disjoint subgraphs $\mathbf{A}$ and $\mathbf{B}$, there exists a vertex in $\mathcal{R}$ that is related to every point in $\mathbf{A}$ but to no point in $\mathbf{B}$. This property can also be expressed by formulas, so $\mathcal{R}$ is $\aleph_0$-categorical.
- The Urysohn sphere is separably categorical. This will follow from the Ryll-Nardzewski theorem.
- On the other hand, the Urysohn space (when made bounded, see remark 3.11) is not separably categorical. Indeed, in this shrinked Urysohn space, no two points are at distance 1, which corresponds to an infinite distance in $\mathbb{U}$. However, the theory cannot prevent it (we recall that in metric model theory, there is no negation): it still contains the formula $\sup_x \sup_y d(x, y) = 1$. Note that the theory of the Urysohn space says that all maps of the form $\dfrac{f}{1+f}$, where f is a Katětov map for d, are realized. Since these maps are Katětov maps with respect to the shrinked metric, we may still apply the Katětov construction with these maps in place of the usual Katětov maps. For instance, we build another model of the theory of $\mathbb{U}$ as follows. Start from a metric space consisting of two points at distance 1 (this corresponds to having a point at infinity). Then, apply the new Katětov construction to this metric space. The resulting structure then has the same theory as the Urysohn space, but cannot be isomorphic to it.

We now present a most powerful characterization of categoricity: a topological condition on the space of types, which is due to Ryll-Nardzewski ([**RN**]) in the classical case and to Henson, as well as Ben Yaacov and Usvyatsov ([**BU1**, fact 1.14]) in the continuous setting. This characterization has a wide array of consequences, most important among which is a description of the action of the automorphism group and strong homogeneity properties.

1. Ryll-Nardzewski

Let $\mathcal{L}$ be a language and let T be an $\mathcal{L}$-theory.

1.1. Principal types. We start by describing a specific class of types, *principal types*, which will happen to be the only types that countably or separably categorical structures realize.

Definition 6.4. A type p in $S(T)$ is said to be **principal** if in every model $\mathbf{M}$ of T, the set of realizations of p in $\mathbf{M}$ is definable.

It can be shown (see [**BBHU**, lemma 12.3]) that it suffices that the set of realizations of p in *some* model is definable and non-empty for p to be principal. Even better, this implies that the set of realizations of p is definable and non-empty in every model. Applying this observation to a sufficiently saturated model, we obtain the following.

Proposition 6.5. Principal types are realized in every model.

In the next section, we prove a strong converse to this proposition. To do so, we will again place ourselves in a sufficiently saturated model, in order to get a grasp on the metric on the space of types (see subsection 5.3 of chapter 3). The essential result is the following topological characterization of principal types.

Theorem 6.6. Let p be a type in $S_n(T)$. Then p is principal if and only the logic topology and the metric topology coincide at p.

Proof. Let $\mathbf{N}$ be a sufficiently saturated model of T. The discussion above guarantees that p is principal if and only the set of realizations of p in $\mathbf{N}$ is definable (by saturation, it is already non-empty).

$\Rightarrow$] Assume p is principal. By proposition 3.65, for every integer m, there exists an $\mathcal{L}$-formula φ_m and $\delta_m > 0$ such that the condition $\varphi_m = 0$ is in p and every type $q \in S_n(T)$ that contains the condition $\varphi_m \leqslant \delta_m$ satisfies $d(p,q) \leqslant 2^{-m}$. Then the d-ball $B(p, 2^{-m})$ contains the open neighborhood $[\varphi_m < \delta_m]$ of p in the logic topology. Since, in addition, the metric topology is finer than the logic topology (proposition 3.59), it follows that the two topologies coincide at p.

$\Leftarrow$] Conversely, suppose that $[\varphi < r]$ is an open neighborhood of p contained in the d-ball $B(p, 2^{-m})$. Then, by definition, there exists δ, with $0 < \delta < r$, such that the condition $\varphi \leqslant \delta$ belongs to p. Let φ_m be the formula $\max(0, \varphi - \delta)$ and let δ_m be any real such that $0 < \delta_m < r - \delta$. Then the condition $\varphi_m = 0$ is in p and for every type q in $S_n(T)$ that contains the condition $\varphi_m \leqslant \delta_m$, we have $d(p,q) \leqslant 2^{-m}$. Thus, by proposition 3.65 again, the type p is principal. □

When the language is classical, the metric topology is discrete, so p is principal if and only if $\{p\}$ is clopen in $S_n(T)$. Consequently, principal types are also called **isolated**. Moreover, since $S_n(T)$, endowed with the logic topology, is compact, theorem 6.6 implies that if the space of types is not metrically compact, it contains non-principal types. In particular, in the classical case, if the space of types is infinite, then it contains a non-principal type.

As we have seen, every model of the theory realizes all principal types. Conversely, principal types are the only ones with this property.

THEOREM 6.7. (Omitting types) Let p be a type in $S(T)$. Then the following are equivalent.

(1) p is principal.
(2) p is realized in every model of T.

PROOF. (1) $\Rightarrow$ (2)] This is proposition 6.5.

(2) $\Rightarrow$ (1)] Suppose that p is not principal. Then, by theorem 6.6, we find a d-ball $B(p, \epsilon)$ around p whose interior for the logic topology is empty. In other words, for every formula and every positive δ, the open set $[\varphi < \delta]$ is either empty or contains a type q such that $d(q,p) \geqslant \epsilon$.

From there, an application of the compactness theorem gives a model of T in which every realized type q satisfies $d(q,p) \geqslant \epsilon$ (see [**BBHU**, theorem 12.16]). Thus, in this model, p cannot be realized. □

1.2. Atomic models. In view of the previous theorem, it is interesting to study models which realize *only* principal types. Such models are called **atomic**. We prove the uniqueness of countable or separable atomic models. In fact, we prove a stronger result, which yields that such atomic models are (approximately) homogeneous. To clarify the proof, we start with the classical case.

PROPOSITION 6.8. Assume that the language is classical and let $\mathbf{M}$ and $\mathbf{N}$ be two countable atomic models of T. Let $\bar{a} \in \mathbf{M}^n$ and $\bar{b} \in \mathbf{N}^n$ be two tuples that satisfy the same type. Then there exists an isomorphism from $\mathbf{M}$ to $\mathbf{N}$ sending $\bar{a}$ to $\bar{b}$.

PROOF. Since $\bar{a}$ and $\bar{b}$ have the same type, the map f sending $\bar{a}$ to $\bar{b}$ is a partial isomorphism. By back-and-forth, we extend f to a global isomorphism. Let us explain how the first step forward works, the others are similar. Pick a new point a in $\mathbf{M}$ and consider the type p of $(a, \bar{a})$. Atomicity ensures that p is principal, hence isolated in $S_{n+1}(T)$. This means that there exists a formula φ such that $[\varphi] = \{p\}$.

Now $\mathbf{M}$ satisfies the condition $\exists x, \varphi(x, \bar{a})$. In other words, the condition $\exists x, \varphi(x, \bar{x})$ belongs to the type of $\bar{a}$, hence to the type of $\bar{b}$. Let then b be a point in $\mathbf{N}$ such that $\mathbf{N} \vDash \varphi(b, \bar{b})$. Since $[\varphi] = \{p\}$, the type of $(b, \bar{b})$ is the same as that of $(a, \bar{a})$. In particular, we may extend f to a partial isomorphism by putting $f(a) = b$. □

In the metric case, the homogeneity needs relaxing, as usual.

Proposition 6.9. Let $\mathbf{M}$ and $\mathbf{N}$ be two separable atomic models of T. Let $\bar{a} \in \mathbf{M}^n$ and $\bar{b} \in \mathbf{N}^n$ be two tuples that satisfy the same type. Then, for every positive ϵ, there exists an isomorphism $f : \mathbf{M} \to \mathbf{N}$ such that $d(f(\bar{a}), \bar{b}) < \epsilon$.

Proof. The argument is essentially the same as in the classical case, with the small difference that we need to keep track of the error. As before, we proceed by back-and-forth on dense subsets of $\mathbf{M}$ and $\mathbf{N}$ and we extend $\bar{a} \mapsto \bar{b}$, up to ϵ, to an isomorphism. Consider a new point a in $\mathbf{M}$ and the type p of $(a, \bar{a})$. Also, denote by q the type of $\bar{a}$ (and $\bar{b}$).

Since p and q are realized in $\mathbf{M}$, they are principal. Thus, the sets of all their realizations is definable, say by definable predicates $P(x, \bar{x})$ and $Q(\bar{x})$.

Consider now the following definable predicate:

$$F(\bar{x}) = \inf_x |Q(\bar{x}) - P(x, \bar{x})|.$$

Observe that $Q(\bar{a}) = 0$ and $F(\bar{a}) = 0$. We deduce that $Q(\bar{b}) = 0$ and $F(\bar{b}) = 0$ too, because the tuples $\bar{a}$ and $\bar{b}$ have the same type. It follows in particular that $\inf_x P(x, \bar{b}) = 0$. Therefore, there exists b_1 in $\mathbf{N}$ such that $P(b_1, \bar{b}) < \dfrac{\epsilon}{2}$.

But P is the distance to the set of realizations of p, so there is a realization $(c_1, \bar{c}_1)$ of p such that $d(c_1, b_1) < \dfrac{\epsilon}{2}$ and $d(\bar{c}_1, \bar{b}) < \dfrac{\epsilon}{2}$. We now replace f with the partial isomorphism that sends $(a, \bar{a})$ to $(c_1, \bar{c}_1)$. For the next step, we do the same thing with precision $\dfrac{\epsilon}{4}$ and so on. □

As mentioned before, the two previous propositions imply in particular that atomic models are **(approximately) homogeneous**: any two tuples with the same type can be sent one onto the other (or arbitrarily close to one another) by an automorphism. We would like to stress, however, that this holds for tuples that really have the same type, not only the same quantifier-free type: this property is weaker than (approximate) ultrahomogeneity.

As a corollary, we obtained the promised characterizations of countably and separably categorical structures.

Theorem 6.10. (The Ryll-Nardzewksi theorem; Henson, Ben Yaacov-Usvyatsov) The following are equivalent.

(1) The theory T is separably categorical.
(2) Every type in $S(T)$ is principal.
(3) Every model of T is atomic.
(4) The logic topology and the metric topology on $S_n(T)$ coincide, for every n.
(5) For every n, the space $(S_n(T), d)$ is compact.

Proof. (1) $\Rightarrow$ (2)] Suppose that $p \in S(T)$ is not principal. By the compactness theorem, T admits a model in which p is realized. Also, by the omitting types theorem (6.7), there exists a model of T in which p is not realized. Two such models can be chosen to be separable, thanks to the Löwenheim-Skolem theorem (see [**BBHU**, proposition 7.3]). But they cannot be isomorphic, contradicting the separable categoricity.

(2) $\Rightarrow$ (3)] Immediate.

(3) $\Rightarrow$ (1)] Proposition 6.9 guarantees that any two separable models of T, which are atomic by assumption, are isomorphic, so T is separably categorical.

The equivalence between (2), (4) and (5) follows from theorem 6.6 and the ensuing discussion. □

2. Action of the automorphism group

The Ryll-Nardzewski theorem establishes tight links between model theoretic properties of a separably categorical structure and topological properties of its automorphism group. In the next chapter, we will see that there is actually a perfect correspondence between the two.

2.1. Definability. First, we prove that in separably categorical structures, definability amounts to invariance under the action of the automorphism group.

PROPOSITION 6.11. Let $\mathbf{M}$ be a separably categorical metric structure and G its automorphism group. Let $P : \mathbf{M}^n \to [0,1]$ be a continuous predicate on $\mathbf{M}$. Then P is definable in $\mathbf{M}$ if and only if P is G-invariant.

PROOF. $\Rightarrow]$ If P is definable, there is a sequence $(\varphi_k)_{k\geqslant 1}$ of formulas which converges uniformly to P. Now G preserves (interpretations of) formulas so P is also G-invariant.

$\Leftarrow]$ Suppose that P is G-invariant. If $\bar{a}$ and $\bar{b}$ have the same type in $\mathbf{M}^n$, then, since $\mathbf{M}$ is approximately homogeneous (proposition 6.9) and P is continuous, the G-invariance of P gives that $P(\bar{a}) = P(\bar{b})$.

Thus, P induces a metrically continuous map $\Phi : S_n(T) \to [0,1]$ on types, defined by $\Phi(p) = P(\bar{a})$ for $\bar{a} \in \mathbf{M}^n$ of type p. Since, by the Ryll-Nardzewski theorem (6.10), every type is realized in $\mathbf{M}$, the map Φ is well-defined.

Now, by the Ryll-Nardzewski theorem again, the logic topology and the metric topology on $S_n(T)$ coincide. This implies that Φ is continuous for the logic topology as well. Thus, by theorem 3.63, the predicate P is definable. □

REMARK 6.12. The same holds for predicates in an infinite number of variables. In fact, if $\mathbf{M}^\omega$ is endowed with d^ω (where d^ω is defined just below), then the Ryll-Nardzewski theorem can be reformulated as follows: a metric structure $\mathbf{M}$ is separably categorical if and only if the space $(S_\omega(\mathbf{M}), d^\omega)$ of types in infinitely many variables is compact. Thus, the proof above readily adapts to an infinite number of variables.

We introduce an item of notation that we will use freely in the remainder of the thesis.

NOTATION 6.13. If ρ is a bounded pseudometric on a structure $\mathbf{M}$, then $(\mathbf{M}, \rho)$ will denote the quotient metric space induced by ρ. We write $\widehat{(\mathbf{M}, \rho)}$ for its completion. For such a ρ, let ρ^ω be the pseudometric on $\mathbf{M}^\omega$ defined by

$$\rho^\omega(a, a') = \sum_{n<\omega} \frac{1}{2^n} \rho(a_n, a'_n).$$

When ρ is a metric, so is ρ^ω, which then induces the product topology on $\mathbf{M}^\omega$.

2.2. Oligomorphicity. On a slightly different note, categoricity translates as a dynamical property of the automorphism group.

If a group G acts on a set X, we consider the **diagonal action** of G on X^n by

$$g \cdot (x_1, ..., x_n) = (g \cdot x_1, ..., g \cdot x_n).$$

DEFINITION 6.14. Let G be a group acting by isometries on a complete metric space X.

- The action of G on X is called **oligomorphic** if for every n, the diagonal action of G on X^n only admits finitely many orbits.
- It is called **approximately oligomorphic** if for every positive ϵ and every n, there exists finitely many elements $\bar{a}_1, ..., \bar{a}_m$ of X^n such that for every $\bar{x}$ in X^n, there is a group element g and an i such that $d(\bar{x}, g \cdot \bar{a}_i) < \epsilon$.

REMARK 6.15. Consider the space $X^n /\!\!/ G$ of orbit closures of X^n under G. We endow $X^n /\!\!/ G$ with the following pseudometric:

$$d(\overline{G \cdot \bar{a}}, \overline{G \cdot \bar{b}}) = \inf\{d(\bar{x}, \bar{y}) : \bar{x} \in \overline{G \cdot \bar{a}}, \bar{y} \in \overline{G \cdot \bar{b}}\}.$$

Since G acts on X by isometries, this pseudometric is the distance between orbits, so it is actually a metric. Moreover, the space $X^n /\!\!/ G$ is complete when X is. We will write $[\bar{a}]$ for the closure of the orbit of $\bar{a}$.

Now the action of G on X is approximately oligomorphic if and only if for every n, the space $X^n /\!\!/ G$ is totally bounded, hence compact.

Theorem 6.16. Let $\mathbf{M}$ be a separable model of T. Then the following are equivalent.

(1) The structure $\mathbf{M}$ is separably categorical.
(2) The action of $\mathrm{Aut}(\mathbf{M})$ on $\mathbf{M}$ is approximately oligomorphic.

Proof. (1) $\Rightarrow$ (2)] By the Ryll-Nardzewski theorem and proposition 6.9, the structure $\mathbf{M}$ is approximately homogeneous. This implies that two tuples with the same type are in the same orbit closure. Since automorphisms preserve types, the converse also holds, hence the space $\mathbf{M}^n \sslash \mathrm{Aut}(\mathbf{M})$ is $S_n(T)$ endowed with the metric topology. But the Ryll-Nardzewski theorem ensures that $(S_n(T), d)$ is compact, so the action of $\mathrm{Aut}(\mathbf{M})$ on $\mathbf{M}$ is approximately oligomorphic.

(2) $\Rightarrow$ (1)] We show that the space $(S_n(T), d)$ is compact. By the Ryll-Nardzewski theorem, this will complete the proof. Consider the space X of realized types:

$$X = \{p \in S_n(T) : p \text{ is realized in } \mathbf{M}\}.$$

We will first show that (X, d) is compact and then that $X = S_n(T)$.

Since any two tuples that are in the same orbit have the same type, the space (X, d) is the continuous image of $\mathbf{M}^n \sslash \mathrm{Aut}(\mathbf{M})$ under the map $\bar{a} \mapsto \mathrm{tp}(\bar{a})$. Hence, the approximate oligomorphicity gives that (X, d) is compact.

But the logic topology is coarser than the metric topology, so X is also compact in the logic topology. In particular, X is closed in the logic topology. Thus, it suffices to show that X is dense in $S_n(T)$ (for the logic topology). To do so, consider a non-empty basic open set $[\varphi < r]$ in $S_n(T)$ and let q be a type in that open set. Then there exists δ, with $0 \leqslant \delta < r$ such that the condition $\varphi \leqslant \delta$ belongs to q. It follows that the theory contains the formula $\inf_x \max(0, \varphi(x) - \delta) = 0$, and therefore that $\mathbf{M}$ satisfies this condition. Thus, for every positive $\epsilon < r - \delta$, we can find a tuple $\bar{a}$ in $\mathbf{M}^n$ such that $\varphi^{\mathbf{M}}(a) \leqslant \delta + \epsilon < r$. Finally, the type of $\bar{a}$ is in X as well as in our open set $[\varphi < r]$, which completes the proof. $\square$

Remark 6.17. Note that the proof actually shows that in general, the space of types that are realized in some fixed model of T is dense in $S_n(T)$.

The previous proof provides a very nice example of the juggling between topologies in the topometric space of types, which we mentioned at the end of section 5 of chapter 3.

Approximate oligomorphicity turns out to be an intrinsic condition on the automorphism group: Roelcke precompactness. First, approximate oligomorphicity implies Roelcke precompactness. In particular, the automorphism group of a separably categorical structure is Roelcke-precompact.

Theorem 6.18. (Rosendal, see [**R3**, theorem 5.2]) Let G act continuously by isometries on a complete metric space X. If the action of G on X is approximately oligomorphic, then G is Roelcke-precompact.

Independently, Rosendal ([**R4**, proposition 1.22]) and Ben Yaacov and Tsankov ([**BT1**, theorem 2.4]) established a converse to the previous theorem.

Theorem 6.19. Let G be a Polish group. Then the following are equivalent.

(1) G is Roelcke-precompact.
(2) Whenever G acts continuously by isometries on a complete metric space X and $X \sslash G$ is compact, the action is approximately oligomorphic.
(3) There exists a separably categorical metric structure whose automorphism group is G.

Proof. (1) $\Rightarrow$ (2)] We proceed by induction. Assume $X^n \sslash G$ is compact. We prove that $X^{n+1} \sslash G$ is compact. More precisely, we prove that it can be covered with finitely many balls of radius 2ϵ, for every positive ϵ.

We cover $X^n \sslash G$ with finitely many balls of radius ϵ, say centered at $[\bar{y}_1]$, ..., $[\bar{y}_m]$. By assumption, $X \sslash G$ is also compact, so we can cover it with balls of radius ϵ centered at $[z_1]$, ..., $[z_p]$. Let V be a symmetric neighborhood of 1 in G such that for all j and k, we have

$d(y_j, V \cdot y_j) < \epsilon$ and $d(z_k, V \cdot z_k) < \epsilon$. Now, by Roelcke precompactness, there exists a finite subset F of G such that $G = VFV$. Consider the finite subset of X^{n+1} defined by

$$W = \{(\bar{y}_j, f \cdot z_k) : j \in \{1, ..., m\}, k \in \{1, ..., p\}, f \in F\}.$$

We show that $X^{n+1} /\!\!/ G$ is covered by balls of radius 2ϵ centered at the closure of orbits of elements of W. Let $(\bar{x}, x_{n+1})$ be in X^{n+1}. There exists group elements g, g_{n+1} and $\bar{y}_j$, z_k such that $d(\bar{x}, g \cdot \bar{y}_j) < \epsilon$ and $d(x_{n+1}, g_{n+1} \cdot z_k) < \epsilon$. Also, write $g^{-1} g_{n+1} = v f v'$, with v, v' in V and f in F. Now, we have

$$\begin{aligned} d([\bar{x}, x_{n+1}], [\bar{y}_j, f \cdot z_k]) &\leqslant d([\bar{x}, x_{n+1}], [g \cdot \bar{y}_j, g_{n+1} \cdot z_k]) + d([g \cdot \bar{y}_j, g_{n+1} \cdot z_k], [\bar{y}_j, f \cdot z_k]) \\ &< \epsilon + d([v^{-1} \cdot y_j, f v' \cdot z_k], [\bar{y}_j, f \cdot z_k]) \\ &< 2\epsilon, \end{aligned}$$

which completes the proof.

(2) $\Rightarrow$ (3)] Consider the hat structure $\widehat{G}$. As we have seen in theorem 5.3, G is the automorphism group of $\widehat{G}$. Besides, the space $\widehat{G} /\!\!/ G$ is a singleton, so it is compact. Thus, condition (2) gives that the action of G on $\widehat{G}$ is approximately oligomorphic. By the Ryll-Nardzewski theorem, this means the structure $\widehat{G}$ is separably categorical.

(3) $\Rightarrow$ (1)] This is a consequence of the previous theorem (and of the Ryll-Nardzewski theorem). □

The Roelcke precompactness of several Polish groups was already known, which allows us to recover separable categoricity for the associated structures.

EXAMPLES 6.20. The following automorphism groups act transitively and are Roelcke-precompact, so, by item (2) of theorem 6.19, the structures are separably categorical.

- The isometry group of the Urysohn sphere (Uspenskij, [**U5**], see section 3 in chapter 4).
- The unitary group of the separable Hilbert space (Uspenskij, [**U3**]).
- The group of measure preserving bijections of the interval (Glasner, [**G3**]).

3. Homogeneity

Whereas countably categorical structures are homogeneous, separably categorical ones are only approximately homogeneous. This brings up the question of *exact* homogeneity again. We observe that exact homogeneity amounts to remaining categorical whenever we name a finite tuple in the structure.

PROPOSITION 6.21. Let $\mathbf{M}$ be a separably categorical metric structure. The following are equivalent.

- The structure $\mathbf{M}$ is exactly homogeneous.
- For every tuple $\bar{a}$ in $\mathbf{M}$, the structure $(\mathbf{M}, \bar{a})$ is again separably categorical.

PROOF. $\Rightarrow$] Suppose $(\mathbf{M}', \bar{a}')$ is a separable structure that has the same theory as $(\mathbf{M}, \bar{a})$. Then $\mathbf{M}$ and $\mathbf{M}'$ have the same theory, so, by separable categoricity, there exists an isomorphism $f : \mathbf{M}' \to \mathbf{M}$. This isomorphism sends $\bar{a}'$ to some $\bar{a}''$ in $\mathbf{M}$. Now, since $\mathbf{M}$ is exactly homogeneous and $\bar{a}$ and $\bar{a}''$ have the same type, there exists an automorphism g of $\mathbf{M}$ sending $\bar{a}$ to $\bar{a}''$. Thus, the map $f^{-1} \circ g$ is the desired isomorphism between $(\mathbf{M}, \bar{a})$ and $(\mathbf{M}', \bar{a}')$, hence the structure $(\mathbf{M}, \bar{a})$ is separably categorical.

$\Leftarrow$] Conversely, let $\bar{a}$ and $\bar{a}'$ have the same type in $\mathbf{M}$. Then the two structures $(\mathbf{M}, \bar{a})$ and $(\mathbf{M}, \bar{a}')$ have the same theory. By categoricity, they are thus isomorphic, which means there exists an automorphism of $\mathbf{M}$ sending $\bar{a}$ to $\bar{a}'$. □

In particular, proposition 6.8 implies that this condition always holds in classical structures.

CHAPTER 7

Reconstruction of separably categorical structures

Elle n'a qu'à ouvrir l'espace de ses bras.

Francis Cabrel[1]

1. Reconstruction up to interdefinability **95**

2. Reconstruction up to bi-interpretability **96**

2.1. Interpretations 96

2.2. From interpretations to group homomorphisms 97

2.3. A special structure: a group with a hat 98

2.4. Reconstruction 100

Categoricity offers an ideal setting for reconstruction: a lot of model-theoretic information on a categorical structure is retained by the action of its automorphism group on it, thanks to the Ryll-Nardzewski theorem. A fact we will use extensively is that definability is exactly invariance under the action of the automorphism group (proposition 6.11).

In this chapter, we focus on a reconstruction result due to Ahlbrandt and Ziegler ([**AZ**]) which states that countably categorical structures are determined, up to bi-interpretability, by their automorphism groups (regarded as topological groups). We extend Ahlbrandt and Ziegler's result to the continuous setting. More precisely, we introduce the notion of an interpretation between metric structures and prove that two separably categorical structures are bi-interpretable if and only if their automorphism groups are topologically isomorphic. This is joint work with Itaï Ben Yaacov ([**BK3**]).

This guarantees that every model-theoretic property of separably categorical structures will translate into a topological property of their automorphism groups. Ben Yaacov and Tsankov ([**BT1**]), and then Ibarlucía ([**I**]), are precisely studying model-theoretic properties directly on groups.

Although our result encompasses its classical counterpart, the proof we give is fundamentally metric and is quite different from the original one. Indeed, we apply Melleray's construction of the hat structure (see section 2 in chapter 5), which provides a canonical way to make a metric structure out of any Polish group. The heart of the reconstruction consists in showing that every separably categorical metric structure is in fact bi-interpretable with the hat structure of its automorphism group.

1. Reconstruction up to interdefinability

We begin by reconstructing separably categorical structures up to interdefinability, mirroring Ahlbrandt and Ziegler's theorem 1.1 (in [**AZ**]). The proof is exactly the same as in the discrete setting.

Definition 7.1. Let $\mathbf{M}$ and $\mathbf{N}$ be two structures on the same universe. We say that $\mathbf{M}$ and $\mathbf{N}$ are **interdefinable** if they have the same definable predicates.

[1] *Je l'aime à mourir*

Proposition 7.2. Let $\mathbf{M}$ and $\mathbf{N}$ be two separably categorical metric structures on the same universe, in languages $\mathcal{L}_{\mathbf{M}}$ and $\mathcal{L}_{\mathbf{N}}$ respectively. Then $\mathbf{M}$ and $\mathbf{N}$ are interdefinable if and only if their automorphism groups are equal.

Proof. $\Rightarrow$] Assume that $\mathbf{M}$ and $\mathbf{N}$ are interdefinable and let R be a relation in $\mathcal{L}_{\mathbf{N}}$. Since it is definable in $\mathbf{N}$, it is definable in $\mathbf{M}$ as well, so it is $\mathrm{Aut}(\mathbf{M})$-invariant. Thus $\mathrm{Aut}(\mathbf{M})$ preserves every relation in $\mathcal{L}_{\mathbf{N}}$. Similarly, if F is a function symbol in $\mathcal{L}_{\mathbf{N}}$, then the distance predicate $d(F(x), y)$ is definable in $\mathbf{N}$, hence in $\mathbf{M}$. Thus, $\mathrm{Aut}(\mathbf{M}) \subseteq \mathrm{Aut}(\mathbf{N})$. Symmetrically, we obtain that $\mathrm{Aut}(\mathbf{N}) \subseteq \mathrm{Aut}(\mathbf{M})$.

$\Leftarrow$] Conversely, assume that $\mathrm{Aut}(\mathbf{M}) = \mathrm{Aut}(\mathbf{N})$ and let P be definable in $\mathbf{M}$. Then, by proposition 6.11, it is $\mathrm{Aut}(\mathbf{M})$-invariant hence $\mathrm{Aut}(\mathbf{N})$-invariant by assumption. Thus, P is definable in $\mathbf{N}$ and the two structures have the same definable predicates. $\square$

2. Reconstruction up to bi-interpretability

2.1. Interpretations. In the classical setting, an **interpretation** of a structure $\mathbf{M}$ in an other structure $\mathbf{N}$ is an embedding of $\mathbf{M}$ into a definable quotient of a finite power of $\mathbf{N}$, that is, into the imaginaries of $\mathbf{N}$. As Ben Yaacov and Usvyatsov pointed out in [**BU2**], the right definition of imaginaries in metric structures should allow classes of infinite tuples and this is also true for interpretations.

Definition 7.3. Let $\mathbf{M}$ and $\mathbf{N}$ be two metric structures in languages $\mathcal{L}_{\mathbf{M}}$ and $\mathcal{L}_{\mathbf{N}}$ respectively. An **interpretation** of $\mathbf{M}$ in $\mathbf{N}$ consists of the following data:

- a definable pseudometric ρ on $\mathbf{N}^\omega$ and
- an isometric map $\varphi : (\mathbf{M}, d_{\mathbf{M}}) \to \widehat{(\mathbf{N}^\omega, \rho)}$

such that

- the predicate $P : \mathbf{N}^\omega \to [0, 1]$ defined by $P(x) = \rho(x, \varphi(\mathbf{M}))$ is definable in $\mathbf{N}^\omega$ and
- for every formula F in $\mathcal{L}_{\mathbf{M}}$, the formula $P_F : \varphi(\mathbf{M})^r \to [0, 1]$ defined by $P_F(x) = F(\varphi^{-1}(x))$ is definable in $\mathbf{N}^\omega$, that is, there exists a definable ρ-invariant predicate on $\mathbf{N}^\omega$ that induces P_F.

To verify the last condition, it suffices to check it on relation symbols in $\mathcal{L}_{\mathbf{M}}$ and on predicates of the form $(x, y) \mapsto d(x, F(y))$, where F is a function symbol in $\mathcal{L}_{\mathbf{M}}$.

Remark 7.4. If $\mathbf{M}$ and $\mathbf{N}$ are classical structures, they can be made into discrete metric structures. Then every interpretation of $\mathbf{M}$ in $\mathbf{N}$ (in the metric sense, as defined above) induces a classical interpretation of $\mathbf{M}$ in $\mathbf{N}$. To see this, given a metric interpretation φ of $\mathbf{M}$ in $\mathbf{N}$, use the continuity of the associated pseudometric to choose a big enough n such that the elements in the image $\varphi(\mathbf{M})$ (which is discrete) are determined by their restriction to the first n coordinates. Then the equivalence relation on $\mathbf{N}^n$ induced by restriction of ρ is well-defined and definable, and it yields an interpretation of $\mathbf{M}$ in $\mathbf{N}$.

If $\mathbf{M}$, $\mathbf{N}$ and $\mathbf{K}$ are metric structures, $\varphi : (\mathbf{M}, d_{\mathbf{M}}) \to \widehat{(\mathbf{N}^\omega, \rho_{\mathbf{N}})}$ is an interpretation of $\mathbf{M}$ in $\mathbf{N}$ and $\psi : (\mathbf{N}, d_{\mathbf{N}}) \to \widehat{(\mathbf{K}^\omega, \rho_{\mathbf{K}})}$ is an interpretation of $\mathbf{N}$ in $\mathbf{K}$, then we can **compose the interpretations** ψ and φ as follows.

Taking the power of ψ, we get an isometric map $\psi^\omega : (\mathbf{N}^\omega, d^\omega_{\mathbf{N}}) \to (\widehat{\mathbf{K}^{\omega\times\omega}, \rho^\omega_{\mathbf{K}}}) = (\widehat{\mathbf{K}^\omega, \rho^\omega_{\mathbf{K}}})$. Now, since $\psi(\mathbf{N})$ is definable in $(\widehat{\mathbf{K}^\omega, \rho_{\mathbf{K}}})$, the image $\psi^\omega(\mathbf{N}^\omega)$ is definable in $(\widehat{\mathbf{K}^\omega, \rho^\omega_{\mathbf{K}}})$ too. Besides, $\rho_{\mathbf{N}}$ is a definable pseudometric, so its pushforward by ψ^ω also is. Then, it extends to a definable pseudometric ρ on $(\widehat{\mathbf{K}^\omega, \rho^\omega_{\mathbf{K}}})$ (this is [**B4**, proposition 3.6]). Thus, the isometric map $\psi^\omega \circ \varphi : (\mathbf{M}, d_{\mathbf{M}}) \to (\widehat{\mathbf{K}^\omega, \rho})$ is an interpretation of $\mathbf{M}$ in $\mathbf{K}$.

DEFINITION 7.5. Let $\mathbf{M}$ and $\mathbf{N}$ be two metric structures. We say that $\mathbf{M}$ and $\mathbf{N}$ are **bi-interpretable** if there exist interpretations φ of $\mathbf{M}$ in $\mathbf{N}$ and ψ of $\mathbf{N}$ in $\mathbf{M}$ such that $\psi \circ \varphi$ and $\varphi \circ \psi$ are definable.

In the rest of this section, we argue that interpretations between separably categorical structures correspond to continuous homomorphisms between their automorphism groups.

2.2. From interpretations to group homomorphisms. The first side of this correspondence is not too surprising, for it amounts to saying one can get information on the automorphism group from the structure. The process is however nicely functorial.

PROPOSITION 7.6. Let $\mathbf{M}$ and $\mathbf{N}$ be two metric structures and φ an interpretation of $\mathbf{M}$ in $\mathbf{N}$. Then φ induces a homomorphism $\mathrm{Ind}(\varphi)$ of topological groups from $\mathrm{Aut}(\mathbf{N})$ to $\mathrm{Aut}(\mathbf{M})$.

PROOF. Let g be an automorphism of $\mathbf{N}$ and a an element of $\mathbf{M}$. We put $\mathrm{Ind}(\varphi)(g)(a) = \varphi^{-1}(g(\varphi(a)))$. Then $\mathrm{Ind}(\varphi)$ is the conjugation by φ so it is a group homomorphism. And since φ and φ^{-1} are continuous, it is easy to see that $\mathrm{Ind}(\varphi)$ is continuous. $\square$

REMARK 7.7. If $\mathbf{N}$ is separably categorical, then the automorphism group of $\mathbf{N}$ acts approximately oligomorphically on $\mathbb{N}^\omega$. In particular, any structure that is interpretable in a separably categorical one is itself separably categorical. Indeed, if φ is an interpretation of $\mathbf{M}$ in a separably categorical structure $\mathbf{N}$, then the group $\mathrm{Ind}(\varphi)(\mathrm{Aut}(\mathbf{N}))$ acts approximately oligomorphically on $\varphi(\mathbf{M})$. It follows that the whole automorphism group of $\mathbf{M}$ acts approximately oligomorphically on $\mathbf{M}$, hence $\mathbf{M}$ is separably categorical, by the Ryll-Nardzewski theorem. That is the reason why it is necessary to impose an oligomorphicity restriction in theorems 7.12 and 7.14.

The map $\varphi \mapsto \mathrm{Ind}(\varphi)$ is functorial: it respects composition.

LEMMA 7.8. Let $\mathbf{M}$, $\mathbf{N}$ and $\mathbf{K}$ be metric structures, φ an interpretation of $\mathbf{M}$ in $\mathbf{N}$ and ψ an interpretation of $\mathbf{N}$ in $\mathbf{K}$. Then $\mathrm{Ind}(\psi \circ \varphi) = \mathrm{Ind}(\varphi) \circ \mathrm{Ind}(\psi)$.

LEMMA 7.9. Let $\mathbf{M}$ be a separably categorical metric structure and φ an interpretation of $\mathbf{M}$ in itself. Then φ is definable in $\mathbf{M}$ if and only if $\mathrm{Ind}(\varphi) = \mathrm{id}_{\mathrm{Aut}(\mathbf{M})}$.

PROOF. $\Rightarrow$] If φ is definable, then φ is $\mathrm{Aut}(\mathbf{M})$-equivariant. Then, if $g \in \mathrm{Aut}(\mathbf{M})$ and $a \in M$, we have $\mathrm{Ind}(\varphi)(g)(a) = \varphi^{-1}(g(\varphi(a))) = \varphi^{-1}(\varphi(g(a))) = g(a)$ and thus $\mathrm{Ind}(\varphi)$ is the identity.

$\Leftarrow$] If $\mathrm{Ind}(\varphi)$ is the identity, the same computation shows that φ is $\mathrm{Aut}(\mathbf{M})$-equivariant. Since φ is continuous (it is isometric), this implies that φ is definable (by proposition 6.11). $\square$

This lemma will yield the first direction of theorem 7.15.

2.3. A special structure: a group with a hat. We now proceed to the second part of the correspondence: the actual reconstruction. This comes down to a canonical structure built from the automorphism group and with which the structure is bi-interpretable: the hat structure (see section 2 of chapter 5).

Let $\mathbf{M}$ be a separable metric structure and let G be its automorphism group. Fix a dense sequence $\xi \in \mathbf{M}^\omega$. Then, the metric on G given by $d_\xi(g,h) = d(g\xi, h\xi)$ is a compatible left-invariant metric, from which we can define the hat structure on G.

PROPOSITION 7.10. If $\mathbf{M}$ is separably categorical, then the structure $\widehat{G}$ (obtained from this particular metric d_ξ) is interpretable in $\mathbf{M}$.

PROOF. Consider the map $\psi : g \mapsto g\xi$ from (G, d_ξ) to $\overline{G \cdot \xi} \subseteq \mathbf{M}^\omega$. It is isometric so it extends to the left completion of G. Then ψ is an interpretation of $\widehat{G}$ in $\mathbf{M}$.

Indeed, the predicate $P : x \mapsto d(x, \psi(\widehat{G})) = d(x, \overline{G \cdot \xi})$ on $\mathbf{M}^\omega$ is G-invariant so it is definable in $\mathbf{M}$ by proposition 6.11. Moreover, if C is an orbit closure and $R = R_C$ is the associated predicate in $\widehat{G}$, we have, for x in $\psi(\widehat{G})$:

$$\begin{aligned} P_R(gx) &= R(\psi^{-1}(gx)) \\ &= R(g\psi^{-1}(x)) \\ &= R(\psi^{-1}(x)) \text{ because } R \text{ is invariant by the automorphism group} \\ &= P_R(x), \end{aligned}$$

so P_R is definable, which completes the proof. □

REMARK 7.11. In fact, since the image of ψ is dense, $\overline{G \cdot \xi}$ is exactly the left completion of G and from now on, we identify $\widehat{G}$ with $\overline{G \cdot \xi}$.

The above proposition, along with remark 7.7, implies that if $\mathbf{M}$ is separably categorical, then so is $\widehat{G}$. And in that case, if d_L is any other compatible left-invariant metric, then the associated hat structure is bi-interpretable with $\widehat{G}$: the two metrics generate the same topology so they are continuous with respect to each other, and their left-invariance implies, by proposition 6.11, that they are definable from each other. Thus, the identity map from the d_L-hat structure to the quotient of $\widehat{G}$ by d_L is an interpretation. Therefore, all the hat structures obtained from G are bi-interpretable and we will therefore identify them.

Moreover, if $\mathbf{M}$ is separably categorical, then the structure $\mathbf{M}$ is also interpretable in $\widehat{G}$. In fact, we have the following more general result which will be the key ingredient in the proof of theorem 7.14.

THEOREM 7.12. Let $\mathbf{N}$ be a metric structure and let H be a subgroup of $\mathrm{Aut}(\mathbf{N})$ which acts approximately oligomorphically on $\mathbf{N}$. Then $\mathbf{N}$ is interpretable in $\widehat{H}$.

PROOF. Let ζ be a dense sequence in $\mathbf{N}$. Then $\widehat{H} = \overline{H \cdot \zeta}$. Now the assumption ensures that the space $\mathbf{N} /\!\!/ H$ of orbit closures of $\mathbf{N}$ by H is compact.

The intuition for the proof is to say that $\mathbf{N}$ is not far from being the product $\widehat{H} \times \mathbf{N} /\!\!/ H$ and moreover that compact spaces should be interpreted in every structure. As a matter of fact, we will build a particular system of representatives of $\mathbf{N} /\!\!/ H$ that $\widehat{H}$ will interpret.

We begin by building a tree T representing this compact quotient $\mathbf{N} /\!\!/ H$. For this, we will choose representatives, within ζ, of a dense sequence of orbit closures that witnesses the compactness of this quotient, and T will be the tree of their indices in ζ. More precisely, we build the tree by induction: first, there exist $\zeta_{n_1}, ..., \zeta_{n_k}$ in ζ such that the balls of radius $\frac{1}{2}$ centered in the closures of the orbits of $\zeta_{n_1}, ..., \zeta_{n_k}$ cover all of the quotient. The *indices* of those elements constitute the first level of our tree. For the next step, we cover each of the balls $B(\zeta_{n_i}, \frac{1}{2})$ in $\mathbf{N}$ with a finite number of balls of radius $\frac{1}{4}$ centered in elements of ζ so that the second level of our tree consists of the indices of those centers, and so on (vertices at level $n+1$ come from a covering of an open ball at level n, and a vertex at level n is related by an edge to all corresponding vertices at level $n+1$).

The construction ensures that for every infinite branch of T, the sequence $(\zeta_{\sigma(i)})$ converges in $\mathbf{N}$. Moreover, every orbit closure corresponds to an infinite branch of T (maybe even several): for every a in $\mathbf{N}$, there exists an infinite branch σ of T such that the limit of the sequence $(\zeta_{\sigma(i)})$ is in the closure of the orbit of a. Let $[T]$ be the set of infinite branches of T.

We now embed $\mathbf{N}$ isometrically into (the completion of) a quotient of $\overline{H \cdot \zeta} \times [T]$, which we identify with $\widehat{H} \times [T]$. This will give the base map for our interpretation.

Endow the set $\overline{H \cdot \zeta} \times [T]$ with the following pseudometric

$$\rho((x, \sigma), (y, \tau)) = \lim_{i \to \infty} d(x_{\sigma(i)}, y_{\tau(i)}).$$

Since for every branch σ in $[T]$, the sequence $(\zeta_{\sigma(i)})$ converges, this is also true of every $(x_{\sigma(i)})$ with x in $\overline{H \cdot \zeta}$, so ρ is well-defined.

We now define a map $\varphi : (\overline{H \cdot \zeta} \times [T], \rho) \to \mathbf{N}$ by $\varphi(x, \sigma) = \lim_{i \to \infty} x_{\sigma(i)}$. By definition of ρ, the map φ is isometric. In addition, the image of φ is dense in $\mathbf{N}$. Indeed, let a be an element of $\mathbf{N}$ and $\epsilon > 0$. There exists a branch σ in $[T]$ such that $(\zeta_{\sigma(i)})$ converges to some a' in $\mathbf{N}$ which is in the same H-orbit closure as a, that is, there exists $h \in H$ such that $d(h(a'), a) < \epsilon$, so $d(\varphi(h\zeta, \sigma), a) < \epsilon$, hence the density.

Thus, the isometric map φ can be extended to an isometry from the completion of $(\overline{H \cdot \zeta} \times [T], \rho)$ onto $\mathbf{N}$. Then its inverse, call it $\tilde{\varphi}$, is the desired isometric map between $\mathbf{N}$ and the completion of $(\widehat{H} \times [T], \rho)$. This was the first step in our construction.

In order to see $\tilde{\varphi}$ as an interpretation of $\mathbf{N}$ in $\widehat{H}$, it remains to interpret $[T]$ in $\widehat{H}$, in other words, to code the branches of T in a power of $\widehat{H}$ (that is $\overline{H \cdot \zeta}$ via the identification of remark 7.11). The map $\tilde{\varphi}$ will then induce a map $\mathbf{N} \to \widehat{H} \times \widehat{H}^{\omega}$, which will be the desired interpretation.

A branch can be coded by a sequence of zeroes and ones[2]. Then we code[3] each bit by a pair of elements of $\overline{H \cdot \zeta}$. Consider the pseudometric on $\overline{H \cdot \zeta} \times \overline{H \cdot \zeta}$ defined by

$$\delta((x, x'), (y, y')) = |d(x_0, x'_0) - d(y_0, y'_0)|,$$

which compares the differences between the first coordinates of the two sequences of the pair. This is a definable pseudometric and we code the bit 0 by the δ-class of (ζ, ζ) and the bit 1 by

[2]There are many ways of doing so; we pick one. For instance, we may say that given a branch of T, we follow the levels of T one by one, and we put a 1 in our sequence when we hit an element of our branch and a 0 otherwise.

[3]There are also many ways of coding zeroes and ones in a power of $\overline{H \cdot \zeta}$. Here we go for a method which compares two sequences of a pair in a very simple way.

the δ-class of $(\zeta, h_0\zeta)$ where h_0 is some element of H that does not fix ζ_0. Note that the code is invariant under the action of H.

Finally, we identify branches of T with their codes in $(\widehat{H}^2, \delta)^\omega$ and we transfer the pseudometric ρ on $\widehat{H} \times [T]$ to a definable pseudometric $\tilde{\rho}$ on $\widehat{H} \times (\widehat{H}^2, \delta)^\omega$. Note that the elements of $(\widehat{H}^2, \delta)^\omega$ that code a branch of $[T]$ may not cover the whole of $(\widehat{H}^2, \delta)^\omega$, but we extend $\tilde{\rho}$ to $(\widehat{H}^2, \delta)^\omega$ all the same (this is [**B4**, proposition 3.6]).

So we can now rewrite the map $\tilde{\varphi}$ as a map from $\mathbf{N}$ to the completion of $(\widehat{H} \times (\widehat{H}^2, \delta)^\omega, \tilde{\rho})$. The oligomorphicity of the action of H on $\mathbf{N}$ implies that the structure $\widehat{H} = \overline{H \cdot \zeta}$, whose automorphism group is H, is separably categorical. Since $\tilde{\rho}$ is invariant under the action of H, proposition 6.11 then yields that the pseudometric $\tilde{\rho}$ is definable in $\widehat{H}$. Besides, since $\tilde{\varphi}$ is H-equivariant, the predicates $P(x) = \rho(x, \tilde{\varphi}(\mathbf{N}))$ and $P_F(x) = F(\tilde{\varphi}^{-1}(x))$ are definable in $\widehat{H}$. Therefore, this new map $\tilde{\varphi}$ is an interpretation of $\mathbf{N}$ in $\widehat{H}$. □

Corollary 7.13. If $\mathbf{M}$ is separably categorical and $G = \mathrm{Aut}(\mathbf{M})$, then the structures $\mathbf{M}$ and $\widehat{G}$ are bi-interpretable.

Proof. Theorem 7.12 implies in particular that $\mathbf{M}$ is interpretable in $\widehat{G}$. Thus, it suffices to show that the compositions of the interpretations constructed in theorem 7.12 and 7.10 are definable. Both interpretations respected the actions of the automorphism groups so proposition 6.11 and remark 7.7 allow us to conclude. □

2.4. Reconstruction. We are now ready to complete the reconstruction.

Theorem 7.14. Let $\mathbf{M}$ and $\mathbf{N}$ be two metric structures, with $\mathbf{M}$ separably categorical. Let $f : \mathrm{Aut}(\mathbf{M}) \to \mathrm{Aut}(\mathbf{N})$ be a continuous group homomorphism whose image is closed and acts approximately oligomorphically on $\mathbf{N}$. Then $\mathbf{N}$ is interpretable in $\mathbf{M}$.

Proof. Set $G = \mathrm{Aut}(\mathbf{M})$ and $H = f(G)$. Since H acts approximately oligomorphically on $\mathbf{N}$, theorem 7.12 implies that $\mathbf{N}$ is interpretable in $\widehat{H}$. And by proposition 7.10, the structure $\widehat{G}$ is interpretable in $\mathbf{M}$. It then suffices to show that $\widehat{H}$ is interpretable in $\widehat{G}$.

Now, since H is closed, H is topologically isomorphic to the quotient of G by the closed normal subgroup $\mathrm{Ker}(f)$. If d_L is a left-invariant metric on G, then we can endow G with the following left-invariant pseudometric

$$d'_L(g_1, g_2) = \inf\{d_L(g_1k_1, g_2k_2) : k_1, k_2 \in \mathrm{Ker}(f)\}.$$

Since $\mathrm{Ker}(f)$ is normal, this indeed defines a pseudometric, which induces a compatible metric on H. Then $\widehat{(H, d'_L)}$, which we identify with $\widehat{H}$ (see subsection 3.3), is the quotient[4] of $\widehat{G}$ by the definable pseudometric d'_L and is thus interpretable in $\widehat{G}$. □

Theorem 7.15. Let $\mathbf{M}$ and $\mathbf{N}$ be separably categorical metric structures. Then $\mathbf{M}$ and $\mathbf{N}$ are bi-interpretable if and only if their automorphism groups are isomorphic as topological groups.

[4] Here, we do not even need to go to a power of $\widehat{G}$ to interpret $\widehat{H}$.

PROOF. $\Rightarrow$] Assume that φ and ψ are interpretations that witness the bi-interpretability of $\mathbf{M}$ and $\mathbf{N}$. Then lemma 7.9 implies that $\mathrm{Ind}(\varphi \circ \psi) = \mathrm{id}_{\mathrm{Aut}(\mathbf{N})}$ and $\mathrm{Ind}(\psi \circ \varphi) = \mathrm{id}_{\mathrm{Aut}(\mathbf{M})}$. But $\mathrm{Ind}(\varphi \circ \psi) = \mathrm{Ind}(\varphi) \circ \mathrm{Ind}(\psi)$ so $\mathrm{Ind}(\varphi) = \mathrm{Ind}(\psi)^{-1}$ and $\mathrm{Ind}(\psi)$ is an isomorphism of topological groups between $\mathrm{Aut}(\mathbf{M})$ and $\mathrm{Aut}(\mathbf{N})$. Note that for this direction, we do not need the categoricity of the structures.

$\Leftarrow$] By corollary 7.13, $\mathbf{M}$ is bi-interpretable with $\widehat{\mathrm{Aut}(\mathbf{M})}$ and $\mathbf{N}$ with $\widehat{\mathrm{Aut}(\mathbf{N})}$. Now if the two groups are isomorphic as topological groups, then their associated hat structures are bi-interpretable (by the discussion following remark 7.11). $\square$

EXAMPLE 7.16. In [**B6**], it is shown, by an explicit computation, that the probability algebra $\mathbf{M}$ of the unit interval is bi-interpretable with the space $\mathbf{N}$ of $[0, 1]$-valued random variables, identified up to equality almost everywhere and endowed with the L^1 metric. Our reconstruction theorem allows us to recover this result in a more abstract way. Indeed, the probability algebra of $[0, 1]$ is separably categorical, thus its automorphism group $G = \mathrm{Aut}(\mu)$ is Roelcke-precompact ([**R3**, theorem 5.2]).

Moreover, G is also the automorphism group of $\mathbf{N}$. We will show that the space of orbit closures of $\mathbf{N}$ under the action of G can be identified with the space of probability measures on $[0, 1]$. It follows that the space of orbit closures of $\mathbf{N}$ is compact. This suffices, by theorem 6.19, to get that the action of G on $\mathbf{N}$ is approximately oligomorphic, hence that the structure $\mathbf{N}$ is also separably categorical. Theorem 7.15 then applies, proving that $\mathbf{M}$ and $\mathbf{N}$ are bi-interpretable.

Let us now see how to identify the space of orbit closures of $\mathbf{N}$ under the action of G with the space of probability measures on $[0, 1]$. From a measurable map f in $\mathbf{N}$, we will build an element g of G such that $f \circ g$ is increasing. The resulting map will then be the repartition function of some probability measure on I.

Define the desired increasing map by

$$h(r) = \inf \{t \in [0, 1] : \mu(\{x \in I : f(x) < t\}) > r\}.$$

By definition, we have that

$$\mu(\{x \in I : f(x) \leqslant h(r)\}) \geqslant r \text{ and } \mu(\{x \in I : f(x) < h(r)\}) \leqslant r.$$

To build the automorphism g, we specify the set $A_r = \{x \in I : g(x) \leqslant r\}$ for every rational point r in $[0, 1]$: we will set

$$A_r = \{x \in I : f(x) < h(r)\} \cup B_r,$$

with B_r being a subset of $\{x \in I : f(x) = h(r)\}$ of adequate measure (such as to bring the measure of A_r up to r). In order to choose the set B_r, we intersect the set $\{x \in I : f(x) = h(r)\}$ with bigger and bigger intervals until we get a subset of measure $m(r) = r - \mu(\{x \in [0, 1] : f(x) < h(r)\})$. More precisely, let

$$s(r) = \inf \{t \in I : \mu([0, t[\cap \{x \in I : f(x) = h(r)\}) > m(r)\}$$

and let $B_r = [0, s(r)] \cap \{x \in I : f(x) = h(r)\}$.

The family $\{A_r : r \in [0,1] \cap \mathbb{Q}\}$ is increasing and the union of all the A_r's has measure 1. Therefore, we may define g as follows (on a full measure subset):

$$g(s) = \inf\{r \in [0,1] \cap \mathbb{Q} : s \in A_r\}.$$

As desired, we have that $f \circ g = h$. Moreover, it is easy to see that g is a measure-preserving automorphism of I. This completes the proof.

Branch 2

Automatic continuity

CHAPTER 8

Automatic continuity for infinite powers of Polish groups

Comme des inconnus qui n'ont rien à se dire
[...] nous restons côte à côte.
Charles Aznavour[1]

1.	**Automatic continuity**	**106**
1.1.	The Steinhaus property	106
1.2.	Better reasons than discreteness to be Steinhaus: ample generics	107
1.3.	Mimicking ample generics	108
2.	**The juxtaposed structure**	**109**
3.	**The extension property**	**110**
4.	**The existence property**	**111**
5.	**Isolation**	**111**
5.1.	Relevant tuples	111
5.2.	Direct strong isolation	112
5.3.	Indirect strong isolation	114
6.	**Connected groups with ample generics**	**115**
7.	**Perspectives**	**117**

In the last chapter, we have seen how to reconstruct some automorphism groups from the associated structures. Here, we tackle a different kind of reconstruction, although it will still be read on the properties of the underlying structure. Namely, we are interested in recovering the topology on the group from its algebraic structure, by means of the *automatic continuity property*. A separable topological group G has the automatic continuity property if every group homomorphism from G to any separable group is continuous.

However strong this may seem, many Polish groups satisfy the automatic continuity property; we refer the reader to Rosendal's survey **[R2]** for more details. We are interested in finding more of those, by looking at infinite powers of Polish groups that satisfy the automatic continuity property. Such powers do not always have the automatic continuity property, even (if not especially) in the simplest of cases. Yet, they do when the Polish groups in question have *ample generics*, a very strong topological property. We prove that they also do with a weaker requirement: in the very particular framework, introduced by Sabok ([**S1**]) and Malicki ([**M1**]), where automatic continuity of the automorphism group is witnessed by specific combinatorial properties of the structure.

Moreover, in the course of a discussion on this question with François Le Maître, we discovered connected Polish groups with ample generics. This answers a question of Kechris and Rosendal (see theorem 8.54), who asked whether the only Polish groups with ample generics were subgroups of S_∞. Malicki has simultaneously answered this question in [**M1**], though with different examples.

[1] *Comme des étrangers*

1. Automatic continuity

DEFINITION 8.1. Let G be a separable topological group. We say that the group G satisfies the **automatic continuity property** if every group homomorphism from G to a separable topological group is continuous.

Note that the separability assumption on the range group is necessary: without it, we can always endow the group G with the discrete topology and the identity map of G will fail to be continuous (when G is not discrete).

PROPOSITION 8.2. A finite product of groups that all satisfy the automatic continuity property also satisfies the automatic continuity property.

PROOF. Let G_1, ..., G_n be topological groups that satisfy the automatic continuity property. Let H be a separable group and let $\varphi : G_1 \times ... \times G_n \to H$ be a group homomorphism. For each i, consider the group homomorphism $\varphi_i : G_i \to H$ defined by $\varphi_i(g_i) = \varphi(1, ..., 1, g_i, 1, ..., 1)$. Since each G_i satisfies the automatic continuity property, all of the homomorphisms φ_i are continuous. Since there are finitely many groups in the product, we can write φ as $(g_1, ..., g_n) \mapsto \varphi_1(g_1)...\varphi_n(g_n)$ so φ is continuous. □

However, the automatic continuity property does not carry over to infinite products in general. The following will be our companion (non-)example throughout this chapter.

EXAMPLE 8.3. The group $\mathbb{Z}/2\mathbb{Z}$ is discrete. Thus, it satisfies the automatic continuity property. However, the group $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ does not have the automatic continuity property. Indeed, let $\mathcal{U}$ be any non-principal ultrafilter on $\mathbb{N}$. It corresponds to a normal subgroup $H_{\mathcal{U}}$ of $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ of index 2, and since $\mathcal{U}$ is non-principal, the subgroup $H_{\mathcal{U}}$ is not open. But then the group homomorphism from $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ into $\mathbb{Z}/2\mathbb{Z}$ of kernel $H_{\mathcal{U}}$ cannot be continuous.

The general question we would like to address is the following.

QUESTION 8.4. If G is a group with the automatic continuity property, when is it the case that the group $G^{\mathbb{N}}$ also has the automatic continuity property?

In this chapter, we only touch upon this question. We answer it for Polish groups, seen as automorphism groups, in the particular case when automatic continuity results from certain combinatorial properties of the structure.

1.1. The Steinhaus property.

Rosendal and Solecki introduced in [**RS**] a very useful tool to prove the automatic continuity property.

DEFINITION 8.5. Let G be a topological group. We say that the group G is **Steinhaus** if there exists an integer k such that for every symmetric countably syndetic[2] subset W of G, W^k contains an open neighborhood of the identity. We also say that G is k-**Steinhaus**.

THEOREM 8.6. (Rosendal-Solecki, [**RS**, proposition 2]) Let G be a separable topological group. If G is Steinhaus, then G satisfies the automatic continuity property.

PROOF. Assume that G is k-Steinhaus. Let H be a separable topological group and let $\varphi : G \to H$ be a group homomorphism. It suffices to show that φ is continuous at 1_G. Let U be an open neighborhood of 1_H. By continuity of multiplication in H, we can find a symmetric open neighborhood V of 1_H such that $V^k \subseteq U$.

[2]A subset W of G is said to be **countably syndetic** if there exists a sequence $(g_n)_{n\in\mathbb{N}}$ of elements of G such that $G = \bigcup_{n\in\mathbb{N}} g_n W$.

Then the set $W = \varphi^{-1}(V)$ is symmetric and countably syndetic in G. Indeed, since H is separable, so is $\varphi(G)$, hence we can write $\varphi(G) \subseteq \bigcup_{n\in\mathbb{N}} \varphi(g_n)V$. Thus, we have

$$\begin{aligned} G &= \varphi^{-1}(\varphi(G)) \\ &\subseteq \bigcup_{n\in\mathbb{N}} \varphi^{-1}(\varphi(g_n)V) \\ &= \bigcup_{n\in\mathbb{N}} g_n\varphi^{-1}(V). \end{aligned}$$

Now, since the group G is k-Steinhaus, it follows that W^k contains an open neighborhood of 1_G. But W^k is contained in $\varphi^{-1}(V^k)$, which is in turn contained in $\varphi^{-1}(U)$, by our choice of V. Thus, $\varphi^{-1}(U)$ contains an open neighborhood of 1_G, which completes the proof. $\square$

Example 8.7. Since the group $\mathbb{Z}/2\mathbb{Z}$ is discrete, it is Steinhaus. However, the group $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ is not (otherwise, it would have the automatic continuity property).

It is unclear whether a finite product of Steinhaus groups also is Steinhaus. But since the Steinhaus property was introduced with the sole aim of proving the automatic continuity property, it does not matter too much in view of proposition 8.2.

We would like to argue that the group $\mathbb{Z}/2\mathbb{Z}$ has the automatic continuity property for the wrong reason (a trivial reason, discreteness).

1.2. Better reasons than discreteness to be Steinhaus: ample generics. Kechris and Rosendal introduced in [**KR**] the property of having *ample generics* for a topological group, and they proved that if a topological group has ample generics, then it satisfies the automatic continuity property ([**KR**, theorem 6.24]).

Definition 8.8. A subset of a Polish space X is called **comeager** if it contains a countable intersection of dense open subsets of X.

Definition 8.9. Let G be a topological group. We say that G has **ample generics** if for every positive integer n, the diagonal conjugacy action of G on G^n, which is given by

$$g \cdot (g_1, ..., g_n) = (g^{-1}g_1g, ..., g^{-1}g_ng),$$

admits a comeager orbit.

Remark 8.10. By the Effros theorem (see [**G1**, theorem 3.2.4]), comeager orbits are in fact G_δ.

Many closed subgroups of S_∞ have ample generics: the automorphism groups of the random graph, of the rational Urysohn space, of the infinitely splitting regular rooted tree. Also, if a Fraïssé class satisfies two combinatorial properties, namely the extension property and the free amalgamation property (see sections 3 and 4 for a definition), then the automorphism group of its Fraïssé limit has ample generics (see [**M6**, theorem 4.5]; the result follows from [**KR**, proposition 6.4]).

However, bigger groups often fail to have ample generics. For instance, in the groups $\mathrm{Iso}(\mathbb{U})$, $\mathrm{Aut}(\mu)$ and $\mathcal{U}(\ell^2)$, every conjugacy class is meager. Actually, Kechris and Rosendal asked in [**KR**, question 6.13 (1)] whether there exist Polish groups with ample generics that are not subgroups of S_∞. With Le Maître ([**KLM**]), we exhibited an example of such a group (see theorem 8.54).

For our purposes, ample generics come out as particularly powerful, for they do carry to infinite powers.

Proposition 8.11. Let G be a topological group with ample generics. Then $G^{\mathbb{N}}$ also has ample generics.

PROOF. Let n be an integer. We can naturally identify the group $(G^{\mathbb{N}})^n$ with $(G^n)^{\mathbb{N}}$. Since G has ample generics, there exists a tuple $\bar{\varphi}$ in G^n whose orbit is comeager in G^n. Now consider the constant sequence $\bar{f} = (\bar{\varphi})_{i\in\mathbb{N}}$ in $(G^n)^{\mathbb{N}}$. We prove that the orbit of $\bar{f}$ under the diagonal action of $G^{\mathbb{N}}$ is comeager in $(G^n)^{\mathbb{N}}$.

The orbit of $\bar{\varphi}$ is dense G_δ: $G \cdot \bar{\varphi} = \bigcap_{k\in\mathbb{N}} U_k$, where each U_k is a dense open subset of G^n. Then we have

$$\begin{aligned} G^{\mathbb{N}} \cdot \bar{f} &= \{(\bar{g}_i)_{i\in\mathbb{N}} \in (G^n)^{\mathbb{N}} : \text{ for all } i \text{ in } \mathbb{N}, \bar{g}_i \in G \cdot \bar{\varphi}\} \\ &= \bigcap_{i\in\mathbb{N}} \{(\bar{g}_i)_{i\in\mathbb{N}} \in (G^n)^{\mathbb{N}} : \bar{g}_i \in G \cdot \bar{\varphi}\} \\ &= \bigcap_{i\in\mathbb{N}} \bigcap_{k\in\mathbb{N}} \{(\bar{g}_i)_{i\in\mathbb{N}} : \bar{g}_i \in U_k\} \\ &= \bigcap_{i\in\mathbb{N}} \bigcap_{k\in\mathbb{N}} A_{i,k}. \end{aligned}$$

Since $(G^n)^{\mathbb{N}}$ is endowed with the product topology, each of the $A_{i,k}$'s is open and dense in $(G^n)^{\mathbb{N}}$, hence the orbit of $\bar{f}$ is comeager and $G^{\mathbb{N}}$ has ample generics. □

We will see in section 6 a generalization of this fact to the group of G-valued random variables.

1.3. Mimicking ample generics. As mentioned before, ample generics fail for quite a number of big Polish groups. Yet, it is still possible to circumvent their absence: before they knew that the group $\mathrm{Homeo}(2^{\mathbb{N}})$ had ample generics[3], Rosendal and Solecki ([**RS**, theorem 13]) managed to prove the automatic continuity property for both $\mathrm{Homeo}(2^{\mathbb{N}})$ and $\mathrm{Homeo}(2^{\mathbb{N}})^{\mathbb{N}}$. Drawing inspiration from their arguments, Sabok then introduced in [**S1**] a set of properties of *exactly* ultrahomogeneous metric structures that imply the Steinhaus property for their (big) automorphism groups. These conditions include the *existence property*, which is in some way similar to the free amalgamation property, and the extension property. Later, Malicki proposed in [**M1**] a slightly different set of properties, designed to imply not only the automatic continuity property but also several other consequences of ample generics (see [**KR**]). In the light of proposition 8.11, this set of properties that mimics ample generics is a reasonable condition to consider for our problem.

Their results are the following, with the different properties to be specified and discussed later on.

THEOREM 8.12. (Sabok, Malicki) Let $\mathbf{M}$ be an exactly ultrahomogeneous metric structure. Assume that $\mathbf{M}$ has the extension property, the existence property and an isolation property. Then the automorphism group of $\mathbf{M}$ is Steinhaus and thus satisfies the automatic continuity property.

REMARK 8.13. Again, the question arises of exact ultrahomogeneity and of a possible finitary characterization for it...

COROLLARY 8.14. The following groups have the automatic continuity property.

- $\mathrm{Aut}(\mu)$ (Ben Yaacov-Berenstein-Melleray, [**BBM**, theorem 6.2]).
- $\mathcal{U}(\ell^2)$ (Tsankov, [**T3**]).
- $\mathrm{Iso}(\mathbb{U})$ and $\mathrm{Iso}(\mathbb{U}_1)$ (Sabok, [**S1**, section 8]).

We would like to investigate these properties and study how they behave with respect to products. In order to do that, given a metric structure and its automorphism group G, we need to exhibit a structure of which $G^{\mathbb{N}}$ is the automorphism group.

[3]This was proved later by Kwiatkowska in [**K7**].

2. The juxtaposed structure

Let $\mathbf{M}$ be a metric structure of diameter smaller than 1 and let G be its automorphism group. The **juxtaposed structure** $\mathbf{M}^*$ of $\mathbf{M}$ consists of countably many copies of the structure $\mathbf{M}$ that do not interact with one another, together with a distinguished element $\star$ (that constitutes the zeroth "copy"). We endow the space $(\{\star\} \times \{0\}) \cup (\mathbf{M} \times \mathbb{N} \setminus \{0\})$ with

- a unary predicate C_n for each copy $\mathbf{M} \times \{n\}$,
- a unary predicate $C_\star$ for the element $(\star, 0)$,
- the metric defined by

$$d((a,i),(b,j)) = \begin{cases} d_{\mathbf{M}}(x,y) & \text{if } i = j \neq 0 \\ 0 & \text{if } i = j = 0 \\ 1 & \text{if } i \neq j, \end{cases}$$

- a predicate P^* for each predicate P in $\mathcal{L}_{\mathbf{M}}$, defined by

$$P^*((a_1,i_1),...,(a_m,i_m)) = \begin{cases} P(a_1,...,a_m) & \text{if } i_1 = ... = i_m \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$

- a function F^* for each definable function $F : \mathbf{M}^m \to \mathbf{M}$ defined by

$$F^*((a_1,i_1),...,(a_m,i_m)) = \begin{cases} (F(a_1,...,a_m), i_1) & \text{if } i_1 = ... = i_m \neq 0 \\ (\star, 0) & \text{otherwise.} \end{cases}$$

Remark 8.15. The additional element $\star$ is designed to define functions. If the structure is relational, we can just take $\mathbf{M}^*$ to be the product space $\mathbf{M} \times \mathbb{N}$ together with the appropriate predicates.

Since there is a predicate for each copy of $\mathbf{M}$, automorphisms of $\mathbf{M}^*$ preserve copies. Hence, as expected, the automorphism group of $\mathbf{M}^*$ is isomorphic to $G^{\mathbb{N}}$. The action of $G^{\mathbb{N}}$ is defined as follows: if $\varphi = (\varphi_n)_{n\in\mathbb{N}}$ is an element of $G^{\mathbb{N}}$ and (x,i) is in $\mathbf{M}^*$, then

$$\varphi(a,i) = (\varphi_i(a), i),$$

with the convention that for every $g \in G$, $g(\star) = \star$.

Remark 8.16. It might seem more natural to consider the actual **product structure** of $\mathbf{M}$, whose universe is $\mathbf{M}^{\mathbb{N}}$, and where predicates and functions work coordinatewise. There is indeed no problem to define functions here. The automorphism group of the product structure of $\mathbf{M}$ is also the product $G^{\mathbb{N}}$. However, the extension property does not carry over to the product structure unless it is in some sense uniform[4]. The homogeneity and the existence property do carry over, though, and the proofs are similar to those for the juxtaposed structure.

Proposition 8.17. Let $\mathbf{M}$ be a metric structure of diameter smaller than 1. If the structure $\mathbf{M}$ is exactly ultrahomogeneous, then so is $\mathbf{M}^*$.

Proof. Let f be an isomorphism between two finite substructures of $\mathbf{M}^*$. Since f preserves the predicates C_n, we can write f as a sequence $(f_n)_{n\in\mathbb{N}}$, where f_n is an isomorphism between finite substructures of $\mathbf{M}$. We apply the ultrahomogeneity of $\mathbf{M}$ to each f_n and extend it to an automorphism φ_n of the whole structure $\mathbf{M}$. Then the sequence $(\varphi_n)_{n\in\mathbb{N}}$ is an automorphism of $\mathbf{M}^*$ which extends f. □

The following proposition gives a description of types in the juxtaposed structure: they are "products" of types in each copy. To simplify the notation, we only state it for pairs, but it works exactly the same for bigger tuples.

[4]The size of the bigger finite set needs to depend only on the size of the smaller one, see definition 8.19.

PROPOSITION 8.18. Let $\mathbf{M}$ be a metric structure of diameter smaller than 1 and let a and b be elements of $\mathbf{M}$. Let i and j be two distinct indices in $\mathbb{N}\setminus\{0\}$ and let p be the quantifier-free type of the tuple $((a,i),(b,j))$ in $\mathbf{M}^*$. Let also p_a and p_b be the quantifier-free types of a and b in $\mathbf{M}$ respectively. Then the set of realizations of p in $\mathbf{M}^*$ is described as follows:

$$p(\mathbf{M}^*) = \{((a',i),(b',j)) : a' \in p_a(\mathbf{M}) \text{ and } b' \in p_b(\mathbf{M})\}.$$

PROOF. Let $\bar{c}$ in $\mathbf{M}^*$ have the same quantifier-free type as $((a,i),(b,j))$. Since the predicates C_i and C_j are in the language, we can write $\bar{c}$ as $((a',i),(b',j))$, with a' and b' in $\mathbf{M}$. Now, if $\theta_{\mathbf{M}}$ is a quantifier-free formula on $\mathbf{M}$, it induces a quantifier-free formula θ on $\mathbf{M}\times\{i\}$. Then $\theta((a',i)) = \theta((a,i))$ by assumption, so $\theta_{\mathbf{M}}(a') = \theta_{\mathbf{M}}(a)$. This implies that a and a' have the same quantifier-free type.

Conversely, let us show that any tuple of the form $((a',i),(b',j))$, where a' and b' have the same quantifier-free type as a and b respectively, has the same quantifier-free type as $((a,i),(b,j))$. To do this, let θ be an atomic quantifier-free formula on $(\mathbf{M}^*)^2$. If θ depends only on its first variable, say θ is a formula on $\mathbf{M}\times\{i\}$, then it is induced by a formula $\theta_{\mathbf{M}}$ on $\mathbf{M}$. We thus have

$$\theta(a',i) = \theta_{\mathbf{M}}(a') = \theta_{\mathbf{M}}(a) = \theta(a,i).$$

If on the contrary, θ depends on its two variables in $\mathbf{M}^*$, then θ is of the form $P(t_1(x,y),...,t_m(x,y))$, where P is a predicate and $t_1(x,y)$, ..., $t_m(x,y)$ are terms. Two cases can occur:

- all the terms $t_i(x,y)$ depend on only one variable (x or y). Then for all i, we have that $t_i((a,i),(b,j)) = t_i((a',i),(b',j))$, hence $\theta((a,i),(b,j)) = \theta((a',i),(b',j))$.
  Actually, since θ depends on both its variables, there must be one term that depends on x, say $t_1(x,y) = t_1(x)$, and one term that depends on y, say $t_2(x,y) = t_2(y)$. Thus, $t_1((a,i)) = t_1((a',i)) \in \mathbf{M}\times\{i\}$ and $t_2((b,j)) = t_2((b',j)) \in \mathbf{M}\times\{j\}$. It follows that $\theta((a,i),(b,j)) = 1 = \theta((a',i),(b',j))$.
- there is one term, say $t_1(x,y)$, that depends on both its variables. Then, the term will involve the value of a function symbol on elements from different copies ; it follows, by induction on the complexity of the term t_1, that there exists an $\mathcal{L}$-term $\tilde{t}_1$ such that both $t_1((a,i),(b,j)) = t_1((a',i),(b',j)) = (\tilde{t}_1(\star),0)$.
  Now, either all terms are of this form, and
  $$\theta((a',i),(b',j)) = \theta((a,i),(b,j)) = P(\tilde{t}_1(\star),...,\tilde{t}_m(\star)),$$
  or there is a term that only depends on one variable, say $t_2(x,y) = t_2(x)$. Then $t_2((a,i),(b,j)) = t_2((a',i),(b',j))$ belongs to $\mathbf{M}\times\{i\}$. Put $(e,i) = t_2((a,i),(b,j))$. It follows that
  $$\theta((a,i),(b,j)) = P((\tilde{t}_1(\star),0),(e,i),...) = 1 = \theta((a',i),(b',j)).$$

Finally, $\theta((a',i),(b',j)) = \theta((a,i),(b,j))$ for all atomic formulas θ. This proves that $((a',i),(b',j))$ has the same quantifier-free type as $((a,i),(b,j))$. $\square$

We now go over the assumptions of theorem 8.12 to see how they carry over to the juxtaposed structure.

3. The extension property

DEFINITION 8.19. Let $\mathbf{M}$ be a metric structure. We say that $\mathbf{M}$ has the **extension property** (Sabok and Malicki say that $\mathbf{M}$ has **locally finite automorphisms**) if for every finitely generated substructure $\mathbf{A}$ of $\mathbf{M}$ and every set P of partial automorphisms of $\mathbf{A}$, there exists a finitely generated substructure $\mathbf{B}$ of $\mathbf{M}$ that contains $\mathbf{A}$ such that every partial automorphism in P extends to a global automorphism of $\mathbf{B}$.

EXAMPLES 8.20. The following structures have the extension property.

- Finite ultrahomogeneous structures.
- The random graph (Hrushovski, [**H2**]).

- The Urysohn space (Solecki, [**S6**]).
- The measure algebra of the standard probability space (Kechris-Rosendal, [**KR**, page 32], see also Sabok, [**S1**, lemma 9.1]).
- The (unit ball of the) separable Hilbert space (see Sabok, [**S1**, lemma 10.2]).

PROPOSITION 8.21. Let $\mathbf{M}$ be a metric structure of diameter smaller than 1 which satisfies the extension property. Then the juxtaposed structure $\mathbf{M}^*$ also satisfies the extension property.

PROOF. Let $\mathbf{A}$ be a finite substructure of $\mathbf{M}^*$. Since $\mathbf{A}$ is finite, it only intersects finitely many copies $\mathbf{M} \times \{n\}$. We then apply the extension property in each of those copies and take the union of the obtained sets (together with the special element $(\star, 0)$). □

4. The existence property

DEFINITION 8.22. Let $\mathbf{M}$ be a metric structure and let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be finitely generated substructures of $\mathbf{M}$ such that $\mathbf{A} \subseteq \mathbf{B} \cap \mathbf{C}$. We say that $\mathbf{B}$ and $\mathbf{C}$ are **independent** over $\mathbf{A}$ if for all automorphisms $f_{\mathbf{B}} : \mathbf{B} \to \mathbf{B}$ and $f_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ that stabilize $\mathbf{A}$ and coincide on $\mathbf{A}$, the map $f_{\mathbf{B}} \cup f_{\mathbf{C}}$ extends to an automorphism of the substructure generated by $\mathbf{B}$ and $\mathbf{C}$.

DEFINITION 8.23. Let $\mathbf{M}$ be a metric structure. We say that $\mathbf{M}$ has the **existence property** (Sabok and Malicki say that $\mathbf{M}$ has the **extension property**) if for all finitely generated substructures $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ such that $\mathbf{A} \subseteq \mathbf{B} \cap \mathbf{C}$, there exists an embedding $f : \mathbf{C} \to \mathbf{M}$ that fixes $\mathbf{A}$ pointwise such that $\mathbf{B}$ and $f(\mathbf{C})$ are independent over $\mathbf{A}$.

EXAMPLES 8.24. Countable structures with the free amalgamation property (see remark 4.7) have the existence property. More generally, structures with a stationary independence relation (in the sense of Tent and Ziegler, [**TZ2**]) have the existence property. In particular, the following structures do.

- The pure infinite set.
- The random graph.
- The Urysohn space and sphere.

NON-EXAMPLE 8.25. Finite structures fail to have the existence property. Indeed, the whole structure is not independent from itself, which is the only substructure isomorphic to it, over the empty set. There is not enough space in the structure to get independence. In particular, this is the case of our favorite non-example: the two-element structure.

PROPOSITION 8.26. Let $\mathbf{M}$ be a metric structure of diameter smaller than 1 which satisfies the existence property. Then the juxtaposed structure $\mathbf{M}^*$ also satisfies the existence property.

PROOF. Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be finite substructures of $\mathbf{M}^*$ such that $\mathbf{A} \subseteq \mathbf{B} \cap \mathbf{C}$. Since $\mathbf{B}$ and $\mathbf{C}$ are finite, they only intersects finitely many copies $\mathbf{M} \times \{n\}$. As for the extension property, we then apply the existence property in each of those copies and take the union of the obtained sets (together with the special element $(\star, 0)$). □

5. Isolation

Sabok and Malicki proposed different definitions for the isolation property in theorem 8.12. However, it is unclear whether Sabok's property carries to the juxtaposed structure. Thus, we will only consider Malicki's isolation conditions.

5.1. Relevant tuples. Malicki's theorem only requires isolation for a sufficiently large family of tuples from the structure. Such families he calls *relevant*.

DEFINITION 8.27. A family R of tuples of $\mathbf{M}$ is called **relevant** if for every tuple $\overline{a}$ in $\mathbf{M}$, there exists a tuple $\overline{b}$ in R such that $G_{\overline{b}} \leqslant G_{\overline{a}}$.

Note that any relevant family of tuples naturally induces a relevant family of tuples of its juxtaposed structure.

PROPOSITION 8.28. Let $\mathbf{M}$ be a metric structure of diameter smaller than 1 and let R be a relevant family of tuples of $\mathbf{M}$. Then the family R^* of all those tuples in $\mathbf{M}^*$ whose projection to the every copy $\mathbf{M} \times \{n\}$ belongs to R is relevant.

PROOF. We only prove it for pairs, but the proof works exactly the same for bigger tuples. Let i and j be two distinct indices in $\mathbb{N}$ and let $\overline{c} = ((a, i), (b, j))$ be a tuple in $\mathbf{M}^*$. The stabilizer of this tuple in $G^{\mathbb{N}}$ is

$$(G^{\mathbb{N}})_{\overline{c}} = \{(\varphi_n) \in G^{\mathbb{N}} : \varphi_i \in G_a \text{ and } \varphi_j \in G_b\}.$$

Now, if both i and j are nonzero, let $\overline{a}' = (a'_1, ..., a'_m)$ and $\overline{b}' = (b'_1, ..., b'_l)$ be tuples in the relevant family R such that $G_{\overline{a}'} \leqslant G_a$ and $G_{\overline{b}'} \leqslant G_b$. Consider the following tuple of R^*: $\overline{c}' = ((a'_1, i), ..., (a'_m, i), (b'_1, j), ..., (b'_l, j))$. Then we have $(G^{\mathbb{N}})_{\overline{c}'} \leqslant (G^{\mathbb{N}})_{\overline{c}}$.

If one of the indices is zero, say $i = 0$, then $G_a = G_\star = G$ so $G^{\mathbb{N}}_{\overline{c}} = G^{\mathbb{N}}_{(b,j)}$. So if $\overline{b}' = (b'_1, ..., b'_l)$ is a tuple in the relevant family such that $G_{\overline{b}'} \leqslant G_b$, the tuple $((b'_1, j), ..., (b'_l, j))$ of R^* satisfies that $(G^{\mathbb{N}})_{\overline{b}'} \leqslant (G^{\mathbb{N}})_{\overline{b}}$, proving that R^* is relevant. $\square$

5.2. Direct strong isolation. We first present one of Malicki's versions of the isolation property needed for theorem 8.12. In fact, we simplify the condition slightly by mentioning only the *local orbit* in the following definition.

DEFINITION 8.29. Let $\mathbf{M}$ be a metric structure and let G be the automorphism group of $\mathbf{M}$. Let $\overline{a}$ be a tuple in $\mathbf{M}$ and let p be the quantifier-free type of $\overline{a}$. Let ϵ be a positive real. We say that $\overline{a}$ is **directly ϵ-strongly isolated** if there exist

- a sequence $(\overline{a}_k)_{k\in\mathbb{N}}$ of tuples of quantifier-free type p,
- a sequence $(G_k)_{k\in\mathbb{N}}$ of subgroups of G, and
- a sequence $(\delta_k)_{k\in\mathbb{N}}$ of positive reals

such that

- $G_k[\overline{a}_k] \subseteq B(\overline{a}, \epsilon)$,
- if $\overline{a}'$ is a tuple of quantifier-free type p in the ball $B(\overline{a}_k, \delta_k)$, then the types $\mathrm{qftp}(\overline{a}'/\overline{a})$ and $\mathrm{qftp}(\overline{a}'/\overline{a}_k)$ are realized in $G_k[\overline{a}_k]$, and
- for every sequence $(g_k)_{k\in\mathbb{N}}$ of automorphisms with $g_k \in G_k$ for all k, there exists an automorphism g in G such that for all k, we have $g \restriction G_k[\overline{a}_k] = g_k \restriction G_k[\overline{a}_k]$.

The last two conditions are conditions of *local relative saturation* and *local relative homogeneity.*

DEFINITION 8.30. We say that a tuple in $\mathbf{M}$ is **directly strongly isolated** if it is directly ϵ-strongly isolated for every positive ϵ.

EXAMPLE 8.31. (Malicki) In the Urysohn space, every tuple is directly strongly isolated.

REMARK 8.32. If the structure $\mathbf{M}$ is discrete, then the condition of being directly strongly isolated is empty. Indeed, if $\bar{a}$ is any tuple in $\mathbf{M}$ and ϵ is any positive real, then $\bar{a}$ is directly ϵ-strongly isolated by the constant sequences $(\bar{a})_{k\in\mathbb{N}}$ and $(\{id_{\mathbf{M}}\})_{k\in\mathbb{N}}$, with any sequence $(\delta_k)_{k\in\mathbb{N}}$ of positive reals. In particular, every tuple in the two-element structure is directly strongly isolated.

We are now ready to state Malicki's theorem in a precise way.

THEOREM 8.33. (Malicki) Let $\mathbf{M}$ be an exactly ultrahomogeneous metric structure. Let R be a relevant family of tuples of $\mathbf{M}$. Assume that $\mathbf{M}$ has the extension property and the existence property, and that every tuple in R is directly strongly isolated. Then the automorphism group of $\mathbf{M}$ is Steinhaus and thus satisfies the automatic continuity property.

REMARK 8.34. If the structure $\mathbf{M}$ is discrete, the theorem extends the result that a Fraïssé structure with both the extension property and the free amalgamation property has the automatic continuity property.

We prove that direct strong isolation carries over to the juxtaposed structure.

PROPOSITION 8.35. Let $\mathbf{M}$ be a metric structure of diameter smaller than 1. Let ϵ be a positive real. Let a and b be two directly ϵ-strongly isolated elements of $\mathbf{M}$. Let i and j be two distinct indices in $\mathbb{N}$. Then the tuple $\bar{c} = ((a,i),(b,j))$ is also directly ϵ-strongly isolated in $\mathbf{M}^*$.

PROOF. Let (a_k), (G_k), (δ_k) and (b_k), (H_k), (ϵ_k) witness the direct ϵ-strong isolation of a and b respectively. We prove that the tuple $\bar{c}$ is then directly ϵ-strongly isolated by the sequences $(\bar{c}_k)$, (K_k) and (η_k), where

- $\bar{c}_k = ((a_k,i),(b_k,j))$,
- $K_k = \{(\varphi_n)_{n\in\mathbb{N}} \in G^{\mathbb{N}} : \varphi_i \in G_k \text{ and } \varphi_j \in H_k\}$ and
- $\eta_k = \min(\delta_k, \epsilon_k)$.

First note that K_k is indeed a subgroup of $G^{\mathbb{N}}$. Besides, any element of $K_k[\bar{c}_k]$ is of the form $((g_k(a_k),i),(h_k(b_k),j))$, with $g_k \in G_k$ and $h_k \in H_k$. The isolation of a and b gives that $g_k(a_k) \in B(a,\epsilon)$ and $h_k(b_k) \in B(b,\epsilon)$, and since we take the supremum distance on tuples, we have that $K_k[c_k]$ is contained in the ball $B(\bar{c},\epsilon)$.

By proposition 8.18, all the tuples $\bar{c}_k$ have the same quantifier-free type as $\bar{c}$. Let now $\bar{c}'$ be a tuple in the ball $B(\bar{c}_k,\eta_k)$ that has the same quantifier-free type as $\bar{c}$. We can write it $\bar{c}' = ((a',i),(b',i))$ and, by proposition 8.18 again, the elements a' and b' have the same quantifier-free type as a and b respectively. We can thus find realizations a_1, a_2 in $G_k[a_k]$ and b_1, b_2 in $H_k[b_k]$ of $\mathrm{qftp}(a'/a)$, $\mathrm{qftp}(a'/a_k)$ and $\mathrm{qftp}(b'/a)$, $\mathrm{qftp}(b'/b_k)$. Now the tuples $\bar{c}_1 = ((a_1,i),(b_1,j))$ and $\bar{c}_2 = ((a_2,i),(b_2,j))$ are realizations of $\mathrm{qftp}(\bar{c}'/\bar{c})$ and $\mathrm{qftp}(\bar{c}'/\bar{c}_k)$ in $K_k[\bar{c}_k]$ (we use proposition 8.18 once again).

Finally, let $(\varphi^k)_{k\in\mathbb{N}}$ be a sequence of automorphisms of $\mathbf{M}^*$, with $\varphi^k \in K_k$ for all k. We can write each φ^k as a sequence $(\varphi^k_n)_{n\in\mathbb{N}}$, with $\varphi^k_i \in G_k$ and $\varphi^k_j \in H_k$. We apply the local relative homogeneity conditions to the sequences $(\varphi^k_i)_{k\in\mathbb{N}}$ and $(\varphi^k_j)_{k\in\mathbb{N}}$ to get automorphisms φ_i and φ_j of $\mathbf{M}$ such that for all k, we have $\varphi_i \restriction G_k[a_k] = \varphi^k_i \restriction G_k[a_k]$ and $\varphi_j \restriction H_k[b_k] = \varphi^k_j \restriction H_k[a_k]$. Then the automorphism $\varphi = (\varphi_n)_{n\in\mathbb{N}}$ of $\mathbf{M}^*$ defined by $\varphi_n = \varphi_i$

$$\varphi_n = \begin{cases} \varphi_i & \text{if } n = i \\ \varphi_j & \text{if } n = j \\ \mathrm{id}_{\mathbf{M}} & \text{otherwise} \end{cases}$$

satisfies that $\varphi \restriction K_k[\bar{c}_k] = \varphi_k \restriction K_k[\bar{c}_k]$, which completes the proof. □

The proof readily adapts to bigger tuples. As a consequence, we obtain that the isolation condition in theorem 8.33 carries over to the product.

COROLLARY 8.36. Let $\mathbf{M}$ be a metric structure of diameter smaller than 1. Let R be a relevant family of tuples of $\mathbf{M}$ and let R^* be the relevant family of tuples of $\mathbf{M}^*$ in proposition 8.28. If every tuple in R is directly strongly isolated, then so is every tuple in R^*.

This finally yields that this better set of reasons to have a Steinhaus automorphism group carries over to the juxtaposed structure.

THEOREM 8.37. Let $\mathbf{M}$ be an exactly ultrahomogeneous metric structure of diameter smaller than 1 and let G be its automorphism group. Let R be a relevant family of tuples of $\mathbf{M}$. Assume that $\mathbf{M}$ has the extension property and the existence property, and that every tuple in R is directly strongly isolated. Then the group $G^{\mathbb{N}}$ is Steinhaus and thus satisfies the automatic continuity property.

COROLLARY 8.38. The group $\mathrm{Iso}(\mathbb{U}_1)^{\mathbb{N}}$ satisfies the automatic continuity property.

REMARK 8.39. The metric on the Urysohn space is not bounded. However, it is equivalent to the following metric

$$d'(x,y) = \frac{d(x,y)}{1+d(x,y)},$$

which is bounded by 1. Moreover, the isometry groups of $(\mathbb{U}, d)$ and $(\mathbb{U}, d')$ are the same. Thus, we can apply the previous results to the Urysohn space too: we also get the automatic continuity property for $\mathrm{Iso}(\mathbb{U})^{\mathbb{N}}$.

5.3. Indirect strong isolation.

Definition 8.40. Let $\mathbf{M}$ be a metric structure and let G be the automorphism group of $\mathbf{M}$. Let $\bar{a}$ be a tuple in $\mathbf{M}$ and let p be the quantifier-free type of $\bar{a}$. Let ϵ be a positive real. We say that $\bar{a}$ is **indirectly ϵ-strongly isolated** if there exist

- a sequence $(\bar{a}_k)_{k\in\mathbb{N}}$ of tuples of quantifier-free type p,
- a sequence $(G_k)_{k\in\mathbb{N}}$ of subgroups of G,
- a sequence $(\delta_k)_{k\in\mathbb{N}}$ of positive reals, and
- a sequence $(X_k)_{k\in\mathbb{N}}$ of metric substructures of M

such that

- X_k is exactly ultrahomogeneous and satisfies the extension property and the existence property,
- X_k is invariant under the action of G_l, for all l in $\mathbb{N}$,
- $G_k \upharpoonright X_k = \mathrm{Aut}(X_k)$ and every element of $\mathrm{Aut}(X_k)$ extends uniquely to an element of G_k,
- $G_k[\bar{a}_k] \subseteq B(\bar{a}, \epsilon)$,
- if $\overline{a}'$ is a tuple of quantifier-free type p in the ball $B(\overline{a}_k, \delta_k)$, then the types $\mathrm{qftp}(\overline{a}'/\overline{a})$ and $\mathrm{qftp}(\overline{a}'/\overline{a}_k)$ are realized in $G_k[\bar{a}_k]$, and
- for every sequence $(g_k)_{k\in\mathbb{N}}$ of automorphisms with $g_k \in G_k$ for all k, there exists an automorphism g in G such that for all k, we have $g \upharpoonright X_k = g_k \upharpoonright X_k$.

Remark 8.41. In the definition of direct isolation, the role of X_k in the local relative homogeneity condition is played by the local orbit $G_k[\bar{a}_k]$, although the local orbit is not necessarily a substructure of $\mathbf{M}$ (let alone an ultrahomogeneous substructure with the extension property and the existence property).

Definition 8.42. We say that a tuple in $\mathbf{M}$ is **indirectly strongly isolated** if it is indirectly ϵ-strongly isolated for every positive ϵ.

Examples 8.43. (Malicki)

- In the measure algebra of a standard probability space X, every partition of X into positive measure sets is indirectly strongly isolated.
- In the Hilbert space, every orthonormal tuple is indirectly strongly isolated.

Here is the indirect version of Malicki's result.

Theorem 8.44. (Malicki) Let $\mathbf{M}$ be an exactly ultrahomogeneous metric structure. Let R be a relevant family of tuples of $\mathbf{M}$. Assume that $\mathbf{M}$ has the extension property and the existence property, and that every tuple in R is indirectly strongly isolated. Then the automorphism group of $\mathbf{M}$ is Steinhaus and thus satisfies the automatic continuity property.

We now prove that indirect strong isolation carries to the juxtaposed structure.

Proposition 8.45. Let $\mathbf{M}$ be a metric structure of diameter smaller than 1. Let ϵ be a positive real. Let a and b be two indirectly ϵ-strongly isolated elements of $\mathbf{M}$. Let i and j be two distinct indices in $\mathbb{N}$. Then the tuple $\bar{c} = ((a, i), (b, j))$ is also indirectly ϵ-strongly isolated in $\mathbf{M}^*$.

Proof. Let (a_k), (G_k), (δ_k), (X_k) and (b_k), (H_k), (ϵ_k), (Y_k) witness the indirect ϵ-strong isolation of a and b respectively. We prove that the tuple $\bar{c}$ is then indirectly ϵ-strongly isolated by the sequences $(\bar{c}_k)$, (K_k), (η_k) and (Z_k), where

- $\bar{c}_k = ((a_k, i), (b_k, j))$,
- $K_k = \{(\varphi_n)_{n\in\mathbb{N}} \in G^{\mathbb{N}} : \varphi_i \in G_k \text{ and } \varphi_j \in H_k\}$,

- $\eta_k = \min(\delta_k, \epsilon_k)$, and
- $Z_k = (X_k \times \{i\}) \cup (Y_k \times \{j\})$.

As in proposition 8.35, we have that K_k is a subgroup of $G^{\mathbb{N}}$, that $K_k[\bar{c}_k]$ is contained in $B(\bar{c}, \epsilon)$, that the tuples $\bar{c}_k$ have the same quantifier-free type as $\bar{c}$ and the local relative saturation property.

Note that Z_k is indeed a substructure of $\mathbf{M}^*$, whose automorphism group is $\mathrm{Aut}(X_k) \times \mathrm{Aut}(Y_k)$.

The proofs of propositions 8.17, 8.21 and 8.26 show that the structure Z_k is exactly ultrahomogeneous and satisfies the extension property and the existence property.

Besides, the restriction of K_k to Z_k is $(G_k \restriction X_k) \times (H_k \restriction Y_k)$, which coincides with $\mathrm{Aut}(X_k) \times \mathrm{Aut}(Y_k) = \mathrm{Aut}(Z_k)$.

Moreover, $K_l(Z_k) = (G_l[X_k] \times \{i\}) \cup (H_l[Y_k] \times \{j\})$. Thus, since X_k and Y_k are invariant under the actions of G_l and H_l respectively, the structure Z_k is invariant under the action of K_l.

Finally, let $(\varphi^k)_{k \in \mathbb{N}}$ be a sequence of automorphisms of $\mathbf{M}^*$, with $\varphi^k \in K_k$ for all k. We can write each φ^k as a sequence $(\varphi^k_n)_{n \in \mathbb{N}}$, with $\varphi^k_i \in G_k$ and $\varphi^k_j \in H_k$. We apply the local relative homogeneity conditions to the sequences $(\varphi^k_i)_{k \in \mathbb{N}}$ and $(\varphi^k_j)_{k \in \mathbb{N}}$ to get automorphisms φ_i and φ_j of $\mathbf{M}$ such that for all k, we have $\varphi_i \restriction X_k = \varphi^k_i \restriction X_k$ and $\varphi_j \restriction Y_k = \varphi^k_j \restriction Y_k$. Then the automorphism $\varphi = (\varphi_n)_{n \in \mathbb{N}}$ of $\mathbf{M}^*$ defined by $\varphi_n = \varphi_i$

$$\varphi_n = \begin{cases} \varphi_i & \text{if } n = i \\ \varphi_j & \text{if } n = j \\ \mathrm{id}_{\mathbf{M}} & \text{otherwise} \end{cases}$$

satisfies that $\varphi \restriction Z_k = \varphi_k \restriction Z_k$, which completes the proof. □

The proof readily adapts to bigger tuples. As a consequence, we obtain that the isolation condition in theorem 8.44 carries to the product.

Corollary 8.46. Let $\mathbf{M}$ be a metric structure of diameter smaller than 1. Let R be a relevant family of tuples of $\mathbf{M}$ and let R^* be the relevant family of tuples of $\mathbf{M}^*$ in proposition 8.28. If every tuple in R is indirectly strongly isolated, then so is every tuple in R^*.

This finally yields that this indirect better set of reasons to have a Steinhaus automorphism group also carries to the juxtaposed structure.

Theorem 8.47. Let $\mathbf{M}$ be an exactly ultrahomogeneous metric structure of diameter smaller than 1 and let G be its automorphism group. Let R be a relevant family of tuples of $\mathbf{M}$. Assume that $\mathbf{M}$ has the extension property and the existence property, and that every tuple in R is indirectly strongly isolated. Then the group $G^{\mathbb{N}}$ is Steinhaus and thus satisfies the automatic continuity property.

Corollary 8.48. The groups $\mathrm{Aut}(\mu)^{\mathbb{N}}$ and $\mathcal{U}(\ell^2)^{\mathbb{N}}$ satisfy the automatic continuity property.

6. Connected groups with ample generics

Recall that when G is a Polish group that has ample generics, then $G^{\mathbb{N}}$ also has ample generics (proposition 8.11). With Le Maître, we noticed that this could be generalized to the group $L^0(X, \mu; G)$ of G-valued random variables on a standard probability space[5] (X, μ). In fact, the group $G^{\mathbb{N}}$ can be viewed as that of G-valued random variables on $\mathbb{N}$. That is what led us to an answer to Kechris and Rosendal's question, asking whether there was a Polish group with ample generics that is not a closed subgroup of S_∞.

Let us first present the space $L^0(X, \mu; Y)$ and establish a few basic lemmas on its topology. Let Y be a Polish space.

[5] Say, the unit interval together with its Lebesgue measure.

Definition 8.49. Let Y be a Polish space. Then $L^0(X,\mu;Y)$ is the set of Lebesgue-measurable maps from X to Y up to equality μ-almost everywhere.

We endow $L^0(X,\mu;Y)$ with the **topology of convergence in measure**, which says that two maps are close in this topology if they are uniformly close on a set of large measure. To be more precise, fix a compatible metric d_Y on Y. Then for $\epsilon > 0$ and $f \in L^0(X,\mu;Y)$, let

$$V_\epsilon(f) := \{g \in L^0(X,\mu;Y) : \mu(\{x \in X : d_Y(f(x),g(x)) < \epsilon\}) > 1-\epsilon\}.$$

The topology generated by the family $(V_\epsilon(f))_{\epsilon,f}$ is the topology of convergence in measure.

This topology does not depend on the choice of a compatible metric d_Y on Y ([**M11**, corollary of proposition 6]). Moreover, it is Polish ([**M11**, proposition 7]) and contractible (see [**K6**, proposition 19.7]): assume $X = [0,1]$ and let $y_0 \in Y$ be an arbitrary point. Then an explicit contraction path is given by

$$\begin{array}{rcl} L^0(X,\mu;Y) \times [0,1] & \to & L^0(X,\mu;Y) \\ (f,t) & \mapsto & f_t : x \mapsto \left\{ \begin{array}{cl} y_0 & \text{if } x > t, \\ f(x) & \text{otherwise.} \end{array} \right. \end{array}$$

The following lemma is an easy consequence of the definition of the topology of convergence in measure.

Lemma 8.50. Let Y be a Polish space and let U be an open subset of Y. Then for every $\epsilon > 0$, the set

$$V_{U,\epsilon} := \{f \in L^0(X,\mu;Y) : \mu(\{x \in X : f(x) \in U\}) > 1-\epsilon\}$$

is open in $L^0(X,\mu;Y)$.

Proof. Let $f \in V_{U,\epsilon}$ and let $A = \{x \in X : f(x) \in U\}$. Fix a compatible metric d_Y on Y. Since U is open, the set A can be written as the increasing union of the sets A_n's, where $A_n = \{x \in A : d_Y(f(x), Y \setminus U) > \frac{1}{n}\}$. By assumption, the set A has measure greater than $1-\epsilon$, so we can find $N \in \mathbb{N}$ such that $\mu(A_N) > 1-\epsilon$. Now, if δ is a positive real such that $\delta < \frac{1}{N}$ and $\delta < \mu(A_N) - (1-\epsilon)$, we see that $V_\delta(f) \subseteq V_{U,\epsilon}$, hence $V_{U,\epsilon}$ is open. $\square$

Given a subset B of Y, let

$$L^0(X,\mu;B) := \{f \in L^0(X,\mu;Y) : \forall x \in X, f(x) \in B\}.$$

Lemma 8.51. Let Y be a Polish space, and let B be a G_δ subset of Y. Then the set $L^0(X,\mu;B)$ is G_δ in $L^0(X,\mu;Y)$.

Proof. Write $B = \bigcap_{n\in\mathbb{N}} U_n$ where each U_n is open. Then clearly $L^0(X,\mu;B) = \bigcap_{n\in\mathbb{N}} L^0(X,\mu;U_n)$. Now, $L^0(X,\mu;U_n) = \bigcap_{k\in\mathbb{N}} V_{U_n,2^{-k}}$, so it is G_δ by the previous lemma, so $L^0(X,\mu;B)$ itself is G_δ. $\square$

Lemma 8.52. Let Y be a Polish space, and let B be dense subset of Y, then $L^0(X,\mu;B)$ is dense in $L^0(X,\mu;Y)$.

Proof. Fix a compatible metric d_Y on Y. Since Y is separable, we can find a countable subset of B which is still dense in Y: in other words, we may as well assume that B is countable. Enumerate B as $\{y_n\}_{n\in\mathbb{N}}$, and fix $\epsilon > 0$ as well as a function $f \in L^0(X,\mu;Y)$. For every $x \in X$, let $n(x)$ be the smallest integer $n \in \mathbb{N}$ such that $d_Y(f(x), y_n) < \epsilon$. It is easily checked that the map $x \mapsto n(x)$ is measurable, so that the function $g : x \mapsto y_{n(x)}$ belongs to $L^0(X,\mu;B)$. But by construction, we actually have $d_Y(f(x),g(x)) < \epsilon$ for *all* $x \in X$, and in particular $g \in V_\epsilon(f)$, which completes the proof. $\square$

What follows is an immediate consequence of the previous two lemmas.

Lemma 8.53. Let Y be a Polish space and let B be a comeager subset of Y. Then $L^0(X,\mu;B)$ is a comeager subset of $L^0(X,\mu;Y)$.

Now, if G is a Polish group, $L^0(X,\mu;G)$ is a connected group (we have seen that it is even contractible), hence it cannot be a topological subgroup of the totally disconnected group S_∞ (except if G is trivial). Together with the following theorem, this yields a family of examples of connected Polish groups with ample generics.

THEOREM 8.54. (with Le Maître) Let G be a Polish group with ample generics. Then the group $L^0(X,\mu;G)$ also has ample generics.

PROOF. We wish to prove that for every n in $\mathbb{N}$, the diagonal action of $L^0(X,\mu;G)$ on $L^0(X,\mu;G)^n$ admits a comeager orbit. Here too, there is a natural identification of $L^0(X,\mu;G)^n$ with $L^0(X,\mu;G^n)$.

Let $\bar\varphi$ be an element of G^n whose orbit is comeager and consider the constant function $\bar f : x \mapsto \bar\varphi$ in $L^0(X,\mu;G^n)$. We show that the orbit of $\bar f$ in $L^0(X,\mu;G^n)$ is comeager.

First, let us remark that the orbit of $\bar f$ is thus described:

$$L^0(X,\mu;G)\cdot \bar f = \{\bar g \in L^0(X,\mu;G^n) : \bar g(x) \in G\cdot\bar\varphi \text{ for } \mu\text{-almost every } x\}.$$

Indeed, if $\bar g$ is in the orbit of $\bar f$, then $\bar g$ is clearly in the set above. Conversely, assume that $\bar g(x)$ is in $G\cdot\bar\varphi$ almost everywhere. There is a Borel subset B of X with measure 1 such that for every x in B, there exists an element h_x in G such that $\bar g(x) = h_x\cdot\bar\varphi$. We would like to find those h_x in a measurable way. For this, we apply the Jankov-von Neumann uniformization theorem (see [**K4**, theorem 18.1]) to the following Borel[6] set

$$S = \{(x,h_x) \in X\times G : [x\in B \text{ and } \bar g(x) = h_x\cdot\bar\varphi] \text{ or } x\notin B\},$$

which projects to the whole space X. We thus obtain a map h in $L^0(X,\mu;G)$ whose graph is contained in S, that is, $\bar g = h\cdot\bar f$, hence $\bar g$ belongs to orbit of $\bar f$.

Now the orbit of $\bar\varphi$ is comeager in G^n so, by lemma 8.53, the set $L^0(X,\mu;G\cdot\bar\varphi)$ is comeager in $L^0(X,\mu;G^n)$. The previous observation thus yields that the orbit of $\bar f$ is comeager in $L^0(X,\mu;G^n)$, which completes the proof.

□

As a consequence, we notably obtain that every Polish group with ample generics embeds in a connected (even contractible) one with ample generics.

REMARK 8.55. We have obtained another example of a connected Polish group with ample generics: the full group of a quasi-measure-preserving hyperfinite equivalence relation (see [**KLM**]). It is interesting to note that it is also a subgroup of $L^0(X,\mu;S_\infty)$. Malicki's examples ([**M1**]) do not seem to be, though.

7. Perspectives

With theorem 8.54 at hand, it is natural to ask whether Malicki's conditions for G carry over to the group $L^0(X,\mu;G)$. However, in this case, the construction of the juxtaposed structure would not make much sense. Rather, the group $L_0([0,1],G)$ is the automorphism group of a randomization of the structure $\mathbf{M}$, which is the counterpart of the product structure of $\mathbf{M}$. This randomization remains exactly ultrahomogeneous if the original is. As in the proof of theorem 8.54, in order to carry properties from the structure over to its randomization, our main tool is the Jankov-van Neumann theorem. But again, we need some amount of uniformity to apply it.

[6]Since g is Lebesgue-measurable, it coincides with a Borel map almost everywhere, so we may assume that g is actually Borel.

Branch 3

Amenability and Ramsey theory

CHAPTER 9

Amenability and convex Ramsey theory

La simplicité réside dans l'alcôve bleue et jaune et mauve et insoupçonnée
de nos rêveries mauves et bleues et jaunes et pourpres et paraboliques
et vice et versa.
Les Inconnus[1]

1. **The metric convex Ramsey property** **123**
2. **The metric convex Ramsey property for the automorphism group** **126**
3. **A criterion for amenability** **129**
4. **Structural consequences** **131**
5. **Concluding remarks** **134**

In this chapter, we take up the results from [**K1**], where we extend a result of Moore ([**M12**, theorem 7.1]) on the amenability of closed subgroups of S_∞ to general Polish groups. A topological group is said to be *amenable* if every continuous action of the group on a compact Hausdorff space admits an invariant probability measure.

Moore's result is the counterpart of a theorem of Kechris, Pestov and Todorčević ([**KPT**]) on extreme amenability. A topological group is said to be *extremely amenable* if every continuous action of the group on a compact Hausdorff space admits a fixed point. In the context of closed subgroups of S_∞, seen as automorphism groups of Fraïssé structures, Kechris, Pestov and Todorčević characterize extreme amenability by a combinatorial property of the associated Fraïssé classes (in the case where its objects are rigid), namely, the Ramsey property. A class $\mathcal{K}$ of structures is said to have *the Ramsey property* if for all structures A and B in $\mathcal{K}$, for all integers k, there is a structure C in $\mathcal{K}$ such that for every coloring of the set of copies of A in C with k colors, there exists a copy of B in C within which all copies of A have the same color (see figure 9.1).

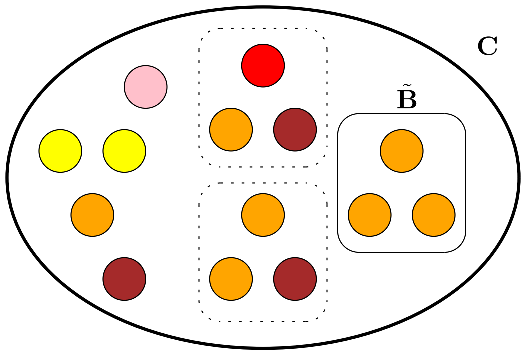


FIGURE 9.1. The Ramsey property

Thus, extreme amenability, which provides fixed points, corresponds to colorings having a "fixed", meaning monochromatic, set. Amenability, on the other side, provides invariant measures. Since a measure is not far from being a barycenter of point masses, the natural mirror image of the Ramsey property in that setting should be for a coloring to have a "monochromatic

[1] *Vice et versa*

convex combination of sets". Indeed, Tsankov (in an unpublished note) and Moore introduced a *convex Ramsey property* and proved that a Fraïssé class has the convex Ramsey property if and only if the automorphism group of its Fraïssé limit is amenable.

Besides, the Kechris-Pestov-Todorčević result was extended to general Polish groups by Melleray and Tsankov in [**MT1**]. They use the framework of continuous logic (see [**BBHU**]) via the observation that every Polish group is the automorphism group of an approximately homogeneous metric structure ([**M5**, theorem 6]), that is of a metric Fraïssé limit (in the sense of [**MT1**]; these were built by Ben Yaacov in [**B5**]). They define an *approximate Ramsey property* for classes of metric structures and then show that a metric Fraïssé class has the approximate Ramsey property if and only if the automorphism group of its Fraïssé limit is extremely amenable.

Here, we "close the diagram" by giving a metric version of Moore's result. We replace the classical notion of a coloring with the metric one (from [**MT1**]) to define a *metric convex Ramsey property*, and we prove the exact analogue of Moore's theorem (theorem 9.22):

THEOREM 9.1. Let $\mathcal{K}$ be a metric Fraïssé class, $\mathbf{K}$ its Fraïssé limit and G the automorphism group of $\mathbf{K}$. Then G is amenable if and only if $\mathcal{K}$ satisfies the metric convex Ramsey property.

From this result, we deduce some interesting structural consequences about amenability. First, we improve the previously known characterization of amenability mentioned below.

If G is a topological group, all minimal continuous actions of G on compact Hausdorff spaces can be captured by a single one: the action of G by translation on its *greatest ambit* $S(G)$ (see [**P3**]). In particular, the topological group G is amenable if and only if the action of G on $S(G)$ admits an invariant Borel probability measure. The greatest ambit of G is none other than the *Samuel compactification*, which is characterized by the property that every right uniformly continuous bounded function on G extends to a continuous function on $S(G)$ and that, conversely, every continuous function on $S(G)$ is the extension of a right uniformly continuous bounded function on G. Thus, amenability can be characterized as follows.

THEOREM 9.2. (see [**P3**, theorem 3.5.12]) Let G be a topological group. Then the following are equivalent.

(1) G is amenable.
(2) There is an invariant *mean*[2] on the space RUCB(G) of right uniformly continuous bounded functions on G.
(3) For every positive integer N and for all $f_1, ..., f_N$ in RUCB(G), there exists a mean Λ on RUCB(G) that is invariant on the orbits of $f_1, ..., f_N$, i.e. for every $j \leqslant N$ and for every g in G, one has $\Lambda(g^{-1} \cdot f_j) = \Lambda(f_j)$.
(4) For every $\epsilon > 0$, every finite subset F of G, every positive integer N and for all $f_1, ..., f_N$ in RUCB(G), there is a finitely supported probability measure μ on G such that for every $j \leqslant N$ and every $h \in F$, one has
$$\left|\int_G f_j d\mu - \int_G f_j d(h \cdot \mu)\right| < \epsilon.$$

The implications (4) $\Rightarrow$ (3) $\Rightarrow$ (2) follow from the weak*-compactness of the space of means on RUCB(G) (which is a consequence of the Banach-Alaoglu theorem), while the implication (2) $\Rightarrow$ (4) follows from an application of the Riesz representation theorem to the Samuel compactification of G and the fact that every Borel probability measure on a compact space can be approximated by finitely supported probability measures. Condition (4) is known as *Day's weak*-asymptotic invariance condition.*

In the course of the proof of theorem 9.1, we provide several reformulations of the metric convex Ramsey property, among which the following (theorem 9.24).

THEOREM 9.3. Let G be a Polish group. Then the following are equivalent.

[2] Positive linear form of norm 1.

(1) G is amenable.
(2) For every $\epsilon > 0$, every finite subset F of G, every left uniformly continuous map $f : G \to [0,1]$, there exist elements $g_1, ..., g_n$ of G and barycentric coefficients $\lambda_1, ..., \lambda_n$ such that for all $h, h' \in F$, one has
$$\left|\sum_{i=1}^{n} \lambda_i f(g_i h) - \sum_{i=1}^{n} \lambda_i f(g_i h')\right| < \epsilon.$$
(3) For every $\epsilon > 0$, every finite subset F of G, every $f \in \mathrm{RUCB}(G)$, there is a finitely supported probability measure μ on G such that for every h in F, one has
$$\left|\int_G f d\mu - \int_G f d(h \cdot \mu)\right| < \epsilon.$$

It constitutes a strengthening of Day's weak*-asymptotic invariance condition for Polish groups: to check that a Polish group is amenable, it suffices to verify Day's condition for a single function. This result was motivated by a similar result obtained by Moore for discrete groups ([**M12**, theorem 2.1]). Besides, the same is true for extreme amenability with *multiplicative* means.

It is interesting that to make this reduction from multiple functions to only one function, we need to express the Polish group as the automorphism group of a metric Fraïssé structure (as per [**M5**]) and then combine multiple colorings into one coloring, whereas it is unclear how to directly combine finitely many right uniformly continuous functions on the group together.

Applying the Riesz representation theorem to the Samuel compactification, as in theorem 9.2, we obtain the following as a corollary (corollary 9.25).

Corollary 9.4. Let G be a Polish group. Then the following are equivalent.

(1) G is amenable.
(2) For every right uniformly continuous bounded function f on G, there exists a mean on $\mathrm{RUCB}(G)$ such that for all $g \in G$, one has $\Lambda(g \cdot f) = \Lambda(f)$.

Another advantage of theorems 9.1 and 9.3 is to express amenability in a finitary way, which allows us to compute its Borel complexity. In [**P3**], it was shown that extreme amenability is equivalent to a Ramsey-theoretic property called *finite oscillation property*, a slight reformulation of which turns out to be a G_δ condition, as observed by Melleray and Tsankov in [**MT2**]. We prove that amenability also is a G_δ condition (corollary 9.27).

From this, a Baire category argument leads to the following sufficient condition for a Polish group to be amenable (corollary 9.29).

Corollary 9.5. Let G be a Polish group such that for every positive $n \in \mathbb{N}$, the set
$$F_n = \{(g_1, ..., g_n) \in G^n : \langle g_1, ..., g_n\rangle \text{ is amenable (as a subgroup of } G)\}$$
is dense in G^n. Then G is amenable.

This is a slight strengthening of the fact that a Polish group whose finitely generated subgroups are amenable is itself amenable (see [**G5**, theorem 1.2.7]), and also admits a direct proof (see remark 9.30).

1. The metric convex Ramsey property

We use the notations of [**MT1**].

Definition 9.6. Let $\mathcal{L}$ be a relational continuous language, $\mathbf{A}$ and $\mathbf{B}$ two finite $\mathcal{L}$-structures and $\mathbf{M}$ an arbitrary $\mathcal{L}$-structure.

- We denote by ${}^{\mathbf{A}}\mathbf{M}$ the set of all embeddings of $\mathbf{A}$ into $\mathbf{M}$. We endow ${}^{\mathbf{A}}\mathbf{M}$ with the metric $\rho_{\mathbf{A}}$ defined by
$$\rho_{\mathbf{A}}(\alpha, \alpha') = \max_{a \in A} d(\alpha(a), \alpha'(a)).$$

- A **coloring** of ${}^{\mathbf{A}}\mathbf{M}$ is a 1-Lipschitz map from $({}^{\mathbf{A}}\mathbf{M}, \rho_{\mathbf{A}})$ to the interval $[0,1]$.
- We denote by $\langle {}^{\mathbf{A}}\mathbf{M} \rangle$ the set of all finitely supported probability measures on ${}^{\mathbf{A}}\mathbf{M}$. We will identify embeddings with their associated Dirac measures.
- If $\kappa : {}^{\mathbf{A}}\mathbf{M} \to [0,1]$ is a coloring, we extend κ to $\langle {}^{\mathbf{A}}\mathbf{M} \rangle$ linearly: if ν in $\langle {}^{\mathbf{A}}\mathbf{M} \rangle$ is of the form $\nu = \sum_{i=1}^{n} \lambda_i \delta_{\alpha_i}$, we set
$$\kappa(\nu) = \sum_{i=1}^{n} \lambda_i \kappa(\alpha_i).$$
- Moreover, we extend composition of embeddings to finitely supported measures bilinearly. Namely, if ν in $\langle {}^{\mathbf{A}}\mathbf{B} \rangle$ and ν' in $\langle {}^{\mathbf{B}}\mathbf{M} \rangle$ are of the form $\nu = \sum_{i=1}^{n} \lambda_i \delta_{\alpha_i}$ and $\nu' = \sum_{j=1}^{m} \lambda'_j \delta_{\alpha'_j}$, we define
$$\nu' \circ \nu = \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda'_j \lambda_i \delta_{\alpha'_j \circ \alpha_i}.$$
- If ν is a measure in $\langle {}^{\mathbf{B}}\mathbf{M} \rangle$, we denote by $\langle {}^{\mathbf{A}}\mathbf{M}(\nu) \rangle$ the set of all finitely supported measures which factor through ν and by ${}^{\mathbf{A}}\mathbf{M}(\nu)$ the set of those which factor through ν via an embedding. More precisely, if $\nu \in \langle {}^{\mathbf{B}}\mathbf{M} \rangle$ is of the form $\sum_{i=1}^{n} \lambda_i \delta_{\beta_i}$, we define
$${}^{\mathbf{A}}\mathbf{M}(\nu) = \left\{ \nu \circ \delta_\alpha : \alpha \in {}^{\mathbf{A}}\mathbf{B} \right\}$$
and
$$\langle {}^{\mathbf{A}}\mathbf{M}(\nu) \rangle = \left\{ \nu \circ \nu' : \nu' \in \langle {}^{\mathbf{A}}\mathbf{B} \rangle \right\}.$$

Throughout the chapter, $\mathcal{K}$ will be a metric Fraïssé class in a relational continuous language and $\mathbf{K}$ will be its Fraïssé limit.

Remark 9.7. We make the assumption that the language is relational to simplify the proofs, but we could also allow functions in the language. In that case, we just need to replace "finite" with "finitely generated" and the proofs are the same.

Definition 9.8. The class $\mathcal{K}$ is said to have **the metric convex Ramsey property** if for every $\epsilon > 0$, for all structures $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{K}$, there exists a structure $\mathbf{C}$ in $\mathcal{K}$ such that for every coloring $\kappa : {}^{\mathbf{A}}\mathbf{C} \to [0,1]$, there is ν in $\langle {}^{\mathbf{B}}\mathbf{C} \rangle$ such that for all $\eta, \eta' \in {}^{\mathbf{A}}\mathbf{C}(\nu)$, one has $|\kappa(\eta) - \kappa(\eta')| < \epsilon$.

Intuition 9.9. In the classical setting, the Ramsey property states that given two structures $\mathbf{A}$ and $\mathbf{B}$, we can find a bigger structure $\mathbf{C}$ such that whenever we color the copies of $\mathbf{A}$ in $\mathbf{C}$, we can find a copy of $\mathbf{B}$ in $\mathbf{C}$ wherein every copy of $\mathbf{A}$ has the same color. Here, the metric convex Ramsey property basically says that we can find a convex combination of copies of $\mathbf{B}$ in $\mathbf{C}$ wherein every compatible convex combination of copies of $\mathbf{A}$ has almost the same color (see figure 9.2).

Remark 9.10. One can replace the assumption $\eta, \eta' \in {}^{\mathbf{A}}\mathbf{C}(\nu)$ with the seemingly stronger one $\eta, \eta' \in \langle {}^{\mathbf{A}}\mathbf{C}(\nu) \rangle$ in the above definition, as is done in [**M12**]. Indeed, the property is preserved under convex combinations.

The following proposition states that the metric convex Ramsey property allows us to stabilize any finite number of colorings at once.

Proposition 9.11. The following are equivalent.

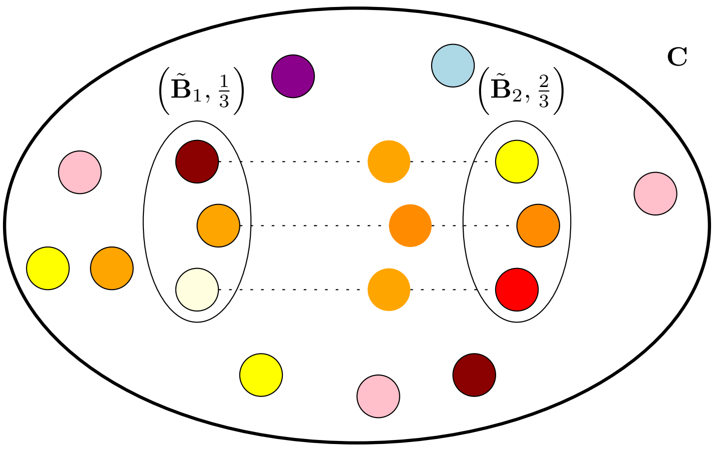


FIGURE 9.2. The orange disks are barycenters of the colors of two corresponding copies of $\mathbf{A}$ in $\tilde{\mathbf{B}}_1$ and $\tilde{B}_2$ with coefficients $1/3$ and $2/3$. The metric convex Ramsey property says that all these disks have almost the same color.

(1) The class $\mathcal{K}$ has the metric convex Ramsey property.
(2) For every $\epsilon > 0$, for all positive integers $N \in \mathbb{N}$ and all structures $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{K}$, there exists a structure $\mathbf{C}$ in $\mathcal{K}$ such that for all colorings $\kappa_1, ..., \kappa_N : {}^{\mathbf{A}}\mathbf{C} \to [0,1]$, there is μ in $\langle {}^{\mathbf{B}}\mathbf{C} \rangle$ such that for all j in $\{1, ..., N\}$ and all η, η' in ${}^{\mathbf{A}}\mathbf{C}(\mu)$, one has $|\kappa_j(\eta) - \kappa_j(\eta')| < \epsilon$.

REMARK. Condition (2) above is equivalent to the metric convex Ramsey property for colorings into $[0,1]^N$, where $[0,1]^N$ is endowed with the supremum metric. It follows that the metric convex Ramsey property is equivalent to the same property for colorings taking values in any convex compact metric space.

PROOF. The second condition clearly implies the first. For simplicity, we prove the other implication for $N = 2$; the same argument carries over for arbitrary N. Let $\mathbf{A}$ and $\mathbf{B}$ structures in $\mathcal{K}$ and $\epsilon > 0$. We apply the metric convex Ramsey property twice consecutively.

We find a structure $\mathbf{C}_1$ in $\mathcal{K}$ witnessing the metric convex Ramsey property for $\mathbf{A}$, $\mathbf{B}$ and ϵ, that is, if $\kappa : {}^{\mathbf{A}}\mathbf{C}_1 \to [0,1]$ is a coloring, then there exists $\nu \in \left\langle {}^{\mathbf{B}}\mathbf{C}_1 \right\rangle$ such that for all α, α' in ${}^{\mathbf{A}}\mathbf{B}$, we have $|\kappa(\nu \circ \delta_\alpha) - \kappa(\nu \circ \delta_{\alpha'})| < \epsilon$. Then we find a structure $\mathbf{C}$ in $\mathcal{K}$ witnessing the metric convex Ramsey property for $\mathbf{A}$, $\mathbf{C}_1$ and ϵ, that is, if $\kappa : {}^{\mathbf{A}}\mathbf{C} \to [0,1]$ is a coloring, then there exists $\nu \in \left\langle {}^{\mathbf{C}_1}\mathbf{C} \right\rangle$ such that for all α, α' in ${}^{\mathbf{A}}\mathbf{C}_1$, we have $|\kappa(\nu \circ \delta_\alpha) - \kappa(\nu \circ \delta_{\alpha'})| < \epsilon$.

We now show that $\mathbf{C}$ has the desired property. To this aim, let $\kappa_1, \kappa_2 : {}^{\mathbf{A}}\mathbf{C} \to [0,1]$ be two colorings. By definition of the structure of $\mathbf{C}$, there exists $\nu \in \left\langle {}^{\mathbf{C}_1}\mathbf{C} \right\rangle$ such that for all α, α' in ${}^{\mathbf{A}}\mathbf{C}_1$, we have $|\kappa_1(\nu \circ \delta_\alpha) - \kappa_1(\nu \circ \delta_{\alpha'})| < \epsilon$.

We then lift the second coloring κ_2 to $\tilde{\kappa}_2 : {}^{\mathbf{A}}\mathbf{C}_1 \to [0,1]$ by putting $\tilde{\kappa}_2(\alpha) = \kappa_2(\nu \circ \delta_\alpha)$. This process corresponds to the classical going color-blind argument: here, instead of forgetting one color, we forget all embeddings that are not channelled through $\mathbf{C}_1$ via ν. The map $\tilde{\kappa}_2$ we obtain is again a coloring. Therefore, we may apply our assumption on $\mathbf{C}_1$ to $\tilde{\kappa}_2$: there exists ν_1 in $\left\langle {}^{\mathbf{B}}\mathbf{C}_1 \right\rangle$ such that for all α, α' in ${}^{\mathbf{A}}\mathbf{C}_1$, we have $|\tilde{\kappa}_2(\nu_1 \circ \delta_\alpha) - \tilde{\kappa}_2(\nu_1 \circ \delta_{\alpha'})| < \epsilon$.

Then $\mu = \nu \circ \nu_1$ is as desired. Indeed, let $\eta, \eta' \in {}^{\mathbf{A}}\mathbf{C}(\mu)$. There exist $\alpha, \alpha' \in {}^{\mathbf{A}}\mathbf{B}$ such that $\eta = \mu \circ \delta_\alpha$ and $\eta' = \mu \circ \delta_{\alpha'}$. Then

$$\begin{aligned}
|\kappa_2(\eta) - \kappa_2(\eta')| &= |\kappa_2(\mu \circ \delta_\alpha) - \kappa_2(\mu \circ \delta_{\alpha'})| \\
&= |\kappa_2(\nu \circ \nu_1 \circ \delta_\alpha) - \kappa_2(\nu \circ \nu_1 \circ \delta_{\alpha'})| \\
&= |\tilde{\kappa}_2(\nu_1 \circ \delta_\alpha) - \tilde{\kappa}_2(\nu_1 \circ \delta_{\alpha'})| \\
&< \epsilon.
\end{aligned}$$

Moreover, whenever $\eta, \eta' \in {}^{\mathbf{A}}\mathbf{C}(\mu)$, they are in ${}^{\mathbf{A}}\mathbf{C}(\nu)$ too, hence the assumption on ν yields that $|\kappa_1(\eta) - \kappa_1(\eta')| < \epsilon$. $\square$

REMARK 9.12. For the sake of simplicity, we state the results for only one coloring at a time; the previous proposition will imply that we can do the same with any finite number of colorings.

We now give an infinitary reformulation of the metric convex Ramsey property, which is what will be used in the proof of theorem 9.22 in showing that amenability implies the metric convex Ramsey property.

PROPOSITION 9.13. The following are equivalent.

(1) The class $\mathcal{K}$ has the metric convex Ramsey property.
(2) For every $\epsilon > 0$, for all structures $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{K}$ and all colorings $\kappa : {}^{\mathbf{A}}\mathbf{K} \to [0,1]$, there exists ν in $\langle {}^{\mathbf{B}}\mathbf{K}\rangle$ such that for all η, η' in ${}^{\mathbf{A}}\mathbf{K}(\nu)$, one has $|\kappa(\eta) - \kappa(\eta')| < \epsilon$.

PROOF. (1) $\Rightarrow$ (2)] Fix $\epsilon > 0$, $\mathbf{A}$ and $\mathbf{B}$ two structures in $\mathcal{K}$ and let $\mathbf{C} \in \mathcal{K}$ witness the metric convex Ramsey property for A, B and ϵ. We may assume that $\mathbf{C}$ is a substructure of $\mathbf{K}$. Now every coloring of ${}^{\mathbf{A}}\mathbf{K}$ restricts to a coloring of ${}^{\mathbf{A}}\mathbf{C}$ so, if ν is the measure given by $\mathbf{C}$ for a coloring κ, then ν satisfies the desired property.

(2) $\Rightarrow$ (1)] We use a standard compactness argument. Suppose that $\mathcal{K}$ does not satisfy the metric convex Ramsey property. We can then find structures $\mathbf{A}$, $\mathbf{B}$ in $\mathcal{K}$ and $\epsilon > 0$ such that for every $\mathbf{C} \in \mathcal{K}$, there exists a *bad* coloring $\kappa_{\mathbf{C}}$ of ${}^{\mathbf{A}}\mathbf{C}$, that is, for all $\nu \in \langle {}^{\mathbf{B}}\mathbf{C}\rangle$, the oscillation of $\kappa_{\mathbf{C}}$ on ${}^{\mathbf{A}}\mathbf{C}(\nu)$ is greater than ϵ.

We fix an ultrafilter $\mathcal{U}$ on the collection of finite subsets of $\mathbf{K}$ such that for every finite $D \subseteq \mathbf{K}$, the set $\{E \subseteq \mathbf{K} \text{ finite} : D \subseteq E\}$ belongs to $\mathcal{U}$. We consider the map $\kappa = \lim\limits_{\mathcal{U}} \kappa_{\mathbf{C}}$ on ${}^{\mathbf{A}}\mathbf{K}$ defined by

$$\kappa(\alpha) = t \Leftrightarrow \forall r > 0, \{\mathbf{C} \subseteq \mathbf{K} \text{ finite} : \kappa_{\mathbf{C}}(\alpha) \in [t-r, t+r]\} \in \mathcal{U}.$$

Note that the assumption on $\mathcal{U}$ implies that for all $\alpha \in {}^{\mathbf{A}}\mathbf{K}$, the set $\{\mathbf{C} \subseteq \mathbf{K} \text{ finite} : \alpha(A) \subseteq C\}$ is in $\mathcal{U}$ so $\kappa_{\mathbf{C}}(\alpha)$ is defined $\mathcal{U}$-everywhere and the above definition makes sense. Besides, since all the $\kappa_{\mathbf{C}}$ are 1-Lipschitz, κ is too and is thus a coloring of ${}^{\mathbf{A}}\mathbf{K}$. We prove that κ contradicts property (2).

Let $\nu \in \langle {}^{\mathbf{B}}\mathbf{K}\rangle$ and write $\nu = \sum_{i=1}^{n} \lambda_i \delta_{\beta_i}$, with the β_i's in ${}^{\mathbf{B}}\mathbf{K}$. Then, for all $i \in \{1, ..., n\}$, the sets $\{\mathbf{C} \subseteq \mathbf{K} \text{ finite} : \beta_i(B) \subseteq \mathbf{C}\}$ belong to $\mathcal{U}$ and so does their intersection U_ν. Furthermore, the set ${}^{\mathbf{A}}\mathbf{K}(\nu)$, which is the same as ${}^{\mathbf{A}}\mathbf{C}(\nu)$ for any $\mathbf{C}$ in U_ν, is finite — note that this is not true of $\langle {}^{\mathbf{A}}\mathbf{K}(\nu)\rangle$ (so choosing the definition of remark 9.10 for the Ramsey property would require an additional appeal to the compactness of $\langle {}^{\mathbf{A}}\mathbf{K}(\nu)\rangle$). For every $\mathbf{C}$ in U_ν, there exist η, η' in ${}^{\mathbf{A}}\mathbf{C}(\nu)$ such that $|\kappa_{\mathbf{C}}(\eta) - \kappa_{\mathbf{C}}(\eta')| \geqslant \epsilon$. So there exist η, η' in ${}^{\mathbf{A}}\mathbf{K}(\nu)$ such that the set $\{\mathbf{C} \subseteq \mathbf{K} \text{ finite} : |\kappa_{\mathbf{C}}(\eta) - \kappa_{\mathbf{C}}(\eta')| \geqslant \epsilon\}$ belongs to $\mathcal{U}$. By definition of κ, this implies that $|\kappa(\eta) - \kappa(\eta')| \geqslant \epsilon$, which shows that (2) fails for ν. As ν was arbitrary, this completes the proof. $\square$

2. The metric convex Ramsey property for the automorphism group

Let G be the automorphism group of $\mathbf{K}$.

In this section, we reformulate the metric convex Ramsey property in terms of properties of G.

DEFINITION 9.14. Let $\mathbf{A}$ be a finite substructure of $\mathbf{K}$. We define a pseudometric $d_{\mathbf{A}}$ on G by

$$d_{\mathbf{A}}(g, h) = \max_{a \in A} d(g(a), h(a)).$$

We will denote by $(G, d_{\mathbf{A}})$ the induced metric quotient space.

REMARK 9.15. The pseudometrics $d_{\mathbf{A}}$, for finite substructures $\mathbf{A}$ of $\mathbf{K}$, generate the topology on G, and hence also the left uniformity.

The pseudometric $d_{\mathbf{A}}$ is the counterpart of the metric $\rho_{\mathbf{A}}$ on ${}^{\mathbf{A}}\mathbf{K}$ for the group side. More specifically, as pointed out in [**MT1**, lemma 3.8], the restriction map $\Phi_{\mathbf{A}} : (G, d_{\mathbf{A}}) \to ({}^{\mathbf{A}}\mathbf{K}, \rho_{\mathbf{A}})$ defined by $g \mapsto g_{\restriction A}$ is distance-preserving and its image $\Phi_{\mathbf{A}}(G)$ is dense in ${}^{\mathbf{A}}\mathbf{K}$. As a consequence, every 1-Lipschitz map $f : (G, d_{\mathbf{A}}) \to [0,1]$ extends uniquely, via $\Phi_{\mathbf{A}}$, to a coloring κ_f of ${}^{\mathbf{A}}\mathbf{K}$, while every coloring κ of ${}^{\mathbf{A}}\mathbf{K}$ restricts to a 1-Lipschitz map $f_\kappa : (G, d_{\mathbf{A}}) \to [0,1]$.

We will need the following lemma to approximate uniformly continuous functions by Lipschitz ones.

LEMMA 9.16. Let $(X, \mathcal{E})$ be a uniform space whose uniformity is generated by a directed family $(d_p)_{p\in\mathcal{P}}$ of pseudometrics. Let $f : (X, \mathcal{E}) \to [0,1]$ be a bounded uniformly continuous map. Then for every positive ϵ, there exists $p \in \mathcal{P}$ and a Lipschitz map $f' : (X, d_p) \to \mathbb{R}$ such that for all x in X, we have $|f(x) - f'(x)| < \epsilon$.

PROOF. Since f is uniformly continuous, there exists an entourage V in the uniformity $\mathcal{E}$ on X such that for all x, y in X, if $(x, y) \in V$, then $|f(x) - f(y)| < \epsilon$. Moreover, as the pseudometrics $(d_p)_{p\in\mathcal{P}}$ generate $\mathcal{E}$, there exist p in $\mathcal{P}$ and $r > 0$ such that for all x, y in X, if $d_p(x, y) < r$, then $(x, y) \in V$.

Now, for a positive integer k, we can define a map $f_k : (X, d_p) \to \mathbb{R}$ by

$$f_k(x) = \inf_{y\in X} f(y) + k d_p(x, y).$$

It is k-Lipschitz as the infimum of k-Lipschitz functions. Note also that f_k is smaller than f.

Take k large enough, so that $\dfrac{1}{k} + \dfrac{\epsilon}{k} < r$ and let x be any element of X. By definition of f_k, there exists an element y of X such that $f(y) + k d_p(x, y) \leqslant f_k(x) + \epsilon$. Since both f and f_k are bounded by 1, this implies that for small enough ϵ, we have $d_p(x, y) \leqslant \dfrac{1}{k} + \dfrac{\epsilon}{k} < r$. Thus, the left uniform continuity of f gives that $|f(x) - f(y)| < \epsilon$. But then, we have

$$\begin{aligned} |f(x) - f_k(x)| = f(x) - f_k(x) &\leqslant f(x) - f(y) - k d_p(x, y) + \epsilon \\ &\leqslant f(x) - f(y) + \epsilon \\ &< 2\epsilon. \end{aligned}$$

Thus, the map f_k is the desired Lipschitz approximation of f. □

PROPOSITION 9.17. The following are equivalent.

(1) The class $\mathcal{K}$ has the metric convex Ramsey property.
(2) For every $\epsilon > 0$, every finite substructure $\mathbf{A}$ of $\mathbf{K}$, every finite subset F of G and every 1-Lipschitz map $f : (G, d_{\mathbf{A}}) \to [0,1]$, there exist elements $g_1, ..., g_n$ of G and barycentric coefficients $\lambda_1, ..., \lambda_n$ such that for all h, h' in F, one has

$$\left| \sum_{i=1}^{n} \lambda_i f(g_i h) - \sum_{i=1}^{n} \lambda_i f(g_i h') \right| < \epsilon.$$

(3) For every $\epsilon > 0$, every finite subset F of G, every left uniformly continuous map $f : G \to [0,1]$, there exist elements $g_1, ..., g_n$ of G and barycentric coefficients $\lambda_1, ..., \lambda_n$ such that for all h, h' in F, one has

$$\left| \sum_{i=1}^{n} \lambda_i f(g_i h) - \sum_{i=1}^{n} \lambda_i f(g_i h') \right| < \epsilon.$$

REMARK 9.18. The finite subset F of G in condition (2) is the counterpart of the structure $\mathbf{B}$ in the Ramsey property: by approximate ultrahomogeneity of the limit $\mathbf{K}$, it corresponds, up to a certain error, to the set of all embeddings of $\mathbf{A}$ into $\mathbf{B}$.

Proof. (1) $\Rightarrow$ (2)] We set $\mathbf{B} = \mathbf{A} \cup \bigcup_{h \in F} h(\mathbf{A})$. Let κ_f be the unique coloring of ${}^{\mathbf{A}}\mathbf{K}$ that extends f. We then apply proposition 9.13 to $\mathbf{A}$, $\mathbf{B}$, ϵ and κ_f: there is ν in $\langle {}^{\mathbf{B}}\mathbf{K} \rangle$ such that for all α, α' in ${}^{\mathbf{A}}\mathbf{B}(\nu)$, we have $|\kappa_f(\nu \circ \delta_\alpha) - \kappa_f(\nu \circ \delta_{\alpha'})| < \frac{\epsilon}{3}$.

Write $\nu = \sum_{i=1}^{n} \lambda_i \delta_{\beta_i}$, with the β_i's in ${}^{\mathbf{B}}\mathbf{K}$. Since the structure $\mathbf{K}$ is a Fraïssé limit, it is approximately ultrahomogeneous. This implies that for each i in $\{1, ..., n\}$, there exists an element g_i of its automorphism group G such that $\rho_{\mathbf{B}}(g_i, \beta_i) < \frac{\epsilon}{3}$. It is straightforward to check, using the triangle inequality and the fact that the coloring κ_f is 1-Lipschitz, that these g_i's and λ_i's have the desired property.

(2) $\Rightarrow$ (3)] Let $f : (G, \mathcal{E}_L) \to [0,1]$ be a left uniformly continuous map and let $\epsilon > 0$. We apply lemma 9.16 to f, ϵ and the family $(d_{\mathbf{A}})$ of pseudometrics which generate $\mathcal{E}_L$ (see remark 9.15). We get a finite substructure $\mathbf{A}$ of $\mathbf{K}$ and a k-Lipschitz map $f_k : (G, d_{\mathbf{A}}) \to [0,1]$ such that for all x in G, we have $|f(x) - f_k(x)| < \epsilon$.

We then apply (2) to $\frac{f_k}{k}$, which is 1-Lipschitz, and to $\frac{\epsilon}{k}$: for every finite subset F of G, there exist elements $g_1, ..., g_n$ of G and barycentric coefficients $\lambda_1, ..., \lambda_n$ such that for all $h, h' \in F$, we have

$$\left| \sum_{i=1}^{n} \lambda_i \frac{1}{k} f_k(g_i h) - \sum_{i=1}^{n} \lambda_i \frac{1}{k} f_k(g_i h') \right| < \frac{\epsilon}{k}$$

hence

$$\left| \sum_{i=1}^{n} \lambda_i f_k(g_i h) - \sum_{i=1}^{n} \lambda_i f_k(g_i h') \right| < \epsilon.$$

Then, for all $h, h' \in F$, the triangle inequality gives

$$\left| \sum_{i=1}^{n} \lambda_i f(g_i h) - \sum_{i=1}^{n} \lambda_i f(g_i h') \right| < 3\epsilon.$$

(3) $\Rightarrow$ (1)] Let $\mathbf{A}$ and $\mathbf{B}$ be two structures in $\mathcal{K}$, let $\epsilon > 0$ and let $\kappa : {}^{\mathbf{A}}\mathbf{K} \to [0,1]$ be a coloring. Since $\mathcal{K}$ is approximately ultrahomogeneous, for every α in ${}^{\mathbf{A}}\mathbf{B}$, we may choose h_α in G such that $\rho_{\mathbf{A}}(h_\alpha, \alpha) < \epsilon$. Let F be the (finite) set of all such h_α's.

Now consider the restriction f_κ of the coloring κ to $(G, d_{\mathbf{A}})$. It is left uniformly continuous from G to $[0,1]$. We apply condition (3) to f_κ, F and ϵ: there exist elements $g_1, ..., g_n$ of G and barycentric coefficients $\lambda_1, ..., \lambda_n$ such that for all $h_\alpha, h_{\alpha'}$ in F, one has

$$\left| \sum_{i=1}^{n} \lambda_i f_\kappa(g_i h_\alpha) - \sum_{i=1}^{n} \lambda_i f_\kappa(g_i h_{\alpha'}) \right| < \epsilon.$$

Set $\nu = \sum_{i=1}^{n} \lambda_i \delta_{g_i} \in \langle {}^{\mathbf{B}}\mathbf{K} \rangle$. Using the triangle inequality and the Lipschitzness of κ, it is now straightforward to check that ν witnesses the metric convex Ramsey property for $\mathbf{A}$, $\mathbf{B}$ and 3ϵ. $\square$

Notice that condition (3) does not depend on the Fraïssé class but only on its automorphism group.

By virtue of remark 9.12, the metric convex Ramsey property is equivalent to condition (3) for any finite number of colorings at once. It is that condition which will imply amenability in theorem 9.22.

Moreover, if G is endowed with a compatible left-invariant metric, Lipschitz functions are uniformly dense in left uniformly continuous bounded ones (by lemma 9.16), so we can replace left uniformly continuous maps by 1-Lipschitz maps in condition (3): we obtain the following.

COROLLARY 9.19. Let d be any compatible left-invariant metric on G. Then the following are equivalent.

- The class $\mathcal{K}$ has the metric convex Ramsey property.
- For every $\epsilon > 0$, every finite subset F of G, every 1-Lipschitz map $f : (G, d) \to [0, 1]$, there exist elements $g_1, ..., g_n$ of G and barycentric coefficients $\lambda_1, ..., \lambda_n$ such that for all h, h' in F, one has
$$\left|\sum_{i=1}^{n} \lambda_i f(g_i h) - \sum_{i=1}^{n} \lambda_i f(g_i h')\right| < \epsilon.$$

3. A criterion for amenability

Given a compact space X, we denote by $P(X)$ the set of all Borel probability measures on X. It is a subset of the dual space of continuous maps on X. Indeed, if μ is in $P(X)$ and f is a continuous function on X, we put $\mu(f) = \int_X f d\mu$. Moreover, if we endow $P(X)$ with the induced weak* topology, it is compact.

If G is a group that acts on X, then one can define an action of G on $P(X)$ by
$$(g \cdot \mu)(f) = \int_X f(g^{-1} \cdot x) d\mu(x).$$

DEFINITION 9.20. A topological group G is said to be **amenable** if every continuous action of G on a compact Hausdorff space X admits a measure in $P(X)$ which is invariant under the action of G.

Although amenability is not preserved under subgroups (not even closed subgroups), it is preserved when taking dense subgroups.

PROPOSITION 9.21. A subgroup of a topological group is amenable (with respect to the induced topology) if and only if its closure is.

PROOF. Let H be a dense subgroup of G. It is straightforward to show that every continuous action of H on a compact Hausdorff space extends to a continuous action of G. Thus, if G is amenable, then so is H. $\square$

We are now ready to prove the main theorem.

THEOREM 9.22. Let $\mathcal{K}$ be a metric Fraïssé class, $\mathbf{K}$ its Fraïssé limit and G the automorphism group of $\mathbf{K}$. Then the following are equivalent.

(1) The topological group G is amenable.
(2) The class $\mathcal{K}$ has the metric convex Ramsey property.

PROOF. (1) $\Rightarrow$ (2)] Suppose G is amenable and let $\mathbf{A}$, $\mathbf{B}$ be structures in the class $\mathcal{K}$, $\epsilon > 0$ and $\kappa_0 : {}^{\mathbf{A}}\mathbf{K} \to [0, 1]$ a coloring. We show that there exists $\nu \in \left\langle {}^{\mathbf{B}}\mathbf{K} \right\rangle$ such that for all $\alpha, \alpha' \in {}^{\mathbf{A}}\mathbf{B}$, we have $|\kappa_0(\nu \circ \delta_\alpha) - \kappa_0(\nu \circ \delta_{\alpha'})| < \epsilon$, which will imply the metric convex Ramsey property (by proposition 9.13). We adapt Moore's proof ([**M12**, (6) $\Rightarrow$ (1) in theorem 7.1]) to the metric setting.

The group G acts on the compact Hausdorff space $[0, 1]^{{}^{\mathbf{A}}\mathbf{K}}$ by $g \cdot \kappa(\alpha) = \kappa(g^{-1} \circ \alpha)$. Denote by Y the orbit of the coloring κ_0 under this action and by X its closure, which is compact Hausdorff. Note that all the functions in X are colorings as well. We consider the restriction of the action to X: the action is continuous. Thus, since G is amenable, there is an invariant probability measure μ on X.

The map $\alpha \mapsto \int_X \kappa(\alpha) d\mu(\kappa)$ is constant on ${}^{\mathbf{A}}\mathbf{K}$. Indeed, the invariance of μ implies that it is constant on every orbit of the action of G on ${}^{\mathbf{A}}\mathbf{K}$. But, by the approximate ultrahomogeneity of $\mathbf{K}$, every such orbit is dense in ${}^{\mathbf{A}}\mathbf{K}$, so our map is constant on the whole of ${}^{\mathbf{A}}\mathbf{K}$ because it is continuous (even 1-Lipschitz). Let r denote this constant value.

Besides, Y being dense in X, the collection of finitely supported probability measures on Y is dense in $P(X)$. In particular, there exist barycentric coefficients $\lambda_1, ..., \lambda_n$ and elements $g_1, ..., g_n$ of G such that for all α in ${}^{\mathbf{A}}\mathbf{B}$, we have $\left|\sum_{i=1}^{n} \lambda_i \kappa_0(g_i^{-1} \circ \alpha) - r\right| < \epsilon$.

Finally, we may assume that $\mathbf{B}$ is a substructure of $\mathbf{K}$, and set $\beta_i = g_i^{-1} \upharpoonright \mathbf{B}$, for i in $\{1, ..., n\}$, and $\nu = \sum_{i=1}^{n} \lambda_i \delta_{\beta_i} \in \langle {}^{\mathbf{B}}\mathbf{K} \rangle$. Then ν as is desired. Indeed, if α, α' are in ${}^{\mathbf{A}}\mathbf{B}$, and thus in ${}^{\mathbf{A}}\mathbf{K}$, we have

$$\begin{aligned}
|\kappa_0(\nu \circ \delta_\alpha) - \kappa_0(\nu \circ \delta_{\alpha'})| &= \left|\sum_{i=1}^{n} \lambda_i \kappa_0(\beta_i \circ \alpha) - \sum_{i=1}^{n} \lambda_i \kappa_0(\beta_i \circ \alpha')\right| \\
&\leqslant \left|\sum_{i=1}^{n} \lambda_i \kappa_0(\beta_i \circ \alpha) - r\right| + \left|r - \sum_{i=1}^{n} \lambda_i \kappa_0(\beta_i \circ \alpha')\right| \\
&= \left|\sum_{i=1}^{n} \lambda_i \kappa_0(g_i^{-1} \circ \alpha) - r\right| + \left|r - \sum_{i=1}^{n} \lambda_i \kappa_0(g_i^{-1} \circ \alpha')\right| \\
&< 2\epsilon.
\end{aligned}$$

(2) $\Rightarrow$ (1)] Conversely, suppose that $\mathcal{K}$ has the metric convex Ramsey property and let G act continuously on a compact Hausdorff space X. We show that X admits an invariant probability measure. Since $P(X)$ is compact, it suffices to show that if $f_1, ..., f_N : X \to [0, 1]$ are uniformly continuous with respect to the unique (see [**P3**, exercise 1.1.3]) uniformity on X, $\epsilon > 0$ and F is a finite subset of G, there exists μ in $P(X)$ such that for all j in $\{1, ..., N\}$ and all h in F, $|h \cdot \mu(f_j) - \mu(f_j)| < \epsilon$.

Fix x in X. For j in $\{1, ..., N\}$, we lift f_j to a map $\tilde{f}_j : G \to [0, 1]$ by setting $\tilde{f}_j(g) = f_j(g^{-1} \cdot x)$. Since the action of G on X is continuous and X is compact, for all x in X, the map $g \mapsto g^{-1} \cdot x$ is left uniformly continuous (see [**P3**, lemma 2.1.5]). It follows that the map $\tilde{f}_j$ is left uniformly continuous.

We then apply proposition 9.17 to $F \cup \{1\}$, ϵ and $\tilde{f}_1, ..., \tilde{f}_N$ to obtain barycentric coefficients $\lambda_1, ..., \lambda_n$ and elements $g_1, ..., g_n$ of G such that for all j in $\{1, ..., N\}$, for all h in F (and $h' = 1$), we have

$$\left|\sum_{i=1}^{n} \lambda_i \tilde{f}_j(g_i h) - \sum_{i=1}^{n} \lambda_i \tilde{f}_j(g_i)\right| < \epsilon.$$

Then $\mu = \sum_{i=1}^{n} \lambda_i \delta_{g_i^{-1} \cdot x}$ is as desired. Indeed, let $j \in \{1, ..., N\}$ and $h \in F$. We have

$$\mu(f_j) = \sum_{i=1}^{n} \lambda_i f_j(g_i^{-1} \cdot x) = \sum_{i=1}^{n} \lambda_i \tilde{f}_j(g_i)$$

and

$$\begin{aligned}
h \cdot \mu(f_j) &= \sum_{i=1}^{n} \lambda_i (h \cdot f_j)(g_i^{-1} \cdot x) \\
&= \sum_{i=1}^{n} \lambda_i f_j(h^{-1} g_i^{-1} \cdot x) \\
&= \sum_{i=1}^{n} \lambda_i \tilde{f}_j(g_i h)
\end{aligned}$$

so finally

$$|h \cdot \mu(f_j) - \mu(f_j)| = \left| \sum_{i=1}^{n} \lambda_i \tilde{f}_j(g_i h) - \sum_{i=1}^{n} \lambda_i \tilde{f}_j(g_i) \right| < \epsilon,$$

which completes the proof. □

EXAMPLE 9.23. Let $\mathcal{K}$ be the class of finite sets with no additional structure. The Fraïssé limit of $\mathcal{K}$ is the countable set $\mathbb{N}$. It is well known that its automorphism group, S_∞, is amenable, as the union of the finite (hence amenable) symmetric groups is dense in S_∞ (see e.g. [**BdlHV**, proposition G.2.2.(iii)]), but not extremely amenable ([**P4**, theorem 6.5]). This class is one of the only examples for which the convex Ramsey property can be shown directly.

Let $\mathbf{A}$ and $\mathbf{B}$ be two finite structures. We prove that the structure $\mathbf{C} = \mathbf{B}$ itself witnesses the convex Ramsey property for $\mathbf{A}$ and $\mathbf{B}$ (with $\epsilon = 0$). Indeed, let $\kappa : {}^{\mathbf{A}}\mathbf{B} \to [0, 1]$ be a coloring. We show that if α is an embedding of $\mathbf{A}$ into $\mathbf{B}$, then the convex combination

$$\frac{1}{\mathrm{Card}(\mathbf{B})!} \sum_{\beta \in \mathrm{Aut}(\mathbf{B})} \kappa(\beta \circ \alpha)$$

does not depend on α. This proves the convex Ramsey property, with all embeddings of $\mathbf{B}$ in itself and coefficients $\dfrac{1}{\mathrm{Card}(\mathbf{B})!}$.

To do this, we use the very strong property of finite sets that $\mathbf{B}$ is ultrahomogeneous with respect to copies of $\mathbf{A}$ inside $\mathbf{B}$, that is, every embedding between two copies of $\mathbf{A}$ in $\mathbf{B}$ extends to an automorphism (here, a bijection) of $\mathbf{B}$. Thus, for every α in ${}^{\mathbf{A}}\mathbf{B}$, we can rewrite the above convex combination as follows:

$$\frac{1}{\mathrm{Card}(\mathbf{B})!} \sum_{\beta \in \mathrm{Aut}(\mathbf{B})} \kappa(\beta \circ \alpha) = \frac{1}{\mathrm{Card}(\mathbf{B})!} \sum_{\substack{\tilde{\mathbf{A}} \subseteq \mathbf{B} \\ \tilde{\mathbf{A}} \cong \mathbf{A}}} \sum_{\substack{\alpha' \in {}^{\mathbf{A}}\mathbf{B} \\ \alpha'(\mathbf{A}) = \tilde{\mathbf{A}}}} \kappa(\alpha'),$$

which is independent of α.

4. Structural consequences

As a consequence of theorem 9.22, of proposition 9.17 and of the fact that every Polish group is the automorphism group of some metric Fraïssé structure ([**M5**, theorem 6]), we obtain the following intrinsic characterization of amenability (and its reformulation in terms of finitely supported measures).

THEOREM 9.24. Let G be a Polish group. Then the following are equivalent.

(1) G is amenable.
(2) For every $\epsilon > 0$, every finite subset F of G, every left uniformly continuous map $f : G \to [0, 1]$, there exist elements $g_1, ..., g_n$ of G and barycentric coefficients $\lambda_1, ..., \lambda_n$ such that for all $h, h' \in F$, one has

$$\left| \sum_{i=1}^{n} \lambda_i f(g_i h) - \sum_{i=1}^{n} \lambda_i f(g_i h') \right| < \epsilon.$$

(3) For every $\epsilon > 0$, every finite subset F of G, every $f \in \mathrm{RUCB}(G)$, there is a finitely supported probability measure μ on G such that for every h in F, one has

$$|\mu(f) - (h \cdot \mu)(f)| < \epsilon.$$

The equivalence of (2) and (3) follows from the fact that inversion exchanges left and right uniformly continuous functions.

We recognize Day's weak*-asymptotic invariance condition with only one function from $\mathrm{RUCB}(G)$ needed to check the amenability of G.

COROLLARY 9.25. Let G be a Polish group. Then the following are equivalent.

(1) G is amenable.
(2) For every right uniformly continuous bounded function on G, there exists a mean on $\mathrm{RUCB}(G)$ such that for all $g \in G$, one has $\Lambda(g \cdot f) = \Lambda(f)$.

PROOF. (1) $\Rightarrow$ (2)] If G is amenable, then the action of G on its Samuel compactification $S(G)$ admits an invariant Borel probability measure μ. The integral against μ gives rise to an invariant mean on the space of all continuous functions on $S(G)$. But continuous functions on the Samuel compactification of G are exactly right uniformly continuous bounded ones, hence condition (2) is satisfied.

(2) $\Rightarrow$ (1)] Since $\mathrm{RUCB}(G)$ is exactly the space of all continuous functions on the Samuel compactification $S(G)$, we can apply the Riesz representation theorem: for each f in $\mathrm{RUCB}(G)$, there exists a Borel probability measure on $S(G)$ such that for all g in G, we have $\mu(g \cdot f) = \mu(f)$.

But since G is dense in $S(G)$, every Borel probability measure on $S(G)$ can be approximated by finitely supported measures on G. Thus, for every $\epsilon > 0$, every finite subset F of G, every $f \in \mathrm{RUCB}(G)$, there is a finitely supported probability measure μ on G such that for every h in F, one has

$$|\mu(f) - (h \cdot \mu)(f)| < \epsilon.$$

Theorem 9.24 then yields that G is amenable. □

Similarly, corollary 9.19 gives the Lipschitz counterpart of theorem 9.24.

THEOREM 9.26. Let G be a Polish group and d a left-invariant metric on G which induces the topology. Then the following are equivalent.
(1) The topological group G is amenable.
(2) For every $\epsilon > 0$, every finite subset F of G, every 1-Lipschitz map $f : (G, d) \to [0, 1]$, there exist elements $g_1, ..., g_n$ of G and barycentric coefficients $\lambda_1, ..., \lambda_n$ such that for all $h, h' \in F$, one has

$$\left| \sum_{i=1}^{n} \lambda_i f(g_i h) - \sum_{i=1}^{n} \lambda_i f(g_i h') \right| < \epsilon.$$

It follows that amenability is a G_δ condition in the following sense (see [**MT2**, theorem 3.1]).

COROLLARY 9.27. Let Γ be a countable group and G a Polish group. Then the set of homomorphisms from Γ to G whose image is an amenable subgroup of G is G_δ in the space of representations of Γ in G, endowed with the topology of pointwise convergence inherited from G^Γ.

PROOF. Let π be a homomorphism from Γ to G and let d be a compatible left-invariant metric on G. By proposition 9.21, the image $\pi(\Gamma)$ is amenable if and only if such is its closure, and its closure is Polish (as a closed subset of a Polish space). Then, by virtue of theorem 9.26, $\overline{\pi(\Gamma)}$ is amenable if and only if for every $\epsilon > 0$, every finite subset F of $\overline{\pi(\Gamma)}$, every 1-Lipschitz function $f : (\overline{\pi(\Gamma)}, d) \to [0, 1]$, there exist elements $g_1, ..., g_n$ of $\overline{\pi(\Gamma)}$ and barycentric coefficients $\lambda_1, ..., \lambda_n$ such that for all h, h' in F, one has

$$\left| \sum_{i=1}^{n} \lambda_i f(g_i h) - \sum_{i=1}^{n} \lambda_i f(g_i h') \right| < \epsilon.$$

Using the same compactness argument as in proposition 9.13, one can show that the condition is equivalent to the following.

$$\forall \epsilon > 0, \forall F \subseteq \overline{\pi(\Gamma)} \text{ finite }, \exists K \subseteq \overline{\pi(\Gamma)} \text{ finite }, \forall f : (KF, d) \to [0, 1] \text{ 1-Lipschitz},$$

$$\exists k_1, ..., k_n \in K, \exists \lambda_1, ..., \lambda_n, \forall h, h' \in F, \left| \sum_{i=1}^{n} \lambda_i f(k_i h) - \sum_{i=1}^{n} \lambda_i f(k_i h') \right| < \epsilon.$$

It is easily seen that this is again equivalent to the following.

$$\forall \epsilon > 0, \forall F \subseteq \Gamma \text{ finite}, \exists K \subseteq \Gamma \text{ finite},$$

$$(*) \begin{cases} \forall f : KF \to [0,1] \text{ such that } \forall \gamma, \gamma' \in KF, |f(\gamma) - f(\gamma')| \leqslant d(\pi(\gamma), \pi(\gamma')), \\ \exists k_1, ..., k_n \in K, \exists \lambda_1, ..., \lambda_n, \forall h, h' \in F, \left| \sum_{i=1}^{n} \lambda_i f(k_i h) - \sum_{i=1}^{n} \lambda_i f(k_i h') \right| < \epsilon. \end{cases}$$

We now prove that, if ϵ, F and K are fixed, the set of representations π satisfying condition $(*)$ above is open, which will imply that the condition is indeed G_δ. We prove that its complement is closed. To that aim, take a sequence (π_k) of representations that do not satisfy condition $(*)$ and assume that (π_k) converges to some representation π. Let $f_k : KF \to [0,1]$ witness that π_k is in the complement. Since KF is finite, maps from KF to $[0,1]$ form a compact set so we may assume that (f_k) converges to some f. Since being 1-Lipschitz is a closed condition, f also satisfies that for all γ, γ' in KF, $|f(\gamma) - f(\gamma')| \leqslant d(\pi(\gamma), \pi(\gamma'))$.

By the choice of f_k, for all $k_1, ..., k_n$ in K and all $\lambda_1, ..., \lambda_n$, there exist h_k, h'_k in F such that

$$\left| \sum_{i=1}^{n} \lambda_i f_k(k_i h_k) - \sum_{i=1}^{n} \lambda_i f_k(k_i h'_k) \right| \geqslant \epsilon.$$

Since F is finite, we may again assume that there are h and h' in F such that for all k, we have $h_k = h$ and $h'_k = h'$. We then take the limit of the above inequality to get that

$$\left| \sum_{i=1}^{n} \lambda_i f(k_i h) - \sum_{i=1}^{n} \lambda_i f(k_i h') \right| \geqslant \epsilon,$$

which implies that π does not satisfy condition $(*)$ either, and thus completes the proof. □

Remark 9.28. The same argument works if, instead of condition (2) of theorem 9.26, we use a version of Day's weak*-asymptotic invariance condition with Lipschitz maps. Thus, corollary 9.27 holds more generally for all topological groups.

This yields the following criterion for amenability, which can however be obtained without the use of Ramsey theory.

Corollary 9.29. Let G be a Polish group such that for every positive n in $\mathbb{N}$, the set

$$F_n = \{(g_1, ..., g_n) \in G^n : \langle g_1, ..., g_n \rangle \text{ is amenable}\}$$

is dense in G^n. Then G is amenable.

Proof. We use a Baire category argument. By virtue of the above corollary applied to the free group $\mathbb{F}_n$ on n generators (identifying $\mathrm{Hom}(\mathbb{F}_n, G)$ with G^n), for all n, the set F_n is dense G_δ in G^n. By the Baire category theorem, the set

$$F = \{(g_k) \in G^{\mathbb{N}} : \forall n, (g_1, ..., g_n) \in F_n\}$$

is dense and G_δ too. Besides, the set of sequences which are dense in G is also dense and G_δ. Then the Baire category theorem gives a sequence (g_k) in their intersection. Thus, the group generated by the g_k's is dense and amenable, hence so is G. □

Note that since compact Hausdorff groups are amenable, it follows in particular that a group in which the tuples that generate a compact subgroup are dense is amenable.

Remark 9.30. The criterion of corollary 9.29 can also be proven directly using the following compactness argument. Let G act continuously on a compact Hausdorff space X. Since the space $P(X)$ is compact, every element of G acts uniformly continuously on $P(X)$. Let F be a finite subset of G and let V be an entourage in the uniformity on $P(X)$. Then there exists a positive ϵ such that for every h in F and every μ in $P(X)$, for all g in G, if $d(g, h) < \epsilon$, then we have that $(g \cdot \mu, h \cdot \mu) \in V$.

Now, if $F = \{h_1, ..., h_n\}$, we approximate $(h_1, ..., h_n)$ by some tuple $(f_1, ..., f_n)$ in F_n with $d(f_i, h_i) < \epsilon$. Since the group generated by the f_i's is amenable, there exists a measure $\mu_{F,V}$ in $P(X)$ which is invariant under the action of every f_i. It follows that for every h_i in F, the pair $(h_i \cdot \mu, \mu) = (h_i \cdot \mu, f_i \cdot \mu)$ belongs to V. Finally, since $P(X)$ is compact, the net $\{\mu_{F,V}\}$ admits a limit point, which is invariant under the action of G.

The same argument works with extreme amenability as well and it allows to slightly simplify the arguments of [**MT2**]: to show that the groups $\mathrm{Iso}(\mathbb{U})$, $U(H)$ and $\mathrm{Aut}(X, \mu)$ are extremely amenable, Melleray and Tsankov use their theorem 7.1 along with the facts that extreme amenability is a G_δ property and that Polish groups are generically $\aleph_0$-generated. This is not necessary, as the core of their proof is basically the above criterion: in each case, they prove that the set of tuples which generate a subgroup that is contained in an extremely amenable group (some $L^0(U(m))$, as it happens) is dense.

5. Concluding remarks

One would expect the characterization of theorem 9.22 to yield new examples of amenable groups or at least simpler proofs of the amenability of known groups. However, proving the convex Ramsey property for a concrete Fraïssé class is quite technical and difficult.

Then, maybe our characterization can be used the other way around, that is, to find new Ramsey-type results. There is also hope that the criterion of corollary 9.29 may lead to (new) examples of amenable groups.

Branch 4

Homogeneity

CHAPTER 10

Homogeneity

Qu'un jour on dise "C'est fini",
Qu'un jour on dise "C'est fini"
Georges Brassens[1]

1.	**Countably categorical case**	**138**
2.	**Somewhere realized types**	**138**
3.	**When types are finitely determined**	**139**
4.	**When they are not**	**142**
4.1.	Extending types over finite sets	142
4.2.	Omitting non-finitely determined types	142
5.	**Perspectives**	**147**
5.1.	Beyond typical finiteness	147
5.2.	Acting only transitively	147
5.3.	Back to metric structures	148

The Urysohn space is ultrahomogeneous, meaning that every isometry between finite subsets extends to a global isometry of $\mathbb{U}$. Even better, Huhunaišvili showed in [**H3**] that it also works with compact subsets. Studying the separability of the Katětov space (see subsection 1.2 of chapter 2), Melleray proved that relatively compact subspaces are actually the only subspaces satisfying this property ([**M3**, théorème 18]). It is natural to ask whether the same is true in other Fraïssé classes (where compact could mean finite). We only study the case of classical Fraïssé classes, which is rich enough to already come unstuck with. In this setting, we give a characterization of this property. This is joint work (in progress) with Isabel Müller and Aristotelis Panagiotopoulos.

Let $\mathcal{K}$ be a classical Fraïssé class in a relational language and let $\mathbf{M}$ be its Fraïssé limit. Let also G be the automorphism group of $\mathbf{M}$.

By $\mathcal{K}_\omega$, we denote the class of all countable structures that embed in $\mathbf{M}$. Equivalently, $\mathcal{K}_\omega$ is the class of all countable structures whose age is contained in $\mathcal{K}$.

Definition 10.1. Let $\mathbf{N}$ be a structure in $\mathcal{K}_\omega$. We say that $\mathbf{N}$ has the **homogeneity property** (**HoP**) if every isomorphism between any two copies of $\mathbf{N}$ in $\mathbf{M}$ extends to an automorphism of $\mathbf{M}$.

The ultrahomogeneity of $\mathbf{M}$ says that all finite sets have the homogeneity property.

Question 10.2. Which structures in $\mathcal{K}_\omega$ satisfy the homogeneity property?

Remark 10.3. Recently, Panagiotopoulos ([**P1**]) studied the following related question: for which Fraïssé structures does the *generic* substructure satisfy HoP? Our approach is somewhat dual to his: here, the Fraïssé structure is fixed in advance and we are interested in characterizing the homogeneity property inside this particular structure.

Compact sets are the metric counterpart of finite sets so in the light of Melleray's result, a first guess as to question 10.2 would be that only finite structures will satisfy HoP. Unfortunately

[1] *Le Roi*

(or rather fortunately; this uncovers some interesting phenomenons), this is not the case. For instance, in the infinitely splitting tree, any infinite branch has the homogeneity property.

EXAMPLE 10.4. Let T_∞ be the infinitely splitting tree in the language $\{d_n : n \in \mathbb{N}\}$ which consists of binary relational symbols to be interpreted as distances between vertices of the tree.

- In T_∞, infinite branches have the homogeneity property.
- On the other hand, if a is any vertex in T_∞ and $X = \{b \in T_\infty : d(a, b) \leqslant 1\}$, then X does not have the homogeneity property.
- More precisely a subset X of T_∞ has the homogeneity property if and only if the algebraic closure of X is a finitely splitting subtree of T_∞. Indeed, as soon as X contains a vertex of infinite degree, we may find isomorphic copies of X where the neighbors of this vertex are either all its neighbors in T_∞ or only a proper subset. But two such copies can never be sent to each other by an automorphism.

Note that in regular finitely splitting trees, all subsets have the homogeneity property.

For the moment, we lack examples of structures where infinite structures satisfy HoP. In fact, trees constitute our leading examples.

We begin with the special case of an $\aleph_0$-categorical Fraïssé limit, where the only substructures that satisfy HoP *are* finite.

1. Countably categorical case

As explained in section 3 of chapter 6, when the structure $\mathbf{M}$ is countably categorical, the homogeneity property can be expressed more intrinsically: the structure $\mathbf{M}$ needs to remain countably categorical after the elements of the substructure have been named. Therefore, only finite structures will satisfy the homogeneity property in this case.

PROPOSITION 10.5. Assume that the structure $\mathbf{M}$ is $\aleph_0$-categorical and let X be a subset of $\mathbf{M}$. Then the following are equivalent.

(1) The set X has the HoP.
(2) The structure $(\mathbf{M}, X)$ is again $\aleph_0$-categorical.
(3) The set X is finite.

PROOF. (1) $\Rightarrow$ (2)] See proposition 6.21.

(2) $\Rightarrow$ (3)] The Ryll-Nardzewski theorem implies that there are only finitely many 1-types over X. But that only happens when X is finite: otherwise, all types "$v = x$", for x in X, are different.

(3) $\Rightarrow$ (1)] The structure $\mathbf{M}$ is ultrahomogeneous, so finite sets satisfy the homogeneity property. □

EXAMPLES 10.6. As a consequence, in the following Fraïssé structures, only finite sets have the homogeneity property.

- The pure set $\mathbb{N}$. In that case, it was easier to get the result directly. Indeed, any infinite set is isomorphic to the whole of $\mathbb{N}$, and we certainly cannot extend an isomorphism from the whole structure to a proper subset!
- The rationals.
- The random graph.

2. Somewhere realized types

To extend an isomorphism, the first method that comes to mind is to carry a back-and-forth argument. An obvious obstacle to the extension would be that at some point during the back-and-forth process, we find an element on one side, the image of whose type by the isomorphism-in-progress is not realized on the other side.

Definition 10.7. Let X be a subset of $\mathbf{M}$ and let p be a type over X. Let f be an embedding of X into $\mathbf{M}$. The **image type** p under f is the type $f(p)$ over $f(X)$ defined by

$$\varphi(\bar{v}, f(\bar{x})) \in f(p) \Leftrightarrow \varphi(\bar{v}, \bar{x}) \in p,$$

for every $\mathcal{L}_X$-formula $\varphi(\bar{v}, \bar{x})$ with parameters $\bar{x}$.

We easily obtain the following.

Proposition 10.8. Let $\mathbf{N}$ be a structure in $\mathcal{K}_\omega$. Let X and X' be two copies of $\mathbf{N}$ in $\mathbf{M}$ and f be an isomorphism between X and X'. Assume that for all finite subsets F and F' of $\mathbf{M}$ such that f extends to an isomorphism $\tilde{f}$ between $X \cup F$ and $X' \cup F'$, the sets $X \cup F$ and $X' \cup F'$ have the same realized quantifier-free 1-types: if p is a quantifier-free type over $X \cup F$, then p is realized if and only if $\tilde{f}(p)$ is realized too. Then we can extend f to an automorphism of $\mathbf{M}$.

This incites us to only consider types that are *somewhere realized*, that is, realized over *some* copy of the parameter structure in $\mathbf{M}$. They are exactly the types in n variables over $\mathbf{N}$ that correspond to a structure of $\mathcal{K}_\omega$, which we obtain by adding at most n elements to the structure $\mathbf{N}$. Thus, other types we do not care about. Moreover, the only types we will consider will be quantifier-free types, for they are the ones that witness isomorphism. Throughout the section, the word "type" will mean "quantifier-free type".

Definition 10.9. Let $\mathbf{N}$ be a structure in $\mathcal{K}_\omega$ and p be a type over $\mathbf{N}$. We call p **somewhere realized** in $\mathbf{M}$, if there is an embedding $f : \mathbf{N} \to \mathbf{M}$ such that $f(p)$ has a realization in $\mathbf{M}$.

Conversely, if a substructure of $\mathbf{M}$ has the homogeneity property, then, as soon as a type over this substructure is realized somewhere, it is realized everywhere.

Proposition 10.10. Let X be a subset of $\mathbf{M}$. If X has the homogeneity property, then the set of realized types over X coincides with the set of somewhere realized types over X.

Proof. Let p be a somewhere realized type over X. There exists an embedding $f : X \to \mathbf{M}$ such that $f(p)$ is realized, say, by some a in $\mathbf{M}$. By the homogeneity property, f extends to an automorphism g of $\mathbf{M}$, so that $g^{-1}(a)$ realizes p. □

In particular, when X has the HoP, the set of somewhere realized types over X must be countable. The analogous observation for the Urysohn space, namely that the space of Katětov maps over X has to be separable, explains how, in Melleray's work, separability considerations came in.

3. When types are finitely determined

The example we gave of an infinite structure with the homogeneity property, an infinite branch, is not finite, but is not far off, to the extent that all types over it are determined by a finite set. We will call such structures *typically finite.*

Definition 10.11. Let $\mathbf{N}$ be a structure in $\mathcal{K}_\omega$ and let p be a somewhere realized type over $\mathbf{N}$. We say that p is **finitely determined** if there exists a finite subset F of $\mathbf{N}$ such that every somewhere realized type over $\mathbf{N}$ whose restriction to F coincides with $p_{\restriction F}$ is in fact equal to p.

Any such set F is called a **support**[2] for the type p.

Remark 10.12. The condition of finite determination looks tremendously like a notion of being an isolated type, for the adequate topology on the space of types that are somewhere realized in $\mathbf{M}$. If p is a somewhere realized type over $\mathbf{N}$, then a basis of neighborhoods of p for this topology is given by all sets of the form

$$\{q \in S(\mathbf{N}) : q \text{ is somewhere realized and } q_{\restriction A} = p_{\restriction A}\},$$

[2]This "support" has nothing to do with the "support" of a Katětov map that we defined in 2.7.

for a finite subset A of $\mathbf{N}$. Exactly how we can turn this topological characterization to our advantage, though, is not clear yet.

EXAMPLE 10.13. Let p be a somewhere (hence everywhere) realized 1-type over an infinite branch in T_∞. The type p specifies the distances to all vertices of the branch. Then a support for p consists in the (unique) closest vertex, together with its two neighbors, which witness the minimality of the distance (see figure 10.13).

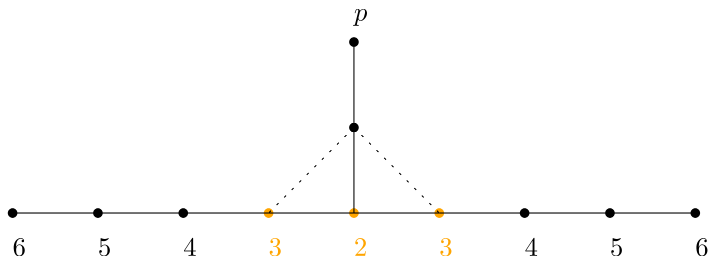


FIGURE 10.1. The type p is determined by three vertices in the infinite branch.

Finite determination is a direct translation of a notion introduced by Melleray for Katětov maps ([**M3**, section 6.2]), that corresponds to *compact determination*: he calls a Katětov map $f \in E(X)$ **saturated**[3] if for every positive ϵ, there exists a compact subset K of X such that for every g in $E(X)$, if $g_{\restriction K} = f_{\restriction K}$, then $d(g, f) \leqslant \epsilon$.

Melleray shows that when X has the collinearity property, that is when $E(X)$ is separable, it is possible to build an isometric copy of X inside the Urysohn space over which all realized types (that is, Katětov maps) are saturated. It follows that the only spaces with the homogeneity property are those over which all Katětov maps are saturated. Together with the observation that non-compact spaces always admit non-saturated Katětov maps, this yields that the only such spaces are compact.

As we have seen, it can happen in our setting that infinite structures have the homogeneity property. However, the first equivalence is still true all the same. We call a structure $\mathbf{N}$ in $\mathcal{K}_\omega$ **typically finite** if all somewhere realized types in finitely many variables over $\mathbf{N}$ are finitely determined. We prove the following theorem (corollary 10.24).

THEOREM 10.14. A structure in $\mathcal{K}_\omega$ has the homogeneity property if and only if it is typically finite.

In this section, we prove the easier direction, that is, typical finiteness always implies the homogeneity property.

REMARK 10.15. If F is a support for a finitely determined type p over $\mathbf{N}$ and f is an embedding of $\mathbf{N}$ into $\mathbf{M}$, then the set $f(F)$ is a support for the type $f(p)$.

Exactly as types over finite sets and as expected, finitely determined types are realized everywhere.

PROPOSITION 10.16. Let $\mathbf{N}$ be a structure in $\mathcal{K}_\omega$ and let p be a somewhere realized type over $\mathbf{N}$. If the type p is finitely determined, then p is realized over every copy of $\mathbf{N}$.

PROOF. Let $F \subseteq \mathbf{N}$ be a support for p. As p is somewhere realized, we find a partial isomorphism $f : \mathbf{N} \to \mathbf{M}$ and a realization $\bar{a}'$ of $f(p)$. By ultrahomogeneity of $\mathbf{M}$, the partial isomorphism $f^{-1}_{\restriction f(F)}$ extends to an automorphism of M which sends $\bar{a}'$ to some tuple $\bar{a}$. Now, $\mathrm{tp}(\bar{a}/F) = p_{\restriction F}$ and p is determined by its restriction to F, hence $\mathrm{tp}(\bar{a}/\mathbf{N}) = p$ and p is realized over $\mathbf{N}$. $\square$

[3]Since there is a certain amount of model theory involved in this thesis, the term "saturated" may be confusing. We therefore opted for the (hopefully) less ambiguous "finitely determined".

Now, we simplify the criterion of proposition 10.8 in the context of finitely determined types.

PROPOSITION 10.17. Let X be a subset of $\mathbf{M}$. Then the following are equivalent.

(1) For every finite subset F of $\mathbf{M}$, all somewhere realized types over $X \cup F$ in finitely many variables are finitely determined.
(2) All somewhere realized types over X in finitely many variables are finitely determined.

PROOF. The direction (1) $\Rightarrow$ (2) is clear. For the other direction, let F be a finite subset of $\mathbf{M}$ and let p be a somewhere realized n-type over $X \cup F$. If F has cardinality m, then p gives rise to an $n+m$-type p' over X by replacing the elements of F with variables. By assumption, this type p' is finitely determined, say by A. We show that $A \cup F$ is a support for p.

To do this, let $\bar{a}$ be an arbitrary realization of $p_{\restriction A \cup F}$. We have to show $\bar{a}$ realizes p. If not, then there is a finite subset B of $X \cup F$ containing $A \cup F$ such that $\bar{a}$ does not realize $p_{\restriction B}$. Now, the type $p_{\restriction B}$ has a realization, say $\bar{a}'$. But then, we have

$$\mathrm{tp}(\bar{a}'F/A) = \mathrm{tp}(\bar{a}F/A) = p'_{\restriction A}$$

while

$$\mathrm{tp}(\bar{a}F/X) \neq p',$$

a contradiction. □

The following proposition shows that finite determination carries over to subtuples.

PROPOSITION 10.18. Let X be a subset of $\mathbf{M}$. Let $\bar{a}$ be a tuple in $\mathbf{M}$ and $\bar{a}'$ be a subtuple of $\bar{a}$. If the type of $\bar{a}$ over X is finitely determined, then the type of $\bar{a}'$ over X is finitely determined.

PROOF. Call p the type of $\bar{a}$ over X and p' the type of $\bar{a}'$ over X. Let F be a finite support for p. We show that F is a finite support for p' as well. To this aim, let p^* be a somewhere realized type over X that coincides with p' on F. Assume, towards a contradiction, that p^* is not equal to p'. Then this is witnessed by a finite subset B of X, with B containing F: $p^*_{\restriction B} \neq p'_{\restriction B}$.

Let $\bar{b}^*$ be a realization of the finite type $p^*_{\restriction B}$ in $\mathbf{M}$ (such a realization exists because p^* is somewhere realized and $\mathbf{M}$ is ultrahomogeneous). Then $\bar{b}^*$ and $\bar{a}'$ have the same type over F, so by ultrahomogeneity of $\mathbf{M}$, there exists an automorphism g of $\mathbf{M}$ that fixes F pointwise and that sends $\bar{a}'$ to $\bar{b}^*$.

Now, consider the image $\bar{b}$ of $\bar{a}$ by g. Then the type of $\bar{b}$ over F is equal to the type of $\bar{a}$ over F, that is, to $p_{\restriction F}$. But $\mathrm{tp}(\bar{b}/X)$ is not equal to p, for the types of the subtuples $\bar{b}^*$ and $\bar{a}'$ already differ. This contradicts the assumption that F is a support for p and completes the proof. □

THEOREM 10.19. Let $\mathbf{N}$ be a structure in $\mathcal{K}_\omega$. If $\mathbf{N}$ is typically finite, then $\mathbf{N}$ has the homogeneity property.

PROOF. Let X and X' be two copies of $\mathbf{N}$ in $\mathbf{M}$ and let f be an isomorphism between X and X'. All somewhere realized types over X and over X' are finitely determined, hence realized. Thus, X and X' have the same realized types, so, by proposition 10.17, we can apply our back-and-forth criterion (proposition 10.8): we obtain that f extends to an automorphism of $\mathbf{M}$, and thus, $\mathbf{N}$ has the homogeneity property. □

Note that the previous theorem, combined with proposition 10.5, yields that in countably categorical structures, the homogeneity property is equivalent to typical finiteness, and even finiteness.

4. When they are not

We are interested in the converse direction: suppose, towards a contradiction, that a structure $\mathbf{N}$ in $\mathcal{K}_\omega$ satisfies the homogeneity property but is not typically finite. Then pick a type p over $\mathbf{N}$ that is somewhere realized but not finitely determined. The idea is to build a copy of $\mathbf{M}$ around $\mathbf{N}$ in which p is not realized, preventing $\mathbf{N}$ from having the homogeneity property. Actually, our construction will give a better result still. We will build a copy of $\mathbf{M}$ that omits *every* non-finitely determined type over $\mathbf{N}$. This is some kind of an atomic model, with respect to the topology defined in remark 10.12.

4.1. Extending types over finite sets. The homogeneity property allows us to extend types over finite sets to finitely determined types, which may not be possible in general. For instance, in the random graph, the type over an infinite complete graph saying "I am not related to any element of A" cannot extend to a finitely determined type.

The following proposition will be used extensively in the proof of theorem 10.22.

Proposition 10.20. Let $\mathbf{N}$ be an infinite structure in $\mathcal{K}_\omega$ that satisfies HoP and let p be a somewhere realized type over a finite subset A of $\mathbf{N}$. Then p extends to a finitely determined type over $\mathbf{N}$.

This is not to say, however, that the finite set A will be a support for the extension.

Proof. Suppose not: every extension of p to $\mathbf{N}$ is non-finitely determined. First note that there exists an extension of p to $\mathbf{N}$ that is somewhere realized. To see this, embed A in $\mathbf{M}$ in such a way that p is realized, say by $\bar{a}$. Since $\mathbf{N}$ is in $\mathcal{K}_\omega$, it embeds into $\mathbf{M}$ as well. Now the ultrahomogeneity of $\mathbf{M}$ enables us to amalgamate $\mathbf{N}$ and $A \cup \{\bar{a}\}$ over A. Pick one such extension, say q_0.

The finite set A is not a support for q_0 so there exists another somewhere realized extension q_1 of p to $\mathbf{N}$ such that $q_0 \neq q_1$. Now since the two types q_0 and q_1 differ, they must differ on a finite set A_1. We may assume that A_1 contains A. We now apply the same argument to each of q_0 and q_1: the finite set A_1 is not a support for q_0 nor q_1 so there exist two somewhere realized extensions q_{00} and q_{01} of $q_0 \restriction A_1$ and two somewhere realized extensions q_{10} and q_{11} of q_1 such that $q_{00} \neq q_{01}$ and $q_{10} \neq q_{11}$. We can then find a finite subset A_2 of $\mathbf{N}$ witnessing these differences, and so on.

This process gives a tree of types that are all different (at each level) and somewhere realized in $\mathbf{M}$. Each of the limit type is somewhere realized in $\mathbf{M}$. Indeed, a type is somewhere realized in $\mathbf{M}$ if and only if it defines a structure in $\mathcal{K}_\omega$. Since every finite substructure defined by our limit types appears in one of the (somewhere realized) types in the sequence, the limit types are somewhere realized. This yields continuum many somewhere realized types over $\mathbf{N}$, contradicting proposition 10.10. $\square$

Remark 10.21. When the Fraïssé limit $\mathbf{M}$ is ω-stable, then the conclusion of the proposition is true for every structure in $\mathcal{K}_\omega$, regardless of the homogeneity property. Recall that a countable structure $\mathbf{N}$ is said to be **ω-stable** if the set $S(\mathbf{N})$ of types over $\mathbf{N}$ is countable. In particular, this is the case of the infinitely splitting tree.

4.2. Omitting non-finitely determined types. We proceed as in the proof of the usual omitting types theorem (see for example [**H1**, theorem 7.2.1]). Since the proof of the following theorem involves both types and quantifier-free types, we will drop our convention that "type" means "quantifier-free type" and we will specify this for each type along the proof.

Theorem 10.22. Let $\mathbf{N}$ in $\mathcal{K}_\omega$ have the homogeneity property. Then there exists an embedding of $\mathbf{N}$ into $\mathbf{M}$ over which all realized quantifier-free types are finitely determined.

Proof. We fix an embedding of $\mathbf{N}$ into $\mathbf{M}$ and we consider the $\mathcal{L}_{\mathbf{N}}$-theory T of $(\mathbf{M}, \mathbf{N})$. Note that T does not depend on the chosen embedding, because the structure $\mathbf{N}$ satisfies the homogeneity property.

Let C be a countable collection of new constant symbols — they will be our *witnesses* — and set $\mathcal{L}^+ = \mathcal{L}_{\mathbf{N}} \cup C$. We want to build a complete $\mathcal{L}^+$-theory T^+ containing T in such a way that in any model of T^+, the structure generated by the constants in C is isomorphic to $\mathbf{M}$ and contains a copy of $\mathbf{N}$ over which all realized quantifier-free types are finitely determined. Following [**H1**], we call this structure the **canonical model** of T^+. Let us introduce a useful item of notation: if $\bar{c}$ is a tuple in C and B is a subset of $\mathbf{N}$, we will denote

$$\operatorname{tp}_{T^+}(\bar{c}/B) = \{\varphi(\bar{x}) \ \mathcal{L}_B\text{-formula } : T^+ \text{ contains } \varphi(\bar{c})\},$$

and we will call this the **type** of $\bar{c}$ over B in T^+.

We will enforce the following properties.

(1) (Witness property) For every $\mathcal{L}^+$-formula $\varphi(x)$, there exists an element c in C such that T^+ contains the formula $(\exists x, \varphi(x)) \to \varphi(c)$.

By the Tarski-Vaught test (see for example [**H1**, theorem 2. 5.1]), this will guarantee that for any model $\mathbf{M}^+$ of T^+, the substructure $\mathbf{C}^{\mathbf{M}^+}$ is an elementary $\mathcal{L}^+$-substructure of $\mathbf{M}^+$.

(2) The canonical model of T^+, seen as an $\mathcal{L}$-structure, is isomorphic to $\mathbf{M}$, that is, it is the Fraïssé limit of $\mathcal{K}$. To guarantee this, we require the following conditions.

(a) The age (with respect to the language $\mathcal{L}$) of the canonical model of T^+ is contained in $\mathcal{K}$: for every $\bar{c}$ in C, there exists $\bar{a}$ in $\mathbf{M}$ such that

$$\operatorname{qftp}_{T^+}(\bar{c}) = \operatorname{qftp}(\bar{a}),$$

that is, for every quantifier-free $\mathcal{L}$-formula $\varphi(\bar{x})$, the theory T^+ contains $\varphi(\bar{c})$ if and only if the structure $\mathbf{M}$ satisfies $\varphi(\bar{a})$.

(b) The age (with respect to the language $\mathcal{L}$) of the canonical model of T^+ contains $\mathcal{K}$: for every tuple $\bar{a}$ in $\mathbf{M}$, there exists $\bar{c}$ in C such that

$$\operatorname{qftp}_{T^+}(\bar{c}) = \operatorname{qftp}(\bar{a}),$$

that is, for every quantifier-free $\mathcal{L}$-formula $\varphi(\bar{x})$, the theory T^+ contains $\varphi(\bar{c})$ if and only if $\mathbf{M}$ satisfies $\varphi(\bar{a})$.

(c) The canonical model of T^+ is $\mathcal{K}$-rich: for every tuple $\bar{c}$ in C and for every tuple $(\bar{a}, \bar{a}')$ in $\mathbf{M}$ such that

$$\operatorname{qftp}(\bar{a}) = \operatorname{qftp}_{T^+}(\bar{c}),$$

there exists a tuple $\bar{c}'$ in C such that

$$\operatorname{qftp}_{T^+}(\bar{c}, \bar{c}') = \operatorname{qftp}(\bar{a}, \bar{a}').$$

We proceed inductively, by building a chain of $\mathcal{L}^+$-theories $(T_i)_{i\in\mathbb{N}}$ and set T^+ to be the union of all the theories $T \cup T_i$. We will ensure that the conditions above are satisfied at the end of the construction. Moreover, in order to get typical finiteness, our induction hypotheses will be the following.

(3) $T \cup T_i$ admits a model.

(4) The set of all parameters from $\mathbf{N}$ and from C that appear in formulas of T_i is finite. Call B_i and C_i the corresponding subsets of $\mathbf{N}$ and C. We will sometimes consider those sets as tuples, in which case we denote them by $\bar{b}_i$ and $\bar{c}_i$.

(5) There exists a tuple $\bar{a}_i$ in $\mathbf{M}$ such that

$$\operatorname{tp}_{T\cup T_i}(\bar{c}_i/B_i) = \operatorname{tp}(\bar{a}_i/B_i)$$

and such that the finite set B_i is a support for $\operatorname{qftp}(\bar{a}_i/\mathbf{N})$.

This condition ensures in particular that the theory T^+ will be complete.

Moreover, we will also ensure that $\displaystyle\bigcup_{i\in\mathbb{N}} C_i = C$ (for this, we enumerate C and whenever we pick new constant symbols, we take the least ones that are outside of C_i).

Observe that since the set of $\mathcal{L}^+$-formulas is countable, condition (1) can be split into countably many subconditions which require countably many steps each. Condition (2a) will be ensured at each step, thanks to item (5) above. As for conditions (2b) and (2c), since the structure $\mathbf{M}$ is countable (up to isomorphism), we also need countably many steps for each one of them. Thus, for every $\varphi \in \mathcal{L}^+$, we pick an infinite set $I_{1,\varphi}$ in $\mathbb{N}$, for all $\bar{c}$ in C and $\bar{a}$, $\bar{a}'$ in $\mathbf{M}$, we pick an infinite set $I_{(2c),(\bar{c},\bar{a},\bar{a}')}$ in $\mathbb{N}$ and we pick an infinite set $I_{(2b)}$ in $\mathbb{N}$ that is indexed by finite tuples in $\mathbf{M}$, and we make all these sets disjoint. At each step contained in those sets, we work towards enforcing the corresponding condition.

First, set T_0 to be the empty theory, so that B_0 and C_0 are empty.

Now assume that T_i has been built. We explain how to build T_{i+1}.

Steps i in $I_{1,\varphi}$: witness property

If the formula φ has parameters outside B_i and C_i, or if the formula $\exists x, \varphi(x)$ is not in $T \cup T_i$, then do nothing (if the formula $\exists x, \varphi(x)$ is false in T^+, then the implication $(\exists x, \varphi(x)) \to \varphi(c_1)$ is always true). Otherwise, pick a new constant symbol c in $C \setminus C_i$. We will add $\varphi(c) = \psi(c, \bar{b}_i)$ to the theory T_i. But in order to preserve item (5) above, we will add more than this.

We consider the two cases separately: either the formula $\exists x, \varphi(x)$ is in T or it is in T_i.

- If it is in T, then write $\varphi(x)$ as $\psi(x, \bar{b}_i)$, for an $\mathcal{L}$-formula ψ. Since the formula $\exists x, \psi(x, \bar{b}_i)$ is in T, there exists a in $\mathbf{M}$ such that $\mathbf{M}$ satisfies $\psi(a, \bar{b}_i)$. Moreover, by item (5) of our induction hypotheses, there exists a tuple $\bar{a}_i$ in $\mathbf{M}$ such that

$$\operatorname{tp}_{T \cup T_i}(\bar{c}_i/\bar{b}_i) = \operatorname{tp}(\bar{a}_i/\bar{b}_i).$$

  Now, since $\mathbf{N}$ has the homogeneity property, proposition 10.20 implies that the quantifier-free type of $(a, \bar{a}_i)$ over $\bar{b}_i$ extends to a finitely determined quantifier-free type q over $\mathbf{N}$, say of support F, with $B_i \subseteq F$. Besides, since the structure $\mathbf{M}$ is ultrahomogeneous, any two realizations of the quantifier-free type $q_{\restriction F}$ in $\mathbf{M}$ have the same complete type (with quantifiers) $\tilde{q}$ over F.

  Thus, it is consistent to add all the formulas that say that

$$\operatorname{tp}_{T \cup T_{i+1}}((c, \bar{c}_i)/F) = \tilde{q}$$

  to the theory T_i. Thus, we set T_{i+1} to be the union of T_i with the set of all those formulas.

  All these formulas have their parameters in the finite set $F \cup C_i \cup \{c\}$. Thus, we set B_{i+1} to be F and C_{i+1} to be $C_i \cup \{c\}$.

  This way, $\psi(c, \bar{b}_i)$ will be in T_{i+1}, hence the formula $(\exists x, \varphi(x)) \to \varphi(c)$ will be in T_{i+1} too.
- If the formula $\exists x, \varphi(x)$ is in T_i. Then we write $\varphi(x)$ as $\psi(x, \bar{b}_i, \bar{c}_i)$, for some $\mathcal{L}$-formula ψ. By item (5), there exists $\bar{a}_i$ in $\mathbf{M}$ such that

$$\operatorname{tp}_{T \cup T_i}(\bar{c}_i/\bar{b}_i) = \operatorname{tp}(\bar{a}_i/\bar{b}_i).$$

  Thus, $\mathbf{M}$ satisfies $\exists x, \psi(x, \bar{b}_i, \bar{a}_i)$ so there exists a in $\mathbf{M}$ such that $\mathbf{M}$ satisfies $\psi(a, \bar{b}_i, \bar{a}_i)$.

  Again, the quantifier-free type of $(a, \bar{a}_i)$ over $\bar{b}_i$ extends to a finitely determined type q over $\mathbf{N}$, say of support F, with $B_i \subseteq F$. Again, by ultrahomogeneity of $\mathbf{M}$, any two realizations of the quantifier-free type $q_{\restriction F}$ in $\mathbf{M}$ have the same complete type $\tilde{q}$ over F. As before, we add all the formulas that say that

$$\operatorname{tp}_{T \cup T_{i+1}}((c, \bar{c}_i)/F) = \tilde{q}.$$

  It follows that the formula $(\exists x, \varphi(x)) \to \varphi(c)$ will indeed be in T_{i+1}.

Steps i in I_{2b}: realizing all structures of $\mathcal{K}$

Each step i in I_{2b} corresponds to a tuple $\bar{a}$ in $\mathbf{M}$. Pick a tuple of new constant symbols $\bar{c}$ in $C \setminus C_i$. By the induction hypotheses, there exists a tuple $\bar{a}_i$ in $\mathbf{M}$ such that

$$\operatorname{tp}_{T \cup T_i}(\bar{c}_i/B_i) = \operatorname{tp}(\bar{a}_i/B_i).$$

Again by proposition 10.20, the quantifier-free type of $(\bar{a}, \bar{a}_i)$ over B_i extends to a finitely determined type q over $\mathbf{N}$, say of support F, with $B_i \subseteq F$. By ultrahomogeneity, any two realizations of the quantifier-free type $q_{\restriction F}$ in $\mathbf{M}$ have the same complete type $\tilde{q}$ over F.

Then we add all the formulas that say that

$$\mathrm{tp}_{T \cup T_{i+1}}((\bar{c}, \bar{c}_i)/F) = \tilde{q}.$$

Steps i in $I_{2c,(\bar{c},\bar{a},\bar{a}')}$: $\mathcal{K}$-richness

Let $\bar{c}$ be a tuple in C and $\bar{a}, \bar{a}'$ be two tuples in $\mathbf{M}$. If at step i, the tuple $\bar{c}$ is not included in C_i or if $\mathrm{qftp}_{T \cup T_i}(\bar{c}) \neq \mathrm{qftp}(\bar{a})$, then do nothing.

Otherwise, for the sake of simplicity, write $\bar{c}_i$ as $(\bar{c}, \bar{d}_i)$.

By the induction hypothesis, there exists a tuple $\bar{a}_i = (\bar{e}, \bar{e}_i)$ in $\mathbf{M}$ such that

$$\mathrm{qftp}((\bar{e}, \bar{e}_i)/B_i) = \mathrm{qftp}_{T \cup T_i}((\bar{c}, \bar{d}_i)/B_i).$$

Now, since the structure $\mathbf{M}$ is $\mathcal{K}$-rich, there exists a tuple $\bar{d}'$ in $\mathbf{M}$ such that

$$\mathrm{qftp}((\bar{e}, \bar{d}')/B_i) = \mathrm{qftp}((\bar{a}, \bar{a}')/B_i).$$

Besides, by proposition 10.20 again, the quantifier-free type of $(\bar{e}, \bar{d}', \bar{e}_i)$ over B_i in $\mathbf{M}$ extends to a finitely determined type q, say of support F. Moreover, by ultrahomogeneity of $\mathbf{M}$, any two realizations of the quantifier-free type $q_{\restriction F}$ have the same complete type $\tilde{q}$ over F.

Pick a tuple $\bar{c}'$ of new constant symbols from $C \setminus C_i$. Then we add all the formulas that say that

$$\mathrm{tp}_{T \cup T_{i+1}}((\bar{c}, \bar{c}', \bar{d}_i)/F) = \tilde{q}.$$

Conclusion

When the theory T^+ has been built, we choose a model $\mathbf{M}^+$ of T^+ (such a model exists by item (3) and the compactness theorem). Consider the $\mathcal{L}^+$-substructure $\mathbf{C}$ generated by the (interpretations of the) constants from C in $\mathbf{M}^+$. Condition (1) ensures that $\mathbf{C}$ satisfies the Tarski-Vaught test in $\mathbf{M}^+$, so that $\mathbf{C}$ is an $\mathcal{L}^+$-elementary substructure of $\mathbf{M}^+$. In particular, $\mathbf{C}$ is a model of T so the structure $\mathbf{C}$ contains $\mathbf{N}$ as an $\mathcal{L}$-substructure.

Moreover, condition (2) ensures that $\mathbf{C}$ is the Fraïssé limit of the class $\mathcal{K}$, so the structure $\mathbf{C}$ is isomorphic, as an $\mathcal{L}$-structure, to $\mathbf{M}$.

Take an isomorphism f from $\mathbf{C}$ to $\mathbf{M}$. We prove that $\mathbf{M}$ realizes only finitely determined quantifier-free types over $f(\mathbf{N})$. To this aim, let $\bar{a}$ be an arbitrary tuple in $\mathbf{M}$ and let $\bar{c}$ in $\mathbf{C}$ be a tuple such that $f(\bar{c}) = \bar{a}$. Denote by p the quantifier-free type of $\bar{a}$ over $f(\mathbf{N})$. Since $\bar{c}$ is finite, there exists an i in $\mathbb{N}$ such that $\bar{c}$ is contained in C_i. By proposition 10.18, we may actually assume that $\bar{c} = \bar{c}_i$.

By item (5) of our induction hypotheses, there exists $\bar{a}_i$ in $\mathbf{M}$ such that

$$\mathrm{tp}_{T^+}(\bar{c}_i/B_i) = \mathrm{tp}(\bar{a}_i/B_i)$$

and such that the quantifier-free type of $\bar{a}_i$ over $\mathbf{N}$ is finitely determined, of support B_i. Now since $\mathbf{N}$ has the homogeneity property, there exists an automorphism g of $\mathbf{M}$ such that $g_{\restriction \mathbf{N}} = f_{\restriction \mathbf{N}}$. In particular, we have $g(B_i) = f(B_i)$.

Now the quantifier-free type $g(\mathrm{qftp}(\bar{a}_i/\mathbf{N}))$ is finitely determined, of support $f(B_i)$. Moreover, we have

$$g(\mathrm{qftp}(\bar{a}_i/B_i)) = g(\mathrm{qftp}(\bar{c}_i/B_i)) = g \circ f^{-1}(\mathrm{qftp}(\bar{a}/f(B_i))) = g \circ f^{-1}(p_{\restriction f(B_i)}).$$

Thus, we obtain that $g \circ f^{-1}(p) = g(\mathrm{qftp}(\bar{a}_i/\mathbf{N}))$. Besides, since $g_{\restriction \mathbf{N}} = f_{\restriction bN}$, we have $g \circ f^{-1}(p) = p$, so p is finitely determined. Finally, f is the desired embedding of $\mathbf{N}$ into $\mathbf{M}$. □

THEOREM 10.23. Let $\mathbf{N}$ be an infinite structure in $\mathcal{K}_\omega$. If $\mathbf{N}$ is not typically finite, then $\mathbf{N}$ does not have the homogeneity property.

PROOF. Assume $\mathbf{N}$ has the homogeneity property. Then, by proposition 10.22, there exists a copy of $\mathbf{N}$ in $\mathbf{M}$ over which all realized types are finitely determined. Moreover, since $\mathbf{N}$ is not typically finite, there is a somewhere realized type that is not finitely determined, so there exists a copy of $\mathbf{N}$ in $\mathbf{M}$ which does not have this property. Consequently, these two copies cannot be mapped one onto the other by an automorphism of $\mathbf{M}$, contradicting the homogeneity property. □

Together with theorem 10.19, this yields the desired equivalence.

COROLLARY 10.24. Let $\mathbf{N}$ be a structure in $\mathcal{K}_\omega$. Then $\mathbf{N}$ satisfies the homogeneity property if and only if $\mathbf{N}$ is typically finite.

Let us conclude this section by the study of typically finite sets in the rational Urysohn space.

PROPOSITION 10.25. Let X be a subset of $\mathbb{QU}$. If X is typically finite, then X is finite. Therefore, the only subsets of $\mathbb{QU}$ that satisfy the homogeneity property are finite.

PROOF. Choose $a_0 \in X$ and $r > 0$ arbitrary. Consider the type p in variable v over X that says

- v is at distance r from a_0 and;
- for all x in X, $d(v,x) = d(v,a_0) + d(a_0,x)$ (in other words — those of Tent and Ziegler **[TZ2]** — v is independent from X over a_0).

This type is well-defined and somewhere realized. Thus, by assumption, p is finitely determined. Let A be a finite support for p. We may assume that a_0 is in A. Note that any type over X that coincides with p on A will be independent from A over a_0. Moreover, by finite determination, p is realized over X, say by y.

Claim. For all b in $X \setminus A$, there are a_1, a_2 in A such that

$$|d(y,a_1) - d(a_1,b)| = d(y,b) = d(y,a_2) + d(a_2,b).$$

Otherwise, take $b \in X \setminus A$ for which that fails. For a in A, consider the sets $I_a := \{t \geqslant 0 : |d(y,a) - d(a,b)| \leqslant t \leqslant d(y,a) + d(a,b)\}$ and note that they form non-trivial closed intervals, as b fails the claim. Denote by I their intersection

$$I = \bigcap_{a \in A} I_a = \{t \geqslant 0 : |d(y,a_1) - d(a_1,b)| \leqslant t \leqslant d(y,a_2) + d(a_2,b) \text{ for all } a_1, a_2 \in A\}.$$

By the assumption on b, the interval I is non-trivial. Now, whenever $t \in I$, we can extend $p_{\restriction A}$ to $A \cup \{b\}$ by $d(v,b) = t$. This gives infinitely many extensions of p to $A \cup \{b\}$ (and thus to X), which contradicts the fact that p is finitely determined by A. Hence, we proved the claim.

Let us now take a closer look at the equation in the claim. First note that we can choose $a_2 = a_0$, as

$$\begin{aligned} d(y,b) &= d(y,a_2) + d(a_2,b) \\ &= d(y,a_0) + d(a_0,a_2) + d(a_2,b) \\ &\geqslant d(y,a_0) + d(a_0,b) \\ &\geqslant d(y,b). \end{aligned}$$

Furthermore, if we had $d(a_1,b) \geqslant d(y,a_1)$, then we would have

$$\begin{aligned} d(a_1,b) &= d(y,b) + d(y,a_1) \\ &= d(y,a_0) + d(a_0,b) + d(y,a_0) + d(a_0,a_1) \\ &\geqslant d(a_1,b) + 2d(y,a_0) \\ &> d(a_1,b), \end{aligned}$$

a contradiction. Thus, $d(y,b) = d(y,a_1) - d(a_1,b)$ which implies that for any $b \in X \setminus A$ we have $d(y,b) \leqslant \max_{a \in A} d(y,a)$ and X is bounded.

Finally, if X is bounded, say by $R > 0$, consider the type q over X, saying "v has distance R to all points in X". By assumption, this type is finitely determined by some $B \subseteq X$. Now consider a second type that agrees with q on B and says in addition "v is independent from X

over B, that is, for all x in X, $d(v,x) = \inf_{b\in B} d(v,b) + d(b,x)$". Finite determination yields that those two types must be equal, which holds only if $X = B$, in which case X is finite. □

Note that we actually showed something stronger than just the finiteness of typically finite sets: for any typically finite set X, there is a type over X that does not admit any proper subset of X as a support.

5. Perspectives

Our characterization of the homogeneity property leaves some questions open and provides several research directions.

5.1. Beyond typical finiteness. With corollary 10.24, it remains to understand typical finiteness in concrete examples. We have succeeded in finding all the substructures with the homogeneity property in a handful of Fraïssé limits which are mentioned. However, it would be nice to find a systematic way to do so. For this, as mentioned before, we lack relevant examples, in particular Fraïssé structures in which typically finite structures may be infinite.

Maybe, then, the right answer to this question lies in a different characterization altogether. Melleray's result for the Urysohn space, as well as our beloved infinitely splitting tree, suggests the idea of compactness. It is not clear which topology to put on structures though[4]; in all likelihood, the right topology should involve the algebraic closure. We are still looking for our kindred topology!

5.2. Acting only transitively. Another question of interest is the study of the following relaxing of the homogeneity property.

Definition 10.26. Let $\mathbf{N}$ be a structure in $\mathcal{K}_\omega$. We say that $\mathbf{N}$ has the **weak homogeneity property** (**wHoP**) if G acts transitively on isomorphic copies of $\mathbf{N}$ in $\mathbf{M}$.

The following proposition gives a sufficient condition for the two properties to coincide; it is essentially contained in [**M3**, théorème 18].

Proposition 10.27. Let $\mathbf{N}$ be a structure in $\mathcal{K}_\omega$. Assume that there exists a copy X of $\mathbf{N}$ inside $\mathbf{M}$ such that all automorphisms of X extend to global automorphisms of $\mathbf{M}$. Then $\mathbf{N}$ has the wHoP if and only if it has the HoP.

Proof. Assume that $\mathbf{N}$ has the weak homogeneity property. Let X' be another copy of $\mathbf{N}$ in $\mathbf{M}$ and let $f : X \to X'$ be an isomorphism. As $\mathbf{N}$ has the weak HoP, there exists an automorphism g of $\mathbf{M}$ that sends X' to X setwise. Then the map $g \circ f$ is an automorphism of X and thereby extends to an automorphism h in $\mathrm{Aut}(\mathbf{M})$. Now, the automorphism $g^{-1} \circ h$ is an extension of f to $\mathbf{M}$, hence $\mathbf{N}$ satisfies the homogeneity property. □

In particular, if it is possible to build Katětov-like towers around structures in $\mathcal{K}_\omega$ (at each step of which automorphisms extend), then the weak homogeneity property and the homogeneity property coincide. This is for instance the case in structures whose age has the free amalgamation property, or more generally in structures that admit a stationary independence relation, as is proved in [**BM2**] and [**M13**].

Actually, we do not know of any example of a structure that has only the weak homogeneity property but not the full one, so that the following questions remain open.

Questions 10.28.
- Are wHoP and HoP equivalent?
- Which structures satisfy the weak homogeneity property?

[4] I thank Todor Tsankov for suggesting one in the course of a very inspiring discussion!

5.3. Back to metric structures. Going from a metric result to a problem about classical Fraïssé structures may not seem very much in tune with the rest of this thesis; and the question arises as to what happens for metric Fraïssé structures? Once again, we encounter the recurring question of characterizing the exact ultrahomogeneity of a structure. Indeed, in the general metric Fraïssé theory, even finite sets need not satisfy the homogeneity property, and outside the separably categorical case, there is no known way to see exact ultrahomogeneity on the Fraïssé class.

So be it, let us restrict ourselves to structures we already know to be exactly ultrahomogeneous. But even then, we come across the same obstacles as in the classical case. Is typical compactness the relevant condition? How should it even be defined? Besides, although compactness seems much more relevant for metric spaces, it is still unclear how to define a topology that accounts for the structure and the homogeneity property. Moreover, here, we lack examples even more critically.

EXAMPLES 10.29. Here is how our two other favorite metric structures behave with regards to the homogeneity property.

- The separable Hilbert space ℓ^2. Substructures are separable Hilbert subspaces. Either they are finite-dimensional, in which case they satisfy HoP, or infinite-dimensional and isomorphic to the whole of ℓ^2, in which case they cannot have the homogeneity property.
- The separable probability measure algebra MALG$([0,1])$. Substructures are measure subalgebras, that have a continuous part and an atomic part.

  As soon as there is a continuous part, HoP is compromised: the continuous part will contain a proper copy of itself. Indeed, since the separable probability measure algebra is unique, MALG$([0,1] \times [0,1])$ is isomorphic to MALG$([0,1])$. It is then easy to find a (proper) copy of MALG$([0,1])$ inside MALG$([0,1] \times [0,1])$, which prevents the homogeneity property. The same reasoning applies to a continuous part of arbitrary finite measure.

  On the other hand, when the substructure is entirely atomic, it has countably many atoms, which form a partition of the interval $[0,1]$. Now any two such partitions in sets of equal measures can be sent one to the other by an automorphism, so atomic substructures satisfy the homogeneity property.

Bibliography

[AZ] G. Ahlbrandt and M. Ziegler, *Quasi-finitely axiomatizable totally categorical theories*, Ann. Pure Appl. Logic **30** (1986), no. 1, 63–82. Stability in model theory (Trento, 1984).

[B1] S. Banach, *Théorie des opérations linéaires*, Chelsea Publishing Co., New York, 1955.

[B2] H. Becker, *Polish group actions: dichotomies and generalized elementary embeddings*, J. Amer. Math. Soc. **11** (1998), no. 2, 397–449.

[B3] I. Ben Yaacov, *Topometric spaces and perturbations of metric structures*, Log. Anal. **1** (2008), no. 3-4, 235–272.

[B4] I. Ben Yaacov, *Definability of groups in $\aleph_0$-stable metric structures*, J. Symbolic Logic **75** (2010), no. 3, 817–840.

[B5] I. Ben Yaacov, *Fraïssé limits of metric structures*, preprint (2012).

[B6] I. Ben Yaacov, *On theories of random variables*, Israel J. Math. **194** (2013), no. 2, 957–1012.

[B7] ———, *On Roelcke-precompact Polish groups which cannot act transitively on a complete metric space*, research notes (2014).

[BBHU] I. Ben Yaacov, A. Berenstein, C. W. Henson, and A. Usvyatsov, *Model theory for metric structures*, Model theory with applications to algebra and analysis. Vol. 2, 2008, pp. 315–427.

[B] G. Birkhoff, *A note on topological groups*, Compositio Math. **3** (1936), 427–430.

[BBM] I. Ben Yaacov, A. Berenstein, and J. Melleray, *Polish topometric groups*, Trans. Amer. Math. Soc. **365** (2013), no. 7, 3877–3897.

[BdlHV] B. Bekka, P. de la Harpe, and A. Valette, *Kazhdan's property (T)*, New Mathematical Monographs, vol. 11, Cambridge University Press, Cambridge, 2008.

[BK1] H. Becker and A. S. Kechris, *Borel actions of Polish groups*, Bull. Amer. Math. Soc. (N.S.) **28** (1993), no. 2, 334–341.

[BK2] ———, *The descriptive set theory of Polish group actions*, London Mathematical Society Lecture Note Series, vol. 232, Cambridge University Press, Cambridge, 1996.

[BK3] I. Ben Yaacov and A. Kaïchouh, *Reconstruction of separably categorical metric structures*, J. Symb. Logic **to appear** (2014).

[BM1] I. Ben Yaacov and J. Melleray, *Gray subsets of Polish spaces*, preprint (2011).

[BM2] D. Bilge and J. Melleray, *Elements of finite order in automorphism groups of homogeneous structures*, Contrib. Discrete Math. **8** (2013), no. 2, 88–119.

[BT1] I. Ben Yaacov and T. Tsankov, *Weakly almost periodic functions, model-theoretic stability, and minimality of topological groups*, preprint (2013).

[BT2] A. Bouziad and J.-P. Troallic, *A precompactness test for topological groups in the manner of Grothendieck*, Topology Proc. **31** (2007), no. 1, 19–30.

[BU1] I. Ben Yaacov and A. Usvyatsov, *On d-finiteness in continuous structures*, Fund. Math. **194** (2007), no. 1, 67–88.

[BU2] I. Ben Yaacov and A. Usvyatsov, *Continuous first order logic and local stability*, Trans. Amer. Math. Soc. **362** (2010), no. 10, 5213–5259.

[C] G. Cantor, *Beiträge zur Begründung der transfiniten Mengenlehre*, Math. Ann. **49** (1897), no. 2, 207–246.

[D] R. W. Dickson, *The farmer's companion; being a complete system of modern husbandry: including the latest improvements and discoveries in theory and practice*, Second edition, Vol. II, Sherwood, Neely, and Jones, London, 1813.

[E1] E. G. Effros, *Transformation groups and C^*-algebras*, Ann. of Math. (2) **81** (1965), 38–55.

[E2] R. Engelking, *General topology*, Second, Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, 1989. Translated from the Polish by the author.

[EH] D. M. Evans and P. R. Hewitt, *Counterexamples to a conjecture on relative categoricity*, Ann. Pure Appl. Logic **46** (1990), no. 2, 201–209.

[ER] P. Erdős and A. Rényi, *Asymmetric graphs*, Acta Math. Acad. Sci. Hungar **14** (1963), 295–315.

[F] R. Fraïssé, *Sur l'extension aux relations de quelques propriétés des ordres*, Ann. Sci. Ecole Norm. Sup. (3) **71** (1954), 363–388.

[G1] S. Gao, *Invariant descriptive set theory*, Pure and Applied Mathematics (Boca Raton), vol. 293, CRC Press, Boca Raton, FL, 2009.

[G2] E. Glasner, *On minimal actions of Polish groups*, Topology Appl. **85** (1998), no. 1-3, 119–125. 8th Prague Topological Symposium on General Topology and Its Relations to Modern Analysis and Algebra (1996).

[G3] ———, *The group* $\mathrm{Aut}(\mu)$ *is Roelcke precompact*, Canad. Math. Bull. **55** (2012), no. 2, 297–302.

[G4] E. E. Granirer, *Extremely amenable semigroups*, Bull. Amer. Math. Soc. **72** (1966), 1028–1032.

[G5] F. P. Greenleaf, *Invariant means on topological groups and their applications*, Van Nostrand Reinhold, 1969.

[G6] V. I. Gurariĭ, *Spaces of universal placement, isotropic spaces and a problem of Mazur on rotations of Banach spaces*, Sibirsk. Mat. Ž. **7** (1966), 1002–1013.

[GK] S. Gao and A. S. Kechris, *On the classification of Polish metric spaces up to isometry*, Memoirs of Amer. Math. Soc. **766** (2003).

[GM1] E. Glasner and M. Megrelishvili, *Representations of dynamical systems on Banach spaces*, Recent progress in general topology. III, 2014, pp. 399–470.

[GM2] M. Gromov and V. D. Milman, *A topological application of the isoperimetric inequality*, Amer. J. Math. **105** (1983), no. 4, 843–854.

[HC] W. Herer and J. P. R. Christensen, *On the existence of pathological submeasures and the construction of exotic topological groups*, Math. Ann. **213** (1975), 203–210.

[H1] F. Hausdorff, *Bemerkung über den Inhalt von Punktmengen*, Math. Ann. **75** (1914), no. 3, 428–433.

[H2] C. W. Henson, *A family of countable homogeneous graphs*, Pacific J. Math. **38** (1971), 69–83.

[H3] G. Hjorth, *Classification and orbit equivalence relations*, Mathematical Surveys and Monographs, vol. 75, American Mathematical Society, Providence, RI, 2000.

[H4] ———, *A universal Polish G-space*, preprint (2008), available at `math/9701202`.

[HHLS] W. Hodges, I. Hodkinson, D. Lascar, and S. Shelah, *The small index property for* ω*-stable* ω*-categorical structures and for the random graph*, J. London Math. Soc. (2) **48** (1993), no. 2, 204–218.

[H1] W. Hodges, *Model theory*, Encyclopedia of Mathematics and its Applications, vol. 42, Cambridge University Press, Cambridge, 1993.

[H2] E. Hrushovski, *Extending partial isomorphisms of graphs*, Combinatorica **12** (1992), no. 4, 411–416.

[H3] G. E. Huhunaišvili, *On a property of Uryson's universal metric space*, Dokl. Akad. Nauk SSSR (N.S.) **101** (1955), 607–610.

[H4] E. V. Huntington, *The continuum as a type of order: an exposition of the modern theory*, Ann. of Math. (2) **7** (1905), no. 1, 15–43.

[HKL] L. Harrington, A. S. Kechris, and A. Louveau, *A Glimm-Effros dichotomy for Borel equivalence relations*, J. Amer. Math. Soc. **3** (1990), no. 4, 903–928.

[HL] B. Herwig and D. Lascar, *Extending partial automorphisms and the profinite topology on free groups*, 1998.

[I] T. Ibarlucía, *The dynamical hierarchy for Roelcke precompact Polish groups*, preprint (2013).

[K1] A. Kaïchouh, *Amenability and Ramsey theory in the metric setting*, Fund. Math. **to appear** (2013).

[K2] S. Kakutani, *Über die Metrisation der topologischen Gruppen*, Proc. Imp. Acad. **12** (1936), no. 4, 82–84.

[K3] M. Katětov, *On universal metric spaces*, General topology and its relations to modern analysis and algebra, VI (Prague, 1986), 1988, pp. 323–330.

[K4] A. S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995.

[K5] ———, *Actions of Polish groups and classification problems*, Analysis and logic (Mons, 1997), 2002, pp. 115–187.

[K6] ———, *Global aspects of ergodic group actions*, Mathematical Surveys and Monographs, vol. 160, American Mathematical Society, Providence, RI, 2010.

[K7] A. Kwiatkowska, *The group of homeomorphisms of the Cantor set has ample generics*, Bull. Lond. Math. Soc. **44** (2012), no. 6, 1132–1146.

[KLM] A. Kaïchouh and F. Le Maître, *A connected Polish group with ample generics*, preprint (2015).

[KPT] A. S. Kechris, V. G. Pestov, and S. Todorčević, *Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups*, Geom. Funct. Anal. **15** (2005), no. 1, 106–189.

[KR] A. S. Kechris and C. Rosendal, *Turbulence, amalgamation, and generic automorphisms of homogeneous structures*, Proc. Lond. Math. Soc. (3) **94** (2007), no. 2, 302–350.

[KST] A. S. Kechris, S. Solecki, and S. Todorcevic, *Borel chromatic numbers*, Adv. Math. **141** (1999), no. 1, 1–44.

[KT] J. Kittrell and T. Tsankov, *Topological properties of full groups*, Ergodic Theory Dynam. Systems **30** (2010), no. 2, 525–545.

[L1] D. Lascar, *The group of automorphisms of a relational saturated structure*, Finite and infinite combinatorics in sets and logic (Banff, AB, 1991), 1993, pp. 225–236.

[L2] M. Ledoux, *The concentration of measure phenomenon*, Mathematical Surveys and Monographs, vol. 89, American Mathematical Society, Providence, RI, 2001.

[L3] W. Lusky, *The Gurarij spaces are unique*, Arch. Math. (Basel) **27** (1976), no. 6, 627–635.

[M1] M. Malicki, *Consequences of the existence of ample generics and automorphism groups of homogeneous metric structures*, preprint (2014).

[M2] S. Mazur, *Über konvexe Mengen in linearen normierten Räumen*, Studia Mathematica **4** (1933), 70–84.

[M3] J. Melleray, *Géométrie de l'espace d'Urysohn et théorie descriptive des ensembles*, Ph.D. Thesis, 2005.

[M4] ———, *Some geometric and dynamical properties of the Urysohn space*, Topology Appl. **155** (2008), no. 14, 1531–1560.

[M5] ———, *A note on Hjorth's oscillation theorem*, J. Symbolic Logic **75** (2010), no. 4, 1359–1365.

[M6] ———, *Baire category methods : some applications to the study of automorphism groups of countable homogeneous structures*, 2011. Notes MALOA.

[M7] ———, *Groupes polonais et méthodes de Baire*, Habilitation à diriger des recherches (2014).

[M8] A. W. Miller, *Vaught's conjecture for theories of one unary operation*, Fund. Math. **111** (1981), no. 2, 135–141.

[M9] B. D. Miller, *Forceless, ineffective, powerless proofs of descriptive dichotomy theorems*, preprint (2009).

[M10] T. Mitchell, *Fixed points and multiplicative left invariant means*, Trans. Amer. Math. Soc. **122** (1966), 195–202.

[M11] C. C. Moore, *Group extensions and cohomology for locally compact groups. III*, Trans. Amer. Math. Soc. **221** (1976), no. 1, 1–33.

[M12] J. T. Moore, *Amenability and Ramsey theory*, Fund. Math. **220** (2013), no. 3, 263–280.

[M13] I. Müller, *Fraïssé structures with universal automorphism groups*, preprint (2015).

[MT1] J. Melleray and T. Tsankov, *Extremely amenable groups via continuous logic*, preprint (2011).

[MT2] ———, *Generic representations of abelian groups and extreme amenability*, Israel J. Math. **198** (2013), no. 1, 129–167.

[N] J. v. Neumann, *Zur allgemeinen Theorie des Masses*, Fundamenta Mathematicae **13** (1929), no. 1, 73–116.

[NR] J. Nešetřil and V. Rödl, *Ramsey classes of set systems*, J. Combin. Theory Ser. A **34** (1983), no. 2, 183–201.

[NVT] L. Nguyen Van Thé, *Structural Ramsey theory of metric spaces and topological dynamics of isometry groups*, Mem. Amer. Math. Soc. **206** (2010), no. 968.

[O] J. C. Oxtoby, *Measure and category*, Second, Graduate Texts in Mathematics, vol. 2, Springer-Verlag, New York-Berlin, 1980. A survey of the analogies between topological and measure spaces.

[P1] A. Panagiotopoulos, *Extendability of automorphisms of generic substructures*, preprint (2014).

[P2] V. Pestov, *Ramsey-Milman phenomenon, Urysohn metric spaces, and extremely amenable groups*, Israel J. Math. **127** (2002), 317–357.

[P3] ———, *Dynamics of infinite-dimensional groups*, University Lecture Series, vol. 40, American Mathematical Society, Providence, RI, 2006. The Ramsey-Dvoretzky-Milman phenomenon, Revised edition of Dynamics of infinite-dimensional groups and Ramsey-type phenomena [Inst. Mat. Pura. Apl. (IMPA), Rio de Janeiro, 2005].

[P4] V. G. Pestov, *On free actions, minimal flows, and a problem by Ellis*, Trans. Amer. Math. Soc. **350** (1998), no. 10, 4149–4165.

[P5] B. J. Pettis, *On continuity and openness of homomorphisms in topological groups*, Ann. of Math. (2) **52** (1950), 293–308.

[P6] J. M. Plotkin, *Who put the "back" in back-and-forth?*, Logical methods (Ithaca, NY, 1992), 1993, pp. 705–712.

[RN] C. Ryll-Nardzewski, *On the categoricity in power* $\leq \aleph_0$, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 545–548.

[R1] R. Rado, *Universal graphs and universal functions*, Acta Arith. **9** (1964), 331–340.

[R2] C. Rosendal, *Automatic continuity of group homomorphisms*, Bull. Symbolic Logic **15** (2009), no. 2, 184–214.

[R3] ———, *A topological version of the Bergman property*, Forum Math. **21** (2009), no. 2, 299–332.

[R4] ———, *Global and local boundedness of Polish groups*, Indiana Univ. Math. J. **62** (2013), no. 5, 1621–1678.

[R5] H. P. Rosenthal, *A characterization of Banach spaces containing* l^1, Proc. Nat. Acad. Sci. U.S.A. **71** (1974), 2411–2413.

[R6] M. Rubin, *On the reconstruction of* $\aleph_0$*-categorical structures from their automorphism groups*, Proc. London Math. Soc. (3) **69** (1994), no. 2, 225–249.

[RS] C. Rosendal and S. Solecki, *Automatic continuity of homomorphisms and fixed points on metric compacta*, Israel J. Math. **162** (2007), 349–371.

[S1] M. Sabok, *Automatic continuity for isometry groups*, preprint (2013).

[S2] K. Schoretsanitis, *Fraïssé theory for metric structures*, Ph.D. Thesis, 2007.

[S3] S. W. Semmes, *Endomorphisms of infinite symmetric groups*, Abstracts Amer. Math. Soc. **426** (1981), no. 2.

[S4] W. Sierpiński, *Sur un espace métrique séparable universel*, Fund. Math. **33** (1945), 115–122.

[S5] S. Solecki, *Actions of non-compact and non-locally compact Polish groups*, J. Symbolic Logic **65** (2000), no. 4, 1881–1894.

[S6] ———, *Extending partial isometries*, Israel J. Math. **150** (2005), 315–331.

[S7] ———, *Notes on a strengthening of the Herwig-Lascar extension theorem*, 2009.

[T1] L. N. V. Thé, *A survey on structural ramsey theory and topological dynamics with the kechris-pestov-todorcevic correspondence in mind*, preprint (201412), available at `1412.3254`.

[T2] J. K. Truss, *Infinite permutation groups. II. Subgroups of small index*, J. Algebra **120** (1989), no. 2, 494–515.

[T3] T. Tsankov, *Automatic continuity for the unitary group*, Proc. Amer. Math. Soc. **141** (2013), no. 10, 3673–3680.

[TZ1] K. Tent and M. Ziegler, *A course in model theory*, Lecture Notes in Logic, vol. 40, Association for Symbolic Logic, La Jolla, CA; Cambridge University Press, Cambridge, 2012.

[TZ2] ———, *On the isometry group of the Urysohn space*, J. Lond. Math. Soc. (2) **87** (2013), no. 1, 289–303.

[U1] P. S. Urysohn, *Sur un espace métrique universel*, Bull. Sci. Math. **51** (1927), 43–64 and 74–96.

[U2] V. V. Uspenskiĭ, *A universal topological group with a countable basis*, Funktsional. Anal. i Prilozhen. **20** (1986), no. 2, 86–87.

[U3] V. V. Uspenskij, *The Roelcke compactification of unitary groups*, Abelian groups, module theory, and topology (Padua, 1997), 1998, pp. 411–419.

[U4] V. V. Uspenskij, *On the group of isometries of the Urysohn universal metric space*, Comment. Math. Univ. Carolin. **31** (1990), no. 1, 181–182.

[U5] ———, *On subgroups of minimal topological groups*, Topology Appl. **155** (2008), no. 14, 1580–1606.

[V1] R. Vaught, *Invariant sets in topology and logic*, Fund. Math. **82** (1974/75), 269–294. Collection of articles dedicated to Andrzej Mostowski on his sixtieth birthday, VII.

[V2] W. A. Veech, *Topological dynamics*, Bull. Amer. Math. Soc. **83** (1977), no. 5, 775–830.

[W] P. Wesolek, *Conjugacy class conditions in locally compact second countable groups*, preprint (2013).

Metric structures and their automorphism groups: reconstruction, homogeneity, amenability and automatic continuity

Abstract: This thesis focuses on the study of Polish groups seen as automorphism groups of metric structures. The observation that every non-archimedean Polish group is the automorphism group of an ultrahomogeneous countable structure has indeed led to fruitful interactions between group theory and model theory. In the framework of metric model theory, introduced by Ben Yaacov, Henson and Usvyastov, this correspondence has been extended to all Polish groups by Melleray. In this thesis, we study various facets of this correspondence.
The relationship between a structure and its automorphism group is particularly close in the setting of $\aleph_0$-categorical structures. Indeed, the Ahlbrandt-Ziegler reconstruction theorem allows one to recover an $\aleph_0$-categorical structure, up to bi-interpretability, from its automorphism group. In a joint work with Itaï Ben Yaacov, we generalize this result to separably categorical metric structures.
Besides, ultrahomogeneous countable structures have the advantage of being completely determined by their finitely generated substructures. In particular, this enabled Moore to give a combinatorial characterization of amenability for non-archimedean Polish groups. We extend this characterization to all Polish groups and we deduce that amenability is a G_δ condition.
Still in a reconstruction perspective, we are interested in the automatic continuity property for Polish groups. Sabok and Malicki introduced conditions of a combinatorial nature on an ultrahomogeneous metric structure that imply the automatic continuity property for its automorphism group. We show that these conditions carry to countable powers, which leads to the groups $\mathrm{Aut}(\mu)^{\mathbb{N}}$, $\mathcal{U}(\ell^2)^{\mathbb{N}}$ and $\mathrm{Iso}(\mathbb{U})^{\mathbb{N}}$ satisfying the automatic continuity property.
Those conditions are a weakening of the property of having ample generics. In a joint work with François Le Maître, we exhibit the first examples of connected groups with ample generics, which answers a question of Kechris and Rosendal.
Finally, in a joint work with Isabel Müller and Aristotelis Panagiotopoulos, we study the relative homogeneity of substructures in an ultrahomogeneous countable structure. We characterize it completely by a property of the types over the substructures: being determined by a finite set.

Structures métriques et leurs groupes d'automorphismes : reconstruction, homogénéité, moyennabilité et continuité automatique

Résumé: Cette thèse porte sur l'étude des groupes polonais vus comme groupes d'automorphismes de structures métriques. L'observation que tout groupe polonais non archimédien est le groupe d'automorphismes d'une structure dénombrable ultrahomogène a en effet mené à des interactions fructueuses entre la théorie des groupes et la théorie des modèles. Dans le cadre de la théorie des modèles métriques, introduite par Ben Yaacov, Henson et Usvyatsov, cette correspondance a été étendue par Melleray à tous les groupes polonais. Dans cette thèse, nous étudions diverses facettes de cette correspondance.
Le lien entre une structure et son groupe d'automorphismes est particulièrement étroit dans le cadre des structures $\aleph_0$-catégoriques. En effet, le théorème de reconstruction d'Ahlbrandt-Ziegler permet de retrouver une structure $\aleph_0$-catégorique, à bi-interprétabilité près, à partir de son groupe d'automorphismes. Dans un travail en commun avec Itaï Ben Yaacov, nous généralisons ce résultat aux structures métriques séparablement catégoriques.
Les structures dénombrables ultrahomogènes ont de plus l'avantage d'être complètement déterminées par leurs sous-structures finiment engendrées. Cela a notamment permis à Moore de donner une caractérisation combinatoire de la moyennabilité des groupes polonais non archimédiens. Nous étendons cette caractérisation à tous les groupes polonais et nous en déduisons que la moyennabilité est une condition G_δ.
Toujours dans une optique de reconstruction, nous nous intéressons à la propriété de continuité automatique pour les groupes polonais. Sabok et Malicki ont introduit des conditions de nature combinatoire sur une structure métrique ultrahomogène qui impliquent la propriété de continuité automatique pour son groupe d'automorphismes. Nous montrons que ces conditions passent à la puissance dénombrable, ce qui a pour conséquence que les groupes $\mathrm{Aut}(\mu)^{\mathbb{N}}$, $\mathcal{U}(\ell^2)^{\mathbb{N}}$ et $\mathrm{Iso}(\mathbb{U})^{\mathbb{N}}$ satisfont la propriété de continuité automatique.
Ces conditions sont un affaiblissement du fait d'avoir des amples génériques. Dans un travail en commun avec François Le Maître, nous exhibons les premiers exemples de groupes connexes qui ont des amples génériques, ce qui répond à une question de Kechris et Rosendal.
Enfin, dans un travail en commun avec Isabel Müller et Aristotelis Panagiotopoulos, nous étudions l'homogénéité relative des sous-structures dans une structure dénombrable ultrahomogène. Nous caractérisons complètement celle-ci à l'aide d'une propriété sur les types au-dessus des sous-structures : le fait d'être déterminés par un sous-ensemble fini.

Image en couverture : Illustration de la propriété de Ramsey convexe (métrique).

ED 512

INFOMATHS

Ecole doctorale

UNIVERSITÉ DE LYON