HOMOGENEOUS ACTIONS ON URYSOHN SPACES.

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Abstract. We show that many countable groups acting on trees, including free products of infinite countable groups and surface groups, are isomorphic to dense subgroups of isometry groups of bounded Urysohn spaces. This extends previous results of the first and last author with Y. Stalder on dense subgroups of the automorphism group of the random graph. In the unbounded case, we also show that every free product of infinite countable groups arises as a dense subgroup of the isometry group of the rational Urysohn space.

Contents

1. Introduction

There is an extensive, and still growing, literature concerning countable groups G admitting faithful actions on $\mathbb{N}$ which are *highly transitive*, that is, such that for any partial bijection σ with finite domain there exists $g \in G$ such that $g \cdot x = \sigma(x)$ for all $x \in \mathrm{dom}(\sigma)$. Such groups are called highly transitive groups. It is clear that the group $\mathfrak{S}_{(\infty)}$ consisting of finitely supported bijections of $\mathbb{N}$ is highly transitive. For a finitely generated example, one can consider the group of permutations of $\mathbb{Z}$ which are translations up to a finite set, or more generally Houghton groups.

The first explicit examples of highly transitive non-amenable groups are free groups $\mathbb{F}_n$ for $2 \leq n \leq +\infty$, as was shown in 1976 by T.P. McDonough [McD77] (see also the work of J.D. Dixon in [Dix90]). The case of a general free product has been studied by A.A.W. Glass and S.H. McCleary in [GM91] and later settled by S.V. Gunhouse [Gun92] and independently by K.K. Hickin [Hic92]: a free product $\Gamma = \Gamma_1 * \Gamma_2$ of two non-trivial countable groups Γ_i, $i = 1, 2$, is highly transitive if and only if Γ is i.c.c.[1], which is equivalent to ask that at least one the Γ_i has cardinal at least 3. In the last few years, many new examples of highly transitive groups have been discovered: non elementary hyperbolic groups with trivial finite radical (V.V. Chaynikov [Cha12]), surface groups (D. Kitroser [Kit12]), $\mathrm{Out}(\mathbb{F}_n)$ for $n \geq 4$ (S. Garian and Y. Glasner [GG13]), many groups acting on trees (P. Fima, S. Moon and Y. Stalder [FMS15a]), and finally acylindrically hyperbolic group with trivial finite radical (M. Hull and D. Osin. [HO16]).

Note that the class of highly transitive countable groups consists of all countable groups which are isomorphic to a dense subgroup of $\mathfrak{S}_\infty$, the group of all permutations of $\mathbb{N}$. It makes sense to ask a similar question for other groups.

Question. Given a Polish group G, what can be said about the class of all its countable dense subgroups?

This question is particularly interesting when G is the isometry group of a *homogeneous* countable metric space, i.e. a countable metric space all whose partial isometries with finite domain extend to global isometries. Indeed, if (X, d) is a homogeneous metric space, then a countable subgroup Γ of $\mathrm{Iso}(X)$ is dense[2] if and only if its natural action on X satisfies the following definition.

Definition 1.1. An action of a countable group Γ on a homogeneous metric space (X, d) is called **homogenous** if every partial isometry of X with finite domain extends to an element of Γ.

If (X, d) is a homogeneous metric space, we denote by $\mathcal{H}(X)$ the class of all countable groups which admit a *faithful* homogeneous isometric action on X. Note that the set $\mathbb{N}$ endowed with the discrete metric is a homogeneous metric space, and its isometry group is $\mathfrak{S}_\infty$, so $\mathcal{H}(\mathbb{N})$ denotes the class of highly transitive groups. In this paper, we are studying the class $\mathcal{H}(X)$ where X is one of the *countable Urysohn spaces*.

To define countable Urysohn spaces, one first fixes a countable set $S \subset [0, +\infty[$, containing 0, and considers the class of all S-metric spaces, that is, the class of all metric spaces whose distance takes its values in S. Under a suitable assumption on S (see the next section), there exists a generic object in this class, which is characterized as being the unique S-metric space which contains an isometric copy of each finite S-metric space (*universality*), and such that any isometry between finite subsets extends to the whole space (*ultrahomogeneity*). We call this space the S-Urysohn space (when $S = \mathbb{Q}$, this space was built by Urysohn in the seminal paper [Ury27]) and denote it by $\mathbb{U}_S$. For instance, when $S = \{0, 1\}$, $\mathbb{U}_S$ is simply a countable

[1] A group is i.c.c. (infinite conjugacy classes) when all the conjugacy classes of nonidentity elements are infinite.

[2] Note that since X is countable, $\mathrm{Iso}(X)$ must be endowed with the topology of pointwise convergence, viewing S as a *discrete* space.

infinite set with the discrete metric; for $S = \{0, 1, 2\}$, $\mathbb{U}_S$ may be identified with Erdös–Radó's random graph $\mathcal{R}$ equipped with the path metric.

Little is known about the class $\mathcal{H}(\mathcal{R})$. It contains $\mathbb{F}_2$ (H.D. Macpherson [Mac86]), a locally finite group (M. Bhattacharjee and H.D. Macpherson [BM05]), many groups acting on trees such as free products of any two infinite countable groups (P. Fima, S.Moon and Y. Stalder [FMS16]). A result of S. Solecki (whose proof appears in a paper by C. Rosendal [Ros11]) yields that the class $\mathcal{H}(\mathbb{U}_S)$ always contains a locally finite group. Our main result relies on the following notion, which was introduced in [FMS15a].

Definition 1.2. A subgroup Σ of a countable group Γ is **highly core-free** if whenever F is a finite subset of Γ, we may find $\gamma \in \Gamma$ such that the map

$$\begin{aligned} \Sigma \times F &\to \Gamma \\ (\sigma, f) &\mapsto \sigma\gamma f \end{aligned}$$

is injective and its range $\Sigma\gamma F$ is disjoint from F.

Easy examples of highly core-free subgroups are provided by the trivial group and any finite subgroup of an i.c.c. group.

Theorem A (see Thm. 6.1)**.** *Let Γ be a countable group acting on a non-trivial tree $\mathcal{T}$. If each edge stabilizer is highly core-free in the two corresponding vertex stabilizers, then $\Gamma \in \mathcal{H}(\mathbb{U}_S)$ for all bounded distance set S.*

Corollary B (see Cor. 4.2 and Cor. 5.2)**.** *Let S be a bounded distance set and Γ_1, Γ_2, H be countably infinite groups with a common subgroup Σ. Let also $\theta : \Sigma \to H$ be an injective group homomorphism. The following holds.*

1. *If Σ is highly core-free in both Γ_1, Γ_2 then $\Gamma_1 \underset{\Sigma}{*} \Gamma_2 \in \mathcal{H}(\mathbb{U}_S)$. In particular:*
   - (a) *For any infinite countable groups Γ_1, Γ_2 one has $\Gamma_1 * \Gamma_2 \in \mathcal{H}(\mathbb{U}_S)$.*
   - (b) *If $\mathcal{S}_g$ is a closed, orientable surface of genus $g > 1$, then $\pi_1(\mathcal{S}_g) \in \mathcal{H}(\mathbb{U}_S)$.*
2. *If Σ and $\theta(\Sigma)$ are both highly core-free in H then $\mathrm{HNN}(H, \Sigma, \theta) \in \mathcal{H}(\mathbb{U}_S)$.*

We also study unbounded Urysohn spaces, for instance when $S = \mathbb{Q}^+$. In that case, we obtain the following result (see Corollary 7.10).

Theorem C. *For any infinite countable groups Γ_1 and Γ_2 one has $\Gamma_1 * \Gamma_2 \in \mathcal{H}(\mathbb{U}_{\mathbb{Q}^+})$.*

The method of proof finds its roots in a paper of Dixon [Dix90], and is an adaptation to the context of S-Urysohn spaces of techniques of P. Fima, S. Moon and Y. Stalder [MS13, FMS15a, FMS16]. Let us describe briefly the argument in the case of a free product of infinite groups $\Gamma * \Lambda$. One first proves that any countable group admits a "sufficiently free" and "sufficiently rich" action on $\mathbb{U}_S$ (notions that are made clear below; this is easier to do in the case when S is bounded, explaining why we have stronger results in that case). Then one uses a Baire category argument: start from two sufficiently rich actions π_1, π_2 of Γ, Λ respectively. Then, for any $\alpha \in \mathrm{Iso}(\mathbb{U}_S)$, one can consider the action π_α of $\Gamma * \Lambda$ which coincides with π_1 on Γ and with $\alpha\pi_2\alpha^{-1}$ on Λ; we then prove that $\{\alpha \colon \pi_\alpha \text{ is faithful}\}$ and $\{\alpha \colon \pi_\alpha(\Gamma * \Lambda) \text{ is dense}\}$ are both dense G_δ in $\mathrm{Iso}(\mathbb{U}_S)$, thus the intersection O of these two sets is nonempty; and for any $\alpha \in O$ the subgroup $\pi_\alpha(\Gamma * \Lambda)$ is dense in $\mathrm{Iso}(\mathbb{U}_S)$ and isomorphic to $\Gamma * \Lambda$.

When S is bounded, this basic strategy can be employed, with some technical modifications, to cover the case of amalgamated free products over highly core-free subgroups, and thus surface groups, as well as the case of free products where one of the factors is finite and the other infinite. Unfortunately, while the basic structure and underlying ideas are the same in all these proofs, technical aspects involving the triangle inequality differ, forcing us to write down several times some very similar arguments.

Looking at the results discussed above, one question in particular begs to be answered: what happens to the class $\mathcal{H}(\mathbb{U}_S)$ when S varies? Do these classes all coincide, can there be nontrivial inclusions, etc.? This question remains largely open; however, we do manage to establish in the last section the following result.

Theorem D (see Thm. 8.1)**.** *The group $\mathfrak{S}_{(\infty)}$ of all finitary permutations of $\mathbb{N}$ does not belong to $\mathcal{H}(\mathbb{U}_S)$ unless $|S| = 2$, that is, unless $\mathrm{Iso}(\mathbb{U}_S) \cong \mathfrak{S}_\infty$.*

Finally, consider the Urysohn space $(\mathbb{U}, d)$, which can be obtained as the metric completion of $\mathbb{U}_{\mathbb{Q}^+}$. Then $\mathrm{Iso}(\mathbb{U})$ is a Polish group for the topology of pointwise convergence induced by the metric d, and a result of Cameron-Vershik yields that $\mathrm{Iso}(\mathbb{U}_{\mathbb{Q}^+})$ is a dense subgroup of $\mathrm{Iso}(\mathbb{U})$ [Cam90]. Theorem C thus yields that every free product of infinite countable groups can be densely embedded in $\mathrm{Iso}(\mathbb{U})$. It is however unclear whether every countable dense subgroup of $\mathrm{Iso}(\mathbb{U})$ belongs to the class $\mathcal{H}(\mathbb{U}_{\mathbb{Q}^+})$. The same question can be asked in the bounded case.

The paper is organized as follows: we first introduce some basic facts about group actions on metric spaces, and develop the machinery we need to deal with the bounded case. We then establish the theorems mentioned above in that case. Once that work is completed, we turn to the unbounded case. While similar in spirit, this case requires some additional work, including the construction of some isometric actions of countable groups on countable metric spaces with interesting combinatorial property. Finally, we prove that $\mathfrak{S}_{(\infty)}$ does not belong to $\mathcal{H}(\mathbb{U}_S)$ unless $|S| = 2$.

2. Preliminaries

Definition 2.1. Consider a countable set $S \subseteq [0, +\infty[$ containing 0 and at least another element. We say that S is an **unbounded distance set** if S is a subsemigroup of the additive semigroup $[0, +\infty[$, meaning that for all $s, t \in S$, we have $s + t \in S$.

We say that S is a **bounded distance set** if there is $M > 0$ such that

$$\forall s, t \in S \quad \min(s + t, M) \in S.$$

Given a bounded or unbounded distance set S, we say that a metric space (X, d) is an **S-metric space** if the metric d takes values in S.

Example 2.2. Every graph can be viewed as a $\{0, 1, 2\}$-metric space by letting $d(x, y) = 1$ when there is an edge between x and y, $d(x, y) = 0$ if $x = y$ and $d(x, y) = 2$ if $x \neq y$ and there is no edge between x and y. Conversely every $\{0, 1, 2\}$-metric space can be seen as a graph equipped with the above metric.

A map between two metric spaces (X, d_X) and (Y, d_Y) is an **isometry** if it is surjective and for all $x_1, x_2 \in X$ we have $d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$. We denote by $\mathrm{Iso}(X, d_X)$ or simply $\mathrm{Iso}(X)$ the group of isometries $X \to X$.

A **partial isometry** between X and Y is a map $\varphi\colon \operatorname{dom}\varphi \subseteq X \to \operatorname{rng}\varphi \subseteq Y$ which is an isometry for the restrictions of d_X and d_Y to the sets $\operatorname{dom}\varphi$ and $\operatorname{rng}\varphi$ respectively.

A partial isometry is **finite** if its domain is. We denote by $P_f(X)$ the set of finite partial isometries between X and X.

Throughout the paper, we will use the following notation : by $A \Subset X$ we mean $A \subseteq X$ and A is finite.

For any bounded or unbounded distance set S, the *S-Urysohn space* $\mathbb{U}_S$ is the unique, up to isometry, countable S-valued metric space which has the **extension property**: given a finite S-metric space (X, d), if (Y, d) is another finite S-valued metric space containing (X, d), any isometric embedding $\rho : X \to \mathbb{U}_S$ extends to an isometric embedding $\tilde{\rho} : Y \to \mathbb{U}_S$.

The *back-and-forth argument* allows one to prove that if (X, d_X) and (Y, d_Y) are two countable S-metric spaces satisfying the extension property, then any finite partial isometry $\varphi : \operatorname{dom} \varphi \Subset X \to \operatorname{rng} \varphi \Subset Y$ extends to an isometry $X \to Y$. Applying this to the empty map, one gets that the S-Urysohn space is indeed unique up to isometry.

Before tackling the case $S = \mathbb{Q} \cap [0, +\infty[$, we will be focusing on bounded metric spaces, since our constructions work best in that case. Thus we make the following

Convention. Until section 7, we assume that $S \subseteq [0, 1]$ and $0, 1 \in S$. We thus have a bounded distance set S satisfying

$$\forall s, t \in S \quad \min(s + t, 1) \in S. \tag{2.1}$$

Remark 2.3. We make the convention that our bounded S-metric spaces have diameter 1 for the sake of simplicity. This does not affect the strength of our results because we are interested in isometry groups, which remain the same when multiplying the metric by a constant.

Remark 2.4. The $\{0, 1\}$-Urysohn space is the set $\mathbb{N}$ equipped with the discrete metric. The $\{0, \frac{1}{2}, 1\}$-Urysohn space is the *Random graph* $\mathcal{R}$ equipped with the metric discussed in Example 2.2. The $\mathbb{Q} \cap [0, 1]$-Urysohn space is called the *rational Urysohn sphere.*

2.1. **Finitely supported extensions and amalgamation.** We now recall the construction of amalgamations of metric spaces and recast the notion of finitely supported extension in this context. The material in this section is standard.

Let (Y, d) be an S-metric space, suppose X is a subset of Y and let $y \in Y$. A subset $F \subseteq X$ is called a **support** for y over X if for all $x \in X$, we have

$$d(x, y) = \min \left(1, \min_{f \in F} \left(d(x, f) + d(f, y) \right) \right).$$

Observe that every subset containing F is then also a support of y.

We say that a point $y \in Y$ is **finitely supported** over X if it has a finite support. Note that every element x of X is finitely supported with support $\{x\}$.

Lemma 2.5. *Suppose (Y, d) is an S-metric space, $X \subseteq Y$ and y_1, $y_2 \in Y$ are finitely supported with respective support F_1 and F_2. Suppose moreover that $d(f_1, f_2) = 1$ for all $f_1 \in F_1$ and all $f_2 \in F_2$. Then $d(y_1, y_2) = 1$.*

Proof. By definition we have

$$\begin{aligned} d(y_1, y_2) &\geq \min_{f_1 \in F_1} \left(d(y_1, f_1) + d(f_1, y_2) \right) \\ &\geq \min_{f_1 \in F_1} \min_{f_2 \in F_2} \left(d(y_1, f_1) + d(f_1, f_2) + d(f_2, y_2) \right) \end{aligned}$$

and since $d(f_1, f_2) = 1$ for all $f_1 \in F_1$ and $f_2 \in F_2$ we conclude $d(y_1, y_2) \geq 1$. Since $S \subseteq [0, 1]$ we have $d(y_1, y_2) \leq 1$ so $d(y_1, y_2) = 1$. $\square$

We say that a metric space (Y, d) is a **finitely supported extension** of a subset X if all its points are finitely supported over X.

Definition 2.6. Suppose (X_1, d_1) and (X_2, d_2) are two S-metric spaces, that $A = X_1 \cap X_2$ and $d_{1\restriction A} = d_{2\restriction A}$. Then we define the **metric amalgam** of (X_1, d_1) and (X_2, d_2) over A as the set $X_1 \cup X_2$ equipped with the metric d which restricts to d_1 on X_1 and d_2 and X_2 and such that for all $x_1 \in X_1$ and all $x_2 \in X_2$ we have

$$d(x_1, x_2) = \min(1, \inf_{a\in A}(d_1(x_1, a) + d_2(a, x_2))).$$

We denote the metric amalgam by $(X_1, d_1) *_A (X_2, d_2)$. Note that we allow amalgamation over the empty set, in which case $d(x_1, x_2) = 1$ for all $x_1 \in X_1$ and all $x_2 \in X_2$. The fact that we have a well-behaved way of amalgamating S-metric spaces over the emptyset is a notable difference between the bounded case and the unbounded case.

Remark 2.7. If $X \subseteq (Y, d)$, a point $y \in Y$ is finitely supported over X with support $F \Subset X$ if and only if $X \cup \{y\}$ is the metric amalgam of $F \cup \{y\}$ and X over F.

It is not clear a priori when the metric amalgam of two S-metric spaces is an S-metric space. However, it is the case when X_1 and X_2 are finitely supported over A.

Proposition 2.8. *Let (X_1, d_1) and (X_2, d_2) be two S-metric spaces which are finitely supported extensions of a common metric subspace $A = X_1 \cap X_2$. Then the metric amalgam $(X_1, d_1) *_A (X_2, d_2)$ is an S-metric space.*

Moreover if F_1 is a support of $x_1 \in X_1$ and F_2 is a support of $x_2 \in X_2$, then $\inf_{a\in A} d_1(x_1, a) + d_2(a, x_2)$ is a minimum which is attained both on F_1 and F_2.

Proof. By equation (2.1), the first part of the proposition follows from the second. By symmetry, we only need to check that if $F_1 \Subset A$ is a support of x_1, then $\inf_{a\in A} d(x_1, a) + d_2(a, x_2)$ is attained on F_1.

By definition for each $a \in A$ we have $d(x_1, a) = \min_{f\in F_1} d(x_1, f) + d(f, a)$. So for each $a \in A$ there is $f \in F_1$ such that

$$d_1(x_1, a) + d_2(a, x_2) = d_1(x_1, f) + d_1(f, a) + d_2(a, x_2)$$

But d_1 and d_2 coincide on A so $d_1(f, a) + d_2(a, x_2) = d_2(f, a) + d_2(a, x_2) \geq d_2(f, x_2)$. We conclude $d_1(x_1, a) + d_2(a, x_2) \geq d_1(x_1, f) + d_2(f, x_2)$, so our infimum is indeed a minimum attained on F_1. □

The construction of metric amalgams makes sense with an arbitrary number of factors: when $(X_i, d_i)_{i\in I}$ is a family of S-metric spaces which are finitely supported over A and for all $i \neq j$ we have $A = X_i \cap X_j$ and $d_{i\restriction A} = d_{j\restriction A}$ then we can form the metric amalgam of $(X_i, d_i)_{i\in I}$ over A as the set $\bigcup_{i\in I} X_i$ equipped with the metric d which coincides with d_i when restricted to X_i, and such that for all $i \neq j$ and all $x_i \in X_i$ and $x_j \in X_j$ we have

$$d(x_i, x_j) = \min(1, \inf_{a\in A}(d_i(x_i, a) + d_j(a, x_j))).$$

Observe that when restricted to $X_i \cup X_j$, the metric is the one of the metric amalgam of X_i and X_j over A. The above proposition has then the following immediate corollary.

Corollary 2.9. *Let $(X_i, d_i)_{i\in I}$ be a family of S-metric spaces such that for all $i \neq j$ we have $A = X_i \cap X_j$ and $d_{i\restriction A} = d_{j\restriction A}$. Suppose that each X_i is finitely supported over A. Then the metric amalgam of $(X_i, d_i)_{i\in I}$ over A is an S-metric space.* □

2.2. **S-Urysohn spaces and one-point extensions.** Let us now see how to build $\mathbb{U}_S$, i.e. how to build a countable S-metric space with the extension property using a slight variation of Katětov's ideas [Kat86].

Given an S-metric space (X,d) and another S-metric space (Y,d) containing X, we say that Y is a **one-point extension** of X if $|Y\setminus X|=1$. A straightforward induction yields that $\mathbb{U}_S$ is characterized by the following version of the extension property: for every one-point extension (Y,d) of every finite S-metric space (X,d), any isometric embedding $\rho\colon X\to\mathbb{U}_S$ extends to an isometric embedding $\tilde{\rho}\colon Y\to\mathbb{U}_S$.

One-point extensions completely determined by the distance to the added point. Such distance functions can be characterized as the functions $f\colon X\to S$ satisfying

$$|f(x_1)-f(x_2)|\leq d(x_1,x_2)\leq f(x_1)+f(x_2)\quad\text{for all } x_1,x_2\in X. \tag{2.2}$$

The functions $f:X\to S$ satisfying the above condition are thus also called one-point extensions of X, and we will often switch between the two point of views.

Definition 2.10. Whenever (X,d) is an S-metric space, we denote by $E_S(X)$ the space of all *finitely supported* one-point extensions of X. By definition these are the functions $f\colon X\to S$ satisfying (2.2) such that there is $F\Subset X$ satisfying for all $x\in X$,

$$f(x)=\min\left(1,\min_{y\in F}(f(y)+d(y,x))\right).$$

Such a subset F is called a *support* for f. Supports are not unique: any finite set containing a support of f is a support of f.

Note that if $F\Subset X$, any one-point extension f of (F,d) extends to a finitely supported one-point extension $\tilde{f}$ of X defined by

$$\tilde{f}(x)=\min\left(1,\min_{y\in F}\left(f(y)+d(y,x)\right)\right).$$

This extension is called the **Katětov extension** of f, and may be viewed as the metric amalgam of $F\cup\{f\}$ and X over F.

A key idea of the *Katětov construction* of the S-Urysohn space is that X embeds isometrically into $E_S(X)$ as the space of trivial one-point extensions, i.e. extensions of the form $\hat{x}=d(x,\cdot)$. Moreover we have $\|f-\hat{x}\|_\infty=f(x)$, so $E_S(X)$ contains as a metric subspace every finitely supported one-point extension of X, and hence every one-point extension of every finite subset of X. Also note that $E_S(X)$ is countable if X was countable.

However the metric induced by $\|\cdot\|_\infty$ on $E_S(X)$ is not the right one for us because we want the one point extensions to be as far from each other as possible. The reason for this will become apparent when we construct for every countable group an action on the S-Urysohn space which is as free as possible.

So for $f,g\in E_S(X)$ we define a new metric still denoted by d, by setting

$$d(f,g):=\begin{cases}\min\left(1,\inf\limits_{x\in X}\left(f(x)+g(x)\right)\right) & \text{if}\quad f\neq g,\\ 0 & \text{if}\quad f=g.\end{cases}$$

Observe that d is the metric obtained by amalgamating all the finitely supported one-point extensions of (X,d) over X, which as a set is still $E_S(X)$. In particular, d is indeed a metric and takes values in S by Corollary 2.9. By construction (X,d) is a metric subspace of $(E_S(X),d)$.

One way of constructing the S-Urysohn space is now to start with an arbitrary countable metric space (X, d), let $X_0 = X$ and then define by induction $X_{n+1} = E_S(X_n)$. The metric space $\bigcup_{n\in\mathbb{N}} X_n$ then satisfies the one-point extension property by construction, and hence it is isometric to the S-Urysohn space $\mathbb{U}_S$. Moreover the construction is equivariant in the sense that if Γ acts on X by isometries, then the action naturally extends to $E_S(X)$ so that in the end we get a Γ-action by isometries on $\mathbb{U}_S$. In order to make the action as free as possible, we will however need to modify further this construction. Let us start by making clear what we mean by as free as possible.

2.3. **Freeness notions for actions on metric spaces.** Let Γ be a countable group. If (X, d) is a metric space, we write $\Gamma \curvearrowright (X, d)$ if Γ acts on X by isometries. A Γ-action by isometries on X is thus a group homomorphism $\Gamma \to \mathrm{Iso}(X, d)$.

We will need the following three notions for an action $\Gamma \curvearrowright (X, d)$ where X is an S-metric space.

Definition 2.11. An action $\Gamma \curvearrowright (X, d)$ is **mixing** if for every $x, y \in X$ and for all but finitely many $\gamma \in \Gamma$ we have $d(\gamma x, y) = 1$.

Observe that if $\Gamma \curvearrowright (X, d)$ is mixing, then for any finite subset $F \subseteq X$, for all but finitely many $\gamma \in \Gamma$ we have $d(\gamma x, y) = 1$ for all $x, y \in F$. Also note that the action of any finite group is mixing. Finally, if $\Gamma \curvearrowright (X, d)$ is mixing then the restriction of the action to every subgroup $\Lambda \leq \Gamma$ is mixing.

Definition 2.12. An action $\Gamma \curvearrowright (X, d)$ is **strongly free** if, for all $\gamma \in \Gamma \setminus \{1\}$ and all $x \in X$, one has $d(\gamma x, x) = 1$.

Note that any strongly free action is obviously free.

Definition 2.13. Let $\Sigma < \Gamma$ be a subgroup. We say that Σ is **highy core-free** for the action $\Gamma \curvearrowright (X, d)$ if for all $F \Subset X$, there exists $g \in \Gamma$ such that $d(gx, u) = 1 = d(\sigma gx, gy)$ for all $x, y \in F$, $u \in \Sigma F$ and $\sigma \in \Sigma \setminus \{1\}$.

Remark 2.14. The fact that $d(\sigma gx, gy) = 1$ for all $x, y \in F, \sigma \in \Sigma \setminus \{1\}$ implies that the Σ-action on ΣgF is conjugate to the Σ-action on $\Sigma \times F$ given by $\sigma(\sigma', x) = (\sigma\sigma', x)$ where we put on $\Sigma \times F$ the metric $\tilde{d}$ given by

$$\tilde{d}((\sigma, x), (\sigma', y)) = \begin{cases} d(x, y) & \text{if } \sigma = \sigma' \\ 1 & \text{else.} \end{cases}$$

Note that if Σ is highly core-free for an action $\Gamma \curvearrowright (X, d)$ then any subgroup of Σ is highly core-free for $\Gamma \curvearrowright (X, d)$.

Our terminology extends that from [FMS15a]: it is easy to check that, if d is the discrete metric then Σ is highly core-free for $\Gamma \curvearrowright (X, d)$ in the sense above if and only if Σ is strongly highly core-free for the action $\Gamma \curvearrowright X$ on the *set* X in the sense of [FMS15a, Definition 1.7]. If moreover the action $\Gamma \curvearrowright X$ is free, then the previous conditions are equivalent to Σ being highly-core-free in Γ (see [FMS15a, Definition 1.1 and Lemma 1.6]). As mentioned in the introduction, easyxamples of highly core-free subgroups are the trivial subgroup of an infinite group and any finite subgroup of an i.c.c. group. A less immediate example is the fact that in the free group over 2 generators $\mathbb{F}_2 = \langle a, b\rangle$, the cyclic subgroup generated by the commutator $[a, b]$ is highly core-free. This fact will allow us to show that surface groups arise as dense subgroups in the isometry group of any bounded S-Urysohn space, following the same strategy as in [FMS15a, Ex. 5.1]. We refer to [FMS15a, Ex. 1.11] for proofs and for more examples of highly core-free subgroups.

Let us now see how these notions behave with respect to $E_S(X)$ (see Definition 2.10). Note that we have an injective group homomorphism $\mathrm{Iso}(X,d) \to \mathrm{Iso}(E_S(X),d)$ defined by $\varphi \mapsto (f \mapsto \varphi(f))$, where $\varphi(f)(x) = f(\varphi^{-1}(x))$. In particular, from any action $\Gamma \curvearrowright X$ we get an action $\Gamma \curvearrowright E_S(X)$ defined by $(\gamma f)(x) = f(\gamma^{-1}x)$ for $\gamma \in \Gamma$ and $f \in E_S(X)$.

Proposition 2.15. *Let $\Gamma \curvearrowright (X,d)$ be an action by isometries. The following holds.*

(1) *If $\Gamma \curvearrowright (X,d)$ is faithful then $\Gamma \curvearrowright E_S(X)$ is faithful.*
(2) *If $\Gamma \curvearrowright (X,d)$ is mixing then $\Gamma \curvearrowright E_S(X)$ is mixing.*
(3) *If $\Sigma < \Gamma$ is highly core-free for $\Gamma \curvearrowright (X,d)$ then it is also highly core-free for $\Gamma \curvearrowright E_S(X)$.*

Proof. (1) is obvious since we have a Γ-equivariant inclusion $X \subseteq E_S(X)$.

(2). Let $f,g \in E_S(X)$, let F be a common finite support for f and g. Since $\Gamma \curvearrowright X$ is mixing, for all but finitely many $\gamma \in \Gamma$ we have $d(\gamma x, y) = 1$ for all $x,y \in F$. Note γF is a support for γf. It now follows from Lemma 2.5 that $d(\gamma f, g) = 1$ for all but finitely many $\gamma \in \Gamma$.

(3). Let $F \subseteq E_S(X)$ be a finite subset. For each $f \in F$ let $Y_f \Subset X$ be a support for f and define $Y = \cup_{f\in F} Y_f \subseteq X$. If $\gamma \in \Gamma$ such that $d(\gamma x, u) = 1$ and $d(\sigma\gamma x, \gamma y) = 1$ for all $x,y \in Y$, $u \in \Sigma Y$ and $\sigma \in \Sigma \setminus \{1\}$ then it follows from Lemma 2.5 that $d(\gamma f, h) = 1$ and $d(\sigma\gamma f, \sigma\gamma g) = 1$ for all $f,g \in F$, $h \in \Sigma F$ and $\sigma \in \Sigma \setminus \{1\}$. □

Observe that we did not mention strong freeness, which indeed does not carry over to $E_S(X)$. For instance, if we let $f \in E_S(X)$ be defined by $f(x) = 1$ for all $x \in X$, then f is actually fixed by the Γ-action. One can tweak the above construction of $\mathbb{U}_S$ to get rid of this obstruction, as we will see in section 3.

2.4. **Homogeneous actions on countable metric spaces.** When all the finite partial isometries of a metric space (X,d) extend to isometries, we say that X is **homogeneous**. By a back-and-forth argument, $\mathbb{U}_S$ is a homogeneous metric space.

The isometry group of the S-Urysohn space $\mathbb{U}_S$ is endowed with the topology of pointwise convergence for the discrete topology on $\mathbb{U}_S$. It then becomes a Polish group, and our main motivation is to understand which countable groups arise as dense subgroups of $\mathrm{Iso}(\mathbb{U}_S)$. Recall from the introduction that this is the same as asking which countable groups admit a *faithful* action by isometries on $\mathbb{U}_S$ such that every finite partial isometry of $\mathbb{U}_S$ extends to an element of Γ. Such actions are called **homogeneous actions**, and we denote by $\mathcal{H}(\mathbb{U}_S)$ the class of countable groups which admit a faithful homogeneous action on $\mathbb{U}_S$.

One can use the methods of K. Tent and M. Ziegler to show that the isometry group of any countable S-Uryshohn space is *topologically simple* (every non-trivial normal subgroup is dense). This yields via the following proposition some restrictions on the class $\mathcal{H}(\mathbb{U}_S)$.

Proposition 2.16. *Let G be a nonabelian infinite topologically simple topological group, and suppose Γ is a dense subgroup. Then every nontrivial normal subgroup of Γ is dense in G and every finite index subgroup of Γ is dense in G. Moreover, Γ is i.c.c. and non-solvable.*

Proof. The first statement follows from the fact that for any normal subgroup $N \leq \Gamma$ its closure $\overline{N}$ is, by density of Γ, normal in G. So either $N = \{1\}$ or N is dense in G. Moreover, any finite index subgroup $\Sigma \leq \Gamma$ has a finite index subgroup $N \leq \Sigma$ which is normal in Γ and non-trivial since it has to be infinite. It follows that any finite index subgroup of Γ is again dense in G.

To prove the second statement, first observe that the center $Z(\Gamma)$ of a dense subgroup $\Gamma \leq G$ must be trivial. Indeed, $Z(\Gamma)$ is a normal subgroup of Γ which is abelian, but G is not, so the center of any dense subgroup of G must be trivial.

Now suppose that $\gamma \in \Gamma$ has a finite conjugacy class and denote by $C(\gamma) := \{\gamma' \in \Gamma \colon \gamma\gamma' = \gamma'\gamma\}$ its centraliser. By definition $C(\gamma)$ has finite index in Γ so by the first statement it is dense and thus has trivial center, from which we deduce that γ is trivial since γ belongs to the center of $C(\gamma)$. We conclude Γ is i.c.c.

Finally, for a group Λ, let us denote by $D(\Lambda)$ the subgroup of Λ generated by all commutators. Recall that $D(\Lambda)$ is normal in Λ. Now define $\Gamma_0 := \Gamma$ and, for $n \geq 0$, $\Gamma_{n+1} := D(\Gamma_n)$. We show by induction that for all $n \geq 0$ the subgroup Γ_n is dense in G, which implies that Γ is not solvable. The case $n = 0$ is true by assumption. Suppose Γ_n is dense in G, then Γ_{n+1} is either trivial or dense in G since it is normal in Γ_n. However, Γ_{n+1} is not trivial since Γ_n, being dense in G, can not be abelian. □

Corollary 2.17. *If $\Gamma \in \mathcal{H}(\mathbb{U}_S)$ for some distance set S then Γ is i.c.c. and non solvable. Furthermore, every finite index subgroup of Γ belongs to $\mathcal{H}(\mathbb{U}_S)$.* □

When $S = \{0, 1\}$, recall that $\mathcal{H}(\mathbb{U}_S)$ is the class of highly transitive groups. Hull and Osin have furthermore shown the following dichotomy: a highly transitive group either contain a copy of the infinite alternating group, or it is *mixed identity free*, which has stronger implications than those of the previous corollary (see [HO16, sec. 5]).

Remark 2.18. We do not know whether in Proposition 2.16, for say a Polish group G, one can remove the hypothesis that G is nonabelian. Indeed, there may exist a topologically simple infinite Polish abelian group, in other words there may exist an infinite Polish group with no nontrivial proper closed subgroup (see the end of section 2 in [**?**], where it is also proved that there is an infinite Polish abelian group all whose locally compact subgroups are trivial).

3. Strongly free actions on the S-Urysohn space

3.1. Equivariant extension of partial isometries.

Definition 3.1. A Γ-action by isometries on an S-metric space X has the **extension property** if every finitely supported one-point extension over a finite union of Γ-orbits is realized in X.

Let us make a few remarks about this definition. First, note that any action with the extension property has infinitely many orbits at distance 1 from each other because the one point extension $f(x) = 1$ is finitely supported.

Also, if Γ is a finite group acting on a metric space which has the extension property, then the action itself has the extension property.

Finally, if $\Gamma \curvearrowright X$ has the extension property, then for every $\Lambda \leq \Gamma$, the Λ-action on X also has the extension property. Indeed if f is a finitely supported one-point extension over ΛF where $F \Subset X$, then the Katětov extension of f to ΓF is also finitely supported (with same support), and hence realized in X.

Definition 3.2. Let Σ be a group and for $k = 1, 2$, let $\pi_k \colon \Sigma \curvearrowright (X_i, d_i)$. A (π_1, π_2)-partial isometry is an isometry between $\pi_1(\Sigma)F_1$ and $\pi_2(\Sigma)F_2$ where $F_1 \Subset X_1$ and $F_2 \Subset X_2$ which is (π_1, π_2)-equivariant.

Lemma 3.3. *Suppose $\Gamma \curvearrowright (X, d)$ is a mixing action and F is a finite subset of X. Then for every $x \in X$, the function $d(x, \cdot)$ is a finitely supported one-point extension of ΓF.*

Proof. Since the action is mixing, there are only finitely many points $y \in \Gamma F$ such that $d(x, y) < 1$. These points form a support for $d(x, \cdot)$. □

The next proposition is the key building block of most of our constructions.

Proposition 3.4. *For $k = 1, 2$, let $\pi_k \colon \Sigma \curvearrowright \mathbb{U}_S$ be actions of a countable group Σ which are strongly free, mixing and have the extension property.*

Then, every (π_1, π_2)-partial isometry φ extends to a (π_1, π_2)-isometry of $\mathbb{U}_S$.

Proof. A straightforward back-and-forth argument yields that we only need to show that for every (π_1, π_2)-partial isometry φ and every $x \in \mathbb{U}_S \setminus \operatorname{dom} \varphi$, there is a (π_1, π_2)-equivariant partial isometry $\tilde{\varphi}$ whose domain contains x and which extends φ. Let f be the one-point extension of $\operatorname{rng} \varphi$ defined by $f(y) = d(\varphi^{-1}(y), x)$ for all $y \in \operatorname{rng} \varphi$.

Because π_1 is mixing, Lemma 3.3 yields f is finitely supported. By the extension property, there is $z \in \mathbb{U}_S$ such that for all $y \in \operatorname{rng} \varphi$ we have $d(\varphi^{-1}(y), x) = d(y, z)$.

Observe that the extension of φ obtained by sending x to z is by construction a partial isometry. By freeness, we may extend φ further to a (π_1, π_2)-equivariant partial isometry $\tilde{\varphi} \colon \operatorname{dom} \varphi \sqcup \pi_1(\Sigma)x \to \operatorname{rng} \varphi \sqcup \pi_2(\Sigma)z$ by letting $\tilde{\varphi}(\pi_1(\sigma)x) = \pi_2(\sigma)z$.

Let us check that $\tilde{\varphi}$ is indeed a partial isometry. First, by strong freeness its restriction to $\pi_1(\Sigma)x$ is isometric. Since φ was also isometric, we only need to check that if $x_1 \in \operatorname{dom} \varphi$ and $x_2 \in \pi_1(\Sigma)x$ then $d(\varphi(x_1), \tilde{\varphi}(x_2)) = d(x_1, x_2)$.

Let $\sigma \in \Sigma$ be such that $x_2 = \pi_1(\sigma)x$. Then

$$\begin{aligned} d(x_1, x_2) &= d(\pi_1(\sigma^{-1})x_1, x) \\ &= d(\varphi(\pi_1(\sigma^{-1})x_1), z) \\ &= d(\pi_2(\sigma^{-1})\varphi(x_1), z) \\ &= d(\varphi(x_1), \pi_2(\sigma)z) \\ &= d(\varphi(x_1), \tilde{\varphi}(x_2)). \end{aligned}$$

So $\tilde{\varphi}$ is indeed a (π_1, π_2)-partial isometry. □

3.2. **Extensions with parameters.** Let (Y, d_Y) be a S-metric space, suppose $X \subseteq Y$ and A is a set. Then we can consider the space $Y \times A$ equipped with the pseudometric

$$d((y, a), (y', a')) = \begin{cases} \min\left(1, \inf_{x \in X} \left(d_Y(y, x) + d_Y(x, y')\right)\right) & \text{if } a' \neq a, \\ d_Y(y, y') & \text{if } a' = a. \end{cases}$$

Then the metric space obtained by identifying elements at distance 0 is denoted by $Y \times_X A$, and the induced metric still by d. Observe that for $x, x' \in X$ and $a, a' \in A$,

$$d((x, a), (x', a')) = \inf_{x'' \in X} \left(d_Y(x, x'') + d_Y(x'', x')\right) = d_Y(x, x')$$

so $Y \times_X A$ is the metric amalgam of $|A|$ copies of Y over X and thus $Y \times_X A$ is a S-metric space as soon as Y is a finitely supported extension of X.

We have the following inequality: for all (y, a) and (y', a') in $Y \times_X A$,

$$d((y,a),(y',a')) \geq d_Y(y,y') \tag{3.1}$$

because by the triangle inequality for all x we have $d_Y(y,x) + d_Y(x,y') \geq d_Y(y,y')$.

We also have for each $a \in A$ an isometry $i_a \colon Y \to Y \times_X A$ which takes y to (y, a). In particular Y embeds in $Y \times_X A$ isometrically.

For all $a \neq a'$ and all $x \in X$ we have $i_a(x) = i_{a'}(x)$ and we thus let $i \colon X \to Y \times_X A$ be the common map.

Remark 3.5. Every isometry g of Y which preserves X naturally extends to $Y \times_X A$ by letting $g(y,a) = (gy, a)$. The construction we need is different, however.

Proposition 3.6. *Suppose that we have a group Γ acting on Y by isometries which preserve the set X, and suppose moreover that Γ acts on A. Then the diagonal action on $Y \times A$ defined by $\gamma(y,a) = (\gamma y, \gamma a)$ induces an isometric action on $(Y \times_X A, d)$ which extends the Γ-action on X.*

Proof. Let us check that every $\gamma \in \Gamma$ defines an isometry for the pseudometric d. Let $a, a' \in A$ and $y, y' \in Y$. If $a = a'$, then $d(\gamma(y,a), \gamma(y',a')) = d_Y(\gamma y, \gamma y') = d((y,a),(y',a'))$. If $a \neq a'$ then $\gamma a \neq \gamma a'$ and thus

$$\begin{aligned} d(\gamma(y,a),\gamma(y',a')) &= \min\left(1, \left(\inf_{x \in X} d_Y(\gamma y, x) + d_Y(x, \gamma y')\right)\right) \\ &= \min\left(1, \left(\inf_{x \in X} d_Y(\gamma y, \gamma x) + d_Y(\gamma x, \gamma y')\right)\right) \end{aligned}$$

because $X = \gamma X$. Now using the fact that γ is an isometry, we conclude

$$\begin{aligned} d(\gamma(y,a),\gamma(y',a')) &= \min\left(1, \left(\inf_{x \in X} d_Y(y, x) + d_Y(x, y')\right)\right) \\ &= d((y,a),(y',a')). \end{aligned}$$

Finally observe that the embedding $i \colon X \to Y \times_X A$ is Γ-equivariant and thus conjugates the Γ-action on X to the Γ-action on $i(X)$. ☐

We call the above action the **diagonal action** of Γ on $Y \times_X A$.

Proposition 3.7. *Let Γ be a group acting by isometries on (Y, d_Y), let $X \subseteq Y$ be Γ-invariant subset and let A be a set also acted upon by Γ. The following hold.*

1. *If the Γ-action on X is faithful, then the Γ-action on $Y \times_X A$ is faithful.*
2. *If the Γ-action on Y is mixing then the Γ-action on $Y \times_X A$ is mixing.*
3. *If $\Sigma \leq \Lambda \leq \Gamma$ and Σ is highly core-free for the Λ-action on Y then Σ is also highly core-free for the Λ-action on $Y \times_X A$.*
4. *If the Γ-action on X is strongly free and the Γ-action on A is free, then the Γ-action on $Y \times_X A$ is strongly free.*

Proof. (1) It is obvious since the isometry $i \colon X \to Y \times_X A$ is Γ-equivariant.

(2) Suppose the Γ-action on Y is mixing. Let $(y_1, a_1), (y_2, a_2) \in Y \times_X A$. By assumption for all but finitely many $\gamma \in \Gamma$ we have $d_Y(\gamma y_1, y_2) = 1$, so by inequality (3.1) for all but finitely $\gamma \in \Gamma$ we have $d(\gamma(y_1, a_1), (y_2, a_2)) = 1$.

(3) Let $\Lambda \leq \Gamma$, let $\Sigma \leq \Lambda$. Suppose Σ is highly core-free for the Λ-action on Y. Let $F \Subset Y \times_X A$ and write $F = F_1 \times \{a_1\} \sqcup \ldots \sqcup F_n \times \{a_n\}$. Let $\tilde{F} = \bigcup_{i=1}^n F_i$, then there is

$\lambda \in \Lambda$ such that $d_Y(\lambda y, u) = 1 = d_Y(\sigma\lambda y, \lambda y')$ for all $y, y' \in F$, $u \in \Sigma F$ and all $\sigma \in \Sigma \setminus \{1\}$. As before by inequation (3.1) we are done.

(4) Suppose the Γ-action on X is strongly free and the Γ-action on A is free. Let $(y, a) \in Y \times_X A$ and let $\gamma \in \Gamma \setminus \{1\}$. Then $\gamma a \neq a$ so

$$\begin{aligned} d((y,a), \gamma(y,a)) &= \inf_{x\in X} (d_Y(y,x) + d_Y(x, \gamma y)) \\ &= \inf_{x\in X} \big(d_Y(y,x) + d_Y(y, \gamma^{-1}x)\big) \\ &\geq \inf_{x\in X} d_Y(x, \gamma^{-1}x). \end{aligned}$$

By strong freeness for all $x \in X$ we have $d_Y(x, \gamma^{-1}x) = 1$. So we have $d((y,a), \gamma(y,a)) = 1$ and the Γ-action on $Y \times_X A$ is thus strongly free. □

Suppose now Γ is a countable group acting on a S-metric space (X, d) by isometries. Then the Γ-action extends to $E_S(X)$, and we define the S-metric space $E_S^\Gamma(X) = E_S(X) \times_X \Gamma$. Then the Γ-action by left translation on itself provides us a Γ-action on $E_S^\Gamma(X)$ which extends the Γ-action on X. We have the following facts.

Proposition 3.8. *The following hold.*

1. *If the Γ-action on X is faithful so is the Γ-action on $E_S^\Gamma(X)$.*
2. *If the Γ-action on X is mixing then so is the Γ-action on $E_S^\Gamma(X)$.*
3. *If $\Sigma \leq \Lambda \leq \Gamma$ and Σ is highly core-free for the Λ-action on X then Σ is also highly core-free for the Λ-action on $E_S^\Gamma(X)$.*
4. *If the Γ-action on X is strongly free then the Γ-action on $E_S^\Gamma(X)$ is strongly free.*

Proof. This is a straightforward application of the previous proposition along with Proposition 2.15. □

3.3. **Induced actions on the S-Urysohn space.** Let (X, d) be an S-metric space on which Γ acts by isometries. Define the sequence of S-metric spaces $X_0^\Gamma = X$ and, for $n \geq 0$, $X_{n+1}^\Gamma = E_S^\Gamma(X_n)$. Consider the inductive limit $X_\infty^\Gamma = \varinjlim X_n^\Gamma$ along with its natural Γ-action.

Proposition 3.9. *With the notations above, the following holds.*

1. $X_\infty^\Gamma \simeq \mathbb{U}_S$.
2. *If the Γ-action on X is mixing then so is the Γ-action on X_∞^Γ.*
3. *If $\Sigma \leq \Lambda \leq \Gamma$ is highly core-free for the Λ-action on X then it is also highly core-free for the Λ-action on X_∞^Γ.*
4. *If the Γ-action on X is strongly free then so is the Γ-action on X_∞^Γ.*
5. *The Γ-action on X_∞^Γ has the extension property.*

Proof. Assertions (2) to (4) follow from Propositions 3.8 and the fact that these conditions hold on X_∞^Γ if and only if they hold on X_n^Γ for each $n \in \mathbb{N}$.

Let us prove (1). It suffices to check that X_∞^Γ has the extension property. Let $F \subset X_\infty^\Gamma$ be a finite subset and $f \in E_S(F)$. Let N be large enough that $F \subset X_N^\Gamma$. Let $\tilde{f}$ be the Katětov extension of f to X_N^Γ, then by definition $\tilde{f}$ is finitely supported so it belongs to $E_S(X_N^\Gamma)$. Moreover for all $x \in X_N^\Gamma$ we have $d(x, \tilde{f}) = \tilde{f}(x)$ where d is the metric on $E_S(X_N^\Gamma)$, in particular $d(x, \tilde{f}) = f(x)$ for all $x \in F$. Since $E_S(X_N^\Gamma)$ embeds isometrically in X_∞^Γ via a map which fixes X_N^Γ, we are done.

Finally, let us prove (5). Let F be a finite subset of X_∞^Γ and let f be a finitely supported one-point extension of ΓF. Pick N large enough that $F \subseteq X_N^\Gamma$. Since X_N^Γ is Γ-invariant we also have $\Gamma F \subseteq X_N^\Gamma$. Let $\tilde{f}$ be the Katětov extension of f to X_N^Γ, observe that $\tilde{f}$ is still finitely supported with the same support as f and thus $\tilde{f} \in E_S(X_N^\Gamma)$. For all $x \in X_N^\Gamma$ we have $d(x, \tilde{f}) = \tilde{f}(x)$ where d is the metric on $E_S(X_N^\Gamma)$, in particular $d(x, \tilde{f}) = f(x)$ for all $x \in \Gamma F$. Since $E_S(X_N^\Gamma)$ embeds isometrically in X_∞^Γ via a Γ-equivariant map which fixes X_N^Γ, we are done. □

Corollary 3.10. *Every finite group admits a unique strongly free action on $\mathbb{U}_S$ up to conjugacy.*

Proof. Let Γ be a finite group, consider the left-action of Γ on itself as an action by isometries where we put on Γ the discrete metric. This action is clearly strongly free. By Proposition 3.9 we can extend this action to a strongly free action of Γ on $(\mathbb{U}_S, d)$. Now observe that any Γ-action on $\mathbb{U}_S$ is automatically mixing and has the extension property. Applying Proposition 3.4 to the empty map, we see that our strongly free action is unique up to conjugacy. □

Corollary 3.11. *Any infinite countable group Γ admits a unique (up to conjugacy) strongly free mixing action with the extension property on $\mathbb{U}_S$. This action has the property that, for any subgroups $\Sigma \leq \Lambda \leq \Gamma$ such that $\Sigma \leq \Lambda$ is a highly core-free subgroup, Σ is highly core-free for $\Lambda \curvearrowright \mathbb{U}_S$.*

Proof. Uniqueness follows from Proposition 3.4 applied to the empty map. To prove existence of such an action, and that it has the property mentioned above, start from Γ acting on $X = \Gamma$ by left translation where we equip X with the discrete metric $d(x, y) = 1$ for $x \neq y$ and $d(x, x) = 0$. It is clear that $\Gamma \curvearrowright X$ is strongly free and mixing. Consider $\Gamma \curvearrowright X_\infty^\Gamma$, then by the previous proposition this action is a strongly free mixing action on the S-Urysohn space with the extension property. Now fix subgroups $\Sigma \leq \Lambda \leq \Gamma$ and suppose that $\Sigma \leq \Lambda$ is a highly core-free subgroup. It is clear that Σ is highly core-free for $\Lambda \curvearrowright X$, since d is the discrete metric and $\Lambda \curvearrowright X$ is free. It follows from item (3) of the above proposition that Σ is highly core-free for $\Lambda \curvearrowright X_\infty^\Gamma$. □

4. Actions of amalgamated free products on bounded Urysohn spaces

Let Γ_1, Γ_2 be two countable groups with a common subgroup Σ and define $\Gamma = \Gamma_1 \underset{\Sigma}{*} \Gamma_2$. Suppose that we have a faithful action $\Gamma \curvearrowright \mathbb{U}_S$ and view Γ as a subgroup of $\mathrm{Iso}(\mathbb{U}_S)$.

Let $Z := \{\alpha \in \mathrm{Iso}(\mathbb{U}_S) \colon \forall \sigma \in \Sigma\ \alpha\sigma = \sigma\alpha\}$. Note that Z is a closed subgroup of $\mathrm{Iso}(\mathbb{U}_S)$, hence a Polish group. Moreover, for all $\alpha \in Z$, there exists a unique group homomorphism $\pi_\alpha \colon \Gamma \to \mathrm{Iso}(\mathbb{U}_S)$ such that

$$\pi_\alpha(g) = \begin{cases} g & \text{if} \quad g \in \Gamma_1, \\ \alpha^{-1} g \alpha & \text{if} \quad g \in \Gamma_2. \end{cases}$$

When Σ is trivial, we have $Z = \mathrm{Iso}(\mathbb{U}_S)$. In this section we prove the following result.

Theorem 4.1. *If $\Gamma \curvearrowright \mathbb{U}_S$ is free with the extension property, $\Sigma \curvearrowright \mathbb{U}_S$ is strongly free, mixing and Σ is highly core-free for $\Gamma_1, \Gamma_2 \curvearrowright \mathbb{U}_S$ then the set:*

$$O = \{\alpha \in Z \colon \pi_\alpha \text{ is homogeneous and faithful}\}$$

is dense G_δ in Z.

Corollary 4.2. *Assume Γ_1, Γ_2 are two countable groups with a commun subgroup Σ, highly core-free in both Γ_1, Γ_2. Then $\Gamma_1 \underset{\Sigma}{*} \Gamma_2$ belongs to $\mathcal{H}(\mathbb{U}_S)$. In particular, the following results hold.*

- *For any infinite countable groups Γ_1, Γ_2 we have $\Gamma_1 * \Gamma_2 \in \mathcal{H}(\mathbb{U}_S)$.*
- *For any i.c.c. groups Γ_1, Γ_2 with a common finite subgroup Σ we have $\Gamma_1 \underset{\Sigma}{*} \Gamma_2 \in \mathcal{H}(\mathbb{U}_S)$.*
- *If Σ_g is a closed, orientable surface of genus $g > 1$, then $\pi_1(\Sigma_g) \in \mathcal{H}(\mathbb{U}_S)$.*

Proof. The first two results are a direct consequence of Theorem 4.1 Corollary 3.11 and the examples of highly core-free subgroups given in [FMS15a, Lem. 1.10]. Moreover, since in the free group over two generators $\mathbb{F}_2 = \langle a, b\rangle$, the cyclic group generated by $[a, b]$ is highly core-free [FMS15a, Ex. 1.11], we also get that $\pi_1(\Sigma_2)$ belongs to $\mathcal{H}(\mathbb{U}_S)$. By covering theory, the latter admits every $\pi_1(\Sigma_g)$ for $g \geq 2$ as a finite index subgroup, so by Corollary 2.17 all such groups belong to $\mathcal{H}(\mathbb{U}_S)$. $\square$

To prove the theorem, we establish two lemmas; the theorem follows by applying the Baire category theorem.

Lemma 4.3. *Assume $\Sigma \curvearrowright \mathbb{U}_S$ is strongly free, mixing and Σ is highly core-free for $\Gamma_1, \Gamma_2 \curvearrowright \mathbb{U}_S$. Then the set $U = \{\alpha \in Z \colon \pi_\alpha$ is homogeneous$\}$ is dense G_δ in Z.*

Proof. For every finite partial isometry φ, consider the following open set

$$U_\varphi := \{\alpha \in Z \colon \exists g \in \Gamma \text{ such that } \pi_\alpha(g)_{\restriction \mathrm{dom}(\varphi)} = \varphi\}.$$

Then $U = \bigcap_{\varphi \in P_f(\mathbb{U}_S)} U_\varphi$ so by the Baire category theorem it suffices to show that U_φ is dense for every finite partial isometry φ. Let $\varphi \in P_f(\mathbb{U}_S)$, $\alpha \in Z$ and $F \Subset \mathbb{U}_S$. We need to prove that there are $\beta \in Z$ and $g \in \Gamma$ such that $\beta_{\restriction F} = \alpha_{\restriction F}$ and $\pi_\beta(g)_{\restriction \mathrm{dom}(\varphi)} = \varphi$.

Since Σ is highly core-free for $\Gamma_1 \curvearrowright \mathbb{U}_S$, there is $g_1 \in \Gamma_1$ such that

$$\forall u \in \Sigma F \; \forall x, x' \in \mathrm{dom}\, \varphi \; \forall \sigma \in \Sigma \setminus \{1\} \quad d(g_1 x, u) = 1 = d(\sigma g_1 x, g_1 x').$$

Define $F' = F \cup g_1 \,\mathrm{dom}(\varphi)$. There exists $h_1 \in \Gamma_1$ such that

$$\forall u \in \Sigma F' \; \forall y, y' \in \mathrm{rng}(\varphi) \; \forall \sigma \in \Sigma \setminus \{1\} \quad d(h_1 y, u) = 1 = d(\sigma h_1 y, h_1 y').$$

Since Σ is highly core-free for $\Gamma_2 \curvearrowright \mathbb{U}_S$, there is $g_2 \in \Gamma_2$ such that

$$\forall u, v \in \alpha F' \; \forall w \in \Sigma F' \; \forall \sigma \in \Sigma \setminus \{1\} \quad d(\sigma g_2 u, g_2 v) = 1 = d(g_2 u, w).$$

Define $A = \Sigma F' \sqcup \Sigma h_1 \,\mathrm{rng}(\varphi)$ and $B = \alpha(\Sigma F') \sqcup \Sigma g_2 \alpha(g_1 \,\mathrm{dom}(\varphi))$.

Note that $\Sigma A = A$ and, since $\alpha \in Z$, $\Sigma B = B$. Since the Σ-action is free, we can define a Σ-equivariant partial isometry $\beta_0 \colon A \to B$ by setting

- $\forall u \in \Sigma F' \quad \beta_0(u) = \alpha(u)$.
- $\forall x \in \mathrm{dom}(\varphi) \; \forall \sigma \in \Sigma \quad \beta_0(\sigma h_1 \varphi(x)) = \sigma g_2 \alpha(g_1 x)$.

Since the Σ action is strongly free and mixing, we may apply Proposition 3.4 to get an extension $\beta \in Z$ of β_0. Note that $\beta_{\restriction F} = \alpha_{\restriction F}$. Letting $g = h_1^{-1} g_2 g_1 \in \Gamma$ we have, for all $x \in \mathrm{dom}(\varphi)$,

$$\pi_\beta(g)x = h_1^{-1}\beta^{-1} g_2 \beta g_1 x = h_1^{-1}\beta^{-1} g_2 \alpha(g_1 x) = h_1^{-1} h_1 \varphi(x) = \varphi(x)$$

as wanted. $\square$

Lemma 4.4. *If $\Gamma \curvearrowright \mathbb{U}_S$ is free with the extension property, $\Sigma \curvearrowright \mathbb{U}_S$ is strongly free and mixing then the set $V = \{\alpha \in Z \colon \pi_\alpha$ is faithful$\}$ is a dense G_δ in Z.*

Proof. We can write V as a countable intersection of open sets $V = \cap_{g\in\Gamma\setminus\{1\}} V_g$, where $V_g = \{\alpha \in Z : \pi_\alpha(g) \neq \mathrm{id}\}$, so by the Baire category theorem it suffices to show that V_g is dense for all $g \in \Gamma \setminus \{1\}$. Since the action $\Gamma \curvearrowright \mathbb{U}_S$ is faithful we have $V_g = Z$ for all $g \in (\Gamma_1 \cup \Gamma_2) \setminus \{1\}$ and it suffices to show that V_g is dense for all g reduced of length ≥ 2.

Let $g = g_{i_n} \dots g_{i_1}$, where $n \geq 2$ and $g_{i_k} \in \Gamma_{i_k} \setminus \Sigma$. Fix $\alpha \in Z$ and $F \Subset \mathbb{U}_S$ and define $F' = F \cup \alpha(F) \Subset \mathbb{U}_S$. Since $\Gamma \curvearrowright \mathbb{U}_S$ has the extension property, there exists $x \in \mathbb{U}_S$ such that $d(x, u) = 1$ for all $u \in \Gamma\widetilde{F}$. Define:

$$\begin{array}{lllllll} A & := & \Sigma F' & \sqcup & (\bigsqcup_{l=1}^n \Sigma g_{i_l} \dots g_{i_1} x) & \sqcup & \Sigma x \quad \text{and} \\ B & := & \alpha(\Sigma F') & \sqcup & (\bigsqcup_{l=1}^n \Sigma g_{i_l} \dots g_{i_1} x) & \sqcup & \Sigma x \end{array}$$

Note that $\Sigma A = A$ and, since $\alpha \in Z$, $\Sigma B = B$. Define the bijection $\gamma_0 : Y \to Y'$ by $\gamma_{0\restriction \Sigma F} = \alpha_{\restriction \Sigma F}$ and $\gamma_{0\restriction A\setminus\Sigma F} = \mathrm{id}$. Note that by definition of x, any element of $A \setminus (\Sigma F)$ (resp. $B \setminus (\alpha(\Sigma F))$ is at distance 1 of any element of ΣF (resp. $\alpha(\Sigma F)$). It follows that γ_0 is an isometry and, since $\alpha \in Z$, we have $\gamma_0\sigma = \sigma\gamma_0$ for all $\sigma \in \Sigma$.

Our assumptions on the action of Σ enable us to apply Proposition 3.4, and find an extension $\gamma \in Z$ of γ_0. Then $\gamma_{\restriction F} = \alpha_{\restriction F}$ and $\pi_\gamma(g_{i_n} \dots g_{i_1})x = g_{i_n} \dots g_{i_1} x \neq x$, since $g = g_{i_n} \dots g_{i_1} \neq 1$ and the Γ-action is free. Hence $\gamma \in V_g$. □

When one of the factors is finite, say Γ_2, then, as explained in [FMS15a], Γ_2 has no highly core-free subgroups. However, we have a similar result in that case.

Theorem 4.5. *Suppose that Γ_2 is finite and $[\Gamma_2 : \Sigma] \geq 2$. If $\Gamma \curvearrowright \mathbb{U}_S$ is strongly free with the extension property and Σ is highly core-free for $\Gamma_1 \curvearrowright \mathbb{U}_S$ then the set*

$$O := \{\alpha \in Z : \pi_\alpha \text{ is homogeneous and faithful}\}$$

is a dense G_δ in Z.

As before, the following corollary follows from Theorem 4.5, Corollary 3.11, as well as the examples of highly core-free subgroups given in [FMS15a].

Corollary 4.6. *For every infinite countable group Γ_1, every finite group Γ_2 with common subgroup $\Sigma < \Gamma_1, \Gamma_2$ such that Σ is highly core-free in Γ_1 and $[\Gamma_2 : \Sigma] \geq 2$, we have $\Gamma_1 \underset{\Sigma}{*} \Gamma_2 \in \mathcal{H}_{\mathbb{U}_S}$. In particular, the following facts hold.*

- *If Γ_1 is countably infinite and Γ_2 is a finite non-trivial then $\Gamma_1 * \Gamma_2 \in \mathcal{H}_{\mathbb{U}_S}$.*
- *If Γ_1 is i.c.c., Γ_2 is finite and $[\Gamma_2 : \Sigma] \geq 2$ then $\Gamma_1 \underset{\Sigma}{*} \Gamma_2 \in \mathcal{H}_{\mathbb{U}_S}$.*

In order to prove Theorem 4.5, we prove the analogue of Lemma 4.3. The theorem then follows as before from the combination of Lemmas 4.7 and 4.4.

Lemma 4.7. *Assume $\Gamma \curvearrowright \mathbb{U}_S$ is strongly free with the extension property, Σ is highly core-free for $\Gamma_1 \curvearrowright \mathbb{U}_S$, Γ_2 is finite and $[\Gamma_2 : \Sigma] \geq 2$. Then the set $U := \{\alpha \in Z : \pi_\alpha$ is homogeneous$\}$ is dense G_δ in Z.*

Proof. We again have to prove that, given $\varphi \in P_f(\mathbb{U}_S)$, $\alpha \in Z$ and $F \Subset \mathbb{U}_S$, we can find $\gamma \in Z$ and $g \in \Gamma$ such that $\gamma_{\restriction F} = \alpha_{\restriction F}$ and $\pi_\gamma(g)_{\restriction \mathrm{dom}(\varphi)} = \varphi$.

Since Σ is highly core-free for $\Gamma_1 \curvearrowright \mathbb{U}_S$ and Γ_2 is finite, we may find $g_1, h_1 \in \Gamma_1$ such that

- $\forall u \in \Gamma_2(F \cup \alpha(F) \cup \mathrm{dom}(\varphi)) \ \forall x \in \mathrm{dom}(\varphi) \quad d(g_1 x, u) = 1.$
- $\forall \sigma \in \Sigma \ \setminus \{1\} \ \forall x, y \in \mathrm{dom}(\varphi) \quad d(\sigma g_1 x, y) = 1.$
- $\forall u \in \Gamma_2 F \ \sqcup \Gamma_2 g_1 \mathrm{dom}(\varphi) \ \forall y \in \mathrm{rng}(\phi) \quad d(h_1 y, u) = 1$
- $\forall \sigma \in \Sigma \setminus \{1\} \ \forall x, y \in \mathrm{dom}(\varphi) \quad d(\sigma h_1 \varphi(x), h_1 \varphi(y)) = 1$

Using the fact that Γ_2 is finite, acts strongly freely and $\mathbb{U}_S$ has the extension property, it is straighforward to build inductively a finite set A and an isometry $\psi\colon \operatorname{dom}(\varphi) \to A$ such that

- $\forall h \in \Gamma_2 \setminus \{1\}\ \forall a, a' \in A\ d(ha, a') = 1$
- $\forall a \in A\ \forall u \in \Gamma_2\alpha(F) \quad d(u, a) = 1$

Take $g_2 \in \Gamma_2 \setminus \Sigma$. Then the sets $\alpha(\Sigma F)$, ΣA, and $\Sigma g_2 A$ are pairwise disjoint. Define

$$B = \Sigma F \sqcup \Sigma g_1 \operatorname{dom}(\varphi) \sqcup \Sigma h_1 \operatorname{rng}(\varphi) \text{ and } C = \alpha(\Sigma F) \sqcup \Sigma A \sqcup \Sigma g_2 A.$$

Note that $\Sigma B = B$, $\Sigma C = C$ and the bijection $\gamma_0\colon B \to C$ defined by $\gamma_0(u) = \alpha(u)$ for $u \in \Sigma F$, $\gamma_0(\sigma g_1 x) = \sigma\psi(x)$ and $\gamma_0(\sigma h_1\varphi(x)) = \sigma g_2\psi(x)$, for all $\sigma \in \Sigma$ and for all $x \in \operatorname{dom}(\varphi)$ is, by construction, an isometry such that $\gamma_0\sigma = \sigma\gamma_0$ for all $\sigma \in \Sigma$. By Proposition 3.4 there exists an extension $\gamma \in Z$ of γ_0. Note that $\gamma_{\restriction F} = \alpha_{\restriction F}$. Moreover, with $g = h_1^{-1} g_2 g_1 \in \Gamma$ and $x \in \operatorname{dom}(\varphi)$ we have

$$\pi_\gamma(g)(x) = h_1^{-1}\gamma^{-1}g_2\gamma g_1 x = h_1^{-1}\gamma^{-1}g_2\psi(x) = h_1^{-1}\gamma^{-1}\gamma h_1\varphi(x) = \varphi(x).$$

□

5. Actions of HNN extensions on bounded Urysohn spaces

Let $\Sigma < H$ be a finite subgroup of a countable group H and $\theta\colon \Sigma \to H$ be an injective group homomorphism. Define $\Gamma = \operatorname{HNN}(H, \Sigma, \theta)$ the HNN-extension and let $t \in \Gamma$ be the "stable letter" i.e. Γ is the universal group generated by H and t with the relations $t\sigma t^{-1} = \theta(\sigma)$ for all $\sigma \in \Sigma$. For $\epsilon \in \{-1, 1\}$, we write

$$\Sigma_\epsilon := \begin{cases} \Sigma & \text{if } \epsilon = 1, \\ \theta(\Sigma) & \text{if } \epsilon = -1. \end{cases}$$

Suppose that we have a faithful action $\Gamma \curvearrowright \mathbb{U}_S$ and view $\Gamma < \operatorname{Iso}(\mathbb{U}_S)$. Define the closed (hence Polish space) subset $Z = \{\alpha \in \operatorname{Iso}(\mathbb{U}_S)\colon \theta(\sigma) = \alpha\sigma\alpha^{-1} \text{ for all } \sigma \in \Sigma\} \subseteq \operatorname{Iso}(\mathbb{U}_S)$ and note that it is non-empty (since $t \in Z$). By the universal property of Γ, for each $\alpha \in Z$ there exists a unique group homomorphism $\pi_\alpha\colon \Gamma \to \operatorname{Iso}(\mathbb{U}_S)$ such that

$$\pi_{\alpha\restriction H} = \operatorname{id}_H \text{ and } \pi_\alpha(t) = \alpha.$$

In this section we prove the following result.

Theorem 5.1. *Assume that $\Gamma \curvearrowright \mathbb{U}_S$ is free with the extension property, $\Sigma_\epsilon \curvearrowright \mathbb{U}_S$ is strongly free, mixing and Σ_ϵ is highly core-free for $H \curvearrowright \mathbb{U}_S$ for all $\epsilon \in \{-1, 1\}$ then the set:*

$$O = \{\alpha \in \operatorname{Iso}(\mathbb{U}_S)\colon \pi_\alpha \text{ is faithful and homogeneous}\}$$

is a dense G_δ in Z.

Corollary 5.2. *For any subgroup Σ of a countable group H such that $\Sigma_\epsilon < H$ is highly core-free for all $\epsilon \in \{-1, 1\}$, we have HNN$(H, \Sigma, \theta) \in \mathcal{H}_{\mathbb{U}_S}$.*

As is by now customary, we separate the proof in two lemmas.

Lemma 5.3. *Assume that for $\epsilon \in \{-1, 1\}$ the action $\Sigma_\epsilon \curvearrowright \mathbb{U}_S$ is strongly free, mixing and Σ_ϵ is highly core-free for $H \curvearrowright \mathbb{U}_S$. Then the set $U = \{\alpha \in Z\colon \pi_\alpha \text{ is homogeneous}\}$ is a dense G_δ in Z.*

Proof. We apply the Baire category theorem as before: given $\varphi \in P_f(\mathbb{U}_S)$, $\alpha \in Z$ and $F \Subset \mathbb{U}_S$, we show that there exists $\gamma \in Z$ and $g \in \Gamma$ such that $\gamma\restriction F = \alpha_{\restriction F}$ and $\pi_\gamma(g)_{\restriction \operatorname{dom}(\varphi)} = \varphi$.

Since Σ and $\theta(\Sigma)$ are highly core-free for $H \curvearrowright \mathbb{U}_S$, we produce $h_1, h_2 \in H$ such that

- $\forall u \in \Sigma F\ \forall x, x' \in \mathrm{dom}(\varphi)\ \forall \sigma \in \Sigma \setminus \{1\} \quad d(h_1x, u) = 1 = d(\sigma h_1 x, h_1 x')$.
- $\forall u \in \alpha(\Sigma F \sqcup \Sigma h_1 \mathrm{dom}(\varphi))\ \forall y \in \mathrm{rng}(\varphi)\ \forall \sigma \in \Sigma \setminus \{1\}\ d(h_2 y, u) = 1$.
- $\forall y, y' \in \mathrm{rng}(\varphi)\ \forall \sigma \in \Sigma \setminus \{1\}\ d(\theta(\sigma)h_2y, h_2y') = 1$.

Define $Y = \Sigma h_1 \mathrm{dom}(\varphi) \sqcup \Sigma\alpha^{-1}(h_2 \mathrm{rng}(\varphi)) \sqcup \Sigma F$ and observe that

$$\alpha(Y) = \theta(\Sigma)\alpha(h_1 \mathrm{dom}(\varphi)) \sqcup \theta(\Sigma)(h_2 \mathrm{rng}(\varphi)) \sqcup (\theta(\Sigma)\alpha(F))$$

Note that $\Sigma Y = Y$ and $\theta(\Sigma)\alpha(Y) = \alpha(Y)$. Define the bijection $\gamma_0 \colon Y \to \alpha(Y)$ by setting $\gamma_{0\restriction \Sigma F} = \alpha_{\restriction \Sigma F}$ and

$$\forall x \in \mathrm{dom}(\varphi)\ \forall \sigma \in \Sigma \quad \gamma_0(\sigma h_1 x) = \theta(\sigma)h_2\varphi(x) \text{ and } \gamma_0(\sigma\alpha^{-1}(h_2\varphi(x))) = \theta(\sigma)\alpha(h_1x).$$

By construction γ_0 is an isometry such that $\gamma_0\sigma = \theta(\sigma)\gamma_0$ for all $\sigma \in \Sigma$. By Proposition 3.4 there exists an extension $\gamma \in Z$ of γ_0. Note that $\gamma_{\restriction F} = \alpha_{\restriction F}$ moreover, with $g = h_2^{-1}th_1 \in \Gamma$ we have, for all $x \in \mathrm{dom}(\varphi)$, $\pi_\gamma(g)x = h_2^{-1}\gamma h_1 x = h_2^{-1}\gamma_0(h_1x) = h_2^{-1}h_2\varphi(x) = \varphi(x)$. □

Lemma 5.4. *Assume $\Gamma \curvearrowright \mathbb{U}_S$ is free, has the extension property and $\Sigma_\epsilon \curvearrowright \mathbb{U}_S$ is strongly free and mixing for all $\epsilon \in \{-1; 1\}$. Then the set $V = \{\alpha \in Z \colon \pi_\alpha \text{ is faithful}\}$ is a dense G_δ in Z.*

Proof. Once again we prove that $V_g = \{\alpha \in Z \colon \pi_\alpha(g) \neq \mathrm{id}\}$ is dense for all $g \in \Gamma \setminus \{1\}$, and we only need to prove it for $g \notin H$. Write $g = h_n t^{\epsilon_n} \dots t^{\epsilon_1} h_0$ a reduced expression for g, where $n \geq 1$, $h_k \in H$ and $\epsilon_k \in \{-1, 1\}$. Fix $\alpha \in Z$ and $F \Subset \mathbb{U}_S$. For $l \in \{1, \dots, n\}$ we defined subsets $H_l \subset \Gamma$:

$$H_1 = \begin{cases} \Sigma h_0 & \text{if} \quad \epsilon_1 = 1 \\ \Sigma t^{-1} h_0 & \text{if} \quad \epsilon_1 = -1 \end{cases} \quad \text{and, for } l \geq 2,\ H_l = \begin{cases} \Sigma h_{l-1} t^{\epsilon_{l-1}} \dots t^{\epsilon_1} h_0 & \text{if} \quad \epsilon_l = 1 \\ \Sigma t^{-1} h_{l-1} t^{\epsilon_{l-1}} \dots t^{\epsilon_1} h_0 & \text{if} \quad \epsilon_l = -1 \end{cases}$$

Observe that $\Sigma H_l = H_l$ for all l. Let $F' := F \cup \alpha(F) \Subset \mathbb{U}_S$ and, for $l \in \{1, \dots, n\}$, $H'_l := tH_l \subset \Gamma$. Using the extension property, we find $x \in \mathbb{U}_S$ such that $d(x, u) = 1$ for all $u \in \Gamma F'$ and define

$$Y := \Sigma F \sqcup (\sqcup_{l=1}^n H_l x) \quad \text{and} \quad Y' := \theta(\Sigma)\alpha(F) \sqcup (\sqcup_{l=1}^n H'_l x).$$

We may then define an isometry $\gamma_0 \colon Y \to Y'$ by setting $\gamma_{0\restriction \Sigma F} = \alpha_{\restriction \Sigma F}$ and, for all $l \in \{1, \dots, n\}$, $\gamma_0 \restriction Y_l = t_{\restriction Y_l}$.

We have by construction $\Sigma Y = Y$, $\theta(\Sigma)Y' = Y'$ and $\gamma_0\sigma = \theta(\sigma)\gamma_0$ for all $\sigma \in \Sigma$. Thus there exists, by Proposition 3.4, an extension $\gamma \in Z$. Then γ satisfies $\gamma_{\restriction F} = \alpha_{\restriction F}$ and $\pi_\gamma(g)x = h_n\gamma^{\epsilon_n} \dots \gamma^{\epsilon_1} h_0 x = h_n t^{\epsilon_n} \dots t^{\epsilon_1} h_0 x = gx \neq x$ since $g \neq 1$ and the Γ-action is free. It follows that $\gamma \in V_g$. □

6. Actions of groups acting on trees on bounded Urysohn spaces

We record here that, just as in [FMS15a], the previous results apply to groups acting on trees. The reasoning is exactly the same as in [FMS15a], but since the argument is short we kept it for the reader's convenience.

Let Γ be a group acting without inversion on a non-trivial tree. By [Ser77], the quotient graph $\mathcal{G}$ can be equipped with the structure of a graph of groups $(\mathcal{G}, \{\Gamma_p\}_{p \in \mathrm{V}(\mathcal{G})}, \{\Sigma_e\}_{e \in \mathrm{E}(\mathcal{G})})$ where each $\Sigma_e = \Sigma_{\overline{e}}$ is isomorphic to an edge stabilizer, each Γ_p is isomorphic to a vertex stabilizer and such that Γ is isomorphic to the fundamental group $\pi_1(\Gamma, \mathcal{G})$ of this graph of groups i.e., given a fixed maximal subtree $\mathcal{T} \subseteq \mathcal{G}$, the group Γ is generated by the groups Γ_p for $p \in \mathrm{V}(\mathcal{G})$ and the edges $e \in \mathrm{E}(\mathcal{G})$ with the relations

$$\overline{e} = e^{-1}, \quad s_e(x) = er_e(x)e^{-1} \quad \forall x \in \Sigma_e \quad \text{and} \quad e = 1 \forall e \in \mathrm{E}(\mathcal{T}),$$

where $s_e\colon \Sigma_e \to \Gamma_{s(e)}$ and $r_e = s_{\overline{e}}\colon \Sigma_e \to \Gamma_{r(e)}$ are respectively the source and range group monomorphisms.

Theorem 6.1. *Assume Γ_p is countably infinite for all $p \in \mathrm{V}(\mathcal{G})$, and $s_e(\Sigma_e)$ is highly core-free in $\Gamma_{s(e)}$ for all $e \in \mathrm{E}(\mathcal{G})$. Then $\Gamma \in \mathcal{H}_{\mathbb{U}_S}$.*

Proof. Let e_0 be one edge of $\mathcal{G}$ and $\mathcal{G}'$ be the graph obtained from $\mathcal{G}$ by removing the edges e_0 and $\overline{e_0}$.

Case 1: $\mathcal{G}'$ is connected. It follows from Bass-Serre theory that $\Gamma = \mathrm{HNN}(H, \Sigma, \theta)$ where H is the fundamental group of our graph of groups restricted to $\mathcal{G}'$, $\Sigma = r_{e_0}(\Sigma_{e_0}) < H$ is a subgroup and $\theta\colon \Sigma \to H$ is given by $\theta = s_{e_0} \circ r_{e_0}^{-1}$. By hypothesis H is countably infinite and, since $\Sigma < \Gamma_{r(e_0)}$ (resp. $\theta(\Sigma) < \Gamma_{s(e_0)}$) is a highly core-free subgroup, $\Sigma < H$ (resp. $\theta(\Sigma) < H$) is also a highly core-free subgroup. Thus we may apply Theorem 5.1 to conclude that $\Gamma \in \mathcal{H}_{\mathbb{U}_S}$.

Case 2: $\mathcal{G}'$ is not connected. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the two connected components of $\mathcal{G}'$ such that $s(e_0) \in \mathrm{V}(\mathcal{G}_1)$ and $r(e_0) \in \mathrm{V}(\mathcal{G}_2)$. Bass-Serre theory implies that $\Gamma = \Gamma_1 *_{\Sigma_{e_0}} \Gamma_2$, where Γ_i is the fundamental group of our graph of groups restricted to $\mathcal{G}_i$, $i = 1, 2$, and Σ_{e_0} is viewed as a highly core-free subgroup of Γ_1 via the map s_{e_0} and as a highly core-free subgroup of Γ_2 via the map r_{e_0} since $s_{e_0}(\Sigma_{e_0})$ is highly core-free in $\Gamma_{s(e_0)}$ and $r_{e_0}(\Sigma_{e_0})$ is highly core-free in $\Gamma_{r(e_0)}$ by hypothesis. Since Γ_1 and Γ_2 are countably infinite and Σ_{e_0} is highly core-free, we may apply Theorem 4.1 to conclude that $\Gamma \in \mathcal{H}_{\mathbb{U}_S}$. $\square$

7. The unbounded case

In this section we explain how to extend some of the above results to the case of the Urysohn space $\mathbb{U}_{\mathbb{Q}}$. While the methods would apply to some other unbounded Urysohn spaces, (for instance they work also for $\mathbb{U}_{\mathbb{N}}$), we chose to focus on the case of the rational Urysohn space in order to improve the clarity of the exposition.

7.1. **Strong disconnection.**

Definition 7.1. An isometric action of Γ on an unbounded metric space (X, d) **strongly disconnects finite sets** if

$$\forall F \Subset X,\ \exists N \in \mathbb{N} \text{ such that } \forall K \geq N,\ \exists \gamma \in \Gamma \text{ satisfying } d(x, \gamma y) = K \text{ for every } x, y \in F.$$

This definition might be a bit hard to grasp at first, as it involves many quantifiers. It is meant to capture the idea that one can find $\gamma \in \Gamma$ so that γF and F are "independent enough", and in the unbounded case a good notion of independence is that all elements of F and γF are at the same distance, and arbitrarily far away.

As in the bounded case, we need to produce sufficiently free actions of a group G on the Urysohn space; and these actions are built by starting from the left action of G on itself and then applying a Katětov-type tower construction. A new feature here is that we first need to produce a suitable metric on G before starting the tower construction.

Lemma 7.2. *Every infinite countable group admits an unbounded left-invariant metric which surjects onto $\mathbb{N}$.*

Proof. If Γ is finitely generated, let S be a finite generating set, then the Cayley metric associated to S is as wanted.

If Γ is not finitely generated write $\Gamma = \bigcup_n \Gamma_n$ where $\Gamma_0 = \{1\}$ and each Γ_n is properly contained in Γ_{n+1}, then consider the left-invariant metric $d(g, h) = \min\{n\colon g^{-1}h \in \Gamma_n\}$.

□

Definition 7.3. Say that two metrics d_1 and d_2 **coincide at scale** K if we have

$$\forall l \leq K \quad d_1(x,y) = l \Leftrightarrow d_2(x,y) = l.$$

Lemma 7.4. *Suppose a countable group Γ acts on a metric space (X,d) with an unbounded orbit, and suppose that d takes values in $\mathbb{N}$. Then for every $N \in \mathbb{N}$ there is a surjective function $f\colon \mathbb{N} \to \mathbb{N}$ so that if we let $\tilde{d} = f \circ d$, then $\tilde{d}$ is a metric which coincides with d at scale N and the Γ-action on $(X, \tilde{d})$ strongly disconnects finite sets.*

Proof. Given a non-decreasing function $\varphi\colon \mathbb{N}\setminus\{0\} \to \mathbb{N}\setminus\{0\}$, we can associate to it a function $f_\varphi\colon \mathbb{N} \to \mathbb{N}$ by defining

$$f_\varphi(m) = \min\{k\colon m \leq \sum_{i=1}^{k} \varphi(i)\}.$$

It can be checked that f_φ is the unique non-decreasing function $\mathbb{N} \to \mathbb{N}$ such that $f(0) = 0$ and for every $n \in \mathbb{N}$ we have $\left|f_\varphi^{-1}(\{n\})\right| = \varphi(n)$. Using the fact that φ is non-decreasing, it is straightforward to check that f_φ is subadditive. We then see that $f_\varphi \circ d$ is still a metric as soon as d is a metric with values in $\mathbb{N}$. Also observe that if $\varphi(n) = 1$ for all $n \leq N$, then $f_\varphi(n) = n$ for all $n \leq N$. We then set $d_\varphi := f_\varphi \circ d$.

We use this construction twice; begin by picking x_0 with an unbounded orbit. We first set $\varphi(i) = 1$ for all $i \leq N$. Then, we inductively define $\varphi(N + j)$ and find elements $\gamma_k \in \Gamma$ in such a way that (defining the empty sum as being equal to 0)

$$\forall k \geq 0 \quad N + \sum_{j=1}^{k} \varphi(N+j) < d(x_0, \gamma_k x_0) < N + \sum_{j=1}^{k+1} \varphi(N+j).$$

Then d and d_φ coincide at scale N, and $f_\varphi(d(x_0, \gamma_k x_0)) = N + k + 1$ for all $k \geq 0$. In particular, for every $n > N$ there exists some $\gamma \in \Gamma$ such that $d_\varphi(x_0, \gamma x_0) = n$. So we may as well assume that d already satisfies this surjectivity condition.

We then let $\varphi(n) = 1$ for $n \leq N$ and $\varphi(n) = 2n$ for $n \geq N+1$. Let us show why d_φ is now as wanted.

Pick $F \Subset \Gamma$, without loss of generality we assume $x_0 \in F$. Let $N' > N$ be such that $N' > 2\mathrm{diam}_d(F)$. By our surjectivity assumption on d there is $\gamma \in \Gamma$ such that

$$d(x_0, \gamma x_0) = N' + \sum_{i=1}^{N'-1} \varphi(i).$$

Then for every $x, y \in F$ we have

$$\begin{aligned}
|d(x,\gamma y) - d(x_0, \gamma x_0)| &\leq |d(x,\gamma y) - d(x, \gamma x_0)| + |d(x, \gamma x_0) - d(x_0, \gamma x_0)| \\
&\leq d(\gamma y, \gamma x_0) + d(x, x_0) \\
&\leq 2\mathrm{diam}(F) \\
&< N'.
\end{aligned}$$

So since $d(x_0, \gamma x_0) = N' + \sum_{i=1}^{N'-1} \varphi(i)$ we conclude that

$$\sum_{i=1}^{N'-1} \varphi(i) < d(x, \gamma y) < \sum_{i=1}^{N'-1} \varphi(i) + 2N' = \sum_{i=1}^{N'} \varphi(i),$$

hence $d_\varphi(x, \gamma y) = N'$ as wanted. $\square$

Note that by rescaling, the above lemma applies for any metric taking values in $\alpha\mathbb{N}$ for some $\alpha > 0$. We extend our definition of strong freeness by saying that an action of a countable group on a metric space (X, d) is **strongly free** if $d(x, \gamma x) \geq 1$ for all $x \in X$.

Proposition 7.5. *Every countable group admits an isometric action on $\mathbb{U}_\mathbb{Q}$ which strongly disconnects finite sets and is strongly free.*

In the proof we make the natural unbounded modification of extensions with parameters (Section 3.2): if we are given a metric space (Y, d), a *nonempty* subset $X \subseteq Y$ and a set A, we let $Y \times_X A$ be the quotient metric space obtained from the pseudo-metric on $Y \times A$ given by

$$d((y,a),(y',a')) = \begin{cases} \inf_{x\in X} d_Y(y,x) + d_Y(x,y') & \text{if } a' \neq a, \\ d_Y(y,y') & \text{if } a' = a. \end{cases}$$

Proof of Theorem 7.5. Start with Γ acting on itself by isometries for an unbounded left-invariant metric provided by lemma 7.2, write it as X_0 and note that this action is strongly free. Then consider the set $E_{1/2}(X_0)$ of finitely supported Katětov extensions of X_0 taking values in $\frac{1}{2}\mathbb{N}$ and equip it with the amalgam distance $d(f, g) = \inf_x f(x) + g(x)$. Then let $X_1 = E_{1/2}(X_0) \times_{X_0} \Gamma$. The diagonal action of Γ on (X_1, d) is still strongly free and has an unbounded orbit since X_0 embeds isometrically, but we may lose strong disconnection.

So we let d_1 be a metric provided by the previous lemma with $N = 2$. At stage n, (X_n, d_n) being constructed, we consider $X_{n+1} = E_{1/n!}(X_n) \times_{X_n} \Gamma$ with a metric d_{n+1} which is the same as d_n at scale 2^{n+1} but for which the diagonal Γ-action strongly disconnects finite sets, which is possible via the previous lemma.

For $x, y \in \bigcup_n X_n =: X_\infty$, since $2^{n+1} \to \infty$ and d_{n+1} coincides with d_n at scale 2^{n+1}, the sequence $d_n(x, y)$ is stationary and we let $d(x, y)$ be the limit. It is easy to check that d is still a metric.

Furthermore, X_∞ has the extension property for $\mathbb{Q}$-valued metrics. Indeed if we have a $\mathbb{Q}$-valued Katětov function on some finite $F \subseteq X_\infty$, we take n large enough so that $2^n \geq \max(\mathrm{diam}(F), \|f\|_\infty)$, f takes values in $\frac{1}{n!}\mathbb{N}$ and F is contained in X_n. We then have that f is realised in the metric space $E_{1/n!}(X_n)$. Since the latter embeds in X_{n+1} in a way which preserves the metric at scale 2^{n+1}, f is actually realised in X_{n+1} and thus in X_∞ since the inclusion $(X_{n+1}, d_{n+1}) \subseteq (X_\infty, d)$ preserves the metric at scale 2^{n+1}.

Let us finally show that the Γ-action on X_∞ strongly disconnects finite sets. Take $F \subseteq X_\infty$ finite, let $n \in \mathbb{N}$ such that $F \subseteq X_n$. Since the Γ-action on (X_n, d_n) strongly disconnects finite sets, we find $N \in \mathbb{N}$ such that for all $k \geq N$, there is $\gamma \in \Gamma$ such that $d_n(x, \gamma y) = k$ for all $x, y \in F$. We may as well assume $N \geq 2^{n+1}$.

Now let $k \geq N$. Observe that the definition of the metrics d_k imply that for every $l \geq N$, there is $\gamma \in \Gamma$ such that $d_l(x, \gamma y) = k$ for all $x, y \in F$ (indeed their restrictions to X_n are obtained by composing d_n with finitely surjective maps $\mathbb{N} \to \mathbb{N}$).

Let l such that $2^{l+1} > k$, there is $\gamma \in \Gamma$ such that $d_l(x, \gamma y) = k$ for all $x, y \in F$. Since d_l and the restriction of d to X_l coincide at scale 2^{l+1}, this implies that $d(x, \gamma y) = k$ for all $x, y \in F$ as wanted. $\square$

7.2. **Dense free products.**

Theorem 7.6. *Let Γ and Λ be two countable infinite groups acting strongly freely on $\mathbb{U}_\mathbb{Q}$. Suppose both actions strongly disconnect finite sets. Then the set of $\alpha \in \mathrm{Iso}(\mathbb{U}_\mathbb{Q})$ so that π_α is faithful and homogeneous is dense G_δ.*

As usual we decompose the proof in two parts.

Lemma 7.7. *Let Γ and Λ be two countable infinite groups acting on $\mathbb{U}_\mathbb{Q}$, suppose their actions strongly disconnect finite sets. Then the set of $\alpha \in \mathrm{Iso}(\mathbb{U}_\mathbb{Q})$ so that π_α is homogeneous is dense G_δ.*

Proof. Let φ be a finite partial isometry, $\beta \in \mathrm{Iso}(\mathbb{U}_\mathbb{Q})$, F a finite subset of $\mathbb{U}_\mathbb{Q}$. We must find α coinciding with β on F such that for some $g \in \Gamma * \Lambda$ we have $\pi_\alpha(g)_{\restriction \mathrm{dom}\,\varphi} = \varphi$.

We first find $\gamma_1 \in \Gamma$ such that every element of $F \cup \beta F$ is at fixed distance K from every element of $\gamma_1 \,\mathrm{dom}\,\varphi$. Let $F' = F \cup \beta F \cup \gamma_1 \,\mathrm{dom}\,\varphi$.

We then find $\gamma_2 \in \Gamma$ and $\lambda \in \Lambda$ such that every element of $\gamma_2^{-1} \,\mathrm{rng}\,\varphi \cup \lambda\gamma_1 \,\mathrm{dom}\,\varphi$ is at distance K' from every element of F' (this is where we use that strong disconnection works for *any* large enough K')

Finally, we define a new finite partial isometry α by letting $\alpha(x) = \beta(x)$ for all $x \in \mathrm{dom}\,\varphi$, then $\alpha(x) = x$ for all $x \in \gamma_1 \,\mathrm{dom}\,\varphi$, and finally $\alpha(\gamma_2^{-1}\varphi(x)) = \lambda\gamma_1 x$ for all $x \in \mathrm{dom}\,\varphi$ (see Fig. 7.2)

$$\begin{tikzcd}
\mathrm{F} \arrow[r, leftrightarrow, dotted, "K"] \arrow[rr, leftrightarrow, dotted, bend left, "K'"] \arrow[d, "\beta"] & \gamma_1 \mathrm{dom}\,\varphi \arrow[r, leftrightarrow, dotted, "K'"] \arrow[d, "\mathrm{id}"] & \gamma_2^{-1} \mathrm{rng}\,\varphi \arrow[d, "\lambda\gamma_1\varphi^{-1}\gamma_2"] \\
\beta(F) \arrow[r, leftrightarrow, dotted, "K"'] \arrow[rr, leftrightarrow, dotted, bend right, "K'"'] & \gamma_1 \mathrm{dom}\,\varphi \arrow[r, leftrightarrow, dotted, "K'"'] & \lambda\gamma_1 \mathrm{dom}\,\varphi
\end{tikzcd}$$

FIGURE 1. α is partial isometry.

We now extend α to an isometry of $\mathbb{U}_\mathbb{Q}$ which we still denote by α. By construction α extends the restriction of β to F. Moreover if we let $g = \gamma_2\lambda\gamma_1$, we have for all $x \in \mathrm{dom}\,\varphi$

$$\pi_\alpha(g)x = \gamma_2\alpha^{-1}\lambda\alpha\gamma_1 x = \gamma_2\alpha^{-1}\lambda\gamma_1 x = \gamma_2\gamma_2^{-1}\varphi(x) = \varphi(x),$$

as wanted. $\square$

Lemma 7.8. *Let Γ, Λ be two groups acting freely on $\mathbb{U}_\mathbb{Q}$, and w be an element of $\Gamma * \Lambda$ which does not belong to Γ. Then, for any $\beta \in \mathrm{Iso}(\mathbb{U}_\mathbb{Q})$ the set $W_f = \{\alpha \colon \pi_\alpha(w) = \beta\}$ does not contain 1 in its interior.*

Proof. We use the fact that for any finite subset $A \Subset \mathbb{U}_\mathbb{Q}$, any $\alpha \in \mathrm{Iso}(A)$ such that $\alpha^2 = 1$, and any $x \notin A$, there exist infinitely many $y \in \mathbb{U}_\mathbb{Q}$ such that one can extend α to an isometric involution of $A \cup \{x, y\}$ by setting $\alpha(x) = y, \alpha(y) = x$.

Fix an open neighborhood U of 1, which we may assume is of the form

$$U = \{\alpha \colon \forall x \in F \ \alpha(x) = x\}$$

for some finite subset F of $\mathbb{U}_{\mathbb{Q}}$. Let $w = \lambda_1\gamma_2\ldots\gamma_n\lambda_n$ be the reduced form of w (one can reduce to the case where w has such a reduced form by multiplying β on the left and/or the right by an element of Γ). Below it will be useful to write $\gamma_1 = 1$.

For $\alpha \in \mathrm{Iso}(\mathbb{U}_{\mathbb{Q}})$, we write

$$w(\alpha) = (\alpha\lambda_1\alpha)\gamma_2(\alpha\lambda_2\alpha)\gamma_3\ldots\gamma_n(\alpha\lambda_n\alpha)$$

Our aim is to find an involution $\alpha \in U$ such that $w(\alpha) \neq \beta$, since this will imply that U is not contained in the interior of W_f. We begin by picking $x \in \mathbb{U}_Q \setminus F$, and find $y \in \mathbb{U}_{\mathbb{Q}} \setminus (\lambda_n^{-1}(F) \cup \{x, \lambda_n^{-1}x, \beta(x), \lambda_n^{-1}\beta(x)\})$ such that setting

$$\forall z \in F \ \alpha(z) = z \ , \quad \alpha(x) = y \ , \quad \alpha(y) = x$$

defines an isometric involution α of $F \cup \{x, y\}$. Now we proceed inductively to extend α to a partial isometric involution such that $\alpha x, \lambda_n\alpha x, \alpha\lambda_n\alpha x, \gamma_n\alpha\lambda_n\alpha x, \ldots, \alpha\lambda_1\alpha\gamma_2\ldots\gamma_n\alpha\lambda_n\alpha x$ are all defined, pairwise distinct, and do not belong to $F \cup \{x, \beta(x)\}$. Then any extension of α to an isometric involution of $\mathbb{U}_{\mathbb{Q}}$ is an element of U such that $w(\alpha) \neq 1$. During the inductive process, we have to deal with two different cases:

(1) We want to define $\alpha(\lambda_k \ldots \lambda_n\alpha x)$ for some $k \geq 1$ (note that our inductive conditions ensure that $\lambda_k \ldots \lambda_n\alpha x$ is not in the domain of α). Set

$$B = \{x, \alpha x, \lambda_n\alpha x, \alpha\lambda_n\alpha x, \gamma_n\alpha\lambda_n\alpha x, \ldots, \lambda_k \ldots \lambda_n\alpha x\} \cup \{\beta(x)\}$$

Then, we can find an element z which is not in $B \cup \gamma_k^{-1}(B) \cup \gamma_k^{-1}(F)$ such that setting

$$\alpha(\lambda_k \ldots \lambda_n\alpha x) = z \ , \quad \alpha(z) = \lambda_k \ldots \lambda_n\alpha x$$

is our desired extension of α (note that in our inductive conditions we require also $\gamma_k z \neq z$ for $k \geq 2$, but this is automatic since nontrivial elements of Γ do not have any fixed points).

(2) We want to define $\alpha(\gamma_k\alpha\lambda_k \ldots \lambda_n\alpha x)$ for some $k \geq 2$: apply the same reasoning as above, replacing γ_k with λ_{k-1} and B with

$$\{x, \alpha x, \lambda_n\alpha x, \alpha\lambda_n\alpha x, \gamma_n\alpha\lambda_n\alpha x, \ldots, \gamma_k\alpha\lambda_k \ldots \lambda_n\alpha x\} \cup \{\beta(x)\} \ .$$

□

Lemma 7.9. *Let Γ and Λ be two countable groups acting freely on $\mathbb{U}_{\mathbb{Q}}$. Then the set of $\alpha \in \mathrm{Iso}(\mathbb{U}_{\mathbb{Q}})$ so that π_α is faithful is dense G_δ.*

Proof. Using the Baire category theorem, it is enough to show that for any $w \in \Gamma * \Lambda \setminus \{1\}$, the set $\{\alpha \colon \pi_\alpha(w) = 1\}$ has empty interior. This is true by assumption if w belongs to either Γ or Λ; if this result is false for some w, then we have some $\alpha \in \mathrm{Iso}(\mathbb{U}_{\mathbb{Q}})$ and a neighborhood V of 1 such that $\pi_{\alpha\beta}(w) = 1$ for any $\beta \in V$.

We can write $w = \gamma_1\lambda_1 \ldots \gamma_n\lambda_n$, with at least one λ_i different from 1. Since

$$\begin{aligned}\pi_{\alpha\beta}(w) &= \gamma_1\lambda_1^{\alpha\beta} \ldots \gamma_n\lambda_n^{\alpha\beta} \\ &= (\gamma_1\alpha)\lambda_1^{\beta}\gamma_2^{\alpha^{-1}}\lambda_2^{\beta} \ldots \gamma_n^{\alpha^{-1}}\lambda_n^{\beta}\alpha^{-1}\end{aligned}$$

we see that $\lambda_1^{\beta}\gamma_2^{\alpha^{-1}}\lambda_2^{\beta} \ldots \gamma_n^{\alpha^{-1}}\lambda_n^{\beta} = \alpha^{-1}\gamma_1^{-1}\alpha$ for all $\beta \in V$ and this contradicts our previous lemma (since the conjugate of the action of Γ by α^{-1} is a free action of Γ). □

Corollary 7.10. *Every free product of infinite groups admits a faithful homogeneous action on the rational Urysohn space.*

8. An example

Having spent quite some time building dense subgroups of isometry groups of countable Urysohn spaces, we now are naturally led to the question: how does the class of dense subgroups of $\mathrm{Iso}(\mathbb{U}_S)$ depend on the set S? We know embarrassingly little about this problem, but we do know that the case where $\mathrm{Iso}(\mathbb{U}_S) = \mathfrak{S}_\infty$ is different from the others.

Recall our convention: we fix $M \in]0, +\infty]$, and S will is a nonempty, countable subset of $[0, +\infty[$ containing 0 and such that

$$\forall x, y \in S \quad \min(x + y, M) \in S \ .$$

Also recall that $\mathfrak{S}_{(\infty)}$ is the (countable) group of all permutations of $\mathbb{N}$ with finite support. By studying its primitive actions, we will prove the following result.

Theorem 8.1. *Let S be a bounded or unbounded distance set. The group $\mathfrak{S}_{(\infty)}$ embeds densely in $\mathrm{Iso}(\mathbb{U}_S)$ if and only if $|S| = 2$.*

Of course when $|S| = 2$, $\mathrm{Iso}(\mathbb{U}_S)$ is isomorphic to $\mathfrak{S}_\infty$ so only one implication above is interesting.

The above theorem shows that the classes of countable groups which are isomorphic to a dense subgroup of $\mathfrak{S}_\infty$ and, say, the automorphism group $\mathrm{Aut}(R)$ of the random graph, are not the same; much remains to be investigated. For instance, we do not know of a countable dense subgroup of $\mathrm{Aut}(R)$ which is not isomorphic to a dense subgroup of $\mathfrak{S}_\infty$, though it seems likely that such groups exist.

Our approach is fairly elementary: assume that $\mathfrak{S}_{(\infty)}$ acts faithfully and homogeneously on $\mathbb{U}_S$. Pick $x \in \mathbb{U}_S$, and consider the associated stabilizer subgroup

$$\Delta = \{\gamma \in \mathfrak{S}_{(\infty)} \colon \gamma x = x\}.$$

Since $\mathrm{Iso}(\mathbb{U}_S)$ acts transitively on $\mathbb{U}_S$, we can identify $\mathfrak{S}_{(\infty)}/\Delta$ with $\mathbb{U}_S$, and the action of $\mathfrak{S}_{(\infty)}$ on $\mathbb{U}_S$ is simply the left-translation action of $\mathfrak{S}_{(\infty)}$ on $\mathfrak{S}_{(\infty)}/\Delta$. Moreover, since the action is faithful the subgroup Δ is core-free and, since the action is homogeneous, the closure of $\mathfrak{S}_{(\infty)}$ in the symmetric group of $\mathfrak{S}_{(\infty)}/\Delta$ is isomorphic to $\mathrm{Iso}(\mathbb{U}_S)$.

Conversely, if there is a core-free subgroup Δ such that the closure of $\mathfrak{S}_{(\infty)}$ in the symmetric group of $\mathfrak{S}_{(\infty)}/\Delta$ is isomorphic to $\mathrm{Iso}(\mathbb{U}_S)$, then obviously $\mathfrak{S}_{(\infty)}$ is isomorphic to a dense subgroup of $\mathbb{U}_S$. Thus, we focus our attention towards understanding what kind of subgroup Δ can arise as point stabilizers.

8.1. Classification of point stabilizers for homogeneous action $\mathfrak{S}_{(\infty)} \curvearrowright \mathbb{U}_S$.

We recall some facts and definitions from permutation group theory.

Definition 8.2. Let X be a countable set, and G be a group acting transitively on X. A *block* for this action is a nonempty set $A \subseteq X$ such that for all $g \in G$ one has either $gA = A$ or $gA \cap A = \emptyset$. A block is *trivial* if it is either a singleton or the whole set X.

The action of G on X is *primitive* if all blocks for this action are trivial (equivalently, there is no nontrivial G-invariant equivalence relation on X).

This property is particularly relevant to us because of the following two facts (the first of which is a standard fact in permutation group theory, while the second is probably well-known. We include the short proofs for the reader's convenience).

Lemma 8.3. *Assume that G is a group that acts transitively on X. Then the action is primitive if and only if the stabilizer of some (any) $x \in X$ is a maximal proper subgroup.*

Proof. Given $x \in X$, denote by G_x its stabilizer. We can identify the action of G on X with the regular action of G on G/G_x. If there exists a subgroup K of G such that $H \subsetneq K \subsetneq G$, then the K-cosets form a family of nontrivial blocks for this action. Conversely, assume that $A \subseteq G/H$ is a block for this action, containing H (seen as an element of G/H). By definition of a block, we see that

$$K := \{g \in G\colon gH \in A\} = \{g \in G\colon gA = A\}$$

so K is a subgroup of G, and if A is nontrivial then $H \subsetneq K \subsetneq G$. □

Lemma 8.4. *The action of* $\mathrm{Iso}(\mathbb{U}_S)$ *on* $\mathbb{U}_S$ *is transitive and primitive.*

Proof. Transitivity of the action is obvious. As for primitivity, assume that A is a block for this action, with two elements x, y such that $d(x, y) = r > 0$. For any z such that $d(y, z) = r$, there exists an isometry α of $\mathbb{U}_S$ such that $\alpha(y) = y$ and $\alpha(x) = z$. Since A is a block, $\alpha(y) = y$ implies $\alpha(A) = A$ so z belongs to A. Now, note that for any $s \in S \cap [0, 2r]$ there exists $z \in \mathbb{U}_S$ with $d(x, z) = s$ and $d(y, z) = r$, so A has an element z such that $d(x, z) = s$. Switching the roles of x and y, we conclude that A contains all z such that $d(x, z) \in [0, 2r]$. From this we deduce that $A = \mathbb{U}_S$. □

It is straightforward to check that transitivity and primitivity are inherited by dense subgroups, so we have the following result.

Proposition 8.5. *If Γ is a dense subgroup of* $\mathrm{Iso}(\mathbb{U}_S)$*, then for any $x \in \mathbb{U}_S$ its stabilizer Γ_x is a maximal proper subgroup of Γ.*

In particular, if $\mathfrak{S}_{(\infty)}$ acts faithfully homogeneously and Δ is the stabilizer of a point, then Δ is a maximal proper subgroup of $\mathfrak{S}_{(\infty)}$. We now turn our attention towards understanding these maximal subgroups; probably the description we obtain is well-known, but we do not know of a reference.

Definition 8.6. Assume that G is a group that acts transitively and primitively on a set X. Assume that H is a subgroup of G which acts transitively but not primitively. We say that H is *almost primitive* if there exists a maximal nontrivial block, and *totally imprimitive* otherwise.

In the classification of maximal subgroups of $\mathfrak{S}_{(\infty)}$, it is natural to distinguish groups with finite/infinite biindex, a notion that we introduce now.

Definition 8.7. The **biindex** of a subgroup H of a group G is the minimal cardinality of a set F such that $HFH = G$, that is, the number of double cosets.

It is straightforward to check that the biindex of H corresponds to the number of orbits for the H-action on G/H.

Maximal subgroups of $\mathfrak{S}_{(\infty)}$ which do not act transitively on $\mathbb{N}$ are easily described.

Lemma 8.8. *Let X be a subset of $\mathbb{N}$ different from $\emptyset$ and $\mathbb{N}$. Then the setwise stabilizer of X is a maximal proper subgroup of $\mathfrak{S}_{(\infty)}$.*

Proof. Since the setwise stabilizer of X is equal to the setwise stabilizer of its complement, we may as well assume that X is infinite.

Given $x_0, x_1 \in \mathbb{N}$ the transposition $(x_0\ \ x_1)$ is the permutation σ defined by $\sigma(x_0) = x_1$, $\sigma(x_1) = x_0$ and $\sigma(x) = x$ for all $x \notin \{x_0, x_1\}$. We use the fact that if $G \subseteq \mathbb{N} \times \mathbb{N}$ is a graph on $\mathbb{N}$, then the set of transpositions of the form $(x\ \ y)$ with $(x, y) \in G$ generates $\mathfrak{S}_{(\infty)}$ if and only if the graph G is connected.

Let Δ be the setwise stabilizer of X, and let us show Δ is a maximal subgroup of $\mathfrak{S}_{(\infty)}$. Observe that the graph of (x, y) such that $(x\ \ y) \in \Delta$ contains both the complete graph on X and the complete graph on $\mathbb{N} \setminus X$.

Now let $g \in \mathfrak{S}_{(\infty)} \setminus \Delta$. Then we find $x \in X$ such that $g(x) \notin X$. Since g has finite support and X is infinite, we find $y \in X$ with $x \neq y$ and $g(y) = y$. Now $g(x\ \ y)g^{-1} = (g(x)\ \ y)$. We conclude that the graph of (x, y) such that $(x\ \ y) \in \langle \Delta, g\rangle$ is connected, which shows that $\langle \Delta, g\rangle = \mathfrak{S}_{(\infty)}$ as wanted. $\square$

Using a famous theorem of Wielandt, it is also not difficult to identify maximal proper subgroups of $\mathfrak{S}_{(\infty)}$ which act transitively on $\mathbb{N}$.

Lemma 8.9. *Let Δ be a maximal proper subgroup of $\mathfrak{S}_{(\infty)}$, and assume that Δ has infinite index and acts transitively on $\mathbb{N}$. Then there is an equivalence relation E on $\mathbb{N}$, all of whose classes have the same cardinality $k \geq 2$, and such that Δ consists of all elements of $\mathfrak{S}_{(\infty)}$ which preserve E. These groups are indeed maximal, and have infinite biindex.*

Proof. Suppose otherwise. If Δ is primitive, Δ must be either $\mathfrak{S}_{(\infty)}$ or the group of even permutations by Wielandt's theorem [DM96, Thm. 3.3D], contradicting the assumption on the index of Δ.

If Δ is almost primitive, let A be a maximal nontrivial block for the action Δ on $\mathbb{N}$. Note that all the maximal blocks are of the form δA, $\delta \in \Delta$, and denote by E the equivalence relation defined by

$$\forall i, j \in \mathbb{N} \quad (iEj) \Leftrightarrow (\exists \delta \in \Delta\ \delta i \in A \text{ and } \delta j \in A).$$

Then all the E-equivalence classes have the same cardinality $|A| \geq 2$. By maximality, Δ must coincide with the group of all elements of $\mathfrak{S}_{(\infty)}$ which are automorphisms of E.

Say that a block δA is perturbed by γ if $\gamma\delta A$ is not equal to some $\delta' A, \delta' \in A$. Let $N(\gamma)$ be the (finite) number of blocks δA which are perturbed by γ. Since Δ respects the equivalence relation E, any element of $\Delta\gamma\Delta$ must perturb as many blocks as γ; there are elements of $\mathfrak{S}_{(\infty)}$ which perturb an arbitrary number of blocks, so the biindex of Δ in $\mathfrak{S}_{(\infty)}$ is infinite.

If Δ is totally imprimitive, we get an increasing sequence of blocks $A_1 \subseteq A_2 \subseteq \ldots$ whose union is equal to $\mathbb{N}$. By maximality, each element with support in some A_i must belong to Δ, so $\Delta = \mathfrak{S}_{(\infty)}$, a contradiction (i.e. there is no proper, maximal,transitive, totally imprimitive subgroup of $\mathfrak{S}_{(\infty)}$).

It remains to prove that groups of finitary permutations which preserve equivalence relations whose classes have fixed cardinality $k \geq 2$ are indeed maximal. Let E be an equivalence relation all whose classes have fixed cardinality $k \geq 2$, let Δ be its automorphism group. As in the previous lemma, consider the graph G of (x, y) such that $(x\ \ y) \in \Delta$; it suffices to show that this graph is connected. First, note that each E-class is contained in a connected component. Then take $\sigma \in \mathfrak{S}_{(\infty)} \setminus \Delta$, by definition there are $x, y \in \mathbb{N}$ with $(x, y) \in E$ such that $(\sigma x, \sigma y) \notin E$. Now observe that for every $z \in \mathbb{N} \setminus [\sigma(x)]_E$ there is $\tau \in \Delta$ such that $\tau(\sigma(y)) = z$ and $\tau(\sigma(x)) = \sigma(x)$. Conjugating the transposition $(x\ \ y)$ by $\tau\sigma$, we conclude that $(\sigma(x), z) \in G$, which proves that G is connected as wanted. $\square$

We can now list the maximal proper subgroups of $\mathfrak{S}_{(\infty)}$; from this description we will then be able to understand the corresponding Schlichting completions.

Theorem 8.10. *Assume that Δ is a maximal proper subgroup of $\mathfrak{S}_{(\infty)}$, of infinite index. Then, either:*

(1) *Δ has finite biindex $k+1$, in which case Δ is the setwise stabilizer of a finite set of integers with cardinality k.*
(2) *Δ has infinite biindex and does not act transitively; then it is the setwise stabilizer of an infinite, coinfinite set of integers.*
(3) *Δ has infinite biindex, and acts transitively; then there exists an equivalence relation E, all of whose classes have the same finite cardinality $k \geq 2$, such that Δ consists of the elements of $\mathfrak{S}_{(\infty)}$ which are automorphisms of E.*

All the groups described above are indeed maximal.

Proof. Assume first that Δ does not act transitively on $\mathbb{N}$. By maximality, we must have exactly two disjoint Δ-orbits, say X and Y; at least one of them (say Y) is infinite. By maximality, Δ is the setwise stabilizer of X, which is indeed maximal by 8.8.

If X is infinite then Δ is the stabilizer of a point for the action of $\mathfrak{S}_{(\infty)}$ on the set Ω of all infinite and co-infinite subsets of $\mathbb{N}$. It follows that Δ has infinite biindex.

If Δ has finite biindex $k+1$, then X is finite of cardinality $K \in \mathbb{N}$ and Δ is the stabilizer of a point for the (transitive) action of $\mathfrak{S}_{(\infty)}$ on the set $[\mathbb{N}]_K$ of all subsets of $\mathbb{N}$ whose cardinality is equal to K. Then the biindex of Δ is equal to $K+1$ (indeed the Δ-orbit of $Y \in [\mathbb{N}]_K$ is determined by the cardinality of its intersection with X) and we conclude $k = K$ as wanted.

The case where Δ acts transitively has already been described in the previous lemma. $\square$

8.2. **Associated Schlichting completions.** As explained at the beginning of this section, we want to understand the closure of $\mathfrak{S}_{(\infty)}$ in the symmetric group of $\mathfrak{S}_{(\infty)})/\Delta$ when Δ is one of the subgroups from the above proposition, in other words we want to understand possible *Schlichting completions* of $\mathfrak{S}_{(\infty)}$.

Definition 8.11. Let Δ be a subgroup of a countable group Γ. The **Schlichting completion** of Γ with respect to Δ is the closure of Γ in the symmetric group of Γ/Δ.

This notion was introduced by Schlichting when Δ was *commensurated* in Γ, and he proved that the completion is then a locally compact group [Sch80].

Note that the Schlichting completion of a group Γ comes with a canonical map from Γ with dense image. This map will always be clear from the context, so we won't explicitely write it down.

We will now define concrete representations of the Schlichting completions of $\mathfrak{S}_{(\infty)}$ with respect to the subgroups that we found in the above section. We will often make use of the following fact without further mention: if G and H be Polish groups and $\pi \colon G \to H$ is an embedding of topological groups, then π has closed image (see [Gao09, Prop. 2.2.1]).

Let us start with item (1) from Theorem 8.10.

Proposition 8.12. *Let Δ be a maximal proper subgroup of $\mathfrak{S}_{(\infty)}$, and assume that Δ has infinite index and biindex $k+1$. Then the Schlichting completion of $\mathfrak{S}_{(\infty)}$ with respect ot Δ is $\mathfrak{S}_\infty$. In particular, $\mathfrak{S}_{(\infty)}$ is a normal subgroup of its Schlichting completion with respect to Δ.*

Proof. By Theorem 8.10 we know that Δ is the stabilizer of a subset of $\mathbb{N}$ of cardinality k. Denote by $[\mathbb{N}]_k$ the set of subsets of $\mathbb{N}$ of cardinality k. Then $\mathfrak{S}_{(\infty)}$ acts transitively on $[\mathbb{N}]_k$, so we can identify $\mathfrak{S}_{(\infty)}/\Delta$ to $[\mathbb{N}]_k$ and the $\mathfrak{S}_{(\infty)}$-action on $\mathfrak{S}_{(\infty)}/\Delta$ extends to the natural $\mathfrak{S}_\infty$-action on $[\mathbb{N}]_k$. Denote by $\pi_k : \mathfrak{S}_\infty \to \mathfrak{S}([\mathbb{N}]_k)$ the associated continuous group morphism.

Let us show π_k is a homeomorphism onto its image. Suppose $\pi_k(\sigma_n) \to \mathrm{id}_{[\mathbb{N}]_k}$, and let $x \in \mathbb{N}$. We find $A, B \in [\mathbb{N}]_k$ such that $A \cap B = \{x\}$, so for large enough n we have $\sigma_n(A) \cap \sigma_n(B) = A \cap B = \{x\}$ and we thus have $\sigma_n(x) = x$. We conclude $\sigma_n \to \mathrm{id}_\mathbb{N}$, so π_k is indeed a homeomorphism onto its image.

We conclude $\pi_k(\mathfrak{S}_\infty)$ is a closed subgroup of $\mathfrak{S}([\mathbb{N}]_k)$ and thus the Schlichting completion of $\mathfrak{S}_{(\infty)}$ with respect to Δ is $\mathfrak{S}_\infty$. □

Let us make explicit the reasoning we used in the above proposition so as to identify a Schlichting completion.

Lemma 8.13. *Let A be a countable set, let $G \leq \mathfrak{S}(A)$ be a closed subgroup acting transitively on A. Suppose Γ is a countable dense subgroup of G. Then for every $x \in A$, the Schlichting completion of Γ with respect to Γ_x is isomorphic to G as a topological group.*

Proof. We have a natural identification of A with Γ/Γ_x which induces an isomorphism $\Phi : \mathfrak{S}(A) \to \mathfrak{S}(\Gamma/\Gamma_x)$. Notice that $\Phi(\Gamma)$ is then dense in $\Phi(G)$, and since the closure of $\Phi(\Gamma)$ is the Schlichting completion of Γ with respect to Γ_x, we get that $\Phi(G)$ is the Schlichting completion of Γ over Γ_x. □

We now move on to case (2) from Theorem 8.10. As it turns out, the study of this group as a Polish group was initiated by Cornulier [dC13]. We need a few preliminary definitions.

Definition 8.14. Let X be an infinite coinfinite subset of $\mathbb{N}$. The **commensurating subgroup** of X is the group $\mathfrak{S}(\mathbb{N}, X)$ of all permutations $\sigma \in \mathfrak{S}_\infty$ such that $\sigma(X) \triangle X$ is finite.

The commensurating subgroup is a Polish non-archimedean group for the topology of pointwise convergence on the countable set $\mathrm{Comm}_X(\mathbb{N})$ of subsets Y of $\mathbb{N}$ **commensurated** to X, i.e. such that $X \triangle Y$ is finite (see). Note that $\mathfrak{S}(\mathbb{N}, X)$ acts transitively on $\mathrm{Comm}_X(\mathbb{N})$.

Define the **transfer character** tr on $\mathfrak{S}(\mathbb{N}, X)$ by

$$tr(\sigma) = |\sigma X \setminus X| - |X \setminus \sigma X| .$$

It can be shown that tr is a continuous homomorphism $\mathfrak{S}(\mathbb{N}, X) \to \mathbb{Z}$ (see [dC13, Prop. 4.H.1]. Its kernel is denoted by $\mathfrak{S}^0(\mathbb{N}, X)$, it contains $\mathfrak{S}_{(\infty)}$ as a dense subgroup [dC13, Prop. 4.H.3, Prop. 4.H.4]. Moreover, $\mathfrak{S}^0(\mathbb{N}, X)$ acts transitively on $\mathrm{Comm}^0_X(\mathbb{N}) := \{Y \in \mathrm{Comm}_X(\mathbb{N}) : |X \setminus Y| = |Y \setminus X|\}$. We have the following basic lemma on the relationship between $\mathrm{Comm}^0_X(\mathbb{N})$ and $\mathrm{Comm}_X(\mathbb{N})$.

Lemma 8.15. *Every element of $\mathrm{Comm}_X(\mathbb{N})$ can be written either as the reunion or the intersection of two elements of $\mathrm{Comm}^0_X(\mathbb{N})$.*

Proof. Let $Y \in \mathrm{Comm}_X(\mathbb{N})$, we write $Y = (X \setminus F_1) \sqcup F_2$ with $F_1 \Subset X$ and $F_2 \Subset \mathbb{N} \setminus X$. If $|F_1| > |F_2|$, we find $F_3 \subseteq \mathbb{N} \setminus (X \sqcup F_2)$ such that $|F_3| = |F_1| - |F_2|$ and $F_4 \subseteq F_1$ such that $|F_4| = |F_2|$. Let $Y_1 = (X \setminus F_1) \sqcup (F_2 \sqcup F_3)$ and $Y_2 = (X \setminus F_4) \sqcup F_2$. Then both Y_1 and Y_2 are in $\mathrm{Comm}^0_X(\mathbb{N})$, and $Y = Y_1 \cap Y_2$. The case $|F_1| \geq |F_2|$ is similar, and one obtains that Y can be obtained as the union of two elements of $\mathrm{Comm}^0_X(\mathbb{N})$. □

Proposition 8.16. *Let X be an infinite coinfinite subset of $\mathbb{N}$. Denote by Δ the group of finitely supported permutations σ such that $\sigma(X) = X$. Then the Schlichting completion of $\mathfrak{S}_{(\infty)}$ with respect to Δ is $\mathfrak{S}^0(\mathbb{N}, X)$. In particular, $\mathfrak{S}_{(\infty)}$ is a normal subgroup of its Schlichting completion with respect to Δ.*

Proof. Let π be the continuous morphism obtained by restricting the $\mathfrak{S}_0(\mathbb{N}, X)$-action to $\mathrm{Comm}^0_X(\mathbb{N})$. By the previous lemma, if $\pi(\sigma_n) \to \mathrm{id}_{\mathrm{Comm}^0_X(\mathbb{N})}$ then $\sigma_n \to \mathrm{id}_{\mathrm{Comm}_X(\mathbb{N})}$ so π is an embedding. So $\mathfrak{S}_0(\mathbb{N}, X)$ may be viewed as a closed subgroup of $\mathfrak{S}(\mathrm{Comm}^0_X(\mathbb{N}))$. Its action on $\mathrm{Comm}^0_X(\mathbb{N})$ is moreover clearly transitive, and the stabilizer of X is our subgroup H. Since it contains $\mathfrak{S}_{(\infty)}$ as a dense subgroup, the desired conclusion follows from Lemma 8.13. ☐

Our third and last possible Schlichting completion (case (3) from Theorem 8.10) arises as follows. Let $k \in \mathbb{N}$ with $k \geq 2$, and consider a partition $\mathcal{P}_k = (A_i)_{i\in\mathbb{N}}$ of $\mathbb{N}$ where each A_i has cardinality k. We then defined the **almost full group** of the partition $\mathcal{P}_k$ as the group

$$A[\mathcal{P}_k] = \{\sigma \in \mathfrak{S}_\infty : \sigma(A_i) = A_i \text{ for all but finitely many } i \in \mathbb{N}\}.$$

The topology on $A[\mathcal{P}_k]$ is obtained by declaring that a basis of neighborhoods of the identity is made of the neighborhoods of the identity of the **full group** of $\mathcal{P}_k$

$$[\mathcal{P}_k] = \{\sigma \in \mathfrak{S}_\infty : \sigma(A_i) = A_i \text{ for all } i \in \mathbb{N}\}$$

equipped with the topology induced by $\mathfrak{S}_\infty$. In particular, G^0_k is an open subgroup of G_k.

Lemma 8.17. *The topology obtained by declaring that a basis of neighborhoods of identity in G_k is formed by neighborhoods of the identity in G^0_k does define a group topology on G_k, which is moreover Polish locally compact. The group $\mathfrak{S}_{(\infty)}$ is a dense subgroup of G_k.*

Proof. To prove that we have a group topology, it suffices to show that for every $g \in G_k$ and every V neighborhood of the identity in G^0_k, the set $gVg^{-1} \cap V$ is still a neighborhood of the identity in G^0_k. We may assume that V is the pointwise stabilizer in G^0_k of a finite reunion $A_{i_1} \sqcup \cdots \sqcup A_{i_k}$, in other words V is the set of all permutations which fix every point in $A_{i_1} \sqcup \cdots \sqcup A_{i_k}$ and leave every A_i invariant. It is clear that gVg^{-1} is the set of all permutations which fix every point in $gA_{i_1} \sqcup \cdots \sqcup gA_{i_k}$ and leave every gA_i invariant. But $g \in G_k$ so there are only finitely many A_i's such that $gA_i \neq A_i$. In particular, we can find $F \Subset \mathbb{N}$ containing $\{i_1, \ldots, i_k\}$ such that $gA_i = A_i$ for all $i \in \mathbb{N}\setminus F$ and $gA_{i_1} \sqcup \cdots \sqcup gA_{i_k} \subseteq \sqcup_{i\in F} A_i$ Then $gVg^{-1} \cap V$ contains the set W of all permutations which fix pointwise every element of $\bigsqcup_{i\in F} A_i$ and leave every A_i invariant. Since W is a neighborhood of the identity in G^0_k, this ends the proof that $gVg^{-1} \cap V$ is still a neighborhood of the identity in G^0_k, and we conclude that our topology is a group topology.

Now observe that since G^0_k is open of countable index in G_k, the associated topology is Polish because G is homeomorphic to the Polish space $\mathbb{N} \times G^0_k$. The fact that the topology is locally compact follows from the fact that G^0_k is compact, since it is a closed permutation group with only finite orbits.

Finally, let us show that $\mathfrak{S}_{(\infty)}$ is dense. Let $g \in A[\mathcal{P}_k]$, we find $F \Subset \mathbb{N}$ such that $gA_i = A_i$ for all $i \notin F$, and then there is $\sigma \in \mathfrak{S}_{(\infty)}$ such that $\sigma g A_i = A_i$ for all $i \in F$. So $\sigma g \in [\mathcal{P}_k]$, and the desired result now follows from the density of the group of finitely supported elements of the full group of $\mathcal{P}_k$ in the full group of $\mathcal{P}_k$. ☐

We now consider a more concrete way to view the topology we defined above so as to apply Lem. 8.13. Given $k \in \mathbb{N}$ let $\mathcal{P}_k = (A_i)_{i\in\mathbb{N}}$ and $\mathcal{Q}_k = (B_i)_{i\in\mathbb{N}}$ be two partitions of $\mathbb{N}$ into subsets of size k, say that they are **almost equal** if for all but finitely many $i \in \mathbb{N}$ we have $A_i = B_i$. Now if we let $A[\mathcal{P}_k]$ be the almost full group of $\mathcal{P}_k$, the $A[\mathcal{P}_k]$-orbit of $\mathcal{P}_k$ is the set of all partitions of $\mathbb{N}$ into subsets of size k which are almost equal to $\mathcal{P}_k$.

Proposition 8.18. *The topology on $A[\mathcal{P}_k]$ is equal to the topology of pointwise convergence on the set of partitions of $\mathbb{N}$ into subsets of size k which are almost equal to $\mathcal{P}_k$.*

Proof. Let $AE(\mathcal{P}_k)$ be the set of partitions of $\mathbb{N}$ into subsets of size k which are almost equal to $\mathcal{P}_k$. Let $\pi : A[\mathcal{P}_k] \to \mathfrak{S}(AE(\mathcal{P}_k))$, observe that π is continuous since $[\mathcal{P}_k]$ is open in $A[\mathcal{P}_k]$. Moreover since $[\mathcal{P}_k]$ is compact it suffices to show that $\pi_{\restriction [\mathcal{P}_k]}$ is injective.

So let $\sigma \in [\mathcal{P}_k]$ and suppose $\pi(\sigma) = \mathrm{id}_{AE(\mathcal{P}_k)}$. Now let $x \in \mathbb{N}$, write $\mathcal{P}_k = (A_i)_{i\in\mathbb{N}}$ and let $i_0 \in \mathbb{N}$ such that $x \in A_{i_0}$. Let $j_0 \neq i_0$, fix $y \in A_{j_0}$. Then define a new partition $\mathcal{Q}_k = (B_i)_{i\in\mathbb{N}}$ by letting

$$B_i = \begin{cases} \{x\} \cup (A_{j_0} \setminus \{y\}) & \text{if } i = i_0 \\ \{y\} \cup (A_{i_0} \setminus \{x\}) & \text{if } i = j_0 \\ A_i & \text{otherwise.} \end{cases}$$

Observe that since σ fixes both $\mathcal{Q}_k$ and $\mathcal{P}_k$, it must fix $A_{i_0} \cap B_{i_0} = \{x\}$, so $\sigma(x) = x$. We conclude $\sigma = 1$, so that $\pi_{\restriction [\mathcal{P}_k]}$ is injective as wanted. □

Corollary 8.19. *Let H_k be the group of finitely supported bijections which fix a partition $\mathcal{P}_k$ of $\mathbb{N}$ into sets of cardinality k. Then the Schlichting completion of $\mathfrak{S}_{(\infty)}$ with respect to H_k is equal to the almost full group $A[\mathcal{P}_k]$. In particular, $\mathfrak{S}_{(\infty)}$ is a normal subgroup of its Schlichting completion with respect to Δ.*

Proof. This follows from the previous lemma along with Lemma 8.13. □

8.3. **Proof of Theorem 8.1.** Combining Proposition 8.12, Proposition 8.16 and Corollary 8.19 with the classification of the maximal proper subgroups of $\mathfrak{S}_{(\infty)}$ of infinite index, we obtain the following proposition.

Proposition 8.20. *Let $\Delta \leq \mathfrak{S}_{(\infty)}$ be a maximal proper subgroup of $\mathfrak{S}_{(\infty)}$ of infinite index. Then $\mathfrak{S}_{(\infty)}$ is a normal subgroup of its Schlichting completion with respect to Δ.* □

Question. Is there a way to prove the above proposition directly, without going to the trouble of identifying the Schlichting completions?

Recall from Proposition 8.5 that if $\mathfrak{S}_{(\infty)} \leq \mathrm{Iso}(\mathbb{U}_S)$ is dense, then the $\mathfrak{S}_{(\infty)}$-stabilizer of any point in $\mathbb{U}_S$ is a maximal proper subgroup of infinite index. As we observed before, $\mathrm{Iso}(\mathbb{U}_S)$ must then be isomorphic to the Schlichting completion of $\mathfrak{S}_{(\infty)}$ with respect to that subgroup. To finish the proof of Theorem 8.1, it is thus enough to prove the following fact.

Proposition 8.21. *Assume that $|S| \geq 3$. Then the conjugacy class of any nonidentity element of $\mathrm{Iso}(\mathbb{U}_S)$ is uncountable. In particular, $\mathrm{Iso}(\mathbb{U}_S)$ does not admit a nontrivial countable normal subgroup.*

Proof. Assume that $g \in \mathrm{Iso}(\mathbb{U}_S)$ has a countable conjugacy class; then its centralizer $C(g)$ is a closed subgroup with countable index, so it is clopen. Hence, there must exist a finite subset F of $\mathbb{U}_S$ such that g commutes with any element h satisfying $h(x) = x$ for all $x \in F$. We want to prove that $g = \mathrm{id}$ and assume for a contradiction that such is not the case.

We first claim that there must exist $x \notin F$ such that $g(x) \neq x$. To see this, pick y such that $g(y) \neq y$. Since $|S| \geq 3$, we can find $s_1 \neq s_2 \in S$ such that the map $f\colon \{y, g(y)\} \to \mathbb{R}$ defined by setting $f(y) = s_1$, $f(g(y)) = s_2$ is Katětov. Thus there exist infinitely many $x \in \mathbb{U}_S$ such that $d(x, y) = s_1$ and $d(x, g(y)) = s_2$; in particular there exists such an x which does not belong to F and since $d(x, y) \neq d(x, g(y))$ we have $g(x) \neq x$.

Next, pick $h \in \mathrm{Iso}(\mathbb{U}_S)$ such that h coincides with the identity on $F \cup \{x\}$ yet $h(g(x)) \neq g(x)$. Then $hg(x) \neq gh(x)$, so h does not commute with g, a contradiction. Hence $g = \mathrm{id}$. □

Another nice consequence of Theorem 8.10 is that $\mathfrak{S}_{(\infty)}$ has only one 2-transitive action on $\mathbb{N}$ up to conjugacy.

Proposition 8.22. *Let $\mathfrak{S}_{(\infty)} \curvearrowright \mathbb{N}$ be a 2-transitive action. Then the action is conjugate to the natural action of $\mathfrak{S}_{(\infty)}$ on $\mathbb{N}$.*

Proof. Let $\Delta = \mathrm{Stab}_{\mathfrak{S}_{(\infty)}}(0)$, then Δ has infinite index and by 2-transitivity its bi-index is 2. This implies that Δ is maximal proper: if $\Delta < \Gamma$ then Γ must contain a Δ-double coset distinct from Δ and is thus equal to $\mathfrak{S}_{(\infty)}$. The desired conclusion now follows from Theorem 8.10. □

Remark 8.23. In [LM18], the second author defines a transitive action of a countable group Γ on a countable set X to be *highly faithful* if for every $F \Subset \Gamma$, there is $x \in X$ such that for all distinct $f_1, f_2 \in F$, $f_1 \cdot x \neq f_2 \cdot x$. The natural action of $\mathfrak{S}_{(\infty)}$ on $\mathbb{N}$ is highly transitive but not highly faithful, so the above proposition yields that $\mathfrak{S}_{(\infty)}$ has no highly transitive highly faithful action, answering a question raised in [LM18].

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