A POLISH METRIC SPACE WHOSE GROUP OF ISOMETRIES INDUCES A UNIVERSAL RELATION FOR POLISH GROUP ACTIONS

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Abstract. We show that there exists a Polish metric space (X, d) such that the action of its isometry group on X produces an equivalence relation which is universal for relations induced by a Borel action of a Polish group on a standard Borel space.

1. Introduction

It is common practice in contemporary mathematics to consider *classification problems*: given a class of structures $\mathcal{A}$, and a notion of equivalence (often, a notion of isomorphism) between elements of $\mathcal{A}$, try to classify elements of $\mathcal{A}$ up to this notion of equivalence, by providing a list of complete invariants. Of course, one might use the equivalence classes themselves as invariants, which is not very satisfactory; one would like to use the simplest possible invariants. Deciding precisely what one means by "simplest possible" here can be tricky; but one can at least try to *compare* the complexity of classification problems, introducing a hierarchy of complexities. One such hierarchy was introduced by Friedman–Stanley in the Borel context [FS89]; let us recall the definitions (we refer the reader to [BK96], [Hjo00] and[Gao09] for background on Borel reducibility theory).

Assume that X, Y are Polish spaces, and that E, F are equivalence relations on X, Y; then one says that E is *Borel reducible to* F is there exists a *Borel* map $f \colon X \to Y$ such that

$$\forall x, x' \in X \quad (xEx') \Leftrightarrow (f(x)Ff(x')) \ .$$

The intuition is that, by applying f, one has reduced the problem of deciding whether two elements are E-equivalent to that of deciding whether two elements are F-equivalent. The fact that f is required to be Borel above is due to the requirement that the reduction be "computable"; other choices are certainly possible, but this one applies well to various classification problems in mathematics, which appear as equivalence relation on Polish spaces, or more generally on standard Borel spaces (see [Kec95] or [Gao09]for background on descriptive set theory). Of particular interest to us here is the class of relations E for which there exists a Polish group G and a Borel action of G on X such that

$$\forall x, x' \in X \quad (xEx') \Leftrightarrow (\exists g \in G \ gx = x') \ .$$

Among all such relations, there exists a *universal* one, that is, a relation induced by a Borel action of a Polish group on a Polish space X and which has the property that all other such actions reduce to it. There are many example of "concrete" classification problems which have exactly this level of complexity: isometry of Polish metric spaces [GK03], linear isometry of separable Banach spaces [Mel07], isomorphism of separable C^*-algebras [Sab], homeomorphism of compact metric spaces [Zie]... In the paper [GK03] where they establish the first result in the previous list, Gao and Kechris ask the following question (Problem 10.7): is there a Polish metric space X and a closed subgroup G of the isometry group $\mathrm{Iso}(X)$ such that the relation induced by the action of G on X is a universal equivalence relation induced by a Borel action of a Polish group?

In this paper, we answer that question in the affirmative, using the Gao–Kechris theorem about the complexity of the isometry problem for Polish metric spaces to prove the following result.

Theorem 1.1. *There exists a Polish metric space X such that the action of the isometry group* $\mathrm{Iso}(X)$ *on X is universal among all relations induced by a Borel action of a Polish group on a Polish space.*

Research supported by ANR project GruPoLoCo (ANR-11-JS01-008).

In particular, relations induced by continuous isometric actions of Polish groups may be, in the sense of Borel reducibility, as complicated as those induced by arbitrary Borel actions of Polish groups on Polish spaces.

The construction is a fairly simple extension of the methods used in [GK03] and [Mel08], and uses various properties of bounded Urysohn spaces.

2. Construction of the action

We begin by looking for a Polish metric space (X,d) and a closed subgroup H of its isometry group such that the action of H on X produces a universal equivalence relation (this is the original question of Gao and Kechris).

From now on, let G denote the isometry group of the Urysohn sphere $\mathbf{U}_1$ (we refer to [GK03] for information about Katětov maps and Urysohn spaces), and d be a left-invariant distance on G. The following fact is standard.

Lemma 2.1. *The completion $\hat{G}$ of (G,d) may naturally be identified with the set of all isometric embeddings of $\mathbf{U}_1$ into itself.*

Proof. Using homogeneity, it is easy to see (using back-and-forth)that any isometric embedding of $\mathbf{U}_1$ into itself is a pointwise limit of isometric bijections of $\mathbf{U}_1$. Conversely, given a Cauchy sequence (g_n) in (G,d), each sequence $(g_n(x))$ is Cauchy, hence (g_n) converges pointwise to some g, which is an isometric embedding of $\mathbf{U}_1$ into itself. ■

Under this identification, the topology on $\hat{G}$ is the pointwise convergence topology, and G acts continuously and isometrically on $\hat{G}$ by left-translation. From this we obtain a continuous map from G to the isometry group of $\hat{G}$, which is a topological group isomorphism onto its image; hence the image is closed, and we see G as a closed subgroup of the isometry group of $\hat{G}$.

Recall that a *Katětov map* f on a metric space (X,d) is a map $f\colon X \to \mathbf{R}^+$ such that

$$\forall x,y \in X \quad |f(x)-f(y)| \le d(x,y) \le f(x)+f(y) \ .$$

These maps correspond to one-point metric extensions of X. Below we say that a Katětov map f from a metric space X of diameter at most 1 to $[0,1]$ is *finitely supported* if there exists a finite subset A of X such that

$$\forall x \in X \quad f(x) = \min(1, \min\{d(x,a)+f(a)\colon a \in A\}) \ .$$

When A is as above, we way that A is a *support* of f (note that there is no unique support: any set containing A will also be a support). We let $E_1(X,\omega)$ denote the set of finitely supported Katětov maps from X to $[0,1]$; this is a separable metric space for the distance $d(f,g)=\sup\{|f(x)-g(x)|\colon x \in X\}$, and X isometrically emebds into $E_1(X,\omega)$ via the map $x \mapsto d(x,\cdot)$. Below we always implictly identify X with a subspace of $E_1(X,\omega)$ in this manner.

Lemma 2.2. *Let (X,d) be a metric space of diameter at most 1; define inductively $X_0 = X$, $X_{n+1} = E_1(X_n,\omega)$. Then for all n and all $z \in X_n$ the restriction of $d(z,\cdot)$ to X belongs to $E_1(X,\omega)$.*

Proof. Proceed by induction; the cases $n = 0,1$ are by definition. So assume the property is valid for some n, and let $f \in X_{n+1} = E_1(X_n,\omega)$. There exists a finite $A \subseteq X_n$ such that for all $x \in X_n$ one has $f(x) = \min(1,\min\{d(x,a)+f(a)\colon a \in A\})$; and by induction, for all $a \in A$ there exists a finite set $B_a \subseteq X$ such that for all $x \in X$ one has $d(a,x) = \min(1,\min\{d(x,b)+d(b,a)\colon b \in B_a\})$. Letting $B = \bigcup_{a\in A} B_a$, which is a finite subset of X, the triangle inequality gives

$$\forall x \in X \ f(x) = \min(1,\min\{f(b)+d(x,b)\colon b \in B\}) \ .$$

■

Proposition 2.3. *The relation induced by the action of G on $\hat{G}$ is universal for relations induced by a Borel action of a Polish group on a standard Borel space.*

We begin by giving the argument without worrying about a Borel coding, which we will turn to afterwards.

Proof. Start from a nonempty Polish space X of diameter at most $\frac{1}{2}$; let Y_X denote a copy of $\mathbf{U}_1$ such that for all $x \in X$ and $y \in Y_X$ one has $d(x,y) = \frac{1}{2}$. Set $Z_X^0 = X \sqcup Y_X$ endowed with this distance. Then, define inductively $Z_X^{n+1} = E_1(Z_X^n, \omega)$, and let Z_X be the completion of $\bigcup_n Z_X^n$.

We claim that the following facts hold:

(i) Any isometry of Z_X^0 must fix X and Y_X setwise; any isometry of X as well as any isometry of Y_X extends to an isometry of Z_X^0.

(ii) Z_X is isometric to $\mathbf{U}_1$, and any isometry of Z_X^0 extends to an isometry of Z_X.

(iii) $X = \{z \in Z_X \colon \forall y \in Y_X \ d(z,y) = \frac{1}{2}\}$.

The first item follows immediately from the definition of the metric on Z_X^0: $z \in Z_X^0$ belongs to Y_X iff there exists $z' \in Z_X^0$ such that $d(z,z') > \frac{1}{2}$ (because the diameter of X is < 1, and Y_X is of diameter 1). This property is invariant under isometries of Z_X^0, which proves that isometries of Z_X^0 must fix X and Y_X setwise.

The second item is standard and well-known (see e.g. [GK03, 2.C]). The last item comes from the fact that for all $z \in Z_X^n$ there exists $y \in Y_X$ such that $d(z,y) = \min(1, \frac{1}{2} + d(z,X))$, which is more than what we need, since X is closed and $\bigcup Z_X^n$ is dense in Z_X. Let us explain why this fact is true: pick some $z \in Z_X^n$, and consider the restriction of the map $d(z,\cdot)$ to Z_X^0. By Lemma 2.2, this map is finitely supported; let S denote a finite support for it, $S_X = S \cap X$ and $S_Y = S \cap Y_X$. The universal property of Y_X ensures that there exists $y \in Y_X$ such that $d(y,s) = 1$ for all $s \in S_Y$. Now, by definition of a support we have

$$\begin{aligned} d(z,y) &= \min(1, \min\{d(z,s) + d(s,y) \colon s \in S\}) \\ &= \min(1, \min\{d(z,s) + d(s,y) \colon s \in S_X\}) \\ &= \min(1, \frac{1}{2} + d(z,X)) \ . \end{aligned}$$

The second equality above is due to the fact that $d(s,y) = 1$ for all $s \in S_Y$, so only S_X may possibly play a role here; and the last equality follows from the fact that $d(s,y) = \frac{1}{2}$ for all $s \in X$.

Now, we identify each Z_X with $\mathbf{U}_1$ and pick an isometric map σ_X from Z_X into itself with image Y_X (of course all of this must be done in a Borel way; we address this concern below, as it is mostly a routine, if a bit cumbersome, book-keeping argument). Let $\cong$ denote the isometry relation between Polish spaces of diameter at most $\frac{1}{2}$, and $\sim$ the equivalence relation induced by the action of G on $\hat{G}$. Note that, if $\sigma_X \sim \sigma_{X'}$ then there exists an isometry g of $\mathbf{U}_1$ such that $g\sigma_X = \sigma_{X'}$, which implies in particular that $g(Y_X) = Y_{X'}$, hence $g(X) = X'$ (because of the third item above; that is the point of this construction: having a copy of X which is "definable" over a copy of $\mathbf{U}_1$ inside $\mathbf{U}_1$) and $X \cong X'$.

Conversely, assume that $X \cong X'$. Starting from an isometry $h \colon X \to X'$, we may follow the construction of $Z_X, Z_{X'}$ to extend h to an isometry h of $\mathbf{U}_1$ which maps X to X' and Y_X to $Y_{X'}$. Then $h\sigma_X\sigma_{X'}^{-1}$ is an isometry of $Y_{X'}$, which extends to an isometry g of $\mathbf{U}_1$ by the second point above. We have $g\sigma_{X'} = h\sigma_X$, proving that $\sigma_X \sim \sigma_{X'}$.

Thus the map $X \mapsto \sigma_X$ is a (Borel) reduction of $\cong$ to $\sim$. As $\cong$ is universal for relations induced by a Borel action of a Polish group, $\sim$ also is and we are done. ∎

To produce a metric space X as promised in Theorem 1.1, we may now simply apply the following lemma.

Lemma 2.4 ([Mel08]). *Let (X,d) be a Polish metric space, and H be a closed subgroup of its isometry group. Then there exists a Polish metric space (Y,d) containing X such that all elements of H extend uniquely to Y, the extension mapping from* $\mathrm{Iso}(H)$ *to* $\mathrm{Iso}(Y)$ *is continuous and all isometries of Y coincide on X with an element of H.*

In particular, the isometry group of Y is naturally identified (as a topological group) with H, and the relation induced by the action of H on X reduces to the action of $\mathrm{Iso}(Y)$ *on Y via the natural inclusion mapping from X to Y.*

This lemma is a reformulation of the argument used to prove [Mel08, Theorem 1]. Actually, the argument of [Mel08] proceeds by proving this lemma in the particular case where H is a Polish group and (X,d) is the completion of H for some compatible left-invariant distance d on H, which is exactly the particular case that we need here. It still seems worth stating the lemma in this slightly greater generality as it might have other uses.

The combination of Proposition 2.3 and Lemma 2.4 proves that Theorem 1.1 holds. We close the paper by sketching briefly how one can make sure that the previous construction is done in a Borel way.

3. A Borel coding of the construction

Given a Polish space X, we denote by $\mathcal{F}(X)$ the standard Borel space of all nonempty closed subsets of X, endowed with the Effros Borel structure (see [Kec95, Section 12.C]). Let $\mathbf{U}_{\frac{1}{2}}$ denote the Urysohn space of diameter $\frac{1}{2}$. We explain how to apply the construction of Proposition 2.3 to an element X of $\mathcal{F}(\mathbf{U}_{\frac{1}{2}})$ so that $X \mapsto \sigma_X$ is Borel (here and below, we reuse the notations of the proof of Proposition 2.3).

Lemma 3.1. *One can define a Borel map $F\colon \mathcal{F}(\mathbf{U}_{\frac{1}{2}}) \to \mathcal{F}(\mathbf{U}_1)$ so that, for all $X \in \mathcal{F}(\mathbf{U}_{\frac{1}{2}})$, there exists an isometry $\varphi_X\colon Z_X \to \mathbf{U}_1$ satisfying $\varphi_X(Y_X) = F(X)$.*

This is very similar to [GK03, 2.2], so we do not give details.

Recall that $\hat{G}$ stands for the left-completion of the isometry group $\mathrm{Iso}(\mathbf{U}_1)$, which we identify with the set of all isometric embeddings of $\mathbf{U}_1$ into itself.

The previous lemma gives us a Borel construction of the image of our desired $\sigma_X \in \hat{G}$. This is enough information for a Borel construction of σ_X.

Lemma 3.2. *Let F be as in the statement of the previous proposition. One may build a Borel map $\sigma\colon \mathcal{F}(\mathbf{U}_{\frac{1}{2}}) \to \hat{G}$ such that for all X $\sigma_X(\mathbf{U}_1) = F(X)$.*

Proof. We fix a sequence of Borel maps $\Phi_n\colon \mathcal{F}(\mathbf{U}_1) \to \mathbf{U}_1$ such that, for any $F \in \mathcal{F}(\mathbf{U}_1)$, $\{\Phi_n(F)\colon n \in \mathbf{N}\}$ is a dense subset of F (see [Kec95, Theorem 12.13]). We also fix a dense subset $\{y_n\}$ of $\mathbf{U}_1$. We use an inductive construction to define sequences of Borel maps $X \mapsto a_{n,X}$ and $X \mapsto b_{n,X}$ in such a way that:

- For all X and n, $a_{n,X} \in \mathbf{U}_1$ and $b_{n,X} \in F(X)$.
- For all X and n, $a_{2n,X} = y_n$ and $\Phi_n(F(X)) \in \{b_{k,X}\colon k \leq 2n+1\}$.
- For all X and n, $a_{n,X} \mapsto b_{n,X}$ is an isometry.

Assuming this construction can indeed be carried out, we then have a unique isometric surjection $\sigma_X\colon \mathbf{U}_1 \to Y_X$, defined by setting $\sigma_X(a_{n,X}) = b_{n,X}$ for all $n \in \mathbf{N}$. It is straightforward to check that, for all $x \in \mathbf{U}_1$ and all open $O \subseteq \mathbf{U}_1$, the set $\{X\colon \sigma_X(x) \in O\}$ is Borel in $\mathcal{F}(\mathbf{U}_{\frac{1}{2}})$, which in turn is equivalent to saying that $X \mapsto \sigma_X$ is Borel.

We conclude by saying a few words about the construction of the maps $X \mapsto a_{n,X}$ and $X \mapsto b_{n,X}$. We may simply set $a_{0,X} = y_0$ and $b_{0,X} = \Phi_0(F(X))$ for all X, which takes care of the initialization. Assume that all maps $a_{i,X}, b_{i,X}$ have been defined up to some rank $n-1$. Consider the case when $n = 2i+1$ is odd (the even case is a bit simpler so we do not give details). For all X such that $\Phi_i(F(X)) \in \{b_{k,X}\colon k < n\}$ (these form a Borel set), simply let $k(X)$ be the smallest integer witnessing that fact and set $b_{n,X} = b_{k(X),X}$, $a_{n,X} = a_{k(X),X}$. Let $\mathcal{X}_n$ denote the set of all the remaining X's, and set $b_{n,X} = \Phi_i(F(X))$ for all $X \in \mathcal{X}_n$. Consider $\Psi_n\colon \mathcal{X}_n \to \mathcal{F}(\mathbf{U}_1)$ defined by

$$\Psi_n(X) = \{z\colon \forall i < n \ d(z, a_{i,X}) = d(b_{n,X}, b_{i,X})\} .$$

This map is well-defined (i.e. $\Psi_n(X)$ is closed and nonempty) and Borel (use the approximate homogeneity of $\mathbf{U}_1$), so we may define $a_{n,X} = \Phi_0(\Psi_n(X))$ and obtain our desired Borel map.

■

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