

DISCRETE RIEMANN SURFACES

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ABSTRACT. We detail the theory of Discrete Riemann Surfaces. It takes place on a cellular decomposition of a surface, together with its Poincaré dual, equipped with a discrete conformal structure. A lot of theorems of the continuous theory follow through to the discrete case, we will define the discrete analogs of period matrices, Riemann's bilinear relations, exponential of constant argument and series. We present the notion of criticality and its relationship with integrability.

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1. INTRODUCTION

Riemann surfaces theory was a major achievement of XIXth century mathematics, setting the framework where modern complex analysis bloomed. Nowadays, surfaces are intensively used in computer science for numeric computations, ranging from visualization to pattern recognition and approximation of partial differential equations. A lot of these computations involve, at the continuous level, analytic functions. But very few algorithms care about this specificity, although analytic functions form a relatively small vector space among the space of functions, problems are usually crudely discretized in a way that doesn't take advantage of analyticity.

The theory of discrete Riemann surfaces aims at filling this gap and setting the theoretical framework in which the notion of discrete analyticity is set on solid grounds.

Most of the results in this paper are a straightforward application of the continuous theory [1, 2] together with the results in [3, 4, 5], to which we refer for details. We define the discrete period matrix, which is twice as large as in the continuous case: the periods of a holomorphic form on the graph and on its dual are in general different, but the continuous limit theorem, given a refining sequence of critical maps, ensures that they converge to the same value. The main tool is the same as in the continuous case, the Riemann bilinear relations.

2. DISCRETE RIEMANN SURFACES

2.1. Discrete Hodge theory. We recall in this section basic definitions and results from [4] where the notion of discrete Riemann surfaces was defined. We are interested in discrete surfaces given by a cellular decomposition \diamond of dimension two, where all faces are *quadrilaterals* (a quad-graph [6, 7, 8]). Its vertices and diagonals define, up to homotopy and away from the boundary, two dual cellular decompositions Γ and Γ^* : The edges in Γ_1^* are dual to edges in Γ_1 , faces in Γ_2^* are dual to vertices in Γ_0 and vice-versa. Their union is denoted the *double* $\Lambda = \Gamma \sqcup \Gamma^*$. A *discrete conformal structure* on Λ is a real positive function ρ on the unoriented edges satisfying $\rho(e^*) = 1/\rho(e)$. It defines a genuine Riemann surface structure on the discrete surface: Choose a length δ and realize each quadrilateral by a lozenge whose diagonals have a length ratio given by ρ . Gluing them together provides a flat riemannian metric with conic singularities at the vertices, hence a conformal structure [9]. It leads to a straightforward discrete version of the *Cauchy-Riemann equation*. A function on the vertices is discrete

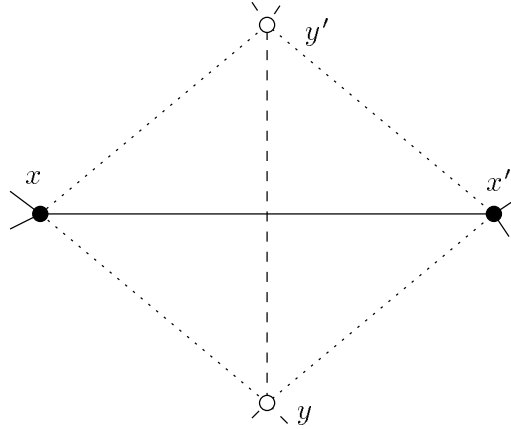
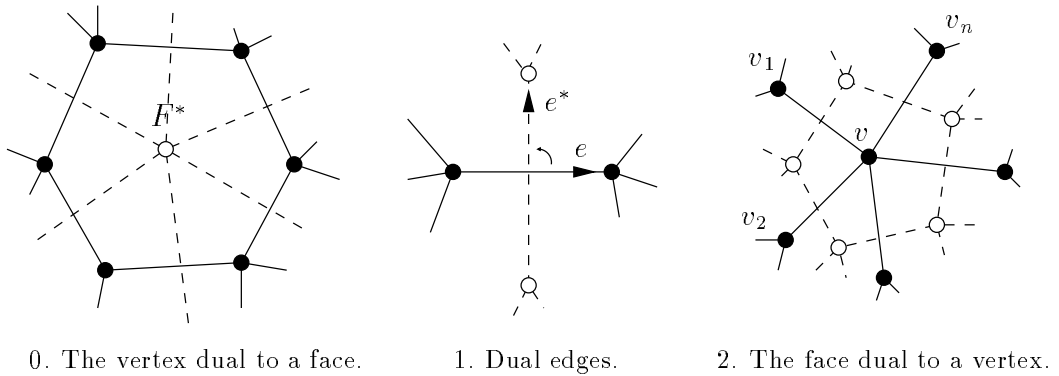


FIGURE 1. The vertices and diagonals of a quadrilateral define a pair of dual edges.



0. The vertex dual to a face.

1. Dual edges.

2. The face dual to a vertex.

FIGURE 2. Duality.

holomorphic iff for every quadrilateral $(x, y, x', y') \in \diamond_2$,

$$(2.1) \quad f(y') - f(y) = i \rho(x, x') (f(x') - f(x)).$$

We recall elements of de-Rham cohomology, doubled in our context: The complex of *chains* $C(\Lambda) = C_0(\Lambda) \oplus C_1(\Lambda) \oplus C_2(\Lambda)$ is the vector space span by vertices, edges and faces. It is equipped with a *boundary* operator $\partial : C_k(\Lambda) \rightarrow C_{k-1}(\Lambda)$, null on vertices and fulfilling $\partial^2 = 0$. The kernel $\ker \partial =: Z_\bullet(\Lambda)$ of the boundary operator are the closed chains or *cycles*. Its image are the *exact* chains. It provides the dual spaces of forms, called *cochains*, $C^k(\Lambda) := \text{Hom}(C_k(\Lambda), \mathbb{C})$ with a

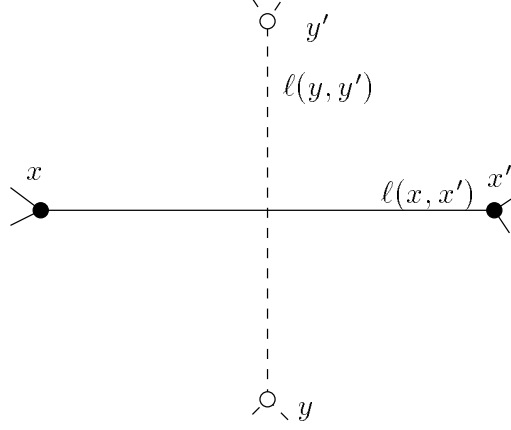


FIGURE 3. The discrete Cauchy-Riemann equation.

coboundary $d : C^k(\Lambda) \rightarrow C^{k+1}(\Lambda)$ defined by Stokes formula:

$$\int_{(x,x')} df := f(\partial(x, x')) = f(x') - f(x), \quad \iint_F d\alpha := \oint_{\partial F} \alpha.$$

A *cocycle* is a closed cochain and we note $\alpha \in Z^k(\Lambda)$.

These spaces are equipped with the canonical scalar product, weighed according to ρ on edges and averaged on the graph and its dual:

$$(\alpha, \beta) := \frac{1}{2} \sum_{e \in \Lambda_1} \rho(e) \left(\int_e \alpha \right) \left(\int_e \beta \right).$$

Duality of complexes allows us to define a *Hodge operator* $*$ on forms by

$$\begin{aligned} * : C^k(\Lambda) &\rightarrow C^{2-k}(\Lambda) \\ C^0(\Lambda) \ni f &\mapsto *f : \iint_F *f := f(F^*), \\ C^1(\Lambda) \ni \alpha &\mapsto *\alpha : \int_e *\alpha := -\rho(e^*) \int_{e^*} \alpha, \\ C^2(\Lambda) \ni \omega &\mapsto *\omega : (*\omega)(x) := \iint_{x^*} \omega. \end{aligned} \tag{2.2}$$

It fulfills $*^2 = (-\text{Id}_{C^k})^k$. The endomorphism $\Delta := -d * d * - * d * d$ is the usual discrete *Laplacian*: Its formula on a function at a vertex $x \in \Gamma_0$ with neighbours $x_1, \dots, x_V \in \Gamma_0$ is the usual weighted averaged

difference:

$$(\Delta(f))(x) = \sum_{k=1}^V \rho(x, x_k) (f(x) - f(x_k)).$$

The space of *harmonic forms* is defined as its kernel .

The Hodge star and the laplacian are real operators. Since $*^2 = -\text{Id}$ on functions, it is natural to consider them on complexified cochains. The discrete holomorphic forms are special complex harmonic forms: a 1-form

$$(2.3) \quad \alpha \in C^1(\Lambda) \text{ is } \textit{holomorphic} \text{ iff } d\alpha = 0 \text{ and } *\alpha = -i\alpha,$$

that is to say if it is closed and of type $(1,0)$. Let d' , resp. d'' the compositions of the exterior derivative with the projection on the space of $(1,0)$, resp. $(0,1)$ -forms, Eq. (2.3) is equivalent to $d'\alpha = 0$. We will note $\alpha \in \Omega^1(\Lambda)$. A function $f : \Lambda_0 \rightarrow \mathbb{C}$ is *holomorphic* iff df is holomorphic, which is equivalent to (2.1) and we note $f \in \Omega^0(\Lambda)$.

In the compact case, $-*d*$ is the adjoint d^* of the coboundary operator d and the Hodge theorem orthogonally decomposes forms into exact, coexact and harmonic,

$$C^k(\Lambda) = \text{Im } d \oplus^\perp \text{Im } d^* \oplus^\perp \text{Ker } \Delta,$$

harmonic forms are the closed and co-closed ones, and harmonic 1-form are the orthogonal sum of holomorphic and anti-holomorphic ones:

$$\text{Ker } \Delta = \text{Ker } d \cap \text{Ker } d^* = \text{Ker } d' \oplus^\perp \text{Ker } d''.$$

2.2. Wedge product. We construct a wedge product on \diamond such that

- the canonical weighted hermitian scalar product reads as expected

$$(\alpha, \beta) = \iint \alpha \wedge *\bar{\beta},$$

- and the coboundary operator d_\diamond on \diamond , is a derivation for this product $\wedge : C^k(\diamond) \times C^l(\diamond) \rightarrow C^{k+l}(\diamond)$.

It is defined by the following formulae, for $f, g \in C^0(\diamond)$, $\alpha, \beta \in C^1(\diamond)$ and $\omega \in C^2(\diamond)$:

$$(2.4) \quad (f \cdot g)(x) := f(x) \cdot g(x) \quad \text{for } x \in \diamond_0,$$

$$(2.5) \quad \int_{(x,y)} f \cdot \alpha := \frac{f(x) + f(y)}{2} \int_{(x,y)} \alpha \quad \text{for } (x, y) \in \diamond_1,$$

$$(2.6) \quad \iint_{(x_1, x_2, x_3, x_4)} \alpha \wedge \beta := \frac{1}{4} \sum_{k=1}^4 \int_{(x_{k-1}, x_k)} \alpha \int_{(x_k, x_{k+1})} \beta - \int_{(x_{k+1}, x_k)} \alpha \int_{(x_k, x_{k-1})} \beta,$$

$$(2.7) \quad \iint_{(x_1, x_2, x_3, x_4)} f \cdot \omega := \frac{f(x_1) + f(x_2) + f(x_3) + f(x_4)}{4} \iint_{(x_1, x_2, x_3, x_4)} \omega$$

for $(x_1, x_2, x_3, x_4) \in \diamond_2$.

A form on \diamond can be *averaged* into a form on Λ : This map A from $C^\bullet(\diamond)$ to $C^\bullet(\Lambda)$ is the identity for functions and defined by the following formulae for 1 and 2-forms:

$$(2.8) \quad \int_{(x,x')} A(\alpha_\diamond) := \frac{1}{2} \left(\int_{(x,y)} + \int_{(y,x')} + \int_{(x,y')} + \int_{(y',x')} \right) \alpha_\diamond,$$

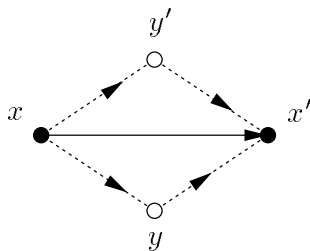
$$(2.9) \quad \iint_{x^*} A(\omega_\diamond) := \frac{1}{2} \sum_{k=1}^d \iint_{(x_k, y_k, x, y_{k-1})} \omega_\diamond,$$

where notations are made clear in Fig. 4. The map A is neither injective nor surjective in the non simply-connected case, so we can neither define a Hodge star on \diamond nor a wedge product on Λ . Its kernel on 1-forms is $\text{Ker}(A) = \text{Vect}(d_\diamond \varepsilon)$, where ε is the biconstant, $+1$ on Γ and -1 on Γ^* . But $d_\Lambda A = A d_\diamond$ so it carries cocycles on \diamond to cocycles on Λ . Its image are these cocycles of Λ verifying that their holonomies along cycles of Λ only depend on their homology on the combinatorial surface. Given a 1-cocycle $\mu \in Z^1(\Lambda)$ with such a property, a corresponding 1-cocycle $\nu \in Z^1(\diamond)$ is built in the following way: Choose an edge $(x_0, y_0) \in \diamond_1$; for an edge $(x, y) \in \diamond_1$ with x and x_0 on the same leaf of Λ , choose two paths λ_{x, x_0} and $\lambda_{y_0, y}$ on the double graph Λ , from x to x_0 and y_0 to y respectively, and define

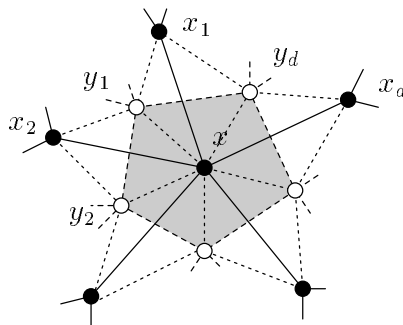
$$(2.10) \quad \int_{(x,y)} \nu := \int_{\lambda_{x, x_0}} \mu + \int_{\lambda_{y_0, y}} \mu - \oint_{[\gamma]} \mu$$

where $[\gamma] = [\lambda_{x,x_0} + (x_0, y_0) + \lambda_{y_0,y} + (y, x)]$ is the class of the full cycle in the homology of the surface. Changing the base points change μ by a multiple of $d_\diamond \varepsilon$.

It follows in the compact case that the dimensions of the harmonic forms on \diamond (the kernel of ΔA) modulo $d\varepsilon$, as well as the harmonic forms on Λ with same holonomies on the graph and on its dual, are twice the genus of the surface, as expected. Unfortunately, the space $\text{Im } A = \mathcal{H}^\perp \oplus \text{Im } d$ is not stable by the Hodge star $*$. We could nevertheless define holomorphic 1-forms on \diamond but their dimension would be much smaller than in the continuous, namely the genus of the surface. Criticality provides conditions which ensure that the space $*\text{Im } A$ is “close” to $\text{Im } A$.



(2.8)



(2.9)

FIGURE 4. Notations.

We construct an *heterogeneous* wedge product for 1-forms: with $\alpha, \beta \in C^1(\Lambda)$, define $\alpha \wedge \beta \in C^1(\diamond)$ by

$$(2.11) \quad \iint_{(x,y,x',y')} \alpha \wedge \beta := \frac{1}{2} \left(\int_{(x,x')} \alpha \int_{(y,y')} \beta + \int_{(y,y')} \alpha \int_{(x',x)} \beta \right).$$

It verifies $A(\alpha_\diamond) \wedge A(\beta_\diamond) = \alpha_\diamond \wedge \beta_\diamond$, the first wedge product being between 1-forms on Λ and the second between forms on \diamond . The usual scalar product on compactly supported forms on Λ reads as expected:

$$(2.12) \quad (\alpha, \beta) = \frac{1}{2} \sum_{e \in \Lambda_1} \rho(e) \left(\int_e \alpha \right) \left(\int_e \bar{\beta} \right) = \iint_{\diamond_2} \alpha \wedge *\bar{\beta}$$

2.3. Energies. The L^2 norm of the 1-form df , called the Dirichlet energy of the function f , is the mean of the usual Dirichlet energies on each independant graph:

(2.13)

$$\begin{aligned} E_D(f) &:= \frac{1}{2} \|df\|^2 = \frac{1}{2} (df, df) = \frac{1}{4} \sum_{(x,x') \in A_1} \rho(x, x') |f(x') - f(x)|^2 \\ &= \frac{E_D(f|_\Gamma) + E_D(f|_{\Gamma^*})}{2}. \end{aligned}$$

Harmonic maps minimize this energy among functions fulfilling certain boundary conditions.

The conformal energy of a map measures its conformality defect, it is null on holomorphic functions:

$$(2.14) \quad E_C(f) := \frac{1}{4} \|df - i * df\|^2.$$

It is related to the Dirichlet energy through the same formula as in the continuous:

$$\begin{aligned} E_C(f) &= \frac{1}{4} (df - i * df, df - i * df) \\ &= \frac{1}{4} \|df\|^2 + \frac{1}{4} \|-i * df\|^2 + \frac{1}{2} \operatorname{Re}(df, -i * df) \\ &= \frac{1}{2} \|df\|^2 + \frac{1}{2} \operatorname{Im} \iint_{\diamond_2} df \wedge \overline{df} \\ (2.15) \quad &= E_D(f) - \mathcal{A}(f) \end{aligned}$$

where the area of the image of the application f in the complex plane has the same formula

$$(2.16) \quad \mathcal{A}(f) = \frac{i}{2} \iint_{\diamond_2} df \wedge \overline{df}$$

as in the continuous case since, for a face $(x, y, x', y') \in \diamond_2$, the algebraic area of the oriented quadrilateral $(f(x), f(x'), f(y), f(y'))$ is given by

$$\begin{aligned} \iint_{(x,y,x',y')} df \wedge \overline{df} &= i \operatorname{Im} \left((f(x') - f(x)) \overline{(f(y') - f(y))} \right) \\ &= -2i \mathcal{A}(f(x), f(x'), f(y), f(y')). \end{aligned}$$

3. PERIOD MATRIX

We use the convention of Farkas and Kra [1], chapter III, to which we refer for details. Consider (\diamond, ρ) a discrete compact Riemann surface.

3.1. Intersection number, on Λ and on \diamond . For a given simple (real) cycle $C \in Z_1(\Lambda)$, we construct a harmonic 1-form η_C such that $\oint_A \eta_C$ counts the algebraic number of times A contains an edge dual to an edge of C : It is the solution of a Neumann problem on the surface cut open along C (see [3] for details). It follows from standard homology technique that η_C depends only on the homology class of C (all the cycles which differ from C by an exact cycle ∂A) and can be extended linearly to all cycles as $\eta_\bullet : H_1(\Lambda) \rightarrow C^1(\Lambda)$; it fulfills, for a closed form θ ,

$$(3.1) \quad \oint_C \theta = \iint_{\diamond} \eta_C \wedge \theta,$$

and a basis of the homology provides a dual basis of harmonic forms on Λ . Beware that if the cycle $C \in Z_1(\Gamma)$ is purely on Γ , then this form $\eta_C|_{\Gamma} = 0$ is null on Γ .

The *intersection number* between two cycles $A, B \in Z_1(\Lambda)$ is defined as

$$(3.2) \quad A \cdot B := \iint_{\diamond} \eta_A \wedge \eta_B.$$

It is obviously linear and antisymmetric, it is an integer number for integer cycles. Let's stress again that the intersection of a cycle on Γ with another cycle on Γ is always null. A cycle $C \in Z_1(\diamond)$ defines a pair of cycles on each graph $C_{\Gamma} \in Z_1(\Gamma)$, $C_{\Gamma^*} \in Z_1(\Gamma^*)$ which are homologous to C on the surface, composed of portions of the boundary of the faces on Λ dual to the vertices of C . They are uniquely defined if we require that they lie "to the left" of C as shown in Fig.5. By the procedure (2.10) applied to $\eta_{C_{\Gamma}} + \eta_{C_{\Gamma^*}}$, we construct a 1-cocycle $\eta_C \in Z^1(\diamond)$ unique up to $d\varepsilon$, and since $\forall \theta, d\varepsilon \wedge \theta = 0$, Eq. (3.2) defines an intersection number on $Z_1(\diamond)$. Unlike the intersection number on Λ , this one has all the usual expected properties. In particular Eq. (3.2) holds for $A, B \in Z_1(\diamond)$.

3.2. Canonical dissection, fundamental polygon. The complex \diamond being connected, consider a maximal tree $T \subset \diamond_1$, that is to say T is a \mathbf{Z}_2 -homologically trivial chain and every edge added to T forms a cycle. A *canonical dissection* or cut-system \aleph of the genus g discrete Riemann surface \diamond is given by a set of oriented edges $(e_k)_{1 \leq k \leq 2g}$ such that the cycles $\aleph \subset (T \cup e_k)$ form a basis of the homology group $H_1(\diamond)$ verifying, for $1 \leq k, \ell \leq g$

$$(3.3) \quad \aleph_k \cdot \aleph_{\ell} = 0, \quad \aleph_{k+g} \cdot \aleph_{\ell+g} = 0, \quad \aleph_k \cdot \aleph_{\ell+g} = \delta_{k,\ell}.$$

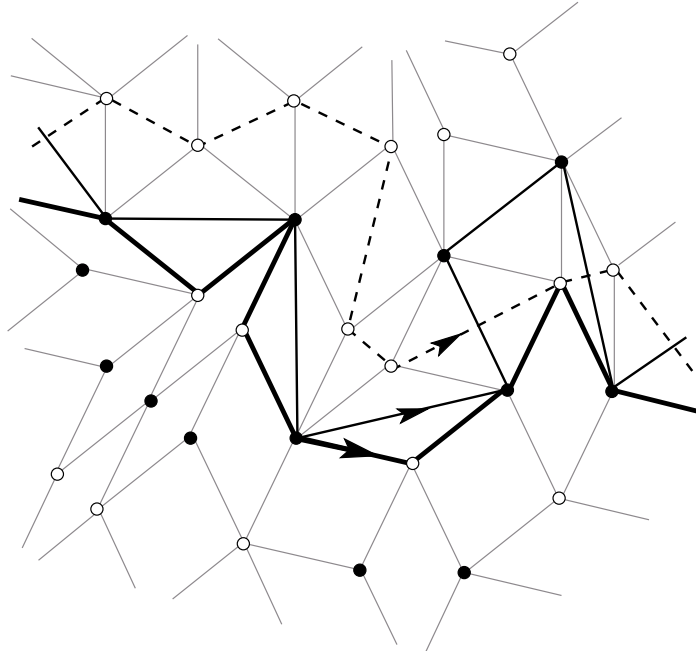


FIGURE 5. A path C on \diamond defines a pair of paths C_Γ and C_{Γ^*} on its left.

They actually form a basis of the fundamental group $\pi_1(\diamond)$ and the defining relation among them is (noted multiplicatively)

$$(3.4) \quad \prod_{k=1}^g \aleph_k \aleph_{k+g} \aleph_k^{-1} \aleph_{k+g}^{-1} = 1.$$

The construction of such a basis is standard and we won't repeat the procedure. What is less standard is the interpretation of Eq. (3.4) in terms of the boundary of a fundamental domain, discretization introduces some subtleties (that can safely be kept in first instance). We end up with the familiar $2g \times 2g$ intersection numbers matrix on \diamond .

Considering $T \cup e_k$ as a rooted graph, we can prune it of all its pending branches, leaving a simple closed loop \aleph_k^- , attached to the origin O by a simple path λ_k (see Fig. 6), yielding the cycle \aleph_k . These three cycles are deformation retract of one another, $\aleph_k^- \subset \aleph_k \subset T \cup e_k$ hence are equal in homology.

In the continuous case, a basis of the homology can be realized by $2g$ simple arcs, transverse to one another and meeting only at the base point. It defines an isometric model of the surface as a fundamental domain homeomorphic to a disc and bordered by $4g$ arcs to identify pairwise. In the discrete case, by definition, the set $\diamond \setminus \aleph$ of the cellular

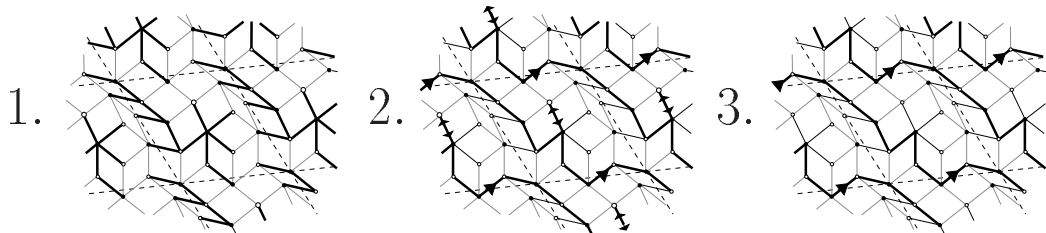


FIGURE 6. 1. A maximal rooted tree in a quadrilateral decomposition of the torus. 2. An additional edge defines a rooted cycle \aleph_1 , pruned of its dangling trees. 3. Its unrooted version, the simple loop \aleph_1^- .

complex minus the edges taking part into the cycles basis is homeomorphic to a disc hence the surface is realized as a polygonal *fundamental domain* \mathcal{M} whose boundary edges are identified pairwise.

But it is sometimes impossible to choose a basis of the homology verifying (3.3) by simple discrete cycles which are transverse to one another. For instance, if the path λ_k is not empty, the cycle \aleph_k is not even simple. Moreover, some edges may belong to several cycles. In this case, the edges on the boundary of this fundamental polygon can not be assigned a unique element of the basis or its inverse, and therefore can not be grouped into only $4g$ continuous paths to identify pairwise but more than $4g$.

In fact, the information contained into the basis \aleph is more than simply this polygon, the set of edges composing the concatenated cycle

$$(3.5) \quad (\aleph_1, \aleph_{g+1}, \aleph_1^{-1}, \aleph_{g+1}^{-1}, \aleph_2, \dots, \aleph_g^{-1}, \aleph_{2g}^{-1})$$

encodes a cellular complex \mathcal{M}_+ which is *not* a combinatorial surface and consists of the fundamental polygon \mathcal{M} plus some *dangling trees*, corresponding to the edges which belong to more than one cycle or participate more than once in a cycle (the paths λ_k), as exemplified in Fig.7. By construction, the edge e_k belongs to the cycle \aleph_k only, hence these trees are in fact without branches, simple paths whose only leaf is the base point O . To retrieve the surface, the edges of this structure \mathcal{M}_+ are identified group-wise, an edge participating k times in cycles will have $[k/2] + 2$ representatives to identify together, two on the fundamental polygon and the rest as edges of dangling trees.

Eliminating repetition, that is to say looking at (3.5) not as a *sequence* of edges but as a simplified *cycle* (or a simplified word in edges), thins \mathcal{M}_+ into \mathcal{M} , pruning away the dangling paths. The fundamental

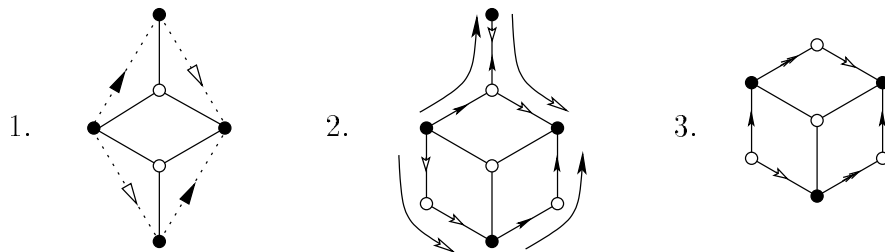


FIGURE 7. Three different fundamental polygons of a decomposition of the torus ($g = 1$) by three quadrilaterals: 1. The standard fundamental domain where the $4g$ paths are not adapted to \diamond . 2. \mathcal{M}_+ is composed of edges of \diamond composing $4g$ arcs (which may have portions in common) to identify pairwise, each edge corresponds to an element of the basis \aleph or its inverse, except for edges of “dangling trees” which are associated with two such elements. 3. \mathcal{M} is composed of edges of \diamond composing more than $4g$ arcs to identify pairwise, there is no correspondence with a basis of cycles.

polygon boundary loses its structure as $4g$ arcs to be identified pairwise, in general a basis cycle will be disconnected around the fundamental domain and a given edge can not be assigned to a particular cycle. This peculiarity gives a more complex yet well defined meaning to the contour integral formula for a 1-form θ defined on the boundary edges of \mathcal{M}_+ ,

$$(3.6) \quad \oint_{\partial \mathcal{M}} \theta = \sum_{k=1}^{2g} \oint_{\aleph_k} \theta + \oint_{\aleph_k^{-1}} \theta.$$

This basis gives rise to cycles \aleph^Γ and \aleph^{Γ^*} whose homology classes form a basis of the group for each respective graph, that we compose into \aleph^Λ defined by

$$(3.7) \quad \begin{aligned} \aleph_k^\Lambda &= \aleph_k^\Gamma, & \aleph_{k+g}^\Lambda &= \aleph_k^{\Gamma^*}, \\ \aleph_{k+2g}^\Lambda &= \aleph_{k+g}^{\Gamma^*}, & \aleph_{k+3g}^\Lambda &= \aleph_{k+g}^\Gamma, \end{aligned}$$

for $1 \leq k \leq g$ so that while the intersection numbers matrix on \diamond is given by the $2g \times 2g$ matrix

$$(3.8) \quad (\aleph_k \cdot \aleph_\ell)_{k,\ell} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

the intersection numbers matrix on Λ is the $4g \times 4g$ matrix with the same structure

$$(3.9) \quad (\aleph_k^\Lambda \cdot \aleph_\ell^\Lambda)_{k,\ell} = \begin{pmatrix} \Gamma & \Gamma^* & \Gamma^* & \Gamma \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \end{pmatrix} \begin{matrix} \Gamma \\ \Gamma^* \\ \Gamma^* \\ \Gamma \end{matrix}.$$

3.3. Bilinear relations.

Proposition 3.1. *Given a canonical dissection \aleph , for two closed forms $\theta, \theta' \in Z^1(\diamond)$,*

$$(3.10) \quad \iint_{\diamond} \theta \wedge \theta' = \sum_{j=1}^g \left(\oint_{\aleph_j} \theta \oint_{\aleph_{j+g}} \theta' - \oint_{\aleph_{j+g}} \theta \oint_{\aleph_j} \theta' \right);$$

for two closed forms $\theta, \theta' \in Z^1(\Lambda)$,

$$(3.11) \quad \iint_{\diamond} \theta \wedge \theta' = \sum_{j=1}^{2g} \left(\oint_{\aleph_j^\Lambda} \theta \oint_{\aleph_{j+2g}^\Lambda} \theta' - \oint_{\aleph_{j+2g}^\Lambda} \theta \oint_{\aleph_j^\Lambda} \theta' \right).$$

Proof 3.1. Each side is bilinear and depends only on the cohomology classes of the forms. Decompose the forms onto the cohomology basis (α_k) . On Λ , use Eq (3.15) for the LHS and the duality property Eq. (3.14) for the RHS. On \diamond , use their counterparts. \diamond

Notice that for a harmonic form $\theta \in \mathcal{H}^1(\Lambda)$, the form $*\theta$ is closed as well, therefore its norm is given by

$$(3.12) \quad \theta \in \mathcal{H}^1(\Lambda) \implies \|\theta\|^2 = \sum_{j=1}^{2g} \left(\oint_{\aleph_j} \theta \oint_{\aleph_{j+2g}} *\bar{\theta} - \oint_{\aleph_{j+2g}} \theta \oint_{\aleph_j} *\bar{\theta} \right).$$

3.4. Basis of harmonic forms, basis of holomorphic forms. We define α^Λ , the basis of real harmonic 1-forms, dual to the homology basis \aleph^Λ , as described in Sec. 3.1,

$$(3.13) \quad \begin{aligned} \alpha_k^\Lambda &:= \eta_{\aleph_{k+2g}^\Lambda} & \text{and} \\ \alpha_{k+2g}^\Lambda &:= -\eta_{\aleph_k^\Lambda} & \text{for } 1 \leq k \leq 2g \end{aligned}$$

which verify

$$(3.14) \quad \begin{aligned} \oint_{\aleph_k^\Lambda} \alpha_\ell &= \delta_{k,\ell}, \\ \oint_{\aleph_{k+2g}^\Lambda} \alpha_{\ell+2g} &= \delta_{k,\ell}, \end{aligned}$$

and dually, the intersection matrix elements are given by

$$(3.15) \quad \aleph_k^\Lambda \cdot \aleph_\ell^\Lambda = \iint_{\diamond} \alpha_k^\Lambda \wedge \alpha_\ell^\Lambda = (\alpha_k^\Lambda, - * \alpha_\ell^\Lambda).$$

On \diamond , the elements $\alpha_k^\diamond := \eta_{\aleph_{k+g}}$ and $\alpha_{k+g}^\diamond := -\eta_{\aleph_k}$ for $1 \leq k \leq g$, defined up to $d\varepsilon$, verify $A(\alpha_k^\diamond) = \alpha_k^\Lambda + \alpha_{k+g}^\Lambda$, $A(\alpha_{k+g}^\diamond) = \alpha_{k+2g}^\Lambda + \alpha_{k+3g}^\Lambda$ and form a basis of the cohomology on \diamond dual to \aleph as well,

$$(3.16) \quad \begin{aligned} \alpha_k^\diamond &:= \eta_{\aleph_{k+g}} & \text{and} \\ \alpha_{k+g}^\diamond &:= -\eta_{\aleph_k} & \text{for } 1 \leq k \leq g, \end{aligned}$$

they fulfill the first identity in Eq.(3.15) but the second is meaningless in general since $*$ can not be defined on \diamond . We will drop the mention Λ when no confusion is possible.

Proposition 3.2. *The matrix of inner products on Λ ,*

$$(3.17) \quad (\alpha_k, \alpha_\ell)_{k,\ell} = \iint_{\diamond} \alpha_k \wedge * \bar{\alpha}_\ell = \begin{cases} + \oint_{\aleph_{k+2g}} * \alpha_\ell, & 1 \leq k \leq 2g, \\ - \oint_{\aleph_{k-2g}} * \alpha_\ell, & 2g < k \leq 4g. \end{cases} =: \begin{pmatrix} A & D \\ B & C \end{pmatrix}$$

is a real symmetric positive definite matrix.

Proof 3.2. It is real because the forms are real, and symmetric because the scalar product (2.12) is skew symmetric. Definition Eq. (3.13) and Eq. (3.1) lead to the integral formulae. Positivity follows from the bilinear relation Eq. (3.11): for $\theta = \sum_{k=1}^{4g} \xi_k \alpha_k$, with $\xi_k \in \mathbf{C}$, $\sum_{k=1}^{4g} |\xi_k|^2 > 0$,

$$(3.18) \quad \begin{aligned} \|\theta\|^2 &= \sum_{j=1}^{2g} \left[\int_{\aleph_j} \theta \int_{\aleph_{2g+j}} * \bar{\theta} - \int_{\aleph_{2g+j}} \theta \int_{\aleph_{2j}} * \bar{\theta} \right] \\ &= \sum_{k,\ell=1}^{4g} \xi_k \bar{\xi}_\ell \sum_{j=1}^{2g} \left[\int_{\aleph_j} \alpha_k \int_{\aleph_{2g+j}} * \alpha_\ell - \int_{\aleph_{2g+j}} \alpha_k \int_{\aleph_{2j}} * \alpha_\ell \right] \\ &= \sum_{k,\ell=1}^{4g} \xi_k \bar{\xi}_\ell (\alpha_k, \alpha_\ell) > 0. \end{aligned}$$

◇

The form α_k is supported by only one of the two graphs Γ or Γ^* , the form $*\alpha_k$ is supported by the other one, and the wedge product $\theta_\Gamma \wedge \theta'_\Gamma = 0$ is null for two 1-forms supported by the same graph. Therefore the matrices A and C are $g \times g$ -block diagonal and B is

anti-diagonal.

$$(3.19) \quad A = \begin{pmatrix} A_\Gamma & 0 \\ 0 & A_{\Gamma^*} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & B_{\Gamma^*,\Gamma} \\ B_{\Gamma,\Gamma^*} & 0 \end{pmatrix}, \quad C = \begin{pmatrix} C_{\Gamma^*} & 0 \\ 0 & C_\Gamma \end{pmatrix}.$$

The matrices of intersection numbers (3.9) and of inner products differ only by the Hodge star $*$. Because $*$ preserves harmonic forms and the inner product, we get its matrix representation in the basis α ,

$$(3.20) \quad * = \begin{pmatrix} -D & A \\ -C & B \end{pmatrix}$$

and because $*^2 = -1$,

$$(3.21) \quad B^2 - C \cdot A + I = 0$$

$$(3.22) \quad A \cdot B = {}^t B \cdot A$$

$$(3.23) \quad C \cdot {}^t B = B \cdot C.$$

On \diamond , while the Hodge star $*$ can not be defined, we can obviously consider the following positive scalar product on the classes of closed forms modulo $d\varepsilon$, to which the set (α_k^\diamond) belong:

$$\begin{aligned} (\alpha^\diamond, \beta^\diamond) &:= (A(\alpha^\diamond), A(\beta^\diamond)) \\ &= \sum_{\substack{(x,y,x',y') \in \diamond_2 \\ \rho=\rho(x,x'), \rho^*=\rho(y,y')}} {}^t \begin{pmatrix} \int_{(x,y)} \alpha \\ \int_{(y,x')} \alpha \\ \int_{(x',y')} \alpha \\ \int_{(y',x)} \alpha \end{pmatrix} \cdot \begin{pmatrix} +\rho+\rho^* & +\rho-\rho^* & -\rho-\rho^* & -\rho+\rho^* \\ +\rho-\rho^* & +\rho+\rho^* & -\rho+\rho^* & -\rho-\rho^* \\ -\rho-\rho^* & -\rho+\rho^* & +\rho+\rho^* & +\rho-\rho^* \\ -\rho+\rho^* & -\rho-\rho^* & +\rho-\rho^* & +\rho+\rho^* \end{pmatrix} \cdot \begin{pmatrix} \int_{(x,y)} \bar{\beta} \\ \int_{(y,x')} \bar{\beta} \\ \int_{(x',y')} \bar{\beta} \\ \int_{(y',x)} \bar{\beta} \end{pmatrix}. \end{aligned}$$

and it yields

$$(3.24) \quad (\alpha_k^\diamond, \alpha_\ell^\diamond)_{k,\ell} = \begin{pmatrix} A_\Gamma + A_{\Gamma^*} & {}^t B_{\Gamma\Gamma^*} + {}^t B_{\Gamma^*\Gamma} \\ B_{\Gamma\Gamma^*} + B_{\Gamma^*\Gamma} & C_\Gamma + C_{\Gamma^*} \end{pmatrix},$$

which, in general, can not be understood as the periods of a set of forms on \diamond along the basis \aleph .

Let's decompose the space of harmonic forms into two orthogonal supplements,

$$(3.25) \quad \mathcal{H}^1(\Lambda) = \mathcal{H}_\parallel^1 \oplus^\perp \mathcal{H}_\perp^1$$

where the first vector space are the harmonic forms whose holonomies on one graph are equal to their holonomies on the dual, that is to say

$$(3.26) \quad \mathcal{H}_\parallel^1 := \text{Vect} (\alpha_k + \alpha_{k+g}, 1 \leq k \leq g \text{ or } 2g < k \leq 3g).$$

Definition (3.13) and Eq. (3.1) imply that

$$(3.27) \quad \mathcal{H}_\perp^1 = \text{Vect} (*\alpha_k - *\alpha_{k+g}, 1 \leq k \leq g \text{ or } 2g < k \leq 3g).$$

These elements in the basis $(\alpha_k + \alpha_{k+g}, ; \alpha_k - \alpha_{k+g})$ for $1 \leq k \leq g$ and $2g < k \leq 3g$, are represented by the following invertible matrix:

$$(3.28) \quad \begin{pmatrix} I & 0 & {}^t B_{\Gamma\Gamma^*} - {}^t B_{\Gamma^*\Gamma} & A_\Gamma - A_{\Gamma^*} \\ 0 & I & C_\Gamma - C_{\Gamma^*} & B_{\Gamma\Gamma^*} - B_{\Gamma^*\Gamma} \\ 0 & 0 & {}^t B_{\Gamma\Gamma^*} + {}^t B_{\Gamma^*\Gamma} & A_\Gamma + A_{\Gamma^*} \\ 0 & 0 & C_\Gamma + C_{\Gamma^*} & B_{\Gamma\Gamma^*} + B_{\Gamma^*\Gamma} \end{pmatrix}.$$

It implies in particular that the lower right $g \times g$ block is invertible, therefore so is Eq. (3.24).

3.5. Period matrix.

Proposition 3.3. *The matrix $\Pi = C^{-1} \cdot (i - B)$ is the period matrix of the basis of holomorphic forms*

$$(3.29) \quad \zeta_k := (i - *) \sum_{\ell=1}^{2g} C_{k,\ell}^{-1} \alpha_{\ell+2g}$$

in the canonical dissection \aleph , that is to say

$$(3.30) \quad \oint_{\aleph_k} \zeta_\ell = \begin{cases} \delta_{k,\ell} & \text{for } 1 \leq k \leq 2g, \\ \Pi_{k-2g,\ell} & \text{for } 2g < k \leq 4g, \end{cases}$$

and Π is symmetric, with a positive definite imaginary part.

The proof is essentially the same as in the continuous case [1] and we include it for completeness.

Proof 3.3. Let $\omega_j := \alpha_j + i * \alpha_j$ for $1 \leq j \leq 4g$. These holomorphic forms fulfill

$$(3.31) \quad P_{k,j} := \frac{1}{2}(\omega_k, \omega_j) = (\alpha_k, \alpha_j) + i(\alpha_k, - * \alpha_j)$$

$$(3.32) \quad = \begin{cases} -i \int_{\aleph_{j+2g}} \omega_k, & 1 \leq j \leq 2g, \\ i \int_{\aleph_{j-2g}} \omega_k, & 2g < j \leq 4g. \end{cases}$$

P is the period matrix of the forms (ω) in the homology basis \aleph . The first $2g$ forms $(\omega_j)_{1 \leq j \leq 2g}$ are a basis of holomorphic forms. It has the right dimension and they are linearly independent:

$$(3.33) \quad \begin{aligned} \sum_{j=1}^{2g} (\lambda_j + i\mu_j)(\alpha_j + i * \alpha_j) &= \sum_{j=1}^{2g} \left((\lambda_j + \sum_{k=1}^{2g} \mu_k B_{j,k}) \alpha_j + \sum_{k=1}^{2g} \mu_k C_{j,k} \alpha_{2g+j} \right) \\ &+ i \sum_{j=1}^{2g} \left((\mu_j + \sum_{k=1}^{2g} \lambda_k B_{j,k}) \alpha_j + \sum_{k=1}^{2g} \lambda_k C_{j,k} \alpha_{2g+j} \right) \end{aligned}$$

is null, for $\lambda, \mu \in \mathbf{R}$ only when $\lambda = \mu = 0$ because C is positive definite. Similarly for the last $2g$ forms. The change of basis iC^{-1} on

them provides the basis of holomorphic forms (ζ) . The last $2g$ rows of P is the $2g \times 4g$ matrix $(B - iI, C)$ hence the periods of (ζ) in \aleph are given by (I, Π) . \diamond

The first identity in Eq.(3.30) uniquely defines the basis ζ and a holomorphic 1-form is completely determined by whether its periods on the first $2g$ cycles of \aleph , or their real parts on the whole set.

Notice that because C is $g \times g$ block diagonal and B is anti-diagonal, Π is decomposed into four $g \times g$ blocks, the two diagonal matrices form iC^{-1} and are pure imaginary, the other two form $-C^{-1} \cdot B$ and are real.

$$(3.34) \quad \Pi = \begin{pmatrix} \Pi_{i^*} & \Pi_r \\ \Pi_{r^*} & \Pi_i \end{pmatrix} = \begin{pmatrix} iC_{\Gamma^*}^{-1} & -C_{\Gamma^*}^{-1} \cdot B_{\Gamma^*, \Gamma} \\ -C_{\Gamma}^{-1} \cdot B_{\Gamma, \Gamma^*} & iC_{\Gamma}^{-1} \end{pmatrix}.$$

Therefore the holomorphic forms ζ_k are real on one graph and pure imaginary on its dual,

$$(3.35) \quad \begin{aligned} 1 \leq k \leq g &\Rightarrow \zeta_k \in C_{\mathbf{R}}^1(\Gamma) \oplus iC_{\mathbf{R}}^1(\Gamma^*) \\ g < k \leq 2g &\Rightarrow \zeta_k \in C_{\mathbf{R}}^1(\Gamma^*) \oplus iC_{\mathbf{R}}^1(\Gamma). \end{aligned}$$

We will call

$$(3.36) \quad \Pi_{\Gamma} = \Pi_r + \Pi_{i^*}$$

the period matrix on the graph Γ the sum of the real periods of ζ_k , $1 \leq k \leq g$, on Γ , with the associated pure imaginary periods on the dual Γ^* , and similarly for ζ_k , $g < k \leq 2g$, the period matrix on Γ^* .

It is natural to ask how close Π_{Γ} and Π_{Γ^*} are from one another, and whether their mean can be given an interpretation. Criticality [3, 4] answers partially the issue:

Theorem 3.1. *In the genus one critical case, the period matrices Π_{Γ} and Π_{Γ^*} are equal to the period matrix Π_{Σ} of the underlying surface Σ . For higher genus, given a refining sequence (\diamond^k, ρ_k) of critical maps of Σ , the discrete period matrices Π_{Γ^k} and $\Pi_{\Gamma^{*k}}$ converge to the period matrix Π_{Σ} .*

Proof 3.1. The genus one case is postponed to Sec. 3.6. The continuous limit comes from techniques in [3, 4], developed in [5] which prove that, given a refining sequence of critical maps, any holomorphic function can be approximated by a sequence of discrete holomorphic functions. Taking the real parts, this implies as well that any harmonic function can be approximated by discrete harmonic functions. In particular, the discrete solutions f_k to a Dirichlet or Neumann problem on a simply connected set converge to the continuous solution f because the latter can be approximated by discrete harmonic functions g_k and

the difference $f_k - g_k$ being harmonic and small on the boundary, converge to zero. In particular, each form in the basis (α_ℓ^\diamond) , provides a solution to the Neumann problem Eq. (3.16) and a similar procedure, detailed afterwards, define a converging sequence of forms ζ_ℓ^\diamond , yielding the result. \diamond

We can try to replicate the work done on Λ on the graph \diamond . A problem is that $A_\Gamma + A_\Gamma$ and $C_\Gamma + C_\Gamma$ need not be positive definite. Moreover, the Hodge star $*$ doesn't preserve the space $(A(\alpha_k^\diamond))$ of harmonic forms with equal holonomies on the graph and on its dual, so we can not define the analogue of $\alpha + i * \alpha$ on \diamond . We first investigate what happens when we can partially define these analogues:

Assume that for $2g < k \leq 3g$, the holonomies of $*\alpha_k$ on Γ are equal to the holonomies of $*\alpha_{k+g}$ on Γ^* , that is to say $C_\Gamma = C_{\Gamma^*} =: \frac{1}{2}C_\diamond$ and $D_{\Gamma\Gamma^*} = D_{\Gamma^*\Gamma} =: \frac{1}{2}D_\diamond$. It implies that the transposes fulfill $B_{\Gamma\Gamma^*} = B_{\Gamma^*\Gamma} =: \frac{1}{2}B_\diamond$ as well. We can then define $\beta_{k-g}^\diamond \in Z^1(\diamond)$ such that $A(\beta_{k-g}^\diamond) = *\alpha_{k+g}$, uniquely up to $d\varepsilon$. The last g columns ${}^t(B_\diamond, C_\diamond)$ of the matrix of scalar product Eq. (3.24) are related to their periods in the homology basis \aleph^\diamond in a way similar to Eq. (3.17). By the same reasoning as before, the forms

$$(3.37) \quad \zeta_k^\diamond = \sum_{\ell=1}^g C_{\diamond, k, \ell}^{-1} (\alpha_{\ell+g}^\diamond - i\beta_{\ell+g}^\diamond), \quad 1 \leq k \leq g$$

verify $A(\zeta_k^\diamond) = \frac{\zeta_k + \zeta_{k+g}}{2}$ and have periods on \aleph^\diamond given by the identity for the first g cycles and the following $g \times g$ matrix, mean of the period matrices on the graph and on its dual:

$$(3.38) \quad \Pi^\diamond = C_\diamond^{-1}(i - B_\diamond) = \frac{\Pi_\Gamma + \Pi_{\Gamma^*}}{2}.$$

The same reasoning applies when the periods of the forms $*\alpha_k$ on the graph and on its dual are not equal but close to one another. In the context of refining sequences, we said that the basis (α_ℓ^\diamond) , converges to the continuous basis of harmonic forms defined by the same Neumann problem Eq. (3.16). Therefore

$$(3.39) \quad C_\Gamma - C_{\Gamma^*} = o(1), \quad B_{\Gamma\Gamma^*} - B_{\Gamma^*\Gamma} = o(1).$$

A harmonic form $\nu_{k+g} = o(1)$ on Γ^* can be added to $*\alpha_{k+g}$ such that there exists $\beta_{k-g}^\diamond \in Z^1(\diamond)$ with $A(\beta_{k-g}^\diamond) = *\alpha_{k+g} + \nu_{k+g}$, yielding forms ζ_k^\diamond , verifying $A(\zeta_k^\diamond) = \frac{1}{2}(\zeta_k + \zeta_{k+g}) + o(1)$ and whose period matrix is $\Pi^\diamond + o(1)$. Since the periods of α_k converge to the same periods as its continuous limit, this period matrix converges to the period matrix Π_Σ of the surface. Which is the claim of Th. 3.1.

In the paper [10], R. Costa-Santos and B. McCoy define a period matrix on a special cellular decomposition Γ of a surface by squares. They don't consider the dual graph Γ^* . Their period matrix is equal to one of the two diagonal blocks of the double period matrix we construct in this case. They don't have to consider the off-diagonal blocks because the problem is so symmetric that their period matrix is pure imaginary.

3.6. Genus one case. Criticality solves partially the problem of having two different $g \times g$ period matrices instead of one since they converge to one another in a refining sequence. However, on a genus one critical torus, the situation is simpler: The overall curvature is null and a critical map is everywhere flat. Therefore the cellular decomposition is the quotient of a periodic cellular decomposition of the plane by two independent periods. They can be normalized to $(1, \tau)$. The continuous period matrix is the 1×1 -matrix τ . A basis of the two dimensional holomorphic 1-forms is given by the real and imaginary parts of dZ on Γ and Γ^* respectively, and the reverse. The discrete period matrix is the 2×2 matrix $\begin{pmatrix} \operatorname{Im} \tau & \operatorname{Re} \tau \\ \operatorname{Re} \tau & \operatorname{Im} \tau \end{pmatrix}$ and the period matrices on the graph and on its dual are both equal to the continuous one.

For illustration purposes, the whole construction, of a basis of harmonic forms, then projected onto a basis of holomorphic forms, yielding the period matrix, can be checked explicitly on the critical maps of the genus 1 torus decomposed by square or triangular/hexagonal lattices:

Consider the critical square (rectangular) lattice decomposition of a torus $\diamond = (\mathbb{Z}e^{i\theta} + \mathbb{Z}e^{-i\theta})/(2pe^{i\theta} + 2qe^{-i\theta})$, with horizontal parameter $\rho = \tan \theta$ and vertical parameter its inverse. Its modulus is $\tau = \frac{q}{p}e^{2i\theta}$. The two dual graphs Γ and Γ^* are isomorphic. An explicit harmonic form α_1^Γ is given by the constant $1/2p$ on horizontal and downwards edges of the graph Γ and 0 on all the other edges. Its holonomies are 1 and 0 on the p , resp. q cycles. Considering $1/2q$ and the dual graph, we construct in the same fashion $\alpha_2^\Gamma, \alpha_1^{\Gamma^*}, \alpha_2^{\Gamma^*}$. The matrix of inner products is

$$(3.40) \quad (\alpha_k, \alpha_\ell)_{k,\ell} = \frac{1}{\sin 2\theta} \begin{pmatrix} \frac{q}{p} & & & \cos 2\theta \\ & \frac{q}{p} & \cos 2\theta & \\ & \cos 2\theta & \frac{p}{q} & \\ \cos 2\theta & & & \frac{p}{q} \end{pmatrix}$$

using $\frac{\rho+1/\rho}{2} = 1/\sin 2\theta$ and $\frac{\rho-1/\rho}{2} = -1/\tan 2\theta$ so that the period matrix is

$$(3.41) \quad \Pi = \frac{q}{p} \begin{pmatrix} i \sin 2\theta & \cos 2\theta \\ \cos 2\theta & i \sin 2\theta \end{pmatrix}.$$

Therefore there exists a holomorphic form which has the same periods on the graph and on its dual, it is the average of the two half forms of Eq. (3.30) and its periods are $(1, \frac{q}{p}e^{2i\theta})$ along the p , resp. q cycles, yielding the continuous modulus. This holomorphic form is simply the normalized fundamental form $\frac{dZ}{pe^{-i\theta}}$.

In the critical triangular/hexagonal lattice, we just point out to the necessary check by concentrating on a tile of the torus, composed of two triangles, pointing up and down respectively. We show that there exists an explicit holomorphic form which has the same shift on the graph and on its dual, along this tile. Let ρ_-, ρ_\setminus and $\rho_/$ the three parameters around a given triangle. Criticality occurs when $\rho_- \rho_\setminus + \rho_\setminus \rho_/ + \rho_/ \rho_- = 1$. The form which is 1 on the rightwards and South-West edges and 0 elsewhere is harmonic on the triangular lattice. Its pure imaginary companion on the dual hexagonal lattice exhibits a shift by $i \rho_\setminus$ in the horizontal direction and $i(\rho_\setminus + \rho_-)$ in the North-East direction along the tile. Dually, on the hexagonal lattice, the form which is $\rho_\setminus \rho_-$ along the North-East and downwards edges and $1 - \rho_\setminus \rho_-$ along the South-East edges, is a harmonic form. Its shift in the horizontal direction is 1, in the North-East direction 0, and its pure imaginary companion on the triangular lattice exhibits a shift by $i \rho_\setminus$ in the horizontal direction and $i(\rho_\setminus + \rho_-)$ in the North-East direction along the tile as before. Hence their sum is a holomorphic form with equal holonomies on the triangular and hexagonal graphs and the period matrix it computes is the same as the continuous one. This simply amounts to pointing out that the fundamental form dz can be explicitly expressed in terms of the discrete conformal data.

4. CRITICALITY AND INTEGRABLE SYSTEM

This theory can be viewed as the simplest (it is linear) of a series of integrable theories [8]. We will present its quadratic counterpart, which leads to another version of discrete analytic functions, based on circle patterns. Along the way, we will see how discrete exponentials and discrete polynomials emerge due to integrable systems theory pieces of technology named the Bäcklund or Darboux transform [8].

4.1. Criticality. Until now, everything has been purely combinatorial, there was no reference to an underlying geometry and no continuous limit. Criticality is what links combinatorics and geometry, and what gives a meaning to approximation theorems.

Definition 4.1. *A discrete conformal map (\diamond, ρ) is critical if there exists a discrete holomorphic map Z such that the quadrilateral faces \diamond_2 can be simultaneously embedded into **rhombi** in the complex plane.*

Because of the Gauss-Bonnet theorem, it is not possible to globally embed a compact surface into the plane, therefore we allow for an atlas of local critical maps with a finite number of fixed local conic singularities. When a continuous limit is taken, their number, angle and position should not change, and the theorem of isolated singularities helps us wipe them out as inessential.

It is a simple calculation to check that if Z is a critical map, any discrete holomorphic function $f \in \Omega(\diamond)$ gives rise, through (2.5) to a holomorphic 1-form $f dZ$.

For a holomorphic function f , the equality $f dZ \equiv 0$ is equivalent to $f = \lambda \varepsilon$ for some $\lambda \in \mathbb{C}$ with ε the biconstant $\varepsilon(\Gamma) = +1$, $\varepsilon(\Gamma^*) = -1$.

Following Duffin [11, 12], we introduce the

Definition 4.2. *For a holomorphic function f , define on a flat simply connected map U the holomorphic functions f^\dagger , the dual of f , and f' , the derivative of f , by the following formulae:*

$$(4.1) \quad f^\dagger(z) := \varepsilon(z) \bar{f}(z),$$

where \bar{f} denotes the complex conjugate, $\varepsilon = \pm 1$ is the biconstant, and

$$(4.2) \quad f'(z) := \frac{4}{\delta^2} \left(\int_O^z f^\dagger dZ \right)^\dagger + \lambda \varepsilon,$$

defined up to ε .

It is an immediate calculation [4] to check the following

Proposition 4.3. *The derivative f' fulfills*

$$(4.3) \quad df = f' dZ.$$

4.2. $\bar{\partial}$ operator. For holomorphic or anti-holomorphic functions, df is, locally on each pair of dual diagonals, proportional to dZ , resp. $d\bar{Z}$, we define ∂ and $\bar{\partial}$ operators (not to be confused with the boundary operator on chains) that decompose the exterior derivative into holomorphic and anti-holomorphic parts yielding

$$df \wedge \bar{df} = (|\partial f|^2 + |\bar{\partial} f|^2) dZ \wedge d\bar{Z}$$

where the derivatives naturally live on faces:

In the continuous theory, for any derivable function f on the complex plane, the derivatives $\partial = \frac{d}{dx} + i \frac{d}{dy}$ and $\bar{\partial} = \frac{d}{dx} - i \frac{d}{dy}$ with respect to $z = x + iy$ and $\bar{z} = x - iy$ yield

$$f(z + z_0) = f(z_0) + z(\partial f)(z_0) + \bar{z}(\bar{\partial} f)(z_0) + o(|z|).$$

These derivatives can be seen as a limit of a contour integral over a small loop γ around z_0 :

$$(\partial f)(z_0) = \lim_{\gamma \rightarrow z_0} \frac{i}{2\mathcal{A}(\gamma)} \oint_{\gamma} f d\bar{z}, \quad (\bar{\partial} f)(z_0) = - \lim_{\gamma \rightarrow z_0} \frac{i}{2\mathcal{A}(\gamma)} \oint_{\gamma} f dZ,$$

which leads to the following definitions in the discrete setup:

$$\begin{aligned} \partial : C^0(\diamond) &\rightarrow C^2(\diamond) \\ f &\mapsto \partial f = [(x, y, x', y') \mapsto \frac{i}{2\mathcal{A}(x, y, x', y')} \oint_{(x, y, x', y')} f d\bar{Z}], \end{aligned}$$

$$\begin{aligned} \bar{\partial} : C^0(\diamond) &\rightarrow C^2(\diamond) \\ f &\mapsto \bar{\partial} f = [(x, y, x', y') \mapsto -\frac{i}{2\mathcal{A}(x, y, x', y')} \oint_{(x, y, x', y')} f dZ]. \end{aligned}$$

A holomorphic function f verifies $\bar{\partial} f \equiv 0$ and (with $Z(u)$ noted simply u)

$$\partial f(x, y, x', y') = \frac{f(y') - f(y)}{y' - y} = \frac{f(x') - f(x)}{x' - x}.$$

Notice that the statement $f = (\int \partial f dz)$ has no meaning, ∂ is not a derivation endomorphism in the space of functions on the vertices of the double.

On the other hand, these differential operators can be extended (see [6]) into operators (the Kasteleyn operator) $\partial_{20}, \bar{\partial}_{20} : C^2(\diamond) \rightarrow C^0(\diamond)$ simply by transposition, $\partial_{20} = -{}^t \partial_{02}$, leading to endomorphisms of $C^0(\diamond) \oplus C^2(\diamond)$. They are such that their composition, restricted to the vertices \diamond_0 , gives back the laplacian:

$$\Delta = \frac{1}{2} (\partial \circ \bar{\partial} + \bar{\partial} \circ \partial).$$

Furthermore, the double derivative $\partial_{20} \circ \bar{\partial}_{02}$ is a well defined endomorphism of $C^0(\diamond)$.

4.3. Discrete exponential.

Definition 4.4. For a constant $\lambda \in \mathbb{C}$, the discrete exponential $\exp(:\lambda: Z)$ is the solution of

$$(4.4) \quad \begin{aligned} \exp(:\lambda: O) &= 1 \\ d \exp(:\lambda: Z) &= \lambda \exp(:\lambda: Z) dZ. \end{aligned}$$

We define its derivatives with respect to the continuous parameter λ :

$$(4.5) \quad Z^{:k:} \exp(:\lambda: Z) := \frac{\partial^k}{\partial \lambda^k} \exp(:\lambda: Z).$$

The discrete exponential on the square lattice was defined by Lelong-Ferrand [13], generalized in [14] and studied independently in [6, 15]. For $|\lambda| \neq 2/\delta$, an immediate check shows that it is a rational fraction in λ at every point: For the vertex $x = \sum \delta e^{i\theta_k}$,

$$(4.6) \quad \exp(:\lambda: x) = \prod_k \frac{1 + \frac{\lambda\delta}{2} e^{i\theta_k}}{1 - \frac{\lambda\delta}{2} e^{i\theta_k}}$$

where (θ_k) are the angles defining $(\delta e^{i\theta_k})$, the set of $(Z$ -images of) \diamond -edges between x and the origin. Because the map is critical, Eq. (4.6) only depends on the end points (O, x) . It is a generalization of a well known formula, in a slightly better version,

$$(4.7) \quad \exp(\lambda x) = \left(1 + \frac{\lambda x}{n}\right)^n + O\left(\frac{\lambda^2 x^2}{n}\right) = \left(\frac{1 + \frac{\lambda x}{2n}}{1 - \frac{\lambda x}{2n}}\right)^n + O\left(\frac{\lambda^3 x^3}{n^2}\right)$$

to the case when the path from the origin to the point $x = \sum_1^n \frac{x}{n} = \sum \delta e^{i\theta_k}$ is not restricted to straight equal segments but to a general path of $O(|x|/\delta)$ segments of any directions.

The integration with respect to λ gives an interesting analogue of $Z^{:-k:} \exp(:\lambda: Z)$. It is defined up to a globally defined discrete holomorphic map. One way to fix it is to integrate from a given λ_0 of modulus $2/\delta$, which is not a pole of the rational fraction, along a path that doesn't cross the circle of radius $2/\delta$ again.

Proposition 4.5. For point-wise multiplication, at every point $x \in \diamond_0$,

$$(4.8) \quad \exp(:\lambda: x) \cdot \exp(:-\lambda: x) = 1.$$

The specialization at $\lambda = 0$ defines monomials:

$$(4.9) \quad Z^{:k:} = Z^{:k:} \exp(:\lambda: Z)|_{\lambda=0}$$

which fulfill $Z^{:k:} = k \int Z^{:k-1:} dZ$. The anti-linear duality \dagger maps exponentials to exponentials:

$$(4.10) \quad \exp(:\lambda:)^\dagger = \exp\left(:\frac{4}{\delta^2 \overline{\lambda}}:\right).$$

In particular, $\exp(:\infty:) = 1^\dagger = \varepsilon$ is the biconstant.

Proof 4.5. The first assertion is immediate.

The derivation of (4.4) with respect to λ yields

$$(4.11) \quad d \frac{\partial^k}{\partial \lambda^k} \exp(:\lambda: Z) = \left(\lambda \frac{\partial^k}{\partial \lambda^k} \exp(:\lambda: Z) + k \frac{\partial^{k-1}}{\partial \lambda^{k-1}} \exp(:\lambda: Z) \right) dZ$$

which implies (4.9).

Derivation of $\exp(:\lambda:)^\dagger$ gives,

$$(4.12) \quad \begin{aligned} (\exp(:\lambda:)^\dagger)' &= \frac{4}{\delta^2} \left(\int_0^z \exp(:\lambda:) dZ \right)^\dagger + \mu \varepsilon \\ &= \frac{4}{\delta^2} \left(\frac{\exp(:\lambda:) - 1}{\lambda} \right)^\dagger + \mu \varepsilon \\ &= \frac{4}{\delta^2 \overline{\lambda}} \exp(:\lambda:)^\dagger + \nu \varepsilon \end{aligned}$$

with μ, ν some constants, so that the initial condition $\exp(:\lambda:O)^\dagger = 1$ at the origin and the difference equation $d \exp(:\lambda:)^\dagger = \frac{4}{\delta^2 \overline{\lambda}} \exp(:\lambda:)^\dagger dZ$ yields the result. \square

Note that it is natural to define $\exp(:\lambda:(x - x_0)) := \frac{\exp(:\lambda:x)}{\exp(:\lambda:x_0)}$ as a function of x with x_0 a fixed vertex. It is simply a change of origin. But apart on a lattice where addition of vertices or multiplication by an integer can be given a meaning as maps of the lattice, there is no easy way to generalize this construction to other discrete holomorphic functions such as $\exp(:\lambda:(x + n y))$ with $x, y \in \diamond_0$ and $n \in \mathbb{Z}$.

4.4. Series. The series $\sum_{k=0}^{\infty} \frac{\lambda^k Z^{:k:}}{k!}$, wherever it is absolutely convergent, coincide with the rational fraction (4.6): Its value at the origin is 1 and it fulfills the defining difference equation (4.4). Using Eq. (4.9), a Taylor expansion of $\exp(:\lambda:x)$ at $\lambda = 0$ gives back the same result. We are now interested in the rate of growth of the monomials.

Direct analysis gives an estimate of $Z^{:k:}$:

Proposition 4.6. *For $x \in \diamond$, at a combinatorial distance $d(x, O)$ of the origin, and any $k \in \mathbb{N}$,*

$$(4.13) \quad \left| \frac{Z^{:k:}(x)}{k!} \right| \leq \left(\frac{\alpha + 1}{\alpha - 1} \right)^{d(x, O)} \left(\alpha \frac{\delta}{2} \right)^k,$$

for any $\alpha > 1$ arbitrarily close to 1.

Corollary 4.7. *The series $\sum_{k=0}^{\infty} \frac{\lambda^k Z^{:k:}}{k!}$ is absolutely convergent for $|\lambda| < \frac{2}{\delta}$.*

Proof 4.6. It is proved by double induction, on the degree k and on the combinatorial distance to the origin.

For $k = 0$, it is valid for any x since $\frac{\alpha+1}{\alpha-1} = 1 + \frac{2}{\alpha-1} > 1$, with equality only at the origin.

Consider $x \in \diamond$ a neighbor of the origin, $Z(x) = \delta e^{i\theta}$, then an immediate induction gives for $k \geq 1$,

$$(4.14) \quad \frac{Z^{:k:}(x)}{k!} = 2 \left(\frac{\delta e^{i\theta}}{2} \right)^k$$

which fulfills the condition Eq. (4.13) for any $k \geq 1$ because $\frac{\alpha+1}{\alpha-1} \alpha^k > 2$. This was done merely for illustration purposes since it is sufficient to check that the condition holds at the origin, which it obviously does.

Suppose the condition is satisfied for a vertex x up to degree k , and for its neighbor y , one edge further from the origin, up to degree $k - 1$. Then,

$$(4.15) \quad \frac{Z^{:k:}(y)}{k!} = \frac{Z^{:k:}(x)}{k!} + \frac{Z^{:k-1:}(x) + Z^{:k-1:}(y)}{(k-1)!} \frac{Z(y) - Z(x)}{2}$$

in absolute value fulfills

$$(4.16) \quad \begin{aligned} \left| \frac{Z^{:k:}(y)}{k!} \right| &\leq \left(\frac{\alpha + 1}{\alpha - 1} \right)^{d(x, O)} \left(\alpha \frac{\delta}{2} \right)^{k-1} \left(\left(\alpha \frac{\delta}{2} \right) + \left(1 + \frac{\alpha + 1}{\alpha - 1} \right) \frac{\delta}{2} \right) \\ &= \left(\frac{\alpha + 1}{\alpha - 1} \right)^{d(x, O)} \left(\alpha \frac{\delta}{2} \right)^k \left(1 + \frac{2}{\alpha - 1} \right) \\ &= \left(\frac{\alpha + 1}{\alpha - 1} \right)^{d(y, O)} \left(\alpha \frac{\delta}{2} \right)^k, \end{aligned}$$

thus proving the condition for y at degree k . It follows by induction that the condition holds at any point and any degree. \square

4.5. Basis. The discrete exponentials form a basis of discrete holomorphic functions on a finite critical map: given any set of pair-wise different reals $\{\lambda_k\}$ of the right dimension, the associated discrete exponentials will form a basis of the space of discrete holomorphic functions. See [8] for the formula

$$(4.17) \quad f(x) = \frac{1}{2i\pi} \int_{\gamma} g(\lambda) \exp(: \lambda : x) d\lambda$$

for a certain fixed contour γ in the space of parameters λ , and the definition of $g(\lambda)$ as a fixed contour integral in \diamond involving f .

The polynomials however don't form a basis in general: the combinatorial surface has to fulfill a certain condition called "combinatorial convexity". A quadrilateral, when traversed from one side to its opposite, define a unique chain of quadrilaterals, that we call a "train-track". The condition we ask is that two different train-tracks have different slopes.

On a combinatorially convex set, the discrete polynomials form a basis as well.

4.6. Continuous limit. In a critical map, where quadrilaterals are mapped to rhombi of side δ , identifying a vertex x with its image $Z(x)$.

The combinatorial distance $d_{\diamond}(x, O)$ is related to the modulus $|x|$ through

$$(4.18) \quad d_{\diamond}(x, O) \frac{\sin \theta_m}{4} \leq \frac{|x|}{\delta} \leq d_{\diamond}(x, O)$$

where θ_m is the minimum of all rhombi angles. When the rhombi don't flatten, the combinatorial distance and the modulus (over δ) are equivalent distances.

Lemma 4.8. *Let $(ABCD)$ be a four sided polygon of the Euclidean plane such that its diagonals are orthogonal and the vertices angles are in $[\eta, 2\pi - \eta]$ with $\eta > 0$. Let (M, M') be a pair of points on the polygon. There exists a path on $(ABCD)$ from M to M' of minimal length ℓ . Then*

$$\frac{MM'}{\ell} \geq \frac{\sin \eta}{4}.$$

It is a straightforward study of a several variables function. If the two points are on the same side, $MM' = \ell$ and $\sin \eta \leq 1$. If they are on adjacent sides, the extremal position with MM' fixed is when the triangle $MM'P$, with P the vertex of $(ABCD)$ between them, is isocel. The angle in P being less than η , $\frac{MM'}{\ell} \geq \sin \frac{\eta}{2} > \frac{\sin \eta}{2}$. If the points are on opposite sides, the extremal configuration is given by Fig. 8.2., where $\frac{MM'}{\ell} = \frac{\sin \eta}{4}$. \square



1. M, M' on adjacent sides.

2. M, M' on opposite sides.

FIGURE 8. The two extremal positions.

A function $f : \diamond_0 \rightarrow \mathbb{C}$ on the combinatorial surface can be extended to a function on the image of the critical map in the complex plane $\hat{f} : U \rightarrow \mathbb{C}$ by stating that $\hat{f}(Z(x)) = f(x)$ for the image of a vertex, and extend it linearly on the segments $[Z(x), Z(y)]$ image of an edge, and harmonically inside each rhombus.

Theorem 4.1. *Let (\diamond_k) a sequence of simply connected critical maps, U the non empty intersection of their images in the complex plane and a holomorphic function $f : U \rightarrow \mathbb{C}$. If the sequence of minimum angles are bounded away from 0 and the sequence of rhombi side lengths (δ_k) converge to 0, then the function f can be approximated by a sequence of discrete holomorphic functions $f_n \in \Omega(\diamond_k)$ converging to f . The convergence is not only pointwise but C^∞ on the intersection of images. Conversely a converging sequence of discrete holomorphic functions converges to a continuous holomorphic function, in particular the discrete polynomials and discrete exponentials with fixed parameters.*

Corollary 4.9. *On a Riemann surface, any 1-form can be approximated by a sequence of discrete holomorphic 1-forms on a refining sequence of critical maps with fixed conic singularities.*

The proof relies on the convergence of polynomials seen as iterated primitives of the constant function.

Lemma 4.10. *Given a sequence of discrete holomorphic functions (f_k) on a refining sequence of critical maps, converging to a holomorphic function f , the sequence of primitives $(\int f_k dZ)$ converges to $\int f(z) dz$. Moreover, in the compact case, if the convergence of the functions is of order $O(\delta_k^2)$, it stays this way for the primitives.*

Proof 4.10. Suppose that we are given a sequence of flat vertices $O_k \in \diamond_k$ where the face containing the fixed flat origin $O \in U$ is adjacent to O_k . For a given integer k , let \hat{F}_k the continuous piecewise

harmonic extension of the discrete primitive $\int_{O_k} f_k dZ$ to U . We want to prove that for any $x \in U$, the following sequence tends to zero

$$(4.19) \quad \left(\left| (\widehat{F}_k(x) - \widehat{F}_k(O)) - \int_O^x f(z) dz \right| \right)_{k \in \mathbb{N}}.$$

For each integer k consider a vertex $x_k \in \diamond_0$ on the boundary of the face of \diamond_2 containing x .

We decompose the difference (4.19) into three parts, inside the face containing the origin O and its neighbor O_k , similarly for x and x_k , and purely along the edges of the graph \diamond^k itself.

$$\begin{aligned} & |(\widehat{F}_k(x) - \widehat{F}_k(O)) - \int_O^x f(z) dz| = \\ & \left| (\widehat{F}_k(x) - \widehat{F}_k(x_k)) + \int_{O_k}^{x_k} f_k dZ + (\widehat{F}_k(O) - \widehat{F}_k(O_k)) - \int_O^x f(z) dz \right| \\ & \leq \left| \widehat{F}_k(x) - \widehat{F}_k(x_k) - \int_{x_k}^x f(z) dz \right| + \left| \int_{O_k}^{x_k} f_k dZ - \int_{O_k}^{x_k} f(z) dz \right| + \\ (4.20) & \left| \widehat{F}_k(O_k) - \widehat{F}_k(O) - \int_O^{O_k} f(z) dz \right|. \end{aligned}$$

On the face of \diamond containing x , the primitive $\xi \mapsto \int_{x_k}^\xi f(z) dz$ is a holomorphic, hence harmonic function as well as $\xi \mapsto \widehat{F}_k(\xi)$. By the maximum principle, the harmonic function $\xi \mapsto \widehat{F}_k(\xi) - \widehat{F}_k(x_k) - \int_{x_k}^\xi f(z) dz$ reaches its maximum on that face, along its boundary. The difference of the discrete primitive along the boundary edge $(x_k, y) \in \diamond_1$ at the point $\xi = (1 - \lambda)x_k + \lambda y$ is equal by definition to

$$(4.21) \quad \widehat{F}_k((1 - \lambda)x_k + \lambda y) - \widehat{F}_k(x_k) = \lambda(y - x_k) \frac{f_k(x_k) + f_k(y)}{2}.$$

The holomorphic f is differentiable with a bounded derivative on U , so averaging the first order expansions at x_k and y , we get

$$\begin{aligned} \int_{x_k}^\xi f(z) dz &= \lambda(y - x_k) \frac{f(x_k) + f(y)}{2} + (y - x_k)^2 \frac{\lambda^2 f'(x_k) + (1 - \lambda)^2 f'(y)}{4} \\ &\quad + o((\xi - x_k)^3) + o((\xi - y)^3) \\ &= \lambda(y - x_k) \frac{f(x_k) + f(y)}{2} + O(\delta_k^2) \end{aligned} \tag{4.22}$$

therefore

$$(4.23) \quad \left| \widehat{F}_k(x) - \widehat{F}_k(x_k) - \int_{x_k}^x f(z) dz \right| = O(\delta_k^2).$$

Similarly for the term around the origin.

By definition of \widehat{f}_k , the 1-form $\widehat{f}_k(z) dz$ along edges of the graph \diamond is equal to the discrete form $f_k dZ$ so that $\int_{O_k}^{x_k} f_k dZ = \int_{O_k}^{x_k} \widehat{f}_k(z) dz$ on a path along \diamond edges. Therefore the difference

$$(4.24) \quad \left| \int_{O_k}^{x_k} f_k dZ - \int_{O_k}^{x_k} f(z) dz \right| \leq \int_{O_k}^{x_k} \left| \widehat{f}_k(z) - f(z) \right| dz$$

is of the same order as the difference $|f_k(z) - f(z)|$ times the length $\ell(\gamma_k)$ of a path on \diamond_k from O_k to x_k . This length is bounded as $\ell(\gamma_k) \leq \frac{4}{\sin \theta_m} |x_k - O_k|$. Since we are interested in the compact case, this length is bounded uniformly and the difference (4.24) is of the same order as the point-wise difference. We conclude that the sequence of discrete primitives converges to the continuous primitive and if the limit for the functions was of order $O(\delta^2)$, it remains of that order. \diamond

The discrete polynomials of degree less than three agree point-wise with their continuous counterpart, $Z^{:2:}(x) = Z(x)^2$.

A simple induction then gives the following

Corollary 4.11. *The discrete polynomials converge to the continuous ones, the limit is of order $O(\delta_k^2)$.*

Which implies the main theorem:

Proof 4.1. On the simply connected compact set U , a holomorphic function f can be written, in a local map z as a series,

$$(4.25) \quad f(z) = \sum_{k \in \mathbb{N}} a_k z^k.$$

Therefore, by a diagonal procedure, there exists an increasing integer sequence $(N(n))_{n \in \mathbb{N}}$ such that the sequence of discrete holomorphic polynomials converge to the continuous series and the convergence is C^∞ .

$$(4.26) \quad \left(\sum_{k=0}^{N(n)} a_k Z^{:k:} \right)_{n \in \mathbb{N}} \rightarrow f.$$

\diamond

4.7. Cross-ratio preserving maps. Once the isometry Z is chosen, holomorphicity of a function f can be written on a quadrilateral $(x, y, x', y') \in \diamond_2$, writing $x = Z(x)$ for a vertex $x \in \diamond_0$, as

$$(4.27) \quad \frac{f(y') - f(y)}{f(x') - f(x)} = \frac{y' - y}{x' - x}$$

and f is understood as a *diagonal ratio preserving map*, and each value at a corner vertex can be linearly solved in terms of the three others.

A quadratic version is given by the *cross-ratio preserving maps*: A function f is said to be *quadratic holomorphic* iff

$$(4.28) \quad \frac{(f(y) - f(x))(f(y') - f(x'))}{(f(x) - f(y'))(f(x') - f(y))} = \frac{(y - x)(y' - x')}{(x - y')(x' - y)}.$$

A rhombic tiling gives rise to two sets of isoradial circle patterns: a set of circles of common radius δ , whose centers are the vertices of Γ and intersections are the vertices of Γ^* and vice-versa. Two interesting families of cross-ratio preserving maps are given by circle patterns with the same combinatorics and intersection angles as one of these two circle patterns.

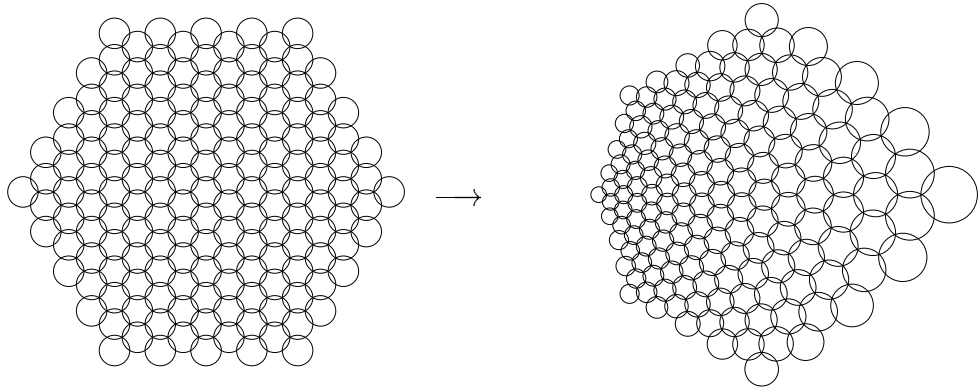


FIGURE 9. A circle pattern with prescribed angles as a cross ratio preserving map.

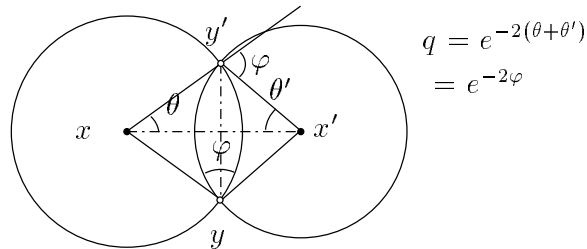


FIGURE 10. The cross-ratio q is given by the intersection angles.

A change of coordinates helps understanding diagonal ratio preserving maps as a linearized version of the cross-ratio preserving maps. We

will say that the function $w : \diamond_2 \rightarrow \mathbb{C}$ solves the *Hirota system* if, around a face $(x, y, x', y') \in \diamond_2$,

$$(4.29) \quad (y-x)w(x)w(y) + (x-y')w(y')w(x) + (y'-x')w(x')w(y') + (x'-y)w(y)w(x') = 0.$$

This is to be understood as a quadratic version of the Morera theorem $\oint f dz = 0$ and w is a half of the derivative of a holomorphic function:

Proposition 4.12. *If w solves the Hirota system, then the function $f : \diamond_2 \rightarrow \mathbb{C}$ defined up to an additive constant by*

$$(4.30) \quad f(y) - f(x) = (y - x)w(x)w(y)$$

is quadratic holomorphic.

Proof 4.12. The function f is well defined because the associated 1-form is closed by definition of the Hirota system. The function w disappears in the cross-ratio of f , leaving the original cross-ratio. \diamond Conversely, a quadratic holomorphic function defines a solution to the Hirota system, unique up to multiplication by λ on Γ , $1/\lambda$ on Γ^* . Concerning circle patterns families, w is real on the centers and unitary on the intersections, and encodes the variation of radius, resp. of direction of the image of the circle:

$$(4.31) \quad f(y) - f(x) = r(x)e^{i\theta(y)}(y - x).$$

Proposition 4.13. *The logarithmic derivative of the Hirota system associated to a family of cross-ratio preserving maps is a diagonal ratio preserving map.*

In other words, for $(1 + \epsilon g)w$ to continue solving the Hirota system at first order, the deformation g must satisfy

$$(4.32) \quad \frac{g(y') - g(y)}{g(x') - g(x)} = \frac{f(y') - f(y)}{f(x') - f(x)}.$$

Proof 4.13. The ϵ contribution of the closeness condition (4.29) for $(1 + \epsilon g)w$ gives

$$(g(x) + g(y))f(x)f(y)(y - x) + (g(y) + g(x'))f(y)f(x')(x' - y) \\ + (g(x') + g(y'))w(x')w(y')(y' - x') + (g(y') + g(x))w(y')w(x)(x - y') = 0,$$

$$\text{which reads, referring to } f: \quad \frac{g(y') - g(y)}{g(x') - g(x)} = \frac{f(y') - f(y)}{f(x') - f(x)}.$$

\diamond

4.8. Baecklund transformation. The Baecklund transformation is a way to associate, to a given solution of an integrable problem, a family of deformed solutions. The two problems under consideration here are the linear and quadratic holomorphicity constraints on each face. They are given by a linear, resp. quadratic algebraic relation between the four values of a solution at the vertices of each face. These relations involve only values supported by the edges of the rhombus, which are equal on opposite sides, namely the complex label $y - x = x' - y'$.

The Baecklund transformation is defined by imposing such constraints over new virtual faces added over each edge, with “vertical edges” labelled by a complex constant λ :

Definition 4.14. *Given a linear holomorphic function $f \in \Omega(\diamond)$, complex numbers $u, \lambda \in \mathbb{C}$, its Baecklund transformation $f_\lambda = B_\lambda^u(f)$ is defined by*

$$(4.33) \quad \begin{aligned} f_\lambda(0) &= u, \\ \frac{f_\lambda(x) - f(y)}{f_\lambda(y) - f(x)} &= \frac{\lambda + x - y}{\lambda + y - x}. \end{aligned}$$

Given a quadratic holomorphic function f , complex numbers $u, \lambda \in \mathbb{C}$, its Baecklund transformation $f_\lambda = B_\lambda^u(f)$ is defined by

$$(4.34) \quad \begin{aligned} f_\lambda(0) &= u, \\ \frac{f_\lambda(y) - f_\lambda(x)}{f_\lambda(x) - f(x)} \frac{f(x) - f(y)}{f(y) - f_\lambda(y)} &= \frac{(y - x)^2}{\lambda^2}. \end{aligned}$$

The right hand sides are the values respectively of the diagonal ratio and cross-ratio of a parallelogram faces of sides $(y - x)$ and λ seen as “over” the edge $(x, y) \in \diamond_1$.

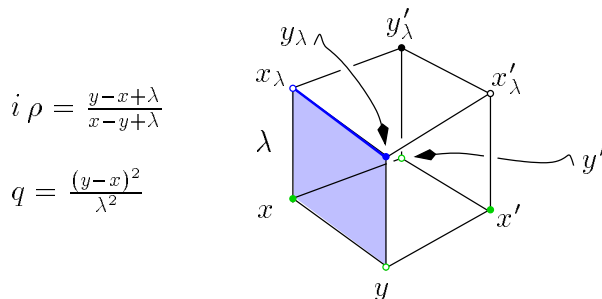


FIGURE 11. The face $(x_\lambda, y_\lambda, x'_\lambda, y'_\lambda)$ “over” the face $(x, y, x', y') \in \diamond_2$.

Proposition 4.15. *This transformation is well defined in the critical case.*

This condition, called *three dimensional consistency* is an overdetermination constraint: if the cube “over” the face $(x, y, x', y') \in \diamond_2$ is split into two hexagons along the cycle $(y, x', y', y'_\lambda, x_\lambda, y_\lambda)$, one can see that, given values at these six vertices, the values at the centers of each hexagons, namely at x'_λ and at x are overdetermined.

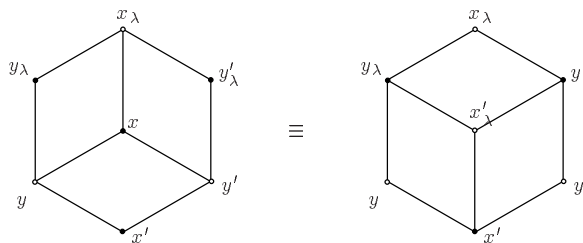


FIGURE 12. The cube split into two hexagons yielding equivalent compatibility constraints.

Therefore only certain values at the six vertices are allowed, defined by two algebraic relations between them. The compatibility condition is that these two algebraic relations are equivalent. It is a simple computation to check it is the case for critical maps.

This transformation verifies

$$(4.35) \quad B_{\lambda^{-1}}^{f(O)} (B_\lambda^u(f)) = f$$

for any (u, λ) . It is an analytic transformation in all the parameters therefore its derivative is a linear map between the tangent spaces, that is to say between diagonal ratio preserving maps,

$$(4.36) \quad dB_\lambda^u(f) : \Omega(f) \rightarrow \Omega(B_\lambda^u(f)).$$

It is not injective and I define the discrete exponential at f as being the direction of this 1-dimensional kernel. It can be characterized as a derivative with respect to the initial value at the origin:

$$(4.37) \quad \exp_u(:\lambda:f) := \frac{\partial}{\partial v} B_{\lambda^{-1}}^v (B_\lambda^u(f)) \Big|_{v=f(O)} \in \ker (dB_\lambda^u(f))$$

because $B_\lambda^u (B_{\lambda^{-1}}^v(g)) = g$ for all λ, g and v .

As in the discrete exponential case, the value of the Baecklund transformation at a given vertex is the image of values at neighbouring vertices by a homography. These homographies can be encoded as

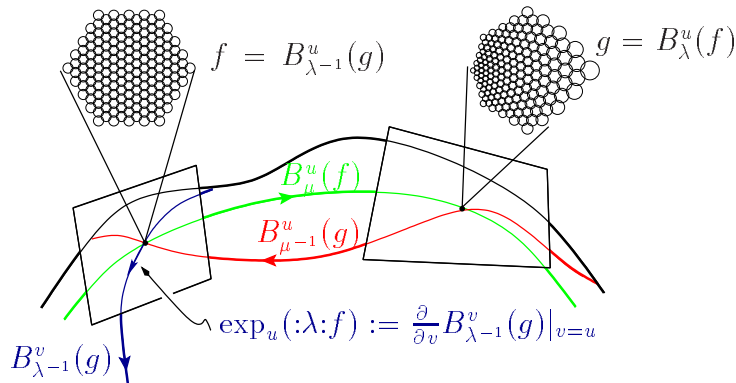


FIGURE 13. The discrete exponential $\exp_u(:\lambda:f)$ is the kernel of the linear transformation $d B_{\lambda}^u(f)$ (here $u = f(O)$).

projective operators $L(e; \lambda) \in GL_2(\mathbb{C})[\lambda]$ lying on the edges $e \in \diamond_2$, called a *zero curvature representation*:

$$(4.38) \quad L((x, y); \lambda) = \begin{pmatrix} \lambda + y - x & -2(y - x)(f(x) + f(y)) \\ 0 & \lambda + x - y \end{pmatrix} \quad \text{for the linear case,}$$

(4.39)

$$L((x, y); \lambda) = \begin{pmatrix} 1 & -(y - x)w(y) \\ -\lambda(y - x)/w(x) & w(y)/w(x) \end{pmatrix} \quad \text{for the Hirota system.}$$

Then we define [16] the *moving frame* $\Psi : \diamond_2 \rightarrow GL_2(\mathbb{C})(\lambda)$ by a prescribed value at the origin and recursively by $\Psi(y; \lambda) = L((x, y); \lambda)\Psi(x; \lambda)$ and its logarithmic derivative with respect to λ

$$(4.40) \quad A(e; \lambda) = \frac{d\Psi(e; \lambda)}{d\lambda} \Psi^{-1}(e; \lambda)$$

is meromorphic in λ for each edge e . We call f , resp. w *isomonodromic* if the positions and orders of the poles don't depend on the edge e . The two points discrete Green function (the discrete logarithm) $G(O, x)$, inverse of the Laplacian in the sense that

$$(4.41) \quad \Delta G(O, \bullet) = \delta_{O, \bullet}$$

can be constructed as the unique isomonodromic solution with some prescribed data [8], which allows us to give an explicit formula for it, recovering results of Kenyon [15]: an integral over a loop in the space

of discrete exponentials:

$$(4.42) \quad G(O, x) = -\frac{1}{8\pi^2 i} \oint_C \exp(:\lambda: x) \frac{\log \frac{\delta}{2} \lambda}{\lambda} d\lambda$$

where the integration contour C contains all the possible poles of the rational fraction $\exp(:\lambda: x)$ but avoids the half line through $-x$. It is real (negative) on half of the vertices and imaginary on the others. Because of the logarithm, this imaginary part is multivalued.

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