L^1 Stability for scalar balance laws. Control of the continuity equation with a non-local flow.

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Institut Camille Jordan, Lyon

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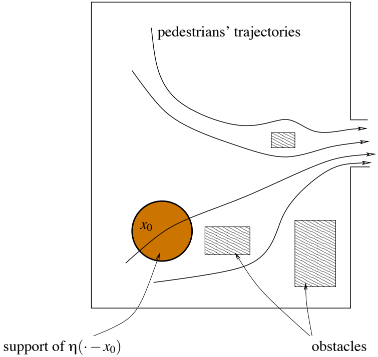
Pedestrian traffic

We consider

$$\partial_t u + \mathrm{Div}(uV(x,u)) = 0\,; \quad u_0 \in (\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^N;\mathbb{R})$$

whith $V(u) = v(\eta *_x u)w(x)$.



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Introduction of the problem

Scalar balance laws :

$$\left\{\begin{array}{lrcl} \partial_t u + \mathrm{Div} f(t,x,u) = F(t,x,u) & (t,x) & \in & \mathbb{R}_+^* \times \mathbb{R}^N \\ u(0,x) = u_0(x) \in \mathbf{L^1} \cap \mathbf{L}^\infty \cap \mathbf{BV} & x & \in & \mathbb{R}^N , \end{array}\right.$$

where $f \in \mathcal{C}^2([0,T] \times \mathbb{R}^N \times \mathbb{R}; \mathbb{R}^N)$, $F \in \mathcal{C}^1([0,T] \times \mathbb{R}^N \times \mathbb{R}; \mathbb{R})$.

- Existence and uniqueness, dependence w.r.t. initial conditions : Kružkov Theorem (1970, Mat. Sb. (N.S.)) ;
- Dependence w.r.t. flow and source ?

Previous Results

Theorem (Kružkov, 1970, Mat. Sb. (N.S.))

We consider the equation

$$\partial_t u + \mathrm{Div} f(t,x,u) = F(t,x,u)\,,$$

with initial condition $u_0 \in \mathbf{L}^1 \cap \mathbf{L}^\infty(\mathbb{R}^N)$. *Under the condition*

$$\textbf{(K)}\ \ \forall A > 0\ ,\ \partial_u f \in \mathbf{L}^\infty(\Omega_A)\,,\ \partial_u(F - \mathrm{div} f) \in \mathbf{L}^\infty(\Omega_A)$$
$$\textit{and}\ \ F - \mathrm{div} f \in \mathbf{L}^\infty(\Omega_A)$$

then there exists a unique weak entropy solution $u \in \mathbf{L}^\infty([0,T];\mathbf{L}^1(\mathbb{R}^N;\mathbb{R}))$ *continuous from the right in time.*

Let $v_0 \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^N;\mathbb{R})$, *then*

$$\left\|(u-v)(t)\right\|_{\mathbf{L}^1} \leq e^{\gamma t}\left\|u_0 - v_0\right\|_{\mathbf{L}^1}\,,$$

where $\gamma = \left\|\partial_u F\right\|_{\mathbf{L}^\infty(\Omega_M)}$.

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Theorem (Lucier, 1986, Math. Comp.)

If $f, g : \mathbb{R} \to \mathbb{R}^N$ are globally lipschitz, then $\exists C > 0$ such that $\forall u_0, v_0 \in \mathbf{L}^1 \cap \mathbf{L}^\infty(\mathbb{R}^N; \mathbb{R})$ initial conditions for

$$\partial_t u + \mathrm{Div} f(u) = 0 \,, \qquad \partial_t v + \mathrm{Div} g(v) = 0 \,.$$

with furthermore $v_0 \in \mathbf{BV}(\mathbb{R}^N; \mathbb{R})$, we have $\forall t \geq 0$,

$$\left\| (u - v)(t) \right\|_{\mathbf{L}^1} \leq \| u_0 - v_0 \|_{\mathbf{L}^1} + C\, t \, \mathrm{TV}(v_0) \, \mathbf{Lip}\,(f - g) \,.$$

Theorem (Chen & Karlsen, 2005, Commun. Pure Appl. Anal.)

With $f(t, x, u) = \lambda(x)\, l(u)$, $g(t, x, v) = \mu(x)\, m(v)$, no source $F = G = 0$,

$$\left\| (u - v)(t) \right\|_{\mathbf{L}^1} \leq \| u_0 - v_0 \|_{\mathbf{L}^1} + C_1\, t \left(\| \lambda - \mu \|_{\mathbf{L}^\infty} + \| \lambda - \mu \|_{\mathbf{W}^{1,1}} + \| l - m \|_{\mathbf{L}^\infty} + \| l - m \|_{\mathbf{W}^{1,\infty}} \right)$$

where $C_1 = C \sup_{[0,T]} \left(\mathrm{TV}(u(t)), \mathrm{TV}(v(t)) \right)$.

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Total Variation

Definition : For $u \in \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}^N;\mathbb{R})$ we denote

$$\begin{aligned} \mathrm{TV}(u) &= \sup\left\{ \int_{\mathbb{R}^N} u \,\mathrm{div}\Psi\,; \quad \Psi \in \mathcal{C}^1_c(\mathbb{R}^N;\mathbb{R}^N)\,, \quad \|\Psi\|_{\mathbf{L}^\infty} \leq 1 \right\}\,; \\ \text{and} & \\ \mathbf{BV}(\mathbb{R}^N;\mathbb{R}) &= \left\{ u \in \mathbf{L}^1_{\mathbf{loc}}; \mathrm{TV}(u) < \infty \right\}\,. \end{aligned}$$

When f and F depend only on u we have

$$u_0 \in \mathbf{L}^\infty \cap \mathbf{BV} \Rightarrow \forall t \geq 0\,, \quad u(t) \in \mathbf{L}^\infty \cap \mathbf{BV}$$

and, denoting $\gamma = \|\partial_u F\|_{\mathbf{L}^\infty(\Omega_M)}$,

$$\mathrm{TV}(u(t)) \leq \mathrm{TV}(u_0) e^{\gamma t}\,.$$

Goal : we want a more general estimate on the total variation.

Estimate on the total variation

Theorem (TV — Colombo, Mercier, Rosini, 2009, Comm. Math. Sciences)

*Assume (f, F) satisfies **(K) + (H1)**. Let $\kappa_0 = (2N+1)\|\nabla_x \partial_u f\|_{\mathbf{L}^\infty(\Omega_M)} + \|\partial_u F\|_{\mathbf{L}^\infty(\Omega_M)}$. If $u_0 \in (\mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^N;\mathbb{R})$, then $\forall t \in [0,T]$, $u(t) \in (\mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^N;\mathbb{R})$ and*

$$\mathrm{TV}(u(t)) \leq \mathrm{TV}(u_0) e^{\kappa_0 t} + N W_N \int_0^t e^{\kappa_0 (t-\tau)} \int_{\mathbb{R}^N} \left\|\nabla_x (F - \mathrm{div} f)(\tau, x, \cdot)\right\|_{\mathbf{L}^\infty(\mathrm{d}u)} \mathrm{d}x \, \mathrm{d}\tau \,.$$

(H1) : $\int_0^T \int_{\mathbb{R}^N} \left\|\nabla_x (F - \mathrm{div} f)\right\|_{\mathbf{L}^\infty(\mathrm{d}u)} \mathrm{d}x\mathrm{d}t < \infty$ and $\nabla_x \partial_u f \in \mathbf{L}^\infty(\Omega_M)$

Remark : In some particular cases, we re-obtain optimal known estimates :

- f, F depending only on u,
- f, F not depending on u.

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then $\forall t \in [0,T]\,,\ u(t) \in (\mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^N;\mathbb{R})$ *and*

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L^1 Stability of the solution

Theorem (Flow/Source... — Colombo, Mercier, Rosini, 2009)

*Assume $(f,F),(g,G)$ satisfy **(K)**, (f,F) satisfies **(H1)** and $(f-g,F-G)$ satisfies **(H2)**. Let $u_0, v_0 \in (\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^N;\mathbb{R})$. We denote*

$$\kappa = 2N\|\nabla_x \partial_u f\|_{\mathbf{L}^\infty(\Omega_M)} + \|\partial_u F\|_{\mathbf{L}^\infty(\Omega_M)} + \left\|\partial_u (F-G)\right\|_{\mathbf{L}^\infty(\Omega_M)}.$$

Let u and v be the solutions associated to (f,F) and (g,G) respectively and with initial conditions u_0 and v_0.

(H2) : $\partial_u(F-G) \in \mathbf{L}^\infty(\Omega_M)$, $\partial_u(f-g) \in \mathbf{L}^\infty(\Omega_M)$ and $\int_0^T \int_{\mathbb{R}^N} \left\|F - G - \mathrm{div}(f-g)\right\|_{\mathbf{L}^\infty(\mathrm{d}u)} \mathrm{d}x\mathrm{d}t < \infty$.

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Theorem (...Flow/Source — Colombo, Mercier, Rosini, 2009)

then $\forall t \in [0, T]$ *:*

$$\begin{aligned}\left\|(u-v)(t)\right\|_{\mathbf{L}^1} \leq{}& e^{\kappa t}\left\|u_0 - v_0\right\|_{\mathbf{L}^1} + \frac{e^{\kappa_0 t} - e^{\kappa t}}{\kappa_0 - \kappa}\mathrm{TV}(u_0)\left\|\partial_u (f-g)\right\|_{\mathbf{L}^\infty} \\ &+ \int_0^t \frac{e^{\kappa_0 (t-\tau)} - e^{\kappa (t-\tau)}}{\kappa_0 - \kappa} \int_{\mathbb{R}^N} \left\|\nabla_x (F - \mathrm{div} f)(\tau, x, \cdot)\right\|_{\mathbf{L}^\infty(\mathrm{d}u)} \mathrm{d}x \mathrm{d}\tau \\ &\quad \times N W_N \left\|\partial_u (f-g)\right\|_{\mathbf{L}^\infty} \\ &+ \int_0^t e^{\kappa(t-\tau)} \int_{\mathbb{R}^N} \left\|((F-G) - \mathrm{div}(f-g))(\tau, x, \cdot)\right\|_{\mathbf{L}^\infty(\mathrm{d}u)} \mathrm{d}x \mathrm{d}\tau \,.\end{aligned}$$

Remark : As before, we re-obtain known estimates in some particular cases

- f, g depend only on u, $F = G = 0$,
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Introduction of the problem

Continuity equation :

$$\partial_t u + \mathrm{Div}(uV(x, u(t))) = 0\,, \qquad u(0,\cdot) = u_0 \in \mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV}\,,$$

where $V : \mathbb{R}^N \times \mathbf{L}^1(\mathbb{R}^N;\mathbb{R}) \to \mathcal{C}^2(\mathbb{R}^N;\mathbb{R})$ is a non-local averaging functional, for example, if $v : \mathbb{R} \to \mathbb{R}$ is a regular function :

- $V(u) = v\left(\int_{\mathbb{R}} u \, \mathrm{d}x\right)$ for a supply-chain [Armbuster et al.]
- $V(u) = v(\eta *_x u)w(x)$ for pedestrian traffic [Colombo et al.].

Goal :

- Existence and uniqueness of an entropy solution ?
- Gâteaux differentiability of the semi-group w.r.t. initial conditions ?

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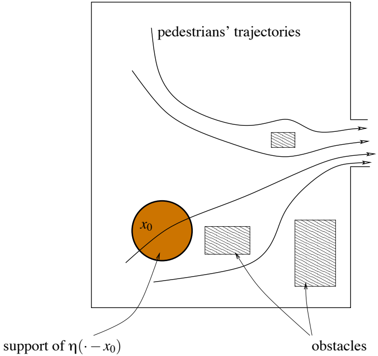

Existence of a solution

Theorem (Traffic — Colombo, Herty, Mercier, 2010, ESAIM-Control Opt. Calc. Var.)

*If V satisfies **(V1)**, then there exists a time $T_{ex} > 0$ and a unique entropy solution*

$$u \in \mathcal{C}^0([0, T_{ex}[; \mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})$$

and we denote $S_t u_0 = u(t, \cdot)$.

We can bound by below the time of existence by

$$T_{ex} \geq \sup \left\{ \sum_n \frac{\ln(\alpha_{n+1}/\alpha_n)}{C(\alpha_{n+1})} \, ; \, (\alpha_n)_n \text{ strict. increasing}, \, \alpha_0 = \|u_0\|_{\mathbf{L}^\infty} \right\}.$$

*If furthermore, V satisfies **(V2)** then*

$$u_0 \in \mathbf{W}^{2,1} \cap \mathbf{L}^\infty \Rightarrow \forall t \in [0, T_{ex}[, \quad u(t) \in \mathbf{W}^{2,1}.$$

Hypotheses

(V1) There exists $C \in \mathbf{L}^\infty_{\mathbf{loc}}(\mathbb{R}_+;\mathbb{R}_+)$ such that $\forall u \in \mathbf{L}^1(\mathbb{R}^N;\mathbb{R})$

$$V(u) \in \mathbf{L}^\infty\,, \qquad \left\|\nabla_x V(u)\right\|_{\mathbf{L}^\infty} \leq C(\|u\|_{\mathbf{L}^\infty})\,,$$
$$\left\|\nabla_x V(u)\right\|_{\mathbf{L}^1} \leq C(\|u\|_{\mathbf{L}^\infty})\,, \qquad \left\|\nabla_x^2 V(u)\right\|_{\mathbf{L}^1} \leq C(\|u\|_{\mathbf{L}^\infty})\,,$$

and $\forall u_1, u_2 \in \mathbf{L}^1(\mathbb{R}^N;\mathbb{R})$

$$\left\|V(u_1) - V(u_2)\right\|_{\mathbf{L}^\infty} \leq C(\|u_1\|_{\mathbf{L}^\infty})\|u_1 - u_2\|_{\mathbf{L}^1}\,,$$
$$\left\|\nabla_x(V(u_1) - V(u_2))\right\|_{\mathbf{L}^1} \leq C(\|u_1\|_{\mathbf{L}^\infty})\|u_1 - u_2\|_{\mathbf{L}^1}\,.$$

(V2) There exists $C \in \mathbf{L}^\infty_{\mathbf{loc}}(\mathbb{R}_+;\mathbb{R}_+)$ such that $\left\|\nabla_x^3 V(u)\right\|_{\mathbf{L}^\infty} \leq C(\|u\|_{\mathbf{L}^\infty})$.

Idea of the proof :

Let us introduce the space $X_\alpha = \mathbf{L^1} \cap \mathbf{BV}(\mathbb{R}^N; [0,\alpha])$ and the application $\mathcal{Q}$ that associates to $w \in \mathcal{X}_\beta = \mathcal{C}^0([0, T[, X_\beta)$ the solution $u \in \mathcal{X}_\beta$ of the Cauchy problem

$$\partial_t u + \mathrm{Div}(uV(w)) = 0\,, \quad u(0,\cdot) = u_0 \in X_\alpha$$

For w_1, w_2, we obtain thanks to the estimate of Thm (Flow/Source)

$$\left\| \mathcal{Q}(w_1) - \mathcal{Q}(w_2) \right\|_{\mathbf{L}^\infty([0,T[,\mathbf{L}^1)} \leq f(T) \|w_1 - w_2\|_{\mathbf{L}^\infty([0,T[,\mathbf{L}^1)}\,,$$

where f is increasing , $f(0) = 0$ and $f \to \infty$ when $T \to \infty$.
Then we apply the Banach Fixed Point Theorem.

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Gâteaux derivative of the semi-group

Classical case : semi-group Lipschitz, not differentiable. Shift differentiability [Bressan, Guerra, Bianchini,...].

Definition : The application $S : \mathbf{L}^1(\mathbb{R}^N;\mathbb{R}) \to \mathbf{L}^1(\mathbb{R}^N;\mathbb{R})$ is said to be $\mathbf{L}^1$ *Gâteaux differentiable in* $u_0 \in \mathbf{L}^1$ *in the direction* $r_0 \in \mathbf{L}^1$ if there exists a linear continuous application $DS(u_0) : \mathbf{L}^1 \to \mathbf{L}^1$ such that

$$\left\| \frac{S(u_0 + hr_0) - S(u_0)}{h} - DS(u_0)(r_0) \right\|_{\mathbf{L}^1} \to 0 \qquad \text{when } h \to 0$$

Formally, we expect the Gâteaux derivative of the semi-group to be the solution of the linearized problem :

$$\partial_t r + \mathrm{Div}(rV(u) + uDV(u)(r)) = 0\,, \quad r(0,\cdot) = r_0\,.$$

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$$\partial_t r + \mathrm{Div}(rV(u) + uDV(u)(r)) = 0\,, \quad r(0,\cdot) = r_0\,.$$

We introduce the hypotheses :
(V3) $V : \mathbf{L}^1 \to \mathcal{C}^2$ is differentiable and there exists $C \in \mathbf{L}^\infty_{\mathbf{loc}}$ such that $\forall u, r \in \mathbf{L}^1$,

$$\left\| V(u+r) - V(u) - DV(u)(r) \right\|_{\mathbf{W}^{2,\infty}} \leq C \left(\|u\|_{\mathbf{L}^\infty} + \|u+r\|_{\mathbf{L}^\infty} \right) \|r\|^2_{\mathbf{L}^1} ,$$
$$\left\| DV(u)(r) \right\|_{\mathbf{W}^{2,\infty}} \leq C(\|u\|_{\mathbf{L}^\infty}) \|r\|_{\mathbf{L}^1} .$$

(V4) There exists $C \in \mathbf{L}^\infty_{\mathbf{loc}}(\mathbb{R}_+; \mathbb{R}_+)$ such that $\forall u, \tilde{u}, r \in \mathbf{L}^1$

$$\left\| \mathrm{div} \left(V(\tilde{u}) - V(u) - DV(u)(\tilde{u} - u) \right) \right\|_{\mathbf{L}^1} \leq C(\|\tilde{u}\|_{\mathbf{L}^\infty} + \|u\|_{\mathbf{L}^\infty})(\|\tilde{u} - u\|_{\mathbf{L}^1})^2$$
$$\left\| \mathrm{div} \left(DV(u)(r) \right) \right\|_{\mathbf{L}^1} \leq C(\|u\|_{\mathbf{L}^\infty}) \|r\|_{\mathbf{L}^1} .$$

We show that the linearised problem has a unique entropy solution :

Theorem (Linearised — Colombo, Herty, Mercier, 2010)

*Assume that V satisfies **(V1), (V2), (V3)**. Let $u \in \mathcal{C}^0([0, T_{ex}[; \mathbf{W}^{1,\infty} \cap \mathbf{W}^{1,1})$, $r_0 \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^N; \mathbb{R})$. Then the linearised problem*

$$\partial_t r + \mathrm{Div}(rV(u) + uDV(u)(r)) = 0\,, \qquad with\ r(0,x) = r_0$$

has a unique entropy solution $r \in \mathcal{C}^0([0, T_{ex}[; \mathbf{L}^1(\mathbb{R}^N; \mathbb{R}))$ and we denote $\Sigma_t^u r_0 = r(t, \cdot)$.
If furthermore $r_0 \in \mathbf{W}^{1,1}$, then $\forall t \in [0, T_{ex}[$, $r(t) \in \mathbf{W}^{1,1}$.

Theorem (Gâteaux Derivative — Colombo, Herty, Mercier, 2010)

*Assume that V satisfies **(V1),(V2),(V3),(V4)**. Let $u_0 \in \mathbf{W}^{1,\infty} \cap \mathbf{W}^{2,1}$, $r_0 \in \mathbf{W}^{1,1} \cap \mathbf{L}^\infty$ and let T_{ex} be the time of existence for the initial problem given by Thm (Trafic).*
Then, for all $t \in [0, T_{ex}[$ the local semi-group of the pedestrian traffic problem is $\mathbf{L}^1$ Gâteaux differentiable in the direction r_0 and

$$DS_t(u_0)(r_0) = \Sigma_t^{S_t u_0} r_0 \,.$$

Idea of the proof : Thm (Flow/Source) allows to compare the solution with initial condition $u_0 + hr_0$ to the solution $u + hr$.

Extrema of a Cost Functional

Let J be a cost functional such that

$$J(\rho_0) = \int_{\mathbb{R}^N} f\left(S_t \rho_0\right)\, \psi(t,x) \mathrm{d}x\,.$$

Proposition (Colombo, Herty, Mercier, 2010)

Let $f \in \mathcal{C}^{1,1}(\mathbb{R};\mathbb{R}_+)$ and $\psi \in \mathbf{L}^\infty(I_{\mathrm{ex}} \times \mathbb{R}^N;\mathbb{R})$. Let us assume that $S\colon I \times (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^N;\mathbb{R}) \to (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^N;\mathbb{R})$ is $\mathbf{L}^1$ Gâteaux differentiable. If $\rho_0 \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^N;\mathbb{R})$ is solution of

$$\mathit{Find}\ \min_{\rho_0} J(\rho_0)\ \textit{s. t.}\ \left\{ S_t\rho_0 \textit{ is solution of (Traffic)} \right\}.$$

then, for all $r_0 \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^N;\mathbb{R})$

$$\int_{\mathbb{R}^N} f'(S_t\rho_0)\, \Sigma_t^{\rho_0} r_0\, \psi(t,x)\, \mathrm{d}x = 0\,.$$

Perspectives

- Derivation with respect to the geometric parameter (speed law).
- Avoid blow-up of the $\mathbf{L}^\infty$ norm.

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