A Class of Non-Local Models for Pedestrian Traffic

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April 5, 2011

Abstract

We present a new class of macroscopic models for pedestrian flows. Each individual is assumed to move towards a fixed target, deviating from the best path according to the instantaneous crowd distribution. The resulting equation is a conservation law with a nonlocal flux. Each equation in this class generates a Lipschitz semigroup of solutions and is stable with respect to the functions and parameters defining it. Moreover, key qualitative properties such as the boundedness of the crowd density are proved. Specific models are presented and their qualitative properties are shown through numerical integrations. In particular, the present models account for the possibility of reducing the evacuation time from a room by carefully positioning obstacles that direct the crowd flow.

2000 Mathematics Subject Classification: 35L65, 90B20.

Keywords: Crowd Dynamics, Macroscopic Pedestrian Model, Non-Local Conservation Laws.

1 Introduction

From a macroscopic point of view, a moving crowd is described by its density $\rho = \rho(t, x)$, so that for any subset A of the plane, the quantity $\int_A \rho(t, x)\,\mathrm{d}x$ is the total number of individuals in A at time t. In standard situations, the number of individuals is constant, so that conservation laws of the type $\partial_t \rho + \mathrm{div}_x (\rho\,\mathbf{v}) = 0$ are the natural tool for the description of crowd dynamics. A key issue is the choice of the speed $\mathbf{v}$, which should describe not only the target of the pedestrians and the modulus of their speed, but also their attitude to adapt their path choice to the crowd density they estimate to find along this path.

Our starting point is the following Cauchy problem for the conservation law

$$\begin{cases} \partial_t \rho + \mathrm{div}\left(\rho\, v(\rho)\, \left(\nu(x) + \mathcal{I}(\rho)\right)\right) = 0\,, \\ \rho(0, x) = \rho_0(x)\,. \end{cases} \tag{1.1}$$

The scalar function $\rho \mapsto v(\rho)$ describes the modulus of the pedestrians' speed, independently from geometrical considerations. In other words, an individual at time t and position $x \in \mathbb{R}^N$ moves at the speed $v\left(\rho(t, x)\right)$ that depends on the density $\rho(t, x)$ evaluated at the same time t and position x. Given that the density is ρ, the vector $\nu(x) + \mathcal{I}(\rho)$ describes the direction

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that the individual located at x follows and has norm (approximately) 1. More precisely, the individual at position x and time t is assumed to move in the direction $\nu(x) + \Big(\mathcal{I}\left(\rho(t)\right)\Big)(x)$.

In situations like the evacuation of a closed space $\Omega \subset \mathbb{R}^N$, it is natural to assume that the first choice of each pedestrian is to follow a path optimal with respect to the visible geometry, for instance the geodesic. As soon as walls or obstacles are relevant, it is necessary to take into consideration the discomfort felt by pedestrians walking along walls or too near to corners, see for instance [19, 21] and the references therein.

The vector $\Big(\mathcal{I}\left(\rho(t)\right)\Big)(x)$ describes the deviation from the direction $\nu(x)$ due to the density distribution $\rho(t)$ at time t. Hence, the operator $\mathcal{I}$ is in general *nonlocal*, so that $\Big(\mathcal{I}\left(\rho(t)\right)\Big)(x)$ depends on all the values of the density $\rho(t)$ at time t in a neighborhood of x. More formally, it depends on all the function $\rho(t) \in \mathbf{L}^1(\mathbb{R}^N; [0, R])$ and not only on the value $\rho(t, x) \in [0, R]$. The case in which $\mathcal{I} = 0$ is equivalent to assume that the paths followed by the individuals are chosen *a priori*, independently from the dynamics of the crowd.

Here we present two specific choices that fit in (1.1). A first criterion assumes that each individual aims at avoiding high crowd densities. Fix a mollifier η. Then, the convolution $(\rho * \eta)$ is an average of the crowd density around x. This leads to the natural choice

$$\mathcal{I}(\rho) = -\varepsilon \, \frac{\nabla(\rho * \eta)}{\sqrt{1 + \left\|\nabla(\rho * \eta)\right\|^2}}\,, \tag{1.2}$$

related to [4], which states that individuals deviate from the optimal path trying to avoid entering regions with higher densities. Through numerical integrations, below we provide examples of solutions to (1.1)–(1.2). They show the interesting phenomenon of *pattern formation*. In the case of a crowd walking along a corridor, coherently with the experimental observation described in the literature, see for instance [17, 18, 20, 28], the solution to (1.1)–(1.2) self-organizes into lanes. The width of these lanes depends on the size of the support of the averaging kernel η. This feature is stable with respect to strong variations in the initial datum and also in the geometry. Indeed, we have lane formation also in the case of the evacuation of a room, when the crowd density sharply increases in front of the door. Section 4.1 is devoted to this property.

A further remarkable property of the model (1.1)–(1.2) is that it captures the following well known, although sometimes counter intuitive phenomenon. The evacuation time through an exit can be reduced by carefully positioning suitable *"obstacles"* that direct the outflow, see for instance [19] and the references therein. Minimizing the evacuation time through the solution of an optimal control problem based on (1.1)–(1.2) provides an alternative, for instance, to the evolutionary strategy described in [19].

From the analytical point of view, we note that the convolution term in (1.2) seems not sufficient to regularize solutions. Indeed, the present analytical framework is devised to consider solutions in $\mathbf{L}^1 \cap \mathbf{BV}$. Both in the case of a crush in front of an exit (Section 4.2) and in the specific example in Section 4.3, numerical simulations highlight that the space gradient of ρ may increase dramatically.

According to (1.2), pedestrians evaluate the crowd density all around their position. When restrictions on the angle of vision are relevant, the following choice is reasonable:

$$\mathcal{I}(\rho) = -\varepsilon \, \frac{\nabla \int_{\mathbb{R}^N} \rho(y)\, \eta(x-y)\, \varphi\big((y-x)\cdot g(x)\big)\, \mathrm{d}y}{\sqrt{1 + \left\|\nabla \int_{\mathbb{R}^N} \rho(y)\, \eta(x-y)\, \varphi\big((y-x)\cdot g(x)\big)\, \mathrm{d}y\right\|^2}}\,. \tag{1.3}$$

Here, η is a fixed mollifier as above and the smooth function φ weights the deviation from the preferred direction $g(x)$.

We note that the constructions in [22, 23] and [14] fit in the present setting. Indeed, there the following choices were considered:

$$\left\{\begin{array}{l}\partial_t\rho - \operatorname{div}\left(\rho\, f^2(\rho)\,\nabla\varphi\right) = 0\\ |\nabla\varphi| = 1/f(\rho)\end{array}\right. \qquad \text{and} \qquad \left\{\begin{array}{l}\partial_t\rho - \operatorname{div}\left(\rho\, f^2(\rho)\,\nabla\varphi\right) = 0\\ -\varepsilon\Delta\varphi + |\nabla\varphi|^2 = 1/\left(f(\rho)+\varepsilon\right)^2 .\end{array}\right.$$

The former admits an immediate interpretation: the direction $\nabla\varphi$ of the speed $\mathbf{v}$ is chosen along the solutions to the eikonal equations, i.e. all pedestrians follow the shortest path, weighing at every instant the length of paths with the amount of people that are moving along it. In the former case, $\nu(x) = 0$ and $\mathcal{I}(\rho)$ is the gradient of the solution to the eikonal equation with $1/f(\rho)$ in the right hand side, while in the latter case $\nu(x) = 0$ and $\mathcal{I}(\rho)$ is the gradient of the solution to an elliptic partial differential equation.

The model introduced in [13, 29] relies on this measure valued conservation law:

$$\partial_t\mu + \operatorname{div}(\mu\, v) = 0 \quad \text{where} \quad v = \nu(x) + \int_{\mathbb{R}^2} f(|x-y|)\,\varphi\left((y-x)\cdot\nu(x)\right)\frac{x-y}{\|x-y\|}\,\mathrm{d}\mu_t\,(y)\,.$$

Here the unknown is a map $\mu\colon [0,T] \to \mathcal{M}(\mathbb{R}^N;\mathbb{R})$ assigning at every time t a positive measure $\mu(t)$ which substitutes the crowd density, in the sense that the amount of people that at time t are in A is $\left(\mu(t)\right)(A)$. Contrary to this model, the present framework is the space of $\mathbf{L}^1$ densities and physical *a priori* $\mathbf{L}^\infty$ bounds on the solutions to (1.1) are rigorously proved, preventing any focusing effect as well as the rise of any Dirac delta. Moreover, below we prove global in time existence of solutions, their continuous dependence from the initial data and their stability with respect to variations in the speed law $\mathbf{v}$.

In [12], the geometric part ν of the speed is chosen *a priori*, while its modulus depend on the density as well as on the gradient of the density:

$$\partial_t\rho + \operatorname{div}\left(\rho\,\varphi(\rho,\nabla\rho)\,\nu(x)\right) = 0\,.$$

On the contrary, the model introduced in [7] postulates a nonlocal dependence of the speed from the density:

$$\partial_t\rho + \operatorname{div}\left(\rho\, v(\rho*\eta)\,\nu(x)\right) = 0\,,$$

which amounts to assume that pedestrians choose their behavior according to evaluations of an *average* of the density around their position, rather than according to the density at their place. The following second order model was presented in [1, 16] and does not fit in (1.1):

$$\left\{\begin{array}{l}\partial_t\rho + \operatorname{div}(\rho v) = 0\\ \partial_t v + (v\cdot\nabla)v = F(\rho,\nabla\rho,v)\,,\end{array}\right.$$

we refer to the review [2] for further details. To underline the variety of analytical techniques with which crowd dynamics has been tackled, we recall the further approaches: optimal transport in [5], the mean field limit in [15], the functional analytic one in [26, 27] and the nonclassical shocks used in [6, 10, 11].

Throughout this work, we set all statements in $\mathbb{R}^N$. In Section 3 we show how the present framework is able to take into consideration the presence of various constraints. In the case of pedestrian dynamics, for instance, this amounts to prove that no individual passes through the walls of a given room, provided the initial datum is assigned inside it. Concerning the dimension, our main applications are referred to crowd dynamics, i.e. $N = 2$. Nevertheless, from the analytical point of view, considering the case of a general N does not add any difficulty. Furthermore, we believe that the present setting can be reasonably applied also to, say, fishes and birds moving in $\mathbb{R}^3$, for example in predator-prey like situations as in [8].

The next section is devoted to the analytical properties of (1.1): well posedness and stability. The general theory is particularized to specific examples in Section 3, where the presence of obstacle (walls) is considered. Sample numerical integrations are provided in Section 4. The final Section 5 collects the analytical proofs. We defer to Appendix A further remarks related to the geometry of the physical domain.

2 Analytical Results

In the following, $N \in \mathbb{N} \setminus \{0\}$ is the (fixed) space dimension. We denote $\mathbb{R}^+ = [0, +\infty[$; the open ball in $\mathbb{R}^N$ centered at x and with radius $r > 0$ is $B(x, r)$ and we let $W_N = \int_0^{\pi/2} (\cos\vartheta)^N \, d\vartheta$. As usual, we denote $S^{N-1} = \left\{x \in \mathbb{R}^N : \|x\| = 1\right\}$.

The density ρ can be defined as the fraction of space occupied by pedestrians, so that ρ turns out to be a nondimensional scalar in $[0, 1]$. Otherwise, it can be useful to think at ρ as measured in, say, individuals$/m^2$ and varying in $[0, R]$, with $R > 0$ being a given maximal density, for example $R = 8$ individuals$/m^2$.

Our first step in the study of (1.1) is the formal definition of solution.

Definition 2.1. Fix a positive T and an initial datum $\rho_0 \in \mathbf{L}^1(\mathbb{R}^N; [0, R])$. A function $\rho \in \mathbf{C}^0\left([0, T]; \mathbf{L}^1(\mathbb{R}^N; \mathbb{R})\right)$ is a *weak entropy solution* to (1.1) if it is a Kružkov solution to the Cauchy problem for the scalar conservation law

$$\begin{cases} \partial_t \rho + \operatorname{div}\left(\rho\, v(\rho)\, w(t,x)\right) = 0 \\ \rho(0,x) = \rho_0(x) \end{cases} \tag{2.1}$$

where $w(t, x) = \nu(x) + \Big(\mathcal{I}\left(\rho(t)\right)\Big)(x)$.

In other words, recalling [24, Definition 1], for all $k \in \mathbb{R}$, for all $\varphi \in \mathbf{C}_c^\infty(]-\infty, T] \times \mathbb{R}^N; \mathbb{R}^+)$,

$$\int_0^T \int_{\mathbb{R}^N} \left[|\rho - k|\, \partial_t \varphi + \Big[\big(\rho\, v(\rho) - k\, v(k)\big)\; w(t,x) \cdot \nabla\varphi - k\, v(k) \operatorname{div} w(t,x)\, \varphi \Big] \operatorname{sgn}(\rho - k) \right] dx\, dt$$
$$+ \int_{\mathbb{R}^N} \left|\rho_0(x) - k\right| \varphi(0,x)\, dx \geqslant 0.$$

On the functions defining the general model (1.1), we introduce the following hypotheses:

(v) $v \in \mathbf{C}^2(\mathbb{R}; \mathbb{R})$ is non increasing, $v(0) = V$ and $v(R) = 0$ for fixed $V, R > 0$.

($\boldsymbol{\nu}$) $\nu \in (\mathbf{C}^2 \cap \mathbf{W}^{1,\infty})(\mathbb{R}^N; \mathbb{R}^N)$ is such that $\operatorname{div} \nu \in (\mathbf{W}^{1,1} \cap \mathbf{W}^{1,\infty})(\mathbb{R}^N; \mathbb{R})$.

(I) $\mathcal{I} \in \mathbf{C}^0\left(\mathbf{L}^1(\mathbb{R}^N; [0, R]); \mathbf{C}^2(\mathbb{R}^N; \mathbb{R}^N)\right)$ satisfies the following estimates:

(I.1) There exists an increasing $C_I \in \mathbf{L}^\infty_{loc}(\mathbb{R}^+; \mathbb{R}^+)$ such that, for all $r \in \mathbf{L}^1(\mathbb{R}^N; [0, R])$,

$$\left\|\mathcal{I}(r)\right\|_{\mathbf{W}^{1,\infty}} \leqslant C_I(\|r\|_{\mathbf{L}^1}) \quad \text{and} \quad \left\|\operatorname{div} \mathcal{I}(r)\right\|_{\mathbf{L}^1} \leqslant C_I(\|r\|_{\mathbf{L}^1}).$$

(I.2) There exists an increasing $C_I \in \mathbf{L}^\infty_{loc}(\mathbb{R}^+;\mathbb{R}^+)$ such that, for all $r \in \mathbf{L}^1(\mathbb{R}^N;[0,R])$,

$$\left\|\nabla \operatorname{div} \mathcal{I}(r)\right\|_{\mathbf{L}^1} \leqslant C_I(\|r\|_{\mathbf{L}^1}) .$$

(I.3) There exists a constant K_I such that for all $r_1, r_2 \in \mathbf{L}^1(\mathbb{R}^N;[0,R])$,

$$\begin{aligned} \left\|\mathcal{I}(r_1) - \mathcal{I}(r_2)\right\|_{\mathbf{L}^\infty} &\leqslant K_I \cdot \|r_1 - r_2\|_{\mathbf{L}^1} , \\ \left\|\mathcal{I}(r_1) - \mathcal{I}(r_2)\right\|_{\mathbf{L}^1} + \left\|\operatorname{div}\left(\mathcal{I}(r_1) - \mathcal{I}(r_2)\right)\right\|_{\mathbf{L}^1} &\leqslant K_I \cdot \|r_1 - r_2\|_{\mathbf{L}^1} . \end{aligned}$$

Furthermore, throughout we denote by q the map $q\colon [0,R] \to \mathbb{R}$ defined by $q(\rho) = \rho\, v(\rho)$.

As a first justification of these conditions, note that they make Definition 2.1 acceptable.

Lemma 2.2. *Fix a positive T. Let **(v)**, **($\boldsymbol{\nu}$)** and **(I.1)** hold. Choose an arbitrary density $r \in \mathbf{C}^0\left([0,T];\mathbf{L}^1(\mathbb{R}^N;[0,R])\right)$ and an initial datum $\rho_0 \in \mathbf{L}^1(\mathbb{R}^N;[0,R])$. Then, the Cauchy problem*

$$\left\{\begin{array}{l} \partial_t \rho + \operatorname{div}\left(\rho\, v(\rho)\, w(t,x)\right) = 0 \\ \rho(0,x) = \rho_0(x) \end{array}\right. \quad \textit{with} \quad w(t,x) = \nu(x) + \Big(\mathcal{I}\left(r(t)\right)\Big)(x) \tag{2.2}$$

satisfies the assumptions of Kružkov Theorem [24, Theorem 5].

This lemma is proved in Section 5 below. In Section 3 we show that the above assumption **(I)** allows to comprehend physically reasonable cases.

The next result is devoted to the proof of existence and uniqueness for (1.1). It is obtained through Banach Fixed Point Theorem.

Theorem 2.3. *Let **(v)**, **($\boldsymbol{\nu}$)** and **(I)** hold. Choose any $\rho_0 \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}^N;[0,R])$. Then, there exists a unique weak entropy solution $\rho \in \mathbf{C}^0\left(\mathbb{R}^+;\mathbf{L}^1(\mathbb{R}^N;[0,R])\right)$ to (1.1). Moreover, ρ satisfies the bounds*

$$\begin{aligned} \left\|\rho(t)\right\|_{\mathbf{L}^1} &= \|\rho_0\|_{\mathbf{L}^1} , \ \textit{for a.e. } t \in \mathbb{R}^+ , \\ \operatorname{TV}(\rho(t)) &\leqslant \operatorname{TV}(\rho_0)\, e^{kt} + t e^{kt} N W_N \|q\|_{\mathbf{L}^\infty([0,R])} \left(\|\nabla \operatorname{div} \nu\|_{\mathbf{L}^1} + C_I(\|\rho_0\|_{\mathbf{L}^1})\right) , \end{aligned}$$

where $k = (2N+1)\left\|q'\right\|_{\mathbf{L}^\infty([0,R])}\left(\|\nabla \nu\|_{\mathbf{L}^\infty} + C_I(\|\rho_0\|_{\mathbf{L}^1})\right)$.

Using the techniques in [7, 9], we now obtain the continuous dependence of the solution to (1.1) from the datum ρ_0 and its stability with respect to v, ν and $\mathcal{I}$ in the natural norms.

Theorem 2.4. *Let **(v)**, **($\boldsymbol{\nu}$)** and **(I)** be satisfied by both systems*

$$\begin{cases} \partial_t \rho + \operatorname{div}\Big[\rho\, v_1(\rho)\, \left(\nu_1(x) + \mathcal{I}_1(\rho)\right)\Big] = 0 \\ \rho(0,x) = \rho_{0,1}(x) \end{cases} \quad \textit{and} \quad \begin{cases} \partial_t \rho + \operatorname{div}\Big[\rho\, v_2(\rho)\, \left(\nu_2(x) + \mathcal{I}_2(\rho)\right)\Big] = 0 \\ \rho(0,x) = \rho_{0,2}(x) \end{cases}$$

with $\rho_{0,1}, \rho_{0,2} \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}^N;[0,R])$. Then, the two solutions ρ_1 and ρ_2 satisfy

$$\begin{aligned} \left\|\rho_1(t) - \rho_2(t)\right\|_{\mathbf{L}^1} \leqslant\ & C(t)\Big(\left\|\rho_{0,1} - \rho_{0,2}\right\|_{\mathbf{L}^1} + \|q_1 - q_2\|_{\mathbf{W}^{1,\infty}} \\ & + \|\nu_1 - \nu_2\|_{\mathbf{L}^\infty} + \left\|\operatorname{div}(\nu_1 - \nu_2)\right\|_{\mathbf{L}^1} + d\left(\mathcal{I}_1, \mathcal{I}_2\right)\Big) \end{aligned}$$

where $d(\mathcal{I}_1,\mathcal{I}_2) = \sup\left\{\left\|\mathcal{I}_1(\rho) - \mathcal{I}_2(\rho)\right\|_{\mathbf{L}^\infty} + \left\|\operatorname{div}\mathcal{I}_1(\rho) - \operatorname{div}\mathcal{I}_2(\rho)\right\|_{\mathbf{L}^1} : \rho \in \mathbf{L}^1(\mathbb{R}^N;[0,R])\right\}$, the map $C \in \mathbf{C}^0(\mathbb{R}^+;\mathbb{R}^+)$ vanishes at $t = 0$ and depends on $\operatorname{TV}(\rho_{0,1})$, $\left\|\rho_{0,1}\right\|_{\mathbf{L}^1}$, $\|\nu_1\|_{\mathbf{L}^\infty}$, $\|\operatorname{div} \nu_1\|_{\mathbf{W}^{1,1}}$, $\|q_1\|_{\mathbf{W}^{1,\infty}}$, $\|q_2\|_{\mathbf{W}^{1,\infty}}$.

Thanks to these stability results, several control problems can be proved to admit a solution through a direct application of Weierstraß Theorem. A possible standard application could be the minimization of the evacuation time from a given room. Describing the actions of a controller able to determine the initial pedestrians' distribution and/or their preferred paths, one is lead to an optimal control problem with the initial datum ρ_0 and the vector field ν as control parameters, for instance. Without any loss of generality, the compact sets on which the optimization is made can be $\left\{\rho \in (\mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}^N;[0,R])\colon \operatorname{TV}(\rho) < M\right\}$ for ρ_0 and $\left\{\nu \in \mathbf{C}^2\left(\mathbb{R}^N;\overline{B(0,1)}\right) : \left\|\nabla^3\nu\right\| \leqslant M\right\}$ for ν, where $M > 0$ is arbitrary.

A different problem solved by the same analytical techniques is that of the dynamic parameter estimation. Once real data are available, one is left with the problem of determining the various parameters entering ν, v or $\mathcal{I}$. Theorem 2.4 ensures the existence of the parameters that allows a best agreement between the solutions to (1.1) and the data.

3 The Models

This section is devoted to the study of specific cases of (1.1). Aiming at real applications, it is necessary to take into consideration the various constraints that are present on the movement of pedestrians. Therefore, we introduce the subset Ω of $\mathbb{R}^N$ which characterizes the region reachable to any individual. The boundary $\partial\Omega$ consists of *walls* that can not be crossed by any individual. On the set Ω we require that

($\boldsymbol{\Omega}$) $\Omega \subseteq \mathbb{R}^N$ is the closure of a non empty connected open set. If $\partial\Omega$ is not empty, there exists a positive r_Ω such that the function

$$\begin{array}{cccc} d_{\partial\Omega} & : & B(\partial\Omega, r_\Omega) \cap \Omega & \to & \mathbb{R}^+ \\ & & x & \mapsto & \inf\left\{d(x,w)\colon w \in \partial\Omega\right\} \end{array}$$

is of class $\mathbf{C}^2\left(B(\partial\Omega, r_\Omega) \cap \Omega; \mathbb{R}^+\right)$.

We do not require that Ω be bounded. This assumption allows to introduce the inward normal

$$n(x) = \nabla d_{\partial\Omega}(x) \tag{3.1}$$

on all the strip $B(\partial\Omega, r_\Omega) \cap \Omega$. Moreover, $\left\|n(x)\right\| = 1$.

For the present models to be acceptable, it is mandatory that no individual enters any wall, provided the initial datum is supported inside the physically admissible space. Analytically, this is described by the following *invariance* property:

(P) The model (1.1) is invariant with respect to Ω if

$$\operatorname{spt}\rho_0 \subset \Omega \qquad \Longrightarrow \qquad \operatorname{spt}\rho(t) \subset \Omega \text{ for all } t \geqslant 0\,.$$

Below, we show that theorems 2.3 and 2.4 can be applied to realistic situations, in the sense that the presence of a physically admissible set Ω can be considered and its invariance in the sense of **(P)** proved. More precisely, we show the following sufficient condition for invariance.

Proposition 3.1. *Let ν, $\mathcal{I}$ and Ω satisfy **($\boldsymbol{\nu}$)**, **(I)** and **($\boldsymbol{\Omega}$)**. If for all $x \in \partial\Omega$ and $\rho \in \mathbf{L}^1(\mathbb{R}^N;[0,R])$ with $\operatorname{spt}\rho \subseteq \Omega$*

$$\Big(\nu(x) + \big(\mathcal{I}(\rho)\big)(x)\Big) \cdot n(x) \geqslant 0 \tag{3.2}$$

*then, property **(P)** holds.*

3.1 The Model (1.1)–(1.2)

The starting point for (1.1)–(1.2) is provided by the degenerate parabolic model

$$\partial_t \rho + \operatorname{div}\left(\rho\, v(\rho)\left(\nu(x) - \varepsilon \frac{\nabla\psi(\rho)}{\sqrt{1+\left\|\nabla\psi(\rho)\right\|^2}}\right)\right) = 0$$

introduced in [4], which fits in (1.1) with $\mathcal{I}(\rho) = -\varepsilon\, \nabla\psi(\rho) \Big/ \sqrt{1+\left\|\nabla\psi(\rho)\right\|^2}$, motivated by the desire of each individual to avoid entering regions occupied by a high crowd density. Here, ψ is a suitable weight function. Assuming that each pedestrian reacts to evaluations of *averages* of the density, we obtain

$$\partial_t \rho + \operatorname{div}\left(\rho\, v(\rho)\left(\nu(x) - \varepsilon \frac{\nabla(\psi(\rho) * \eta)}{\sqrt{1+\left\|\nabla(\psi(\rho) * \eta)\right\|^2}}\right)\right) = 0\,.$$

Here, we avoid the introduction of ψ to limit the analytical technicalities, obtaining (1.1)–(1.2). We assume throughout that $\varepsilon > 0$ is fixed and that the mollifier η satisfies

($\boldsymbol{\eta}$) $\eta \in \mathbf{C}_c^3(\mathbb{R}^N; \mathbb{R}^+)$ with $\int_{\mathbb{R}^N} \eta(x)\, \mathrm{d}x = 1$.

This convolution kernel has a key role: the individual at x deviates from the optimal path considering the crowd present in the region $x - \operatorname{spt}\eta$, when no walls are present. The value $\eta(\xi)$ is the relevance that the individual at x gives to the density $\rho(x-\xi)$ located at $x - \xi$.

The next result shows that the present model fits in the framework described in Section 2 when applied on all of $\mathbb{R}^N$.

Lemma 3.2. *Fix $\varepsilon > 0$ and let η satisfy **($\boldsymbol{\eta}$)**. Then, the operator $\mathcal{I}$ in* (1.2) *satisfies **(I)**.*

When the region Ω reachable by the crowd is restricted by the walls $\partial\Omega$, we intend the convolution restricted to Ω

$$(\rho * \eta)(x) = \int_\Omega \rho(y)\, \eta(x-y)\, \mathrm{d}y \tag{3.3}$$

which coincide with the previous definition in the case $\operatorname{spt}\rho \subset \Omega$. A better choice is described in (A.1). The vector ν is here chosen as a sum

$$\nu = g + \delta\,. \tag{3.4}$$

The former vector g is tangent to the *"optimal"* path, depending only on the geometry of the environment and coherent with it. Hence we assume that

(g) $g \in \mathbf{C}^2(\mathbb{R}^N; S^{N-1})$ satisfies $\nabla g \in \mathbf{L}^\infty(\mathbb{R}^N; \mathbb{R}^{N\times N})$, $\operatorname{div} g \in (\mathbf{W}^{1,1} \cap \mathbf{W}^{1,\infty})(\mathbb{R}^N; \mathbb{R})$ and the invariance condition $g(x) \cdot n(x) \geqslant 0$ holds for all $x \in \partial\Omega$.

The latter vector δ describes the discomfort felt when passing too near to walls or obstacles. Below, we choose

$$\delta(x) = \lambda\, \alpha(x)\, n(x)\,. \tag{3.5}$$

Here, $\lambda \in \mathbb{R}^+$ is a suitable constant and $n(x)$ is the inward normal (3.1). The role of the function α is to confine this discomfort to the region near the walls, i.e. we require

($\boldsymbol{\alpha}$) $\alpha \in \mathbf{C}^2(\mathbb{R}^N;[0,1])$ is such that $\alpha(x) = 0$ whenever $B(x, r_\Omega) \subseteq \Omega$ and $\alpha(x) = 1$ for $x \in \partial\Omega$.

A better choice for the discomfort is discussed in Appendix A.

The present setting (1.1)–(1.2) can be applied also in presence of walls.

Proposition 3.3. *Let $\varepsilon > 0$, **(v)**, **($\boldsymbol{\Omega}$)**, **($\boldsymbol{\eta}$)**, **(g)** and **($\boldsymbol{\alpha}$)** hold. Define ν by* (3.4)*, δ by* (3.5) *and $\mathcal{I}$ by* (1.2)*. Assume moreover that either $\partial\Omega$ is compact, or $\alpha \in (\mathbf{W}^{2,1} \cap \mathbf{W}^{2,\infty})$ and $d_{\partial\Omega} \in (\mathbf{W}^{3,1} \cap \mathbf{W}^{3,\infty})$. Then,* (1.1)–(1.2) *satisfies the assumptions of Theorem 2.3 and Theorem 2.4. Furthermore, if $\lambda \geqslant \varepsilon R \left\|\nabla\eta\right\|_{\mathbf{L}^1}$, then property **(P)** holds.*

3.2 The Model (1.1)–(1.3)

In the framework of (1.1), we now extend the model [13, (2.1)–(2.4)] to take into account the effects of crowd density on the speed modulus, i.e. of $v(\rho)$. Moreover, we interpret the nonlocal term as a weighted average of the density, where the amount of crowd in the direction of the optimal path g is given more importance.

Using the same notation as above, we thus consider the model (1.1) with

$$\mathcal{I}(\rho) = -\varepsilon \frac{\nabla \int_{\mathbb{R}^N} \rho(y)\, \eta(x-y)\, \varphi\big((y-x)\cdot g(x)\big)\, \mathrm{d}y}{\sqrt{1 + \left\|\nabla \int_{\mathbb{R}^N} \rho(y)\, \eta(x-y)\, \varphi\big((y-x)\cdot g(x)\big)\, \mathrm{d}y\right\|^2}} . \tag{3.6}$$

Here, ε, η and v are as in § 3.1 and, in particular, ν is as in (3.4).

The convolution in (3.3) is here weighted by φ, whose argument is essentially the angle between the preferred path g and $y - x$, the point x is the position of the individual while y is the point where the density is evaluated. This term takes into account the preference of each individual to deviate little from the preferred path defined by g. For example, let $\varphi \in \mathbf{C}^\infty(\mathbb{R};[0,1])$ be such that $\varphi \equiv 0$ on $]-\infty, 0]$ and $\varphi \equiv 1$ on $[\vartheta, +\infty[$, where $\vartheta > 0$ is a given parameter. Then, adding the function $\varphi\left((y-x)\cdot g(x)\right)$ into the nonlocal term means that the individual at x reacts to the average density evaluated in the preferred direction $g(x)$.

The denominator is a regularized normalization. Its presence is necessary from the modeling point of view, for coherence with the presence of $v(\rho)$. From the analytical point of view, this normalization makes various expressions slightly more complicate, but all estimates remain doable. We are thus lead to the equation

$$\partial_t \rho + \operatorname{div}\left(\rho\, v(\rho) \left(\nu(x) - \varepsilon \frac{\nabla \int_{\mathbb{R}^N} \rho(y)\, \eta(x-y)\, \varphi\big((y-x)\cdot g(x)\big)\, \mathrm{d}y}{\sqrt{1 + \left\|\nabla \int_{\mathbb{R}^N} \rho(y)\, \eta(x-y)\, \varphi\big((y-x)\cdot g(x)\big)\, \mathrm{d}y\right\|^2}}\right)\right) = 0 \tag{3.7}$$

and we verify that it fits in the analytical framework provided in Section 2.

Lemma 3.4. *Fix $\varepsilon > 0$, $\eta \in \mathbf{C}^3_c(\mathbb{R}^N;\mathbb{R}^+)$ with $\int_{\mathbb{R}^N} \eta(x)\, \mathrm{d}z = 1$, $g \in \mathbf{W}^{3,\infty}(\mathbb{R};[0,1])$ and $\nu \in \mathbf{W}^{3,\infty}(\mathbb{R}^N;\mathbb{R})$. Then, the operator $\mathcal{I}$ defined by* (1.3) *satisfies **(I)**.*

When the crowd's movement is constrained by the walls $\partial\Omega$, as in the preceding section, we intend all integrals in (3.6)–(3.7) restricted to Ω. In particular, we consider now

$$\mathcal{I}(\rho) = -\varepsilon \frac{\nabla \int_{\Omega} \rho(y)\, \eta(x-y)\, \varphi\big((y-x)\cdot g(x)\big)\, \mathrm{d}y}{\sqrt{1 + \left\|\nabla \int_{\Omega} \rho(y)\, \eta(x-y)\, \varphi\big((y-x)\cdot g(x)\big)\, \mathrm{d}y\right\|^2}} , \tag{3.8}$$

with v as in **(v)** and ν as in (3.4). The applicability of theorems 2.3 and 2.4 to (1.1)–(3.8) and the validity of property **(P)** is ensured by the following proposition.

Proposition 3.5. *Let* $\varepsilon > 0$*,* ***(v)****,* ***(Ω)****,* ***(η)*** *and* ***(g)*** *hold, with moreover* $g \in (\mathbf{C}^3 \cap \mathbf{W}^{3,\infty})(\mathbb{R}^N; S^{N-1})$*. Let* $\varphi \in (\mathbf{C}^3 \cap \mathbf{W}^{3,\infty})(\mathbb{R};\mathbb{R})$*. Define* ν *by* (3.4)*,* δ *by* (3.5) *and* $\mathcal{I}$ *by* (3.6)*. Then,* (1.1)–(3.6) *satisfies the assumptions of Theorem 2.3 and Theorem 2.4.*

Moreover, assume that $\varphi' \geqslant 0$ *and call* $\ell = \operatorname{diam}(\operatorname{spt}\eta)$*. Then,*

$$\varphi' \geqslant 0 \qquad \text{and} \qquad \lambda \geqslant R \left\|\eta\right\|_{\mathbf{W}^{1,1}} \left\|\varphi\right\|_{\mathbf{W}^{1,\infty}} \left(1 + \ell \left\|\nabla g\right\|_{\mathbf{L}^\infty}\right)$$

imply that property ***(P)*** *holds.*

3.3 The Model [13, (2.1)–(2.4)]

The model in [13, 29], although set therein in the space $\mathcal{M}(\mathbb{R}^N;\mathbb{R}^+)$ of positive Radon measures, can be seen as a particular case of (1.1) setting

$$\begin{array}{rcl} v(\rho) & = & 1 \\ \nu(x) & = & v_{\mathrm{des}}(x) \end{array} \qquad \text{and} \qquad \mathcal{I}(\rho) = \varepsilon \int_{\mathbb{R}^N} \rho(y)\, \nabla\eta(x-y)\, \varphi\big((y-x)\cdot\nu(x)\big)\, \mathrm{d}y\ , \tag{3.9}$$

v_{des} being the *desired* speed, see [13, formula (2.4)]. If η is radially symmetric, we recover [13, formula (2.6)] with $\eta(x) = \tilde\eta(\|x\|)$ and $\tilde\eta' = f$, see [13] the motivations of this model.

Proposition 3.6. *Let* $\varepsilon > 0$*,* ***(v)****,* ***(ν)*** *and* ***(η)*** *hold. Assume that* $\varphi \in (\mathbf{C}^2 \cap \mathbf{W}^{2,\infty})(\mathbb{R};\mathbb{R})$*. Then,* (1.1)–(3.9) *satisfies the assumptions of Theorem 2.3 and Theorem 2.4.*

In particular, above we prove that if the initial datum ρ_0 is in $\mathbf{L}^1(\mathbb{R}^N;[0,R])$, then the corresponding solution satisfies the same bounds. This ensures that neither focusing to any Dirac delta takes place, nor values of the density above R can be expected. (For the sake of completeness, we note that the conditions $\eta \geqslant 0$ and $\int_{\mathbb{R}^N} \eta\, \mathrm{d}x = 1$ is in the case of (1.1)–(3.9) neither necessary, nor meaningful and can be replaced by $\eta \in (\mathbf{C}^3_c \cap \mathbf{W}^{2,\infty} \cap \mathbf{W}^{3,1})(\mathbb{R}^N;\mathbb{R})$, see the proof in Section 5 for more details.).

4 Qualitative Properties

This section is devoted to sample numerical integrations of (1.1). In all the examples below, we choose as vector field $\nu = \nu(x)$ the geodesic one, computed solving numerically the eikonal equation. This leads to the formation of congested queues near to the door jambs. From the modeling point of view, a more refined choice would consist in choosing ν so that most pedestrians are directed towards the central part of the exit. This choice increases the difficulties neither of the analytical treatment nor of the numerical integration but imposes the introduction of several further parameters.

The algorithm used is the Lax-Friedrichs method with dimensional splitting. A grid (x_i, y_j) for $i = 1,\ldots,n_x$ and $j = 1,\ldots,n_y$ is introduced and the density ρ is approximated through the values ρ_{ij} on this grid. At every time step, the convolution in $\mathcal{I}(\rho)$ is computed through products of the type $A_{ih}\rho_{hk}B_{kj}$, where the matrices A and B depend only on η.

All the examples below are set in $\mathbb{R}^2$, due to obvious visualization problems in higher dimensions. As is well known, the analytical techniques are independent from the dimension as also the numerical algorithm. The time of integration obviously increases with the dimension.

4.1 Lane Formation

A widely detected pattern formed in the context of crowd dynamics is that of *lane formation*, see for instance [17, 18, 20, 28]. This feature has been often related to the specific qualities of each individual, i.e. it has usually been explained from a microscopic point of view. Here, in a purely macroscopic setting, we show that the solutions to (1.1)–(1.2) also display this pattern formation phenomenon, with pedestrian self organizing along lanes.

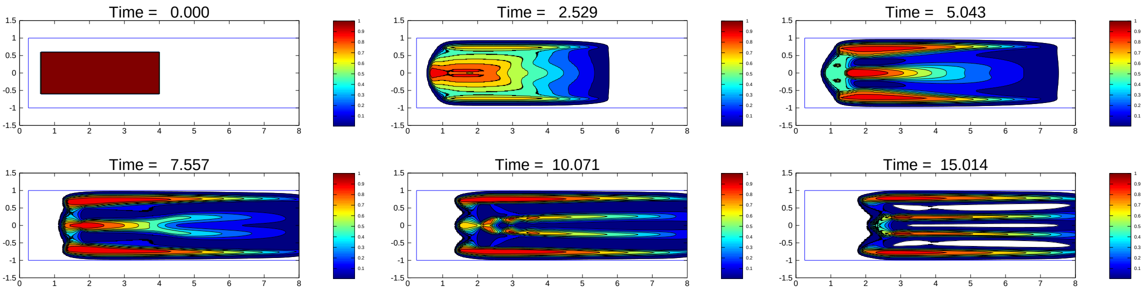

Figure 1: Solution to (1.1)–(1.2)–(3.4)–(4.1) at times $t = 0$, 2.529, 5.043, 7.557, 10.071, 15.014. First 3 lanes are formed, then the middle lane bifurcates forming the fourth lane.

Consider (1.1)–(1.2) with

$$\begin{array}{rclrclrcl} \nu(x) & = & \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \delta(x)\,, & \eta(x) & = & \left[1-\left(\frac{x_1}{r}\right)^2\right]^3 \left[1-\left(\frac{x_2}{r}\right)^2\right]^3 \chi_{[-r,r]^2}(x)\,, & r & = & \frac{4}{5}\,, \\ v(\rho) & = & \frac{1}{2}\,(1-\rho)\,, & \rho_0(x) & = & \chi_{[3/5,4]\times[-3/5,3/5]}(x)\,, & \varepsilon & = & \frac{2}{5}\,, \end{array} \tag{4.1}$$

where $\delta = \delta(x)$ describes the discomfort due to walls: it is a vector normal to the walls, pointing inward, with intensity 3/2 along the walls, decreasing linearly to 0 at a distance 3/10 from the walls. As Figure 1 shows, the initially uniform crowd distribution evolves into a patterned configuration, first with 4 lanes and then with 5 lanes. The number of lanes depends on the size of the support of the convolution kernel η. Indeed, keeping all functions and parameters fixed, but not the parameter r, which determines the size of $\operatorname{spt}\eta$, we obtain patterns differing in the number of lanes, see Figure 2.

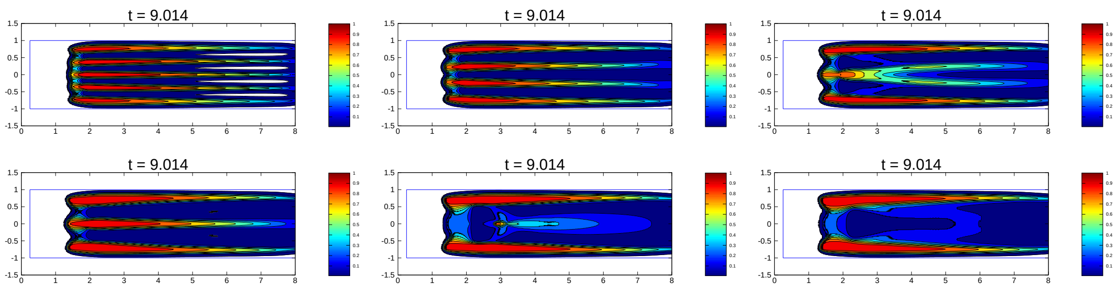

Figure 2: Solution to (1.1)–(1.2)–(3.4)–(4.1) computed at time $t = 9.014$ and with $\operatorname{spt}\eta$ with radius $r = 0.5$, 0.6, 0.8, 0.9, 1.0, 1.4. Note that as r increases, the number of lanes diminishes.

The formation of lanes is a rather stable phenomenon. Indeed, Figure 3 shows the result of the integration of (1.1)–(1.2)–(3.4)–(4.1) computed at time $t = 0$, 5, 10 with $r = 3/5$ (above) and $r = 9/10$ (below) with initial data different from that in (4.1). In both cases, lanes are

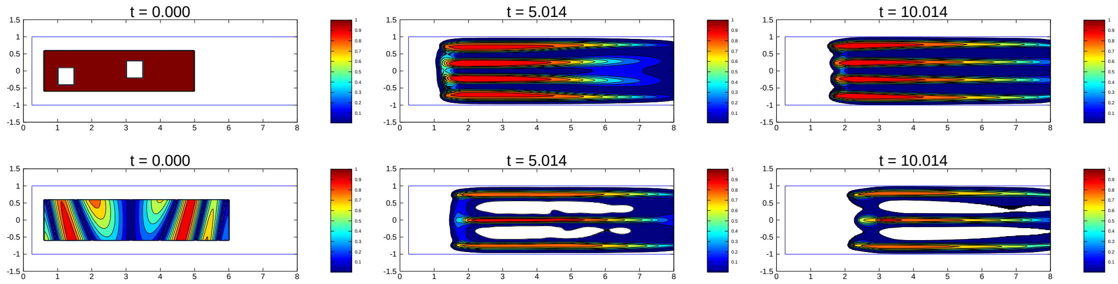

Figure 3: Solution to (1.1)–(1.2)–(3.4)–(4.1) with different initial data at time $t = 0, 5.014, 10.014$ and above with $r = 0.6$, below with $r = 0.9$. Note that above 4 lanes form and below 5, similarly to what obtained in Figure 1.

formed similar to the corresponding situations in Figure 2.

We also note that in the present framework, using the terms in [28, Section 5.4], lanes form also in an *isotropic* setting. Indeed, the integrations in figures 1–2 were obtained with individuals able to see both forward *and* behind.

4.2 Evacuation of a Room

A standard application of macroscopic models for crowd dynamics is the minimization of evacuation times. The present setting applies to general geometries, see the assumption **(Ω)**. Here we show that (1.1) captures reasonable features of the escape dynamics.

We consider a room with an exit, as in Figure 4. The vector $\nu = \nu(x)$ is chosen as the unit vector tangent at x to the geodesic connecting x to the exit. The discomfort $d = d(x)$ is a vector normal to the walls, pointing inward, with intensity 1 along the walls, decreasing linearly to 0 at a distance 1/2 from the walls. The other quantities are in (4.2). Keeping the

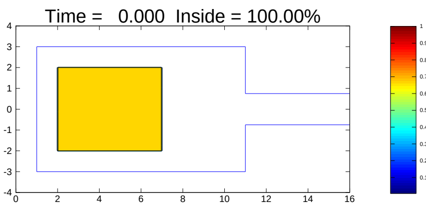

Figure 4: Initial datum and room geometry considered in § 4.2.

$$\begin{aligned} v(\rho) &= 6(1-\rho) \\ \eta(x) &= \left[1-\left(\tfrac{x_1}{r}\right)^2\right]^3 \left[1-\left(\tfrac{x_2}{r}\right)^2\right]^3 \chi_{[-r,r]^2}(x)\,, \\ r &= 0.6 \\ \rho_0(x) &= 0.75\,\chi_{[2,7]\times[-2,2]}(x) \\ \varepsilon &= 0.4 \end{aligned} \tag{4.2}$$

above parameters fixed, as well as the outer walls of the room, we insert various obstacles (columns) to direct the movement of the crowd.

First, Figure 5, first line, shows an integration of the case with two columns that direct people towards the exit. The number of lanes self adapts to the available space, with three lanes merging into one before the bottleneck. On the second line of Figure 5, the insertion of three columns in these positions delays but does not avoid the congestion at the exit. Those individuals that pass through the bottleneck are favored in exiting the room. On the third line of Figure 5, an asymmetric layout partly hinders the lanes' pattern.

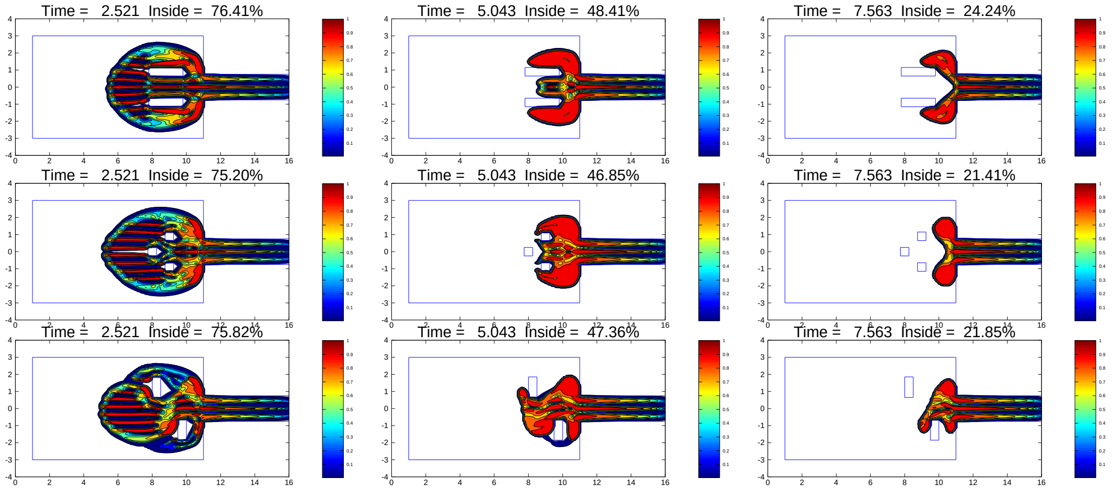

Figure 5: Solution to (1.1)–(1.2)–(3.4)–(4.2) with different geometries, computed at time $t = 2.521$, 5.043 and 7.563.

One of the most relevant problems in the design of escape routes is the planning of suitable devices that organize the movement of the crowd in order to reduce the evacuation time. With the situation in (4.2), using the present framework, we show that the careful introduction of suitable obstacles in suitable locations reduce the evacuation time, see Figure 6. Indeed, the presence of the columns allows to better exploit the exit, avoiding the formation of congested areas at the sides of the door jambs. Nevertheless, the careful positioning of these obstacles does permit to some people to reach the exit from the sides (see Figure 6, first and second picture on the second line) which also contributes to lower the evacuation time. As a result, the time necessary to completely evacuate the room get lower thanks to the 4 columns, although for short times the no-columns configuration is more efficient.

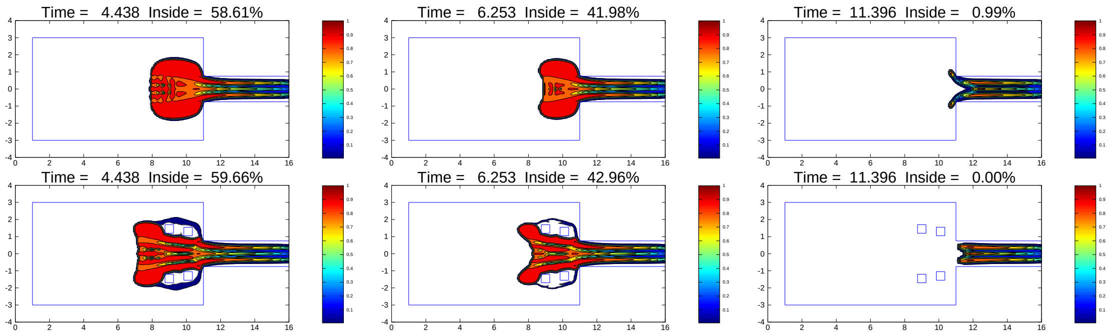

Figure 6: Solution to (1.1)–(1.2)–(3.4)–(4.2) with $\varepsilon = 0.2$, at times $t = 4.438$, 6.253, 11.396. On the first line, no obstacle is present. On the second line, 4 columns direct the crowd flow. Note that the evacuation time in the latter case is *shorter* than in the former one.

4.3 On the Rise of Singularities

A typical feature of conservation laws is the possible rise of singularities, see for instance [3, Example 1.4]. The nonlocal equation (1.1) shares this characteristic. Indeed, assume $\rho = \rho(t, x)$ is a given solution to (1.1), smooth up to time $T > 0$. Then, setting $w(t, x) =$

$\Big(\nu(x) + \big(\mathcal{I}(t)\big)\,(x)\Big)$, simple computations lead to the following equation for the space derivative ρ_j of ρ in the direction x_j, for $j = 1, \ldots, N$:

$$\partial_t \rho_j + (\operatorname{div} \rho_j)\, q'\, w = (\rho_j)^2\, q''\, w + \rho_j\, q'' \sum_{i \neq j} (\partial_i \rho)\, w + \rho_j\, q'\, \operatorname{div} w + q' \sum (\partial_i \rho)\, \partial_j w + q\, \partial_j \operatorname{div} w\,.$$

The first term in the right hand side is quadratic in ρ_j, showing that a blow up of ρ_j may take place in finite time.

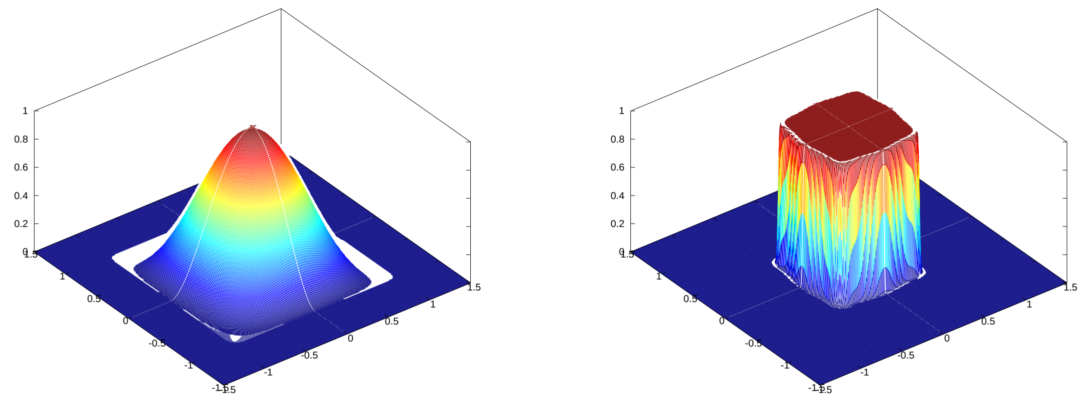

Figure 7: Left, the initial datum and, right, the solution to (1.1)–(1.2)–(4.3) at time $t = 1$. Note the formation of vertical faces in the initially smooth distribution.

Moreover, we consider (1.1) with $\mathcal{I}$ as in (1.2) and

$$\begin{array}{rclrcl} v(\rho) & = & 1 - \rho\,, & \eta(x,y) & = & (1-16x^2)^3\,(1-16y^2)^3\,\chi_{[-1/4,1/4]^2}(x,y)\,, \\ \nu(x) & = & 0\,, & \rho_0(x,y) & = & (1-4x^2/9)^2\,(1-4y^2/9)^2\,\chi_{[-3/2,3/2]^2}(x,y)\,, \end{array} \quad \varepsilon = -1 \qquad (4.3)$$

and we obtain the solution in Figure 7.

5 Technical Details

5.1 Existence and uniqueness

Proof of Lemma 2.2. Let $q(\rho) = \rho\, v(\rho)$. Thanks to $(\boldsymbol{\nu})$, the assumptions on r and **(I.1)**, we have for all $M > 0$

$$\begin{aligned} \partial_\rho f(t,x,\rho) &= q'(\rho)\, \Big(\nu(x) + \mathcal{I}\big(r(t)\big)\,(x)\Big)\,; & \partial_\rho f &\in \mathbf{L}^\infty([0,T] \times \mathbb{R}^N \times [-M,M]; \mathbb{R}^N)\,; \\ \operatorname{div} f(t,x,\rho) &= q(\rho)\, \operatorname{div} \Big(\nu(x) + \mathcal{I}\big(r(t)\big)\,(x)\Big)\,; & \operatorname{div} f &\in \mathbf{L}^\infty([0,T] \times \mathbb{R}^N \times [-M,M]; \mathbb{R})\,. \end{aligned}$$

Thus, we apply Kružkov Theorem [24, Theorem 5 & 5.4] and ensure that (2.2) admits a unique Kružkov solution $\rho \in \mathbf{L}^\infty\Big(\mathbb{R}^+; \mathbf{L}^1(\mathbb{R}^N; \mathbb{R})\Big)$, right continuous in time. □

Lemma 5.1. *Let $\rho_0 \in \mathbf{L}^1(\mathbb{R}^N;[0,R])$, $r \in \mathbf{C}^0\left(\mathbb{R}^+;\mathbf{L}^1(\mathbb{R}^N;[0,R])\right)$. Under hypotheses* **(v)***–***(ν)***–***(I.1)***, the Cauchy problem* (2.2) *admits a unique weak entropy solution ρ, with $\rho \in \mathbf{C}^0\left(\mathbb{R}^+;\mathbf{L}^1(\mathbb{R}^N;[0,R])\right)$ satisfying, for all $t \in \mathbb{R}^+$,*

$$\left\|\rho(t)\right\|_{\mathbf{L}^1} = \left\|\rho_0\right\|_{\mathbf{L}^1}. \tag{5.1}$$

Assume, in addition, that **(I.2)** *is satisfied. Then, $\rho_0 \in \mathbf{BV}(\mathbb{R}^N;[0,R])$ implies $\rho(t) \in \mathbf{BV}(\mathbb{R}^N;[0,R])$ for all time $t \geqslant 0$. Moreover, the following bound is satisfied*

$$\mathrm{TV}\left(\rho(t)\right) \leqslant \left[\mathrm{TV}\left(\rho_0\right) + t\, N W_N \|q\|_{\mathbf{L}^\infty([0,R])}\left(\|\nabla \operatorname{div}\nu\|_{\mathbf{L}^1} + C_I(\|r\|_{\mathbf{L}^\infty([0,T];\mathbf{L}^1)})\right)\right] e^{\kappa_0^* t}, \tag{5.2}$$

where $W_N = \int_0^{\pi/2}(\cos\vartheta)^N\,\mathrm{d}\vartheta$ and the constant κ_0^* is bounded above as follows:

$$\kappa_0^* \leqslant (2N+1)\left\|q'\right\|_{\mathbf{L}^\infty([0,R])}\left(\|\nabla\nu\|_{\mathbf{L}^\infty} + C_I(\|r\|_{\mathbf{L}^\infty([0,T];\mathbf{L}^1)})\right).$$

Proof. Below we set $q(\rho) = \rho\, v(\rho)$, so that $f(t,x,\rho) = q(\rho)\left(\nu(x) + \left(\mathcal{I}\left(r(t)\right)\right)(x)\right)$. The existence of a solution follows from Lemma 2.2. The rest of the proof is obtained through the following steps.

Estimates in $\mathbf{L}^1$ and $\mathbf{L}^\infty$. Since ρ solves a conservation law, (5.1) is immediately satisfied.

Besides, $\rho \equiv 0$ and $\rho \equiv R$ are solutions to (2.2) associated respectively to the constant initial conditions $\rho_0 \equiv 0$ and $\rho_0 \equiv R$. Hence, we can apply the comparison Theorem [24, Theorem 3] and obtain that if the initial condition takes value in $[0,R]$, then the solution takes values in $[0,R]$, so that $\rho \in \mathbf{L}^\infty\left(\mathbb{R}^+;\mathbf{L}^1(\mathbb{R}^N;[0,R])\right)$.

Continuity in time. Thanks to [9, Remark 2.4] or [25, Corollary 2.4], ρ is continuous in time if, for any $T > 0$,

$$\|q\|_{\mathbf{L}^\infty([0,R])}\int_0^T\int_{\mathbb{R}^N}\left|\operatorname{div}\left(\nu(x) + \left(\mathcal{I}\left(r(t)\right)\right)(x)\right)\right|\mathrm{d}x\,\mathrm{d}t < \infty. \tag{5.3}$$

We have, thanks to **(I.1)**, for all $t \geqslant 0$

$$\left\|\operatorname{div}\left(\nu + \mathcal{I}\left(r(t)\right)\right)\right\|_{\mathbf{L}^1} \leqslant \|\operatorname{div}\nu\|_{\mathbf{L}^1} + C_I(\left\|r(t)\right\|_{\mathbf{L}^1}).$$

Hence, condition (5.3) is satisfied under hypotheses **(v)**–**(ν)**–**(I.1)** and we also obtain $\rho \in \mathbf{C}^0\left(\mathbb{R}^+;\mathbf{L}^1(\mathbb{R}^N;[0,R])\right)$.

Estimate in BV. To prove the bound on the TV norm we use [25, Theorem 2.2]. To this aim, we have to check that for any $T > 0$,

$$\nabla\partial_\rho f \in \mathbf{L}^\infty([0,T]\times\mathbb{R}^N\times[0,R];\mathbb{R}^{N\times N}) \quad \text{and} \quad \int_0^T\int_{\mathbb{R}^N}\left\|\nabla\operatorname{div} f(t,x,\cdot)\right\|_{\mathbf{L}^\infty([0,R])}\mathrm{d}x\,\mathrm{d}t < \infty.$$

Note that $\nabla \partial_\rho f(t,x,\rho) = q'(\rho)\, \nabla \left(\nu + \mathcal{I}\left(r(t)\right)\right)$. Hence, thanks to **(v)**–**($\boldsymbol{\nu}$)** and **(I.1)**,

$$\left\|\nabla \partial_\rho f(t,x,\rho)\right\|_{\mathbf{L}^\infty} \leqslant \left\|q'\right\|_{\mathbf{L}^\infty([0,R])} \left(\|\nabla \nu\|_{\mathbf{L}^\infty} + C_I(\left\|r(t)\right\|_{\mathbf{L}^1})\right) ,$$

Thanks to **(v)**–**($\boldsymbol{\nu}$)** and **(I.2)**, we have, for all $t \geqslant 0$

$$\left\|\nabla \operatorname{div} \left(\nu + \mathcal{I}\left(r(t)\right)\right)\right\|_{\mathbf{L}^1} \leqslant \|\nabla \operatorname{div} \nu\|_{\mathbf{L}^1} + C_I(\left\|r(t)\right\|_{\mathbf{L}^1}) .$$

Applying [25, Theorem 2.2], we obtain

$$\begin{array}{rcl}
\kappa_0^* & = & (2N+1)\left\|\nabla \partial_\rho f\right\|_{\mathbf{L}^\infty} \\
& \leqslant & (2N+1)\left\|q'\right\|_{\mathbf{L}^\infty([0,R])} \left(\|\nabla \nu\|_{\mathbf{L}^\infty} + C_I(\|r\|_{\mathbf{L}^\infty([0,T];\mathbf{L}^1)})\right) , \\
\mathrm{TV}\left(\rho(t)\right) & \leqslant & \mathrm{TV}\left(\rho_0\right) e^{\kappa_0^* t} \\
& & + t e^{\kappa_0^* t} N W_N \|q\|_{\mathbf{L}^\infty([0,R])} \left(\|\nabla \operatorname{div} \nu\|_{\mathbf{L}^1} + C_I(\|r\|_{\mathbf{L}^\infty([0,T];\mathbf{L}^1)})\right) ,
\end{array}$$

completing the proof. □

Proof of Theorem 2.3. Thanks to Lemma 5.1, under hypotheses **(v)**–**($\boldsymbol{\nu}$)** and **(I)**, for any $r \in \mathbf{C}^0\left([0,T]; \mathbf{L}^1(\mathbb{R}^N;[0,R])\right)$ there exists a unique solution $\rho \in \mathbf{C}^0\left(\mathbb{R}^+; \mathbf{L}^1(\mathbb{R}^N;[0,R])\right)$ to (2.2) satisfying (5.1)–(5.2). For any time $T > 0$, introduce the space

$$X(T) = \left\{ r \in \mathbf{C}^0\left([0,T]; \mathbf{L}^1(\mathbb{R}^N;[0,R])\right) : \forall\, t \in [0,T], \left\|r(t)\right\|_{\mathbf{L}^1} \leqslant \|\rho_0\|_{\mathbf{L}^1} \right\} .$$

Note that $X(T)$ is a Banach space with respect to the norm $\|r\|_{X(T)} = \sup_{t\in[0,T]} \left\|r(t)\right\|_{\mathbf{L}^1}$. Let $\mathcal{Q}$ be the map that associates to $r \in X(T)$ the solution $\rho \in X(T)$ to (2.2). We want to find a condition on T so that $\mathcal{Q}$ is a contraction.

Let $r_1, r_2 \in X(T)$. To apply [25, Theorem 2.6], we have to check that, for all $t \in [0,T]$,

$$\begin{array}{l}
\partial_\rho(f_1 - f_2) \in \mathbf{L}^\infty([0,T] \times \mathbb{R}^N \times [0,R]; \mathbb{R}^N) , \text{ and} \\
\displaystyle\int_0^T \int_{\mathbb{R}^N} \left\|\operatorname{div}(f_1 - f_2)(t,x,\cdot)\right\|_{\mathbf{L}^\infty([0,R])} \mathrm{d}x\, \mathrm{d}t < +\infty .
\end{array}$$

Thanks to **(I.3)** we have

$$\begin{array}{l}
\left\|\partial_\rho(f_1 - f_2)\right\|_{\mathbf{L}^\infty} = \left\|\left(\mathcal{I}(r_1) - \mathcal{I}(r_2)\right) q'\right\|_{\mathbf{L}^\infty} \leqslant K_I \left\|q'\right\|_{\mathbf{L}^\infty([0,R])} \|r_1 - r_2\|_{\mathbf{L}^\infty([0,T];\mathbf{L}^1)} , \\
\displaystyle\int_0^T \int_{\mathbb{R}^N} \left\|\operatorname{div}(f_1 - f_2)(t,x,\cdot)\right\|_{\mathbf{L}^\infty([0,R])} \mathrm{d}x\, \mathrm{d}t \leqslant T K_I \|q\|_{\mathbf{L}^\infty([0,R])} \|r_1 - r_2\|_{\mathbf{L}^\infty([0,T];\mathbf{L}^1)} .
\end{array}$$

Hence, we get for all $t \geqslant 0$

$$\begin{array}{rl}
& \left\|(\rho_1 - \rho_2)(t)\right\|_{\mathbf{L}^1} \\
\leqslant & t\, K_I \|q\|_{\mathbf{W}^{1,\infty}([0,R])} \|r_1 - r_2\|_{\mathbf{L}^\infty([0,T];\mathbf{L}^1)} \\
& \times \left[1 + e^{\kappa_0^* t} \left(\mathrm{TV}\left(\rho_0\right) + \dfrac{N W_N}{2}\, t\, \|q\|_{\mathbf{L}^\infty([0,R])} \left(\|\nabla \operatorname{div} \nu\|_{\mathbf{L}^1} + C_I(\|r\|_{\mathbf{L}^\infty([0,T];\mathbf{L}^1)})\right)\right)\right] .
\end{array}$$

Denoting

$$\begin{aligned} k &= (2N+1)\left\|q'\right\|_{\mathbf{L}^\infty([0,R])} \left(\|\nabla\nu\|_{\mathbf{L}^\infty} + C_I(\|\rho_0\|_{\mathbf{L}^1})\right) , \\ C_1 &= K_I \, \|q\|_{\mathbf{W}^{1,\infty}([0,R])} , \\ C_2 &= \frac{NW_N}{2} \|q\|_{\mathbf{L}^\infty([0,R])} \left(\|\nabla \operatorname{div} \nu\|_{\mathbf{L}^1} + C_I(\|\rho_0\|_{\mathbf{L}^1})\right) . \end{aligned} \tag{5.4}$$

The above estimate can be written

$$\left\|(\rho_1-\rho_2)(t)\right\|_{\mathbf{L}^1} \leqslant C_1 \, t \, \|r_1-r_2\|_{\mathbf{L}^\infty([0,T];\mathbf{L}^1)} \left(1 + e^{kt} \left(\mathrm{TV}\,(\rho_0) + C_2 t\right)\right) ,$$

We choose now T so that

$$C_1 \, T \, \left(1 + e^{kT} \left(\mathrm{TV}\,(\rho_0) + C_2 \, T\right)\right) = \frac{1}{2}$$

and, applying Banach Fixed Point Theorem, we obtain a unique fixed point for $\mathcal{Q}$ on $X(T)$.

We now iterate the procedure above. To this aim, we use the total variation estimate (5.2)

$$\mathrm{TV}\,\left(\rho(t)\right) \leqslant \left(\mathrm{TV}\,(\rho_0) + 2\, C_2 t\right) e^{kt},$$

and, given T_n, we recursively define T_{n+1} so that

$$C_1 \, (T_{n+1} - T_n) \left(1 + e^{kT_{n+1}} \, \mathrm{TV}\,(\rho_0) + C_2 \, e^{k(T_{n+1}-T_n)} (2T_n e^{kT_n} + T_{n+1} - T_n)\right) = \frac{1}{2}$$

and the above procedure ensures the existence of a fixed point on the interval $[T_n, T_{n+1}]$.

The sequence T_n is strictly increasing. If it is bounded, then the latter relation above yields $0 = 1/2$. Hence, $T_n \to \infty$ when $n \to \infty$, completing the proof. □

5.2 Stability

Proof of Theorem 2.4. Here, we apply [25, Corollary 2.8] to compare the solutions ρ_1 and ρ_2 of (2.4). Let k be as in (5.4). As in the proof of Theorem 2.3, **(v)**–**($\boldsymbol{\nu}$)**–**(I)** ensure that the set of hypotheses [25, **(H1*)**–**(H2*)**–**(H3*)**] are satisfied. Thus, for any $t \geqslant 0$, we obtain

$$\begin{aligned} & \left\|\rho_1(t) - \rho_2(t)\right\|_{\mathbf{L}^1} \\ \leqslant\; & \left\|\rho_{0,1} - \rho_{0,2}\right\|_{\mathbf{L}^1} \\ & + \Big[e^{kt} \, \mathrm{TV}\,(\rho_{0,1}) + NW_N \|q_1\|_{\mathbf{L}^\infty([0,R])} \int_0^t e^{k(t-\tau)} \int_{\mathbb{R}^N} \left\|\nabla \operatorname{div}\left(\nu_1(x) + \mathcal{I}_1(\rho_1)\right)\right\| \mathrm{d}x \, \mathrm{d}\tau \Big] \\ & \qquad \times \int_0^t \left\| q_1'(\rho) \left(\nu_1(x) + \mathcal{I}_1(\rho_1)\right) - q_2'(\rho) \left(\nu_2(x) + \mathcal{I}_2(\rho_2)\right) \right\|_{\mathbf{L}^\infty(\mathbb{R}^N)} \mathrm{d}\tau \\ & + \int_0^t \int_{\mathbb{R}^N} \left\| q_1(\rho) \operatorname{div}\left(\nu_1(x) + \mathcal{I}_1(\rho_1)\right) - q_2(\rho) \operatorname{div}\left(\nu_2(x) + \mathcal{I}_2(\rho_2)\right) \right\|_{\mathbf{L}^\infty([0,R])} \mathrm{d}x \, \mathrm{d}\tau \\ \leqslant\; & \left\|\rho_{0,1} - \rho_{0,2}\right\|_{\mathbf{L}^1} + \Big[\mathrm{TV}\,(\rho_{0,1}) + t \, NW_N \|q_1\|_{\mathbf{L}^\infty([0,R])} \left(\|\nabla \operatorname{div} \nu_1\|_{\mathbf{L}^1} + C_I(\left\|\rho_{0,1}\right\|_{\mathbf{L}^1})\right) \Big] \\ & \qquad \times \Big(t \left\|q_1' - q_2'\right\|_{\mathbf{L}^\infty([0,R])} \left\|\nu_1 + \mathcal{I}_1(\rho_1)\right\|_{\mathbf{L}^\infty} \end{aligned}$$

$$
\begin{aligned}
&\qquad +t\left\|q_2'\right\|_{\mathbf{L}^\infty([0,R])}\left\|\nu_1-\nu_2+\mathcal{I}_1(\rho_1)-\mathcal{I}_2(\rho_1)\right\|_{\mathbf{L}^\infty}\\
&\qquad +\left\|q_2'\right\|_{\mathbf{L}^\infty([0,R])}\int_0^t\left\|\mathcal{I}_2\left(\rho_1(\tau)\right)-\mathcal{I}_2\left(\rho_2(\tau)\right)\right\|_{\mathbf{L}^\infty}\mathrm{d}\tau\Big)\\
&+t\|q_2\|_{\mathbf{L}^\infty([0,R])}\left\|\mathrm{div}\left(\nu_1-\nu_2+\mathcal{I}_1(\rho_1)-\mathcal{I}_2(\rho_1)\right)\right\|_{\mathbf{L}^1}\\
&+t\|q_1-q_2\|_{\mathbf{L}^\infty([0,R])}\left\|\mathrm{div}\left(\nu_1+\mathcal{I}_1(\rho_1)\right)\right\|_{\mathbf{L}^1}+\|q_2\|_{\mathbf{L}^\infty([0,R])}\int_0^t\left|\mathrm{div}\left(\mathcal{I}_2(\rho_1)-\mathcal{I}_2(\rho_2)\right)\right|\mathrm{d}\tau\,.
\end{aligned}
$$

Hence, denoting

$$
\begin{aligned}
a(t) &= \mathrm{TV}\left(\rho_{0,1}\right)+t\,NW_N\,\|q_1\|_{\mathbf{L}^\infty([0,R])}\left(\|\nabla\,\mathrm{div}\,\nu_1\|_{\mathbf{L}^1}+C_I(\left\|\rho_{0,1}\right\|_{\mathbf{L}^1})\right)\\
b(t) &= K_I\left(\left\|q_2'\right\|_{\mathbf{L}^\infty([0,R])}e^{kt}a(t)+\|q_2\|_{\mathbf{L}^\infty([0,R])}\right)\\
\gamma_1 &= \left\|q_1'-q_2'\right\|_{\mathbf{L}^\infty([0,R])}\left(\|\nu_1\|_{\mathbf{L}^\infty}+C_I(\left\|\rho_{0,1}\right\|_{\mathbf{L}^1})\right)+\left\|q_2'\right\|_{\mathbf{L}^\infty([0,R])}\|\nu_1-\nu_2\|_{\mathbf{L}^\infty}\\
&\qquad +\left\|q_2'\right\|_{\mathbf{L}^\infty([0,R])}\sup_{\rho\in\mathbf{L}^1(\mathbb{R}^N;[0,R])}\left\|\mathcal{I}_1(\rho)-\mathcal{I}_2(\rho)\right\|_{\mathbf{L}^\infty}\\
\gamma_2 &= \|q_1-q_2\|_{\mathbf{L}^\infty([0,R])}\left(\|\mathrm{div}\,\nu_1\|_{\mathbf{L}^1}+C_I(\left\|\rho_{0,1}\right\|_{\mathbf{L}^1})\right)\\
&\quad +\|q_2\|_{\mathbf{L}^\infty([0,R])}\left(\left\|\mathrm{div}\left(\nu_1-\nu_2\right)\right\|_{\mathbf{L}^1}+\sup_{\rho\in\mathbf{L}^1(\mathbb{R}^N;[0,R])}\left\|\mathrm{div}\,\mathcal{I}_1(\rho)-\mathrm{div}\,\mathcal{I}_2(\rho)\right\|_{\mathbf{L}^1}\right)
\end{aligned}
$$

we get

$$
\left\|\rho_1-\rho_2(t)\right\|_{\mathbf{L}^1}\leqslant\left\|\rho_{0,1}-\rho_{0,2}\right\|_{\mathbf{L}^1}+\gamma_1\,t\,e^{kt}\,a(t)+t\,\gamma_2+b(t)\int_0^t\left\|\rho_1-\rho_2(\tau)\right\|\,\mathrm{d}\tau\ .
$$

By Gronwall Lemma, we obtain

$$
\begin{aligned}
\left\|\rho_1-\rho_2(t)\right\|_{\mathbf{L}^1} &\leqslant \left\|\rho_{0,1}-\rho_{0,2}\right\|_{\mathbf{L}^1}+\gamma_1\,t\,e^{kt}\,a(t)+t\,\gamma_2\\
&\qquad +b(t)e^{\int_0^t b(u)\mathrm{d}u}\int_0^t e^{-\int_0^\tau b(u)\mathrm{d}u}(\left\|\rho_{0,1}-\rho_{0,2}\right\|_{\mathbf{L}^1}+\gamma_1\tau e^{k\tau}\,a(\tau)+\tau\gamma_2)\,\mathrm{d}\tau\ .
\end{aligned}
$$

Noting that

$$
e^{-\int_0^t b(u)\mathrm{d}u}+b(t)\int_0^t e^{-\int_0^\tau b(u)\mathrm{d}u}\,\mathrm{d}\tau\leqslant\frac{b(t)}{b(0)}
$$

we get

$$
\left\|\rho_1-\rho_2(t)\right\|_{\mathbf{L}^1}\quad\leqslant\quad\left(\left\|\rho_{0,1}-\rho_{0,2}\right\|_{\mathbf{L}^1}+\gamma_1\,t\,e^{kt}\,a(t)+t\,\gamma_2\right)\frac{b(t)}{b(0)}e^{tb(t)}
$$

completing the proof. □

5.3 Driving examples

Proof of Proposition 3.1. Let ρ solve (1.1) in the sense of Definition 2.1. Define $w(t,x) = \nu(x) + \left(\mathcal{I}\left(\rho(t)\right)\right)(x)$. Then, $\rho \in \mathbf{C}^0([0,T];\mathbf{L}^1(\mathbb{R}^N;[0,R]))$ is also a weak solution to (2.1). Then, set as above $q(\rho) = \rho\, v(\rho)$ and, for any $\varphi \in \mathbf{C}_c^\infty(\mathbb{R}\times\mathbb{R}^N;\mathbb{R})$, we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}^N} \left(\rho\,\partial_t\varphi + q(\rho)\, w(t,x)\cdot\nabla\varphi(t,x)\right) \mathrm{d}x\ \mathrm{d}t = 0\,.$$

Fix positive T and M. For $\alpha > 0$ define $Y_\alpha \in \mathbf{C}^\infty(\mathbb{R};\mathbb{R})$ so that $Y_\alpha(s) = 0$ for $s \leqslant 0$, $Y_\alpha(s) = 1$ for $s \geqslant \alpha$ and $Y'_\alpha(s) \geqslant 0$ for all s. Let $\zeta(t) = Y_\alpha(t-\alpha) - Y_\alpha(t-T)$.

Call $K_M = B(0,M)\setminus\Omega$. Set $\mu \in \mathbf{C}_c^\infty([-1,1];[0,1])$ such that $\int_{\mathbb{R}}\mu(r)\ \mathrm{d}r = 1$. For $\vartheta > 0$, let $\mu_\vartheta(x) = \frac{1}{\vartheta^N}\,\mu\left(\|x\|/\vartheta\right)$ and define $\psi = \chi_{K_M} * \mu_\vartheta$, the convolution being over all of $\mathbb{R}^N$. Note that $\nabla\psi = 0$ on $\mathbb{R}^N \setminus B(\partial K_M,\vartheta)$. If $x \in B(\partial K_M,\vartheta)$, we have

$$\nabla\psi(x) = \int_{K_M} \nabla\mu_\vartheta(x-y)\ \mathrm{d}y = -\int_{\partial K_M} \mu_\vartheta(x-y)\,\tilde n(y)\ \mathrm{d}\sigma(y)\ ,$$

where $\tilde n(x)$ is the outer normal to K_M at any $x\in\partial K_M$. Then, we choose $\varphi(t,x) = \zeta(t)\,\psi(x)$. Clearly, $\varphi \in \mathbf{C}_c^\infty(\mathbb{R}\times\mathbb{R}^N;\mathbb{R})$. The definition of weak solution gives

$$0 = \int_{\mathbb{R}}\int_{\mathbb{R}^N} \left(\rho(t,x)\,\zeta'(t)\,\psi(x) + q\left(\rho(t,x)\right)\ w(t,x)\,\zeta(t)\,\nabla\psi(x)\right) \mathrm{d}x\,\mathrm{d}t\ .$$

Letting $\alpha \to 0$, we obtain

$$\begin{aligned}
0 &= \int_{\mathbb{R}^N} \left(\rho(0,x) - \rho(T,x)\right)\psi(x)\ \mathrm{d}x + \int_0^T\int_{\mathbb{R}^N} q\left(\rho(t,x)\right)\ w(t,x)\,\nabla\psi(x)\ \mathrm{d}x\ \mathrm{d}t\\
&= \int_{\mathbb{R}^N} \left(\rho(0,x) - \rho(T,x)\right)\psi(x)\ \mathrm{d}x\\
&\qquad -\int_0^T\int_{\mathbb{R}^N}\int_{\partial K_M} q\left(\rho(t,x)\right) w(t,x)\cdot\tilde n(y)\,\mu_\vartheta(x-y)\ \mathrm{d}\sigma(y)\ \mathrm{d}x\ \mathrm{d}t\\
&= \int_{\mathbb{R}^N} \left(\rho(0,x) - \rho(T,x)\right)\psi(x)\ \mathrm{d}x\\
&\qquad -\int_0^T\int_{\partial K_M}\int_{B(0,1)} q\left(\rho(t,y+\vartheta z)\right)\ w(t,y+\vartheta z)\cdot\tilde n(y)\,\mu(\|z\|)\ \mathrm{d}z\ \mathrm{d}\sigma(y)\ \mathrm{d}t\ .
\end{aligned}$$

That is to say, setting $H_\vartheta(t) = \{(y,z):\ w(t,y+\vartheta z)\cdot\tilde n(y) \leqslant 0\}$ and $\partial\Omega_M = \partial\Omega\cap B(0,M)$

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(\rho(0,x) - \rho(T,x)\right)\psi(x)\ \mathrm{d}x\\
=\ &\int_0^T\int_{\partial K_M}\int_{B(0,1)} q\left(\rho(t,y+\vartheta z)\right)\ w(t,y+\vartheta z)\cdot\tilde n(y)\,\mu(\|z\|)\ \mathrm{d}z\ \mathrm{d}\sigma(y)\ \mathrm{d}t\\
\geqslant\ &\int_0^T\int_{\partial K_M}\int_{B(0,1)} \chi_{H_\vartheta(t)}(y,z)\, q\left(\rho(t,y+\vartheta z)\right)\ w(t,y+\vartheta z)\cdot\tilde n(y)\,\mu(\|z\|)\ \mathrm{d}z\ \mathrm{d}\sigma(y)\ \mathrm{d}t\\
=\ &\int_0^T\int_{\partial \Omega_M}\int_{B(0,1)} \chi_{H_\vartheta(t)}(y,z)\, q\left(\rho(t,y+\vartheta z)\right)\ w(t,y+\vartheta z)\cdot\tilde n(y)\,\mu(\|z\|)\ \mathrm{d}z\ \mathrm{d}\sigma(y)\ \mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
& + \int_0^T \int_{\partial K_M \setminus \partial\Omega_M} \int_{B(0,1)} \chi_{H_\vartheta(t)}(y,z)\, q\left(\rho(t, y+\vartheta z)\right)\, w(t, y+\vartheta z)\cdot \tilde n(y)\, \mu(\|z\|)\, \mathrm{d}z\, \mathrm{d}\sigma(y)\, \mathrm{d}t \\
\geqslant \;& \int_0^T \int_{\partial\Omega_M} \int_{B(0,1)} \chi_{H_\vartheta(t)}(y,z)\, \|q\|_{\mathbf{L}^\infty}\, w(t, y+\vartheta z)\cdot \tilde n(y)\, \mu(\|z\|)\, \mathrm{d}z\, \mathrm{d}\sigma(y)\, \mathrm{d}t + m
\end{aligned}
$$

where m can be arbitrarily small provided M is sufficiently large, since $\rho \in \mathbf{L}^1([0,T]\times \mathbb{R}^N;[0,R])$. Letting $\vartheta \to 0$, $M \to \infty$ and using (3.2), by the Dominated Convergence Theorem we get

$$
0 \leqslant \int_{^c\Omega} \left(\rho(0,x) - \rho(T,x)\right) \mathrm{d}x = -\int_{^c\Omega} \rho(T,x)\, \mathrm{d}x\ ,
$$

since by hypothesis $\operatorname{spt}\rho_0 \subset \Omega$. Besides, the positivity of ρ gives us $-\int_{^c\Omega} \rho(t,x)\,\mathrm{d}x \leqslant 0$. Finally, we have $\int_{^c\Omega}\rho(T,x) = 0$ and $\operatorname{spt}\rho(T) \subset \Omega$. □

Proof of Lemma 3.2. Let $\rho \in \mathbf{L}^1(\mathbb{R}^N;[0,R])$. Note that $\mathcal{I}$ can be rewritten as $\mathcal{I}(\rho) = -\varepsilon\, \rho*(\nabla\eta) \Big/ \sqrt{1+\|\rho*\nabla\eta\|^2}$. By the assumptions on η and the properties of the convolution product, we have

$$
\begin{aligned}
\left\|\mathcal{I}(\rho)\right\|_{\mathbf{L}^\infty} &\leqslant \varepsilon\, \|\rho\|_{\mathbf{L}^1} \|\nabla\eta\|_{\mathbf{L}^\infty}\ , \\
\left\|\mathcal{I}(\rho)\right\|_{\mathbf{L}^1} &\leqslant \varepsilon\, \|\rho\|_{\mathbf{L}^1} \|\nabla\eta\|_{\mathbf{L}^1}\ , \\
\left\|\nabla\mathcal{I}(\rho)\right\|_{\mathbf{L}^\infty} &\leqslant \varepsilon\, \|\rho\|_{\mathbf{L}^1} \|\nabla\eta\|_{\mathbf{W}^{1,\infty}} \left(1 + R^2 \|\nabla\eta\|_{\mathbf{L}^1} \left\|\nabla^2\eta\right\|_{\mathbf{L}^1}\right)\ , \\
\left\|\nabla\mathcal{I}(\rho)\right\|_{\mathbf{L}^1} &\leqslant \varepsilon\, \|\rho\|_{\mathbf{L}^1} \|\nabla\eta\|_{\mathbf{W}^{1,\infty}} \left(1 + R^2 \|\nabla\eta\|_{\mathbf{L}^1} \left\|\nabla^2\eta\right\|_{\mathbf{L}^1}\right)\ .
\end{aligned}
$$

Hence **(I.1)** is satisfied. Let us check now **(I.2)**

$$
\left\|\nabla^2\mathcal{I}(\rho)\right\|_{\mathbf{L}^1} \leqslant \varepsilon\, R^2\, \|\rho\|_{\mathbf{L}^1} \|\nabla\eta\|_{\mathbf{W}^{2,1}} \|\nabla\eta\|^2_{\mathbf{W}^{1,1}} (1 + 4 + 3R^2 \|\nabla\eta\|^2_{\mathbf{W}^{1,1}})\ .
$$

Let $r_1, r_2 \in \mathbf{L}^1(\mathbb{R}^N;[0,R])$. We have:

$$
\begin{aligned}
\left\|\mathcal{I}(r_1) - \mathcal{I}(r_2)\right\|_{\mathbf{L}^\infty} &\leqslant \varepsilon \left(1 + R^2 \|\nabla\eta\|^2_{\mathbf{L}^1}\right) \|\nabla\eta\|_{\mathbf{L}^\infty} \|r_1 - r_2\|_{\mathbf{L}^1}\ , \\
\left\|\mathcal{I}(r_1) - \mathcal{I}(r_2)\right\|_{\mathbf{L}^1} &\leqslant \varepsilon \left(1 + R^2 \|\nabla\eta\|^2_{\mathbf{L}^1}\right) \|\nabla\eta\|_{\mathbf{L}^1} \|r_1 - r_2\|_{\mathbf{L}^1}\ , \\
\left\|\operatorname{div}\left(\mathcal{I}(r_1) - \mathcal{I}(r_2)\right)\right\|_{\mathbf{L}^1} &\leqslant \varepsilon\, \|r_1 - r_2\|_{\mathbf{L}^1} \|\nabla\eta\|_{\mathbf{W}^{1,1}} (1 + 8R^2 \|\nabla\eta\|^2_{\mathbf{W}^{1,1}} + 3R^4 \|\nabla\eta\|^4_{\mathbf{W}^{1,1}})\ ,
\end{aligned}
$$

completing the proof. □

Proof of Proposition 3.3. The fact that **($\boldsymbol{\nu}$)** holds follows through simple computations from **($\boldsymbol{\Omega}$)**, **(g)**, **($\boldsymbol{\eta}$)**, **($\boldsymbol{\alpha}$)** and the assumptions on $\partial\Omega$ or $\alpha, d_{\partial\Omega}$. Condition **(I)** follows from Lemma 3.2.

To prove the invariance property **(P)**, we verify that

$$
\Big(g(x) + \delta(x) + \left(\mathcal{I}(\rho)\right)(x)\Big) \cdot n(x) \geqslant 0
$$

for all $x \in \partial\Omega$. By **(g)**, (3.5) and (1.2), it is sufficient to prove that

$$\lambda \geqslant \varepsilon \sup_{x\in\partial\Omega} \frac{(\rho * \nabla\eta)(x)\cdot n(x)}{\sqrt{1+\left\|(\rho*\nabla\eta)(x)\right\|^2}}\,.$$

Note that

$$\sup_{x\in\partial\Omega} \frac{(\rho * \nabla\eta)(x)\cdot n(x)}{\sqrt{1+\left\|(\rho*\nabla\eta)(x)\right\|^2}} \leqslant \sup_{x\in\partial\Omega}(\rho*\nabla\eta)(x)\cdot n(x) \leqslant \|\rho\|_{\mathbf{L}^\infty}\|\nabla\eta\|_{\mathbf{L}^1}\|n\|_{\mathbf{L}^\infty} \leqslant R\,\|\nabla\eta\|_{\mathbf{L}^1}$$

completing the proof, by Proposition 3.1. □

Proof of Lemma 3.4. Note that $\mathcal{I}$ as $\mathcal{I}(\rho) = \varepsilon\int_{\mathbb{R}^N}\rho(y)\,\nabla\left(\eta(x-y)\,g\left((y-x)\cdot\nu(x)\right)\right)\mathrm{d}y$. Let r_η be such that $B(0,r_\eta)\supset \operatorname{spt}\eta$. Fix $\rho\in\mathbf{L}^1(\mathbb{R}^N;[0,R])$. Let us check **(I.1)**:

$$\begin{aligned}
\left\|\mathcal{I}(\rho)\right\|_{\mathbf{L}^\infty} &\leqslant \varepsilon\,\|\rho\|_{\mathbf{L}^1}\|\eta\|_{\mathbf{W}^{1,\infty}}\|g\|_{\mathbf{W}^{1,\infty}}\left(1+(1+r_\eta)\|\nu\|_{\mathbf{W}^{1,\infty}}\right)\,,\\
\left\|\mathcal{I}(\rho)\right\|_{\mathbf{L}^1} &\leqslant \varepsilon\,\|\rho\|_{\mathbf{L}^1}\|\eta\|_{\mathbf{W}^{1,1}}\|g\|_{\mathbf{W}^{1,\infty}}\left(1+(1+r_\eta)\|\nu\|_{\mathbf{W}^{1,\infty}}\right)\,,\\
\left\|\nabla\mathcal{I}(\rho)\right\|_{\mathbf{L}^\infty} &\leqslant \varepsilon\,\|\rho\|_{\mathbf{L}^1}\|\eta\|_{\mathbf{W}^{2,\infty}}\|g\|_{\mathbf{W}^{2,\infty}}\\
&\quad\times\left(1+2(r_\eta+1)\|\nu\|_{\mathbf{W}^{1,\infty}}+(1+r_\eta)^2\|\nu\|^2_{\mathbf{W}^{1,\infty}}+(2+r_\eta)\|\nabla\nu\|_{\mathbf{W}^{1,\infty}}\right),\\
\left\|\nabla\mathcal{I}(\rho)\right\|_{\mathbf{L}^1} &\leqslant \varepsilon\,\|\rho\|_{\mathbf{L}^1}\|\eta\|_{\mathbf{W}^{2,1}}\|g\|_{\mathbf{W}^{2,\infty}}\\
&\quad\times\left(1+2(r_\eta+1)\|\nu\|_{\mathbf{W}^{1,\infty}}+(1+r_\eta)^2\|\nu\|^2_{\mathbf{W}^{1,\infty}}+(2+r_\eta)\|\nabla\nu\|_{\mathbf{W}^{1,\infty}}\right).
\end{aligned}$$

Passing to **(I.2)**:

$$\begin{aligned}
\left\|\nabla^2\mathcal{I}(\rho)\right\|_{\mathbf{L}^1} &\leqslant \varepsilon\,\|\rho\|_{\mathbf{L}^1}\|\eta\|_{\mathbf{W}^{3,1}}\|g\|_{\mathbf{W}^{3,\infty}}\\
&\quad\times\Big[1+3(1+r_\eta)\|\nu\|_{\mathbf{W}^{1,\infty}}+3(1+r_\eta)^2\|\nu\|^2_{\mathbf{W}^{1,\infty}}+3(2+r_\eta)\|\nabla\nu\|_{\mathbf{W}^{1,\infty}}\\
&\qquad+(1+r_\eta)^3\|\nu\|^3_{\mathbf{W}^{1,\infty}}+3(2+r_\eta)^2\|\nu\|^2_{\mathbf{W}^{2,\infty}}+(3+r_\eta)\left\|\nabla^2\nu\right\|_{\mathbf{W}^{1,\infty}}\Big].
\end{aligned}$$

Let $r_1,r_2\in\mathbf{L}^1(\mathbb{R}^N;[0,R])$. The operator $\mathcal{I}$ is linear in ρ, hence

$$\begin{aligned}
\left\|\mathcal{I}(r_1)-\mathcal{I}(r_2)\right\|_{\mathbf{L}^\infty} &\leqslant \varepsilon\,\|r_1-r_2\|_{\mathbf{L}^1}\|\eta\|_{\mathbf{W}^{1,\infty}}\|g\|_{\mathbf{W}^{1,\infty}}(1+(1+r_\eta)\|\nu\|_{\mathbf{W}^{1,\infty}})\,,\\
\left\|\mathcal{I}(r_1)-\mathcal{I}(r_2)\right\|_{\mathbf{L}^\infty} &\leqslant \varepsilon\,\|r_1-r_2\|_{\mathbf{L}^1}\|\eta\|_{\mathbf{W}^{1,1}}\|g\|_{\mathbf{W}^{1,\infty}}(1+(1+r_\eta)\|\nu\|_{\mathbf{W}^{1,\infty}})\,,\\
\left\|\operatorname{div}\left(\mathcal{I}(r_1)-\mathcal{I}(r_2)\right)\right\|_{\mathbf{L}^\infty} &\leqslant \varepsilon\,\|r_1-r_2\|_{\mathbf{L}^1}\|\eta\|_{\mathbf{W}^{2,1}}\|g\|_{\mathbf{W}^{2,\infty}}\\
&\quad\times\Big[1+2(1+r_\eta)\|\nu\|_{\mathbf{W}^{1,\infty}}+(1+r_\eta)^2\|\nu\|^2_{\mathbf{W}^{1,\infty}}\\
&\qquad+(2+r_\eta)\|\nabla\nu\|_{\mathbf{W}^{1,\infty}}\Big]
\end{aligned}$$

showing that **(I.3)** is satisfied. □

Proof of Proposition 3.5. Fix $\rho \in \mathbf{L}^1(\mathbb{R}^N; [0,R])$. Let us check **(I.1)** and **(I.2)**:

$$
\begin{aligned}
\left\|\mathcal{I}(\rho)\right\|_{\mathbf{L}^\infty} &\leqslant \varepsilon \|\rho\|_{\mathbf{L}^1} \|\eta\|_{\mathbf{W}^{1,\infty}} \|\varphi\|_{\mathbf{W}^{1,\infty}} \left(1 + (1+\ell)\|g\|_{\mathbf{W}^{1,\infty}}\right) , \\
\left\|\mathcal{I}(\rho)\right\|_{\mathbf{L}^1} &\leqslant \varepsilon \|\rho\|_{\mathbf{L}^1} \|\eta\|_{\mathbf{W}^{1,1}} \|\varphi\|_{\mathbf{W}^{1,\infty}} \left(1 + (1+\ell)\|g\|_{\mathbf{W}^{1,\infty}}\right) , \\
\left\|\nabla\mathcal{I}(\rho)\right\|_{\mathbf{L}^\infty} &\leqslant \varepsilon \|\rho\|_{\mathbf{L}^1} \|\eta\|_{\mathbf{W}^{2,\infty}} \|\varphi\|_{\mathbf{W}^{2,\infty}} \\
&\quad \times \left(1 + 2(\ell+1)\|g\|_{\mathbf{W}^{1,\infty}} + (1+\ell)^2\|g\|^2_{\mathbf{W}^{1,\infty}} + (2+\ell)\|\nabla g\|_{\mathbf{W}^{1,\infty}}\right) \\
&\quad \times \left(1 + R^2 \|\eta\|^2_{\mathbf{W}^{1,1}} \|\varphi\|^2_{\mathbf{W}^{1,\infty}} \left(1 + (1+\ell)\|g\|_{\mathbf{W}^{1,\infty}}\right)^2\right) , \\
\left\|\nabla\mathcal{I}(\rho)\right\|_{\mathbf{L}^1} &\leqslant \varepsilon \|\rho\|_{\mathbf{L}^1} \|\eta\|_{\mathbf{W}^{2,1}} \|\varphi\|_{\mathbf{W}^{2,\infty}} \\
&\quad \times \left(1 + 2(\ell+1)\|g\|_{\mathbf{W}^{1,\infty}} + (1+\ell)^2\|g\|^2_{\mathbf{W}^{1,\infty}} + (2+\ell)\|\nabla g\|_{\mathbf{W}^{1,\infty}}\right) \\
&\quad \times \left(1 + R^2 \|\eta\|^2_{\mathbf{W}^{1,1}} \|\varphi\|^2_{\mathbf{W}^{1,\infty}} \left(1 + (1+\ell)\|g\|_{\mathbf{W}^{1,\infty}}\right)^2\right) .
\end{aligned}
$$

Passing to **(I.2)**:

$$
\begin{aligned}
\left\|\nabla^2\mathcal{I}(\rho)\right\|_{\mathbf{L}^1} &\leqslant \varepsilon \|\rho\|_{\mathbf{L}^1} \|\eta\|_{\mathbf{W}^{3,1}} \|\varphi\|_{\mathbf{W}^{3,\infty}} \\
&\quad \times \Big[1 + 3(1+\ell)\|g\|_{\mathbf{W}^{1,\infty}} + 3(1+\ell)^2\|g\|^2_{\mathbf{W}^{1,\infty}} + 3(2+\ell)\|\nabla g\|_{\mathbf{W}^{1,\infty}} \\
&\qquad + (1+\ell)^3\|g\|^3_{\mathbf{W}^{1,\infty}} + 3(2+\ell)^2\|g\|^2_{\mathbf{W}^{2,\infty}} + (3+\ell)\left\|\nabla^2 g\right\|_{\mathbf{W}^{1,\infty}}\Big] \\
&\quad \times \left(1 + R^2 \|\eta\|^2_{\mathbf{W}^{1,1}} \|\varphi\|^2_{\mathbf{W}^{1,\infty}} \left(1 + (1+\ell)\|g\|_{\mathbf{W}^{1,\infty}}\right)^2\right) \\
&\quad + 3\,\varepsilon\, R \|\eta\|_{\mathbf{W}^{1,1}} \|\varphi\|_{\mathbf{W}^{1,\infty}} \left(1 + (1+\ell)\|g\|_{\mathbf{W}^{1,\infty}}\right) \\
&\quad \times \|\rho\|_{\mathbf{L}^1} \|\eta\|_{\mathbf{W}^{2,1}} \|\varphi\|_{\mathbf{W}^{2,\infty}} \\
&\quad \times \left(1 + 2(\ell+1)\|g\|_{\mathbf{W}^{1,\infty}} + (1+\ell)^2\|g\|^2_{\mathbf{W}^{1,\infty}} + (2+\ell)\|\nabla g\|_{\mathbf{W}^{1,\infty}}\right) \\
&\quad \times \left(1 + R^2 \|\eta\|^2_{\mathbf{W}^{1,1}} \|\varphi\|^2_{\mathbf{W}^{1,\infty}} \left(1 + (1+\ell)\|g\|_{\mathbf{W}^{1,\infty}}\right)^2\right) \\
&\leqslant \varepsilon \|\rho\|_{\mathbf{L}^1} \|\eta\|_{\mathbf{W}^{3,1}} \|\varphi\|_{\mathbf{W}^{3,\infty}} \\
&\quad \times \Big[1 + 3(1+\ell)\|g\|_{\mathbf{W}^{1,\infty}} + 3(1+\ell)^2\|g\|^2_{\mathbf{W}^{1,\infty}} + 3(2+\ell)\|\nabla g\|_{\mathbf{W}^{1,\infty}} \\
&\qquad + (1+\ell)^3\|g\|^3_{\mathbf{W}^{1,\infty}} + 3(2+\ell)^2\|g\|^2_{\mathbf{W}^{2,\infty}} + (3+\ell)\left\|\nabla^2 g\right\|_{\mathbf{W}^{1,\infty}} \\
&\qquad + 3\, R \|\eta\|_{\mathbf{W}^{1,1}} \|\varphi\|_{\mathbf{W}^{1,\infty}} \left(1 + (1+\ell)\|g\|_{\mathbf{W}^{1,\infty}}\right) \\
&\qquad \times \left(1 + 2(\ell+1)\|g\|_{\mathbf{W}^{1,\infty}} + (1+\ell)^2\|g\|^2_{\mathbf{W}^{1,\infty}} + (2+\ell)\|\nabla g\|_{\mathbf{W}^{1,\infty}}\right)\Big] \\
&\quad \times \left(1 + R^2 \|\eta\|^2_{\mathbf{W}^{1,1}} \|\varphi\|^2_{\mathbf{W}^{1,\infty}} \left(1 + (1+\ell)\|g\|_{\mathbf{W}^{1,\infty}}\right)^2\right) .
\end{aligned}
$$

Finally, **(I.3)** is proved exactly as in Lemma 3.2.

To prove the property **(P)**, following the same procedure as in the proof of Proposition 3.3, we check that $\Big(\nu(x) + \big(\mathcal{I}(\rho)\big)(x)\Big) \cdot n(x) \geqslant 0$ for all $x \in \partial\Omega$. In fact, if $x \in \partial\Omega$,

$$\begin{aligned}
&\Big(\nu(x) + \big(\mathcal{I}(\rho)\big)(x)\Big) \cdot n(x) \\
= &\left(g(x) + \lambda\, n(x) - \varepsilon \frac{\nabla \int_\Omega \rho(y)\, \eta(x-y)\, \varphi\big((y-x)\cdot g(x)\big)\, \mathrm{d}y}{\sqrt{1 + \left\| \nabla \int_\Omega \rho(y)\, \eta(x-y)\, \varphi\big((y-x)\cdot g(x)\big)\, \mathrm{d}y \right\|^2}} \right) \cdot n(x) \\
\geqslant\ & \lambda(x) + g(x)\cdot n(x) - \varepsilon\, \|\varphi\|_{\mathbf{L}^\infty} \int_\Omega \rho(y) \Big| \nabla\left(\eta(x-y)\right) \cdot n(x) \Big|\, \mathrm{d}y \\
& -\varepsilon \int_\Omega \rho(y)\, \eta(x-y)\, \varphi'\left((y-x)\cdot g(x)\right) \big((y-x)\, \nabla g(x)\, n(x) - g(x)\cdot n(x)\big)\, \mathrm{d}y \\
\geqslant\ & \lambda(x) - \varepsilon\, R\, \|\varphi\|_{\mathbf{L}^\infty} \|\nabla\eta\|_{\mathbf{L}^1} - \varepsilon\, R\, \left\|\varphi'\right\|_{\mathbf{L}^\infty} \int_\Omega \eta(x-y)\, \left|(y-x)\, \nabla g(x)\, n(x)\right| \mathrm{d}y\ . \\
\geqslant\ & \lambda(x) - \varepsilon\, R \Big(\|\varphi\|_{\mathbf{L}^\infty} \|\nabla\eta\|_{\mathbf{L}^1} + \left\|\varphi'\right\|_{\mathbf{L}^\infty} \|\eta\|_{\mathbf{L}^1}\, \ell\, \|\nabla g\|_{\mathbf{L}^\infty} \Big)
\end{aligned}$$

completing the proof, by Proposition 3.1. □

Proof of Proposition 3.6. Denote $\ell = \operatorname{diam} \operatorname{spt} \eta$. Straightforward computations give:

$$\begin{aligned}
\left\|\mathcal{I}(\rho)\right\|_{\mathbf{L}^\infty} &\leqslant \varepsilon\, \|\rho\|_{\mathbf{L}^1} \|\nabla\eta\|_{\mathbf{L}^\infty} \|\varphi\|_{\mathbf{L}^\infty}\ , \\
\left\|\nabla\mathcal{I}(\rho)\right\|_{\mathbf{L}^\infty} &\leqslant \varepsilon\, \|\rho\|_{\mathbf{L}^1} \left(\left\|\nabla^2\eta\right\|_{\mathbf{L}^\infty} \|\varphi\|_{\mathbf{L}^\infty} + \|\nabla\eta\|_{\mathbf{L}^\infty} \left\|\varphi'\right\|_{\mathbf{L}^\infty} \left(\|g\|_{\mathbf{L}^\infty} + \ell\, \|\nabla g\|_{\mathbf{L}^\infty}\right) \right)\ , \\
\left\|\mathcal{I}(\rho)\right\|_{\mathbf{L}^1} &\leqslant \varepsilon\, \|\rho\|_{\mathbf{L}^1} \|\nabla\eta\|_{\mathbf{L}^1} \|\varphi\|_{\mathbf{L}^\infty}\ , \\
\left\|\nabla\mathcal{I}(\rho)\right\|_{\mathbf{L}^1} &\leqslant \varepsilon\, \|\rho\|_{\mathbf{L}^1} \left(\left\|\nabla^2\eta\right\|_{\mathbf{L}^1} \|\varphi\|_{\mathbf{L}^\infty} + \|\nabla\eta\|_{\mathbf{L}^1} \left\|\varphi'\right\|_{\mathbf{L}^\infty} \left(\|g\|_{\mathbf{L}^\infty} + \ell\, \|\nabla g\|_{\mathbf{L}^\infty}\right) \right)\ , \\
\left\|\nabla^2\mathcal{I}(\rho)\right\|_{\mathbf{L}^1} &\leqslant \varepsilon\, \|\rho\|_{\mathbf{L}^1} \|\nabla\eta\|_{\mathbf{W}^{2,1}} \|\varphi\|_{\mathbf{W}^{2,\infty}} \left(1 + 3\|g\|_{\mathbf{L}^\infty} + \|\nabla g\|_{\mathbf{L}^\infty}(3\ell+2) + \ell \left\|\nabla^2 g\right\|_{\mathbf{L}^\infty} \right)
\end{aligned}$$

giving **(I.1)** and **(I.2)**. The proof of **(I.3)** is immediate by the linearity of $\mathcal{I}$ in ρ. □

A Appendix: Geometrical Issues Related to Ω

The framework presented in Section 3 can be adapted to various real situations. For instance, the *"walls"* $\partial\Omega$ may stop the visibility of the individuals. Then, from a modeling point of view, it can be reasonable to introduce the set

$$\Omega_x = \big\{ y \in \Omega \colon x + \sigma(y-x) \in \Omega\,, \forall\, \sigma \in [0,1] \big\} \tag{A.1}$$

of the points in Ω visible from x. Correspondingly, the nonlocal operator $\mathcal{I}$ in (1.2) can be modified intending the convolution as follows:

$$(\rho * \eta)(x) = \int_{\Omega_x} \rho(y)\, \eta(x-y)\, \mathrm{d}y\ . \tag{A.2}$$

The above relation means that the individual at x evaluates an average of the densities $\rho(y)$ at all values y that are visible from x. With these choices, the validity of condition **(I)** essentially depends on the geometry of Ω. In particular, if Ω is convex, then $\Omega_x = \Omega$ and **(I)** holds by Proposition 3.3.

Here we only show how to choose a discomfort so that **(P)** holds for (1.1)–(1.2)–(A.2). The case of the nonlocal operator (1.3) is entirely similar. To this aim, introduce the set

$$H_x = \left\{ y \in \Omega \colon \nabla\eta(x-y) \cdot n(x) \geqslant 0 \right\} . \tag{A.3}$$

Proposition A.1. *Assume that* **(Ω)** *and* **(η)** *hold. Let* $\mathcal{I}$ *be as in (1.2), where the convolution is intended as in* (A.2). *If*

$$\delta(x) = \varepsilon\, R \int_{\Omega_x \cap H_x} \nabla\eta(x-y)\, \mathrm{d}y \qquad \text{for } x \in \partial\Omega \tag{A.4}$$

then $\big(\delta(x) + \mathcal{I}(\rho)(x)\big) \cdot n(x) \geqslant 0$ *for all* $x \in \partial\Omega$ *and* $\rho \in \mathbf{L}^1(\mathbb{R}^N; [0,R])$*, so that* **(P)** *is satisfied.*

If moreover $\eta(x) = \bar\eta(\|x\|)$ *with* $\bar\eta \in \mathbf{C}^3_c([0,+\infty[\,; \mathbb{R}^+)$ *and* $\bar\eta' \leqslant 0$*, then* $\Omega_x \subseteq H_x$ *along* $\partial\Omega$ *and the discomfort can be defined on all* Ω *by*

$$\delta(x) = \varepsilon\, R \int_{\Omega_x} \nabla\eta(x-y)\, \mathrm{d}y$$

satisfying to property **(P)**.

Proof. We have $\mathcal{I}(\rho)(x) = -\varepsilon \int_{\Omega_x} \rho(y) \nabla\eta(x-y)\, \mathrm{d}y \Big/ \sqrt{1 + \|\nabla\rho * \eta\|^2}$. Hence, to ensure that $(\delta(x) + \mathcal{I}(\rho)(x)) \cdot n(x) \geqslant 0$ for all $x \in \partial\Omega$ and $\rho \in \mathbf{L}^1(\mathbb{R}^N; [0,R])$ we require

$$\delta(x) \cdot n(x) \quad \geqslant \quad \varepsilon \sup_{\rho \in \mathbf{L}^1(\mathbb{R}^N;[0,R])} \int_{\Omega_x} \rho(y)\, \nabla\eta(x-y) \cdot n(x)\, \mathrm{d}y \, .$$

In the latter expression, the supremum is attained for $\rho = R\, \chi_{{}_{\Omega_x \cap H_x}}$, since $\Omega_x \cap H_x = \{y \in \Omega_x\, ;\ \nabla\eta(x-y) \cdot n(x) \geqslant 0\}$. It is thus sufficient to define δ so that

$$\delta(x) \cdot n(x) \quad \geqslant \quad \varepsilon\, R \int_{\Omega_x \cap H_x} \nabla\eta(x-y) \cdot n(x)\, \mathrm{d}y \, ,$$

proving that (A.4) implies **(P)**, by Proposition 3.1.

If $\eta(x) = \bar\eta(\|x\|)$, then $\nabla\eta(x) = \bar\eta'(\|x\|)\, \frac{x}{\|x\|}$. Since $\bar\eta' \leqslant 0$,

$$\begin{aligned} H_x \quad &\supseteq \quad \left\{ y \in \Omega_x \colon (x-y) \cdot n(x) \leqslant 0 \right\} \\ &= \quad \Big\{ y \in \Omega_x \colon y = x + a\, n(x) + \omega \text{ with } a \geqslant 0 \text{ and } \omega \in n(x)^\perp \Big\} \\ &\supseteq \quad \Omega_x \, . \end{aligned}$$

The latter inclusion holds since Ω_x is a convex set contained in Ω and containing x. □

The choice (1.2)–(A.2) is appealing from the modeling point of view, but not easily tractable from both the numerical and the analytical points of view, without major restrictions on the geometry of Ω and on η. Therefore, we consider also the following choice.

Proposition A.2. *Assume that* **(Ω)** *and* **(η)** *hold. Let $\mathcal{I}$ be as in (1.2), with the usual convolution* (3.3). *If*

$$\delta(x) = \varepsilon\, R \int_{H_x} \nabla\eta(x-y)\; \mathrm{d}y \qquad \text{for } x \in \partial\Omega$$

then $\big(\delta(x) + \mathcal{I}(\rho)(x)\big) \cdot n(x) \geqslant 0$ for all $x \in \partial\Omega$ and $\rho \in \mathbf{L}^1(\mathbb{R}^N; [0, R])$, so that **(P)** *holds.*

With this choice, a way to define δ on all Ω could be

$$\delta(x) = \varepsilon\, R\, \alpha(x) \int_{H_x} \nabla\eta(x-y)\; \mathrm{d}y\ ,$$

where α is as in **(α)**. The proof is entirely similar to that of Proposition A.1.

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