An Analytical Framework to Describe the Interactions Between Individuals and a Continuum

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Abstract

We consider a discrete set of individual agents interacting with a continuum. Examples might be a predator facing a huge group of preys, or a few shepherd dogs driving a herd of sheeps. Analytically, these situations can be described through a system of ordinary differential equations coupled with a scalar conservation law in several space dimensions. This paper provides a complete well posedness theory for the resulting Cauchy problem. A few applications are considered in detail and numerical integrations are provided.

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1 Introduction

In various situations a small set of individuals interacts with a continuum. Below, we consider a predator (the individual) seeking to split a flock of preys (the continuum). An entirely different case is that of shepherd dogs (the individuals) confining, or steering, sheeps (the continuum). A very famous, albeit fabulous, example comes from the fairy tale [7] of the pied piper, where a musician (the individual) frees a city from rats (the continuum) using his magic flute. These are sample instances that all fit in the analytical framework developed below, but several other situations are conceivable. For instance, the dog-sheeps model can be easily rephrased as police officers trying to confine, or steer, a large crowd of protesters. Similarly, the pied piper case can be seen as a moving light attracting cells such as, for instance, the *chlamydomonas* [15] through their phototactic response, see [8].

From a deterministic point of view, studying these phenomena leads to a dynamical system consisting of ordinary differential equations for the evolution of the individuals and partial differential equations for that of the continuum. Here, motivated by the present applications, we choose scalar conservation laws for the description of the continuum's evolution. In particular, no diffusion is here considered. On one side, this choice makes the analytical treatment technically more difficult, due to the possible singularities arising in the density that describes the continuum. On the other hand, we obtain a framework where all propagation speeds are finite. As a consequence, for instance, a continuum initially confined in a bounded region will

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remain in a (larger but) bounded region at any positive time. This allows to state problems concerning the support of the continuum, such as confinement problems (the rats should leave the city, or the shepherd dogs should confine sheeps inside a given area) or far more complex ones (how can a predator split the support of the density of its preys? How many policeman are necessary to suitably confine a given group of protesters?).

In the current literature, individual vs. continuum interactions have been considered with a great variety of analytical tools, see for instance [2] for a fire confinement problem modeled through differential inclusions, or [3] for a tumor–induced angiogenesis described through a stochastic geometric model. Other examples are provided by the interaction of a fluid (liquid or gas) with a solid body, see [1, 13, 14]: the evolution of the rigid body is described by a system of ordinary differential equations, while the evolution of the fluid is subject to partial differential equations, like Navier-Stokes or Euler equations. Further results are currently available in the case of 1D systems of conservation laws. For instance, a problem motivated by traffic flow is considered in [10]; the piston problem, a blood circulation model and a supply chain model are considered in [1].

Formally, we are thus lead to the dynamical system

$$\begin{cases} \partial_t \rho + \operatorname{div}_x f\left(t, x, \rho, p(t)\right) = 0\,, \\ \dot p = \varphi\left(t, p, A\left(\rho(t)\right)(p)\right)\,, \\ \rho(0,x) = \bar\rho(x)\,, \\ p(0) = \bar p\,. \end{cases} \qquad \begin{array}{rcl} (t,x) & \in & \mathbb{R}^+ \times \mathbb{R}^d\,, \\ \rho & \in & \mathbb{R}^+\,, \\ p & \in & \mathbb{R}^n\,, \end{array} \tag{1.1}$$

where the unknowns are ρ and p. The former one, $\rho = \rho(t)$ is the density describing the macroscopic state of the continuum while the latter, $p = p(t)$, characterizes the state of the individuals[1]. It can be for instance the vector of the individuals' positions or of the individuals' positions and speeds. The dynamics of the continuum is described by the flow f, which in general can be thought as the product $f = \rho\, v$ of the density ρ and a suitable speed $v = v(t, x, \rho, p)$. The vector field φ defines the dynamics of the individuals at time t and it depends from the continuum density $\rho(t)$ through a suitable average $A(\rho(t))$. Our driving example below is the convolution in the space variable, so that for instance $A\left(\rho(t)\right)(p) = \int_{\mathbb{R}^d} \rho(t, p-y)\, \eta(y)\, \mathrm{d}y$, with a smooth compactly supported kernel η.

Below we address and solve the first mathematical questions that arise about (1.1), i.e. the existence and uniqueness of entropy solutions, their stability with respect the data and the equation, and the existence of optimal controls. A first well posedness result, that applies to general initial data, is provided in Theorem 2.2. As usual in this context, see also [4, 5, 9, 11], the hypotheses on f are rather intricate. However, the present framework naturally applies to situations in which the continuum can be supposed initially confined in a bounded region, i.e. ρ vanishes outside a compact subset of $\mathbb{R}^d$. In this case, Corollary 2.3 below applies and the hypotheses on f are greatly simplified.

The present setting lacks any linear structure. Hence, a key role in the analytical techniques employed is played by Banach Contraction Theorem. The necessary estimates are obtained through an *ad hoc* adaptation of results from the standard theories of conservation laws and from Caratheodory differential equations.

The next section presents the analytical well-posedness results. Section 3 is devoted to various applications, while all proofs are deferred to the last section.

[1]We follow for the p.d.e. the standard o.d.e. convention: $p \in \mathbb{R}^n$ is a vector that varies with time, so that $p = p(t)$. Similarly, $\rho \in \mathbf{L^1}(\mathbb{R}^d; \mathbb{R}^+)$ is a function of space which is time dependent and we write $\rho = \rho(t)$.

2 Notation and Analytical Results

We now collect the various assumptions on (1.1) that allow us to prove well posedness, i.e. the existence of solutions, their uniqueness and their stability with respect to data and equations. The hypotheses collected below are essentially those that ensure the well posedness of the conservation law and, separately, of the ordinary differential equation.

Notation. Throughout, we denote $\mathbb{R}^+ = [0, +\infty[$. Let $T_{\max} \in]0, +\infty]$ and call $I = [0, T_{\max}]$ if $T_{\max} < +\infty$, while $I = \mathbb{R}^+$ otherwise. The real parameter R, i.e. the maximal possible density, is fixed and positive. The space dimension is d and the number of components in the individuals' state is n. We denote $B_k(\xi, \delta)$ the closed ball in $\mathbb{R}^k$ centered at $\xi \in \mathbb{R}^k$ with radius $\delta \in \mathbb{R}^+$. For a given compact set K in $\mathbb{R}^n$ and a $T > 0$, we denote $\Omega_T = [0, T] \times \mathbb{R}^d \times [0, R] \times K$.

Flow of the Continuum. At point x and time t, the continuum flows with a flux $f = f\left(t, x, \rho(t, x), p(t)\right)$ that depends on time t, on the space variable x, on the continuum density ρ evaluated at (t, x) and on the state p of the individuals at time t. We require the following regularity:

(f) The flow $f \colon I \times \mathbb{R}^d \times [0, R] \times \mathbb{R}^n \to \mathbb{R}^d$ is such that

- **(f.1)** $f \in \mathbf{C^2}(I \times \mathbb{R}^d \times [0, R] \times \mathbb{R}^n; \mathbb{R}^d)$.
- **(f.2)** For all $(t, x, p) \in I \times \mathbb{R}^d \times \mathbb{R}^n$, $f(t, x, 0, p) = f(t, x, R, p) = 0$.
- **(f.3)** For all $T \in I$ and for all compact subsets $K \subset \mathbb{R}^n$, there exists a constant C_f such that for $t \in [0, T]$, $x \in \mathbb{R}^d$, $\rho \in [0, R]$ and $p \in K$,

$$\left\|\partial_\rho f(t, x, \rho, p)\right\| < C_f\,, \qquad \left|\operatorname{div}_x f(t, x, \rho, p)\right| < C_f\,.$$

- **(f.4)** For all $T \in I$ and for all compact subsets $K \subset \mathbb{R}^n$, there exists a constant C_f such that for $t \in [0, T]$, $x \in \mathbb{R}^d$, $\rho \in [0, R]$ and $p \in K$,

$$\left\|\nabla_x \partial_\rho f(t, x, \rho, p)\right\| < C_f\,.$$

- **(f.5)** For all compact subsets $K \subset \mathbb{R}^n$, there exists a constant C_f such that

$$\int_I \int_{\mathbb{R}^d} \sup_{p \in K, \rho \in [0,R]} \left\|\nabla_x \operatorname{div}_x f(t, x, \rho, p)\right\| \mathrm{d}x\, \mathrm{d}t < C_f\,.$$

- **(f.6)** For all compact subsets $K \subset \mathbb{R}^n$, there exists a constant C_f such that

$$\int_I \int_{\mathbb{R}^d} \sup_{p \in K, \rho \in [0,R]} \left\|\operatorname{div}_x f(t, x, \rho, p)\right\| \mathrm{d}x\, \mathrm{d}t < C_f\,.$$

- **(f.7)** For all $T \in I$ and for all compact subsets $K \subset \mathbb{R}^n$, there exists a constant C_f such that for $t \in [0, T]$, $\rho \in [0, R]$ and $p \in K$,

$$\int_{\mathbb{R}^d} \left\|\nabla_p \operatorname{div}_x f(t, x, \rho, p)\right\| \mathrm{d}x < C_f\,, \qquad \left\|\nabla_p \partial_\rho f(t, x, \rho, p)\right\| < C_f \text{ for all } x \in \mathbb{R}^d.$$

Condition **(f.2)** states that at the maximal density $\rho = R$, the continuum is at congestion and can not move. Assumption **(f.3)** has a key importance. The bound on $\partial_\rho f$ ensures the finite propagation speed of the solution to the partial differential equation, see Proposition 4.4 or [9, Theorem 1]. The bound on $\operatorname{div}_x f$ ensures that the solutions are bounded, similarly to the role of sublinearity in the ordinary differential equation, see **($\boldsymbol{\varphi}$.3)** below.

All these assumptions are satisfied, for instance, by vector fields of the form $f(\rho, x, p) = v(\rho)\,\vec{\mathbf{v}}(x,p)$ with $v \in \mathbf{C^2}([0,R];\mathbb{R})$ and $\vec{\mathbf{v}} \in \mathbf{C^2_c}(\mathbb{R}^d \times \mathbb{R}^n;\mathbb{R}^d)$.

We note that if f does not depend explicitly on t and x, which is a usual situation when dealing with systems of conservation laws in one space dimension, then the above assumptions reduce to only **(f.1)**, **(f.2)**, the bound on $\partial_\rho f$ in **(f.3)** and the bound on $\nabla_p \partial_\rho f$ in **(f.7)**. Moreover, Corollary 2.3 shows that whenever the initial density distribution $\bar\rho$ has compact support, then the requirements on f are reduced, since only **(f.1)**, **(f.2)** and **(f.3)** are necessary.

Speed of the Individuals. At time t, the individuals' state changes with a speed $\varphi = \varphi\left(t, p(t), A\left(\rho(t)\right)\left(p(t)\right)\right)$ that depends on time t, on the individuals' state p at time t and on an average $A(\rho(t))$ of the continuum density ρ evaluated at time t, computed at $p(t)$. On the averaging operator A we require the following conditions.

(A) $A\colon \mathbf{L^1}(\mathbb{R}^d;\mathbb{R}) \to \mathbf{W^{1,\infty}}(\mathbb{R}^n;\mathbb{R}^m)$ is linear and continuous, i.e. there exists a constant C_A such that for all $\rho \in \mathbf{L^1}(\mathbb{R}^d;\mathbb{R})$

$$\|A\rho\|_{\mathbf{W^{1,\infty}}} \le C_A \|\rho\|_{\mathbf{L^1}} .$$

Below, the operator norm is denoted $\|A\|_{\mathcal{L}(\mathbf{L^1},\mathbf{W^{1,\infty}})}$. For instance, in the case p is the position of a single individual, so that $n = d$, a typical example of such an operator A is $A\rho = \rho * \eta$ for a kernel $\eta \in \mathbf{C^1_c}(\mathbb{R}^d;\mathbb{R})$ with $\int_{\mathbb{R}^d} \eta\, dx = 1$, so that $A\left(\rho(t)\right)\left(p(t)\right) = \int_{\mathbb{R}^d} \rho\left(t, p(t) - y\right)\eta(y)\, dy$.

The speed law φ satisfies the assumptions:

($\boldsymbol{\varphi}$) The vector field $\varphi\colon \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is such that

- **($\boldsymbol{\varphi}$.1)** $t \mapsto \varphi(t,p,a)$ is measurable for all $p \in \mathbb{R}^n$ and all $a \in \mathbb{R}^m$;
- **($\boldsymbol{\varphi}$.2)** there exists a $C_\varphi \in \mathbf{L^1}(I;\mathbb{R}^+)$ such that for a.e. $t \in I$, $p_1, p_2 \in \mathbb{R}^n$ and $a_1, a_2 \in \mathbb{R}^m$,

$$\left\|\varphi(t,p_1,a_1) - \varphi(t,p_2,a_2)\right\| \le C_\varphi(t)\left(\|p_1 - p_2\| + \|a_1 - a_2\|\right) ;$$

- **($\boldsymbol{\varphi}$.3)** there exists a $C_\varphi \in \mathbf{L^1}(I;\mathbb{R}^+)$ such that for a.e. $t \in [0,T]$, $p \in \mathbb{R}^n$ and $a \in \mathbb{R}^m$,

$$\left\|\varphi(t,p,a)\right\| \le C_\varphi(t)\left(1 + \|p\|\right) .$$

The dummy variable a is used where it has to be replaced by $A\rho$. These hypotheses are motivated by the theory of Caratheodory ordinary differential equations, see [6, § 1].

The assumptions **(f)**, **(A)** and **($\boldsymbol{\varphi}$)** are satisfied in the applications considered in Section 3.

As a first step in the analytical treatment of (1.1), we rigorously state what we mean by *solution* to (1.1).

Definition 2.1 *Fix* $\bar\rho \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R}^d;[0,R])$ *and* $\bar p \in \mathbb{R}^n$. *A pair* (ρ, p) *with*

$$\rho \in \mathbf{C^0}\left(I; \mathbf{L^1}(\mathbb{R}^d;[0,R])\right) \qquad \text{and} \qquad p \in \mathbf{W^{1,1}}(I;\mathbb{R}^n)$$

is a solution to (1.1) with initial datum $(\bar\rho, \bar p)$ *if* *(i)* *the map* $\rho = \rho(t,x)$ *is a Kružkov solution to the scalar conservation law*

$$\partial_t \rho + \mathrm{div}_x\, f\left(t, x, \rho, p(t)\right) = 0\,,$$

(ii) *the map* $p = p(t)$ *is a Caratheodory solution to the ordinary differential equation*

$$\dot p = \varphi\left(t, p, A\left(\rho(t)\right)(p)\right)\,,$$

(iii) $\rho(0) = \bar\rho$ *and* $p(0) = \bar p$.

For the standard definition of Kružkov solution we refer to [9, Definition 1], for that of Caratheodory solution, see [6, § 1].

We are now ready to state the main result of this work.

Theorem 2.2 *Under conditions **(f)**, **(φ)** and **(A)**, for any initial datum* $\bar p \in \mathbb{R}^n$ *and* $\bar\rho \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R}^d; [0,R])$*, problem (1.1) admits a unique solution in the sense of Definition 2.1. This solution can be extended to all* I*.*

Let now f_1*,* f_2 *satisfy **(f)**;* A_1*,* A_2 *satisfy **(A)** and* φ_1*,* φ_2 *satisfy **(φ)**; in all cases for the same interval* I *and the same parameters or functions* R, C_f, C_A, C_φ*. Then, there exists a function* $\mathcal{K} \in \mathbf{C^0}(I;\mathbb{R}^+)$ *that vanishes at* $t=0$ *such that for any initial data* $(\bar\rho_1, \bar p_1), (\bar\rho_2, \bar p_2) \in (\mathbf{L^1}\cap\mathbf{BV})(\mathbb{R}^d; [0,R]\times\mathbb{R}^n)$*, the solutions* (ρ_1, p_1) *and* (ρ_2, p_2) *to the problems*

$$\left\{\begin{array}{l}\partial_t\rho_1 + \mathrm{div}_x\, f_1\left(t,x,\rho_1,p_1(t)\right) = 0\,,\\ \dot p_1 = \varphi_1\left(t, p_1, A_1\left(\rho_1(t)\right)(p_1)\right)\,,\\ \rho_1(0,x) = \bar\rho_1(x)\,,\\ p_1(0) = \bar p_1\,,\end{array}\right. \quad \text{and} \quad \left\{\begin{array}{l}\partial_t\rho_2 + \mathrm{div}_x\, f_2\left(t,x,\rho_2,p_2(t)\right) = 0\,,\\ \dot p_2 = \varphi_2\left(t, p_2, A_2\left(\rho_2(t)\right)(p_2)\right)\,,\\ \rho_2(0,x) = \bar\rho_2(x)\,,\\ p_2(0) = \bar p_2\,.\end{array}\right. \tag{2.1}$$

satisfy the inequalities

$$\begin{array}{rl} & \left\|(\rho_1-\rho_2)(t)\right\|_{\mathbf{L^1}} \\ \le & \left(1+\mathcal{K}(t)\right)\|\bar\rho_1-\bar\rho_2\|_{\mathbf{L^1}} \\ & +\mathcal{K}(t)\left(\left\|\partial_\rho(f_1-f_2)\right\|_{\mathbf{L^\infty}(\Omega_t)} + \left\|\mathrm{div}\,(f_1-f_2)\right\|_{\mathbf{L^1}(\mathbb{R}^d)\times\mathbf{L^\infty}([0,t]\times[0,R]\times K_t)}\right) \\ & +\mathcal{K}(t)\left(\|\varphi_1-\varphi_2\|_{\mathbf{L^\infty}([0,t]\times K_t\times[0,C_A])} + \|A_1-A_2\|_{\mathcal{L}(\mathbf{L^1},\mathbf{W^{1,\infty}})} + \|\bar p_1 - \bar p_2\|\right)\end{array}$$

and

$$\begin{array}{rl} & \left\|(p_1-p_2)(t)\right\| \\ \le & \left(1+\mathcal{K}(t)\right)\|\bar p_1-\bar p_2\| \\ & +\mathcal{K}(t)\left(\left\|\partial_\rho(f_1-f_2)\right\|_{\mathbf{L^\infty}(\Omega_t)} + \left\|\mathrm{div}\,(f_1-f_2)\right\|_{\mathbf{L^1}(\mathbb{R}^d)\times\mathbf{L^\infty}([0,t]\times[0,R]\times K_t)}\right) \\ & +\mathcal{K}(t)\left(\|\varphi_1-\varphi_2\|_{\mathbf{L^\infty}([0,t]\times K_t\times[0,C_A])} + \|A_1-A_2\|_{\mathcal{L}(\mathbf{L^1},\mathbf{W^{1,\infty}})} + \|\bar\rho_1 - \bar\rho_2\|_{\mathbf{L^1}}\right)\,.\end{array}$$

The proof is deferred to Section 4 and is obtained through Banach Contraction Theorem. The necessary estimates for the convergence are a consequence of [9, Theorem 5], [5, Theorem 2.5] and of adaptations from the standard theories of conservation laws and from Caratheodory differential equations, collected in the lemmas 4.3 and 4.2. Detailed expressions of the various coefficients are also provided presented in Section 4.

In the applications below, the support of the initial data is compact. Thanks to the finite propagation speed typical of conservation laws, this allows the following major simplification in the assumptions of Theorem 2.2.

Corollary 2.3 *Consider problem* (1.1) *with f satisfying **(f.1)**, **(f.2)** and **(f.3)**. Let A satisfy **(A)** and φ satisfy **(φ)**. If $\bar\rho$ vanishes outside a compact set, then problem* (1.1) *admits a unique solution in the sense of Definition 2.1. This solution can be extended to all of I. Moreover, the stability estimates of Theorem 2.2 apply, provided both $\bar\rho_1$ and $\bar\rho_2$ vanish outside a compact set.*

The proof of Corollary 2.3 is described in Section 4.

3 Applications

This section is devoted to sample situations that fit into (1.1) and to which the above analytical results can be applied. While the unknown ρ keeps throughout the meaning of a scalar density, the state p of the individuals is a 4–vector position–speed in § 3.1, it is a vector of several positions in § 3.2 and it becomes the position of a single agent in § 3.3.

Numerical integrations are provided in order to show the qualitative behavior of the solutions. The algorithm used exploits both the Lax–Friedrichs method, see [12, § 12.5], for the partial differential equation and the classical Euler method for the ordinary differential equation.

3.1 Predator and Preys

As a first example, we consider a predator attacking a group of preys. We can think for instance at a hawk pursuing a flock of smaller birds or at a shark attacking a group of sardines. Here, ρ is the density of the preys with $x \in \mathbb{R}^3$, p is now the pair $(P, V) \in \mathbb{R}^6$, where $P \in \mathbb{R}^3$ is the position of the predator, $V \in \mathbb{R}^3$ is its speed and we postulate below an equation for the acceleration $\ddot P = \dot V$ of the predator. Indeed, the framework in Theorem 2.2 allows to consider also second, or higher, order ordinary differential equations for the single agents. The initial density of the preys is assumed to have a compact, connected support. The aim of the predator is to divide this group into smaller groups. Hence, its acceleration is directed along the gradient of the average preys' density, say $\ddot P = \alpha \rho(t) * \nabla\eta$ for a suitable $\alpha > 0$. The preys have a speed $V_{\max}(1-\rho/R)V_0$, for a fixed $V_0 \in \mathbb{R}^2$, and modify it trying to escape from the predator. The resulting speed of the preys is thus

$$v(t,x,\rho,p) = V_{\max}(1-\rho/R)\left(V_0 + B\, e^{-C\|x-p(t)\|}\, \left(x-p(t)\right)\right) \tag{3.1}$$

where B, C are positive constants. The former one is related to the speed at which preys escape the predator and the latter to the distance at which preys feel the presence of the predator. Once again, v is maximal at zero density and vanishes at the maximal density R, which means that the preys can not move when their density is maximal.

Lemma 3.1 *Let $d = 3$, $n = 6$, $m = 3$ and fix a positive R. Assume v is as in* (3.1), *$\eta \in \mathbf{C_c^2}(\mathbb{R}^2, \mathbb{R})$ with $\int_{\mathbb{R}^2} \eta \,\mathrm{d}x = 1$. Denote $p = (P, V)$ and define*

$$f(t,x,\rho,p) = \rho\, v(t,x,\rho,p), \qquad \varphi\left(t, \begin{bmatrix} P \\ V \end{bmatrix}, a\right) = \begin{bmatrix} V \\ \alpha a \end{bmatrix}, \qquad A\rho = \rho * \nabla\eta\,. \tag{3.2}$$

Then, this setting fits in the framework of Corollary 2.3 as soon as $\bar\rho$ vanishes a.e. outside a compact set.

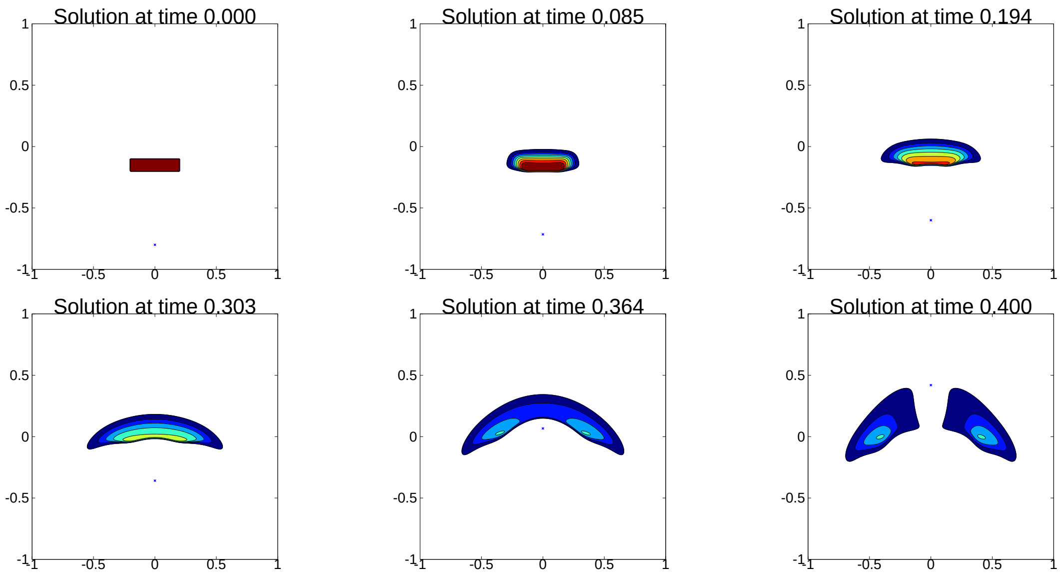

Figure 1: Solution obtained through the numerical integration of (1.1)–(3.1)–(3.2)–(3.4) computed at times 0, 0.091, 0.267, 0.358, 0.449 and 0.491.

Numerical Example: For graphical purposes, we limit the numerical integration to the 2D case. With reference to (1.1)–(3.1)–(3.2), we choose the following parameters

$$\begin{aligned} &V_{\max} = 2\,, \quad C = 5.25\,, \quad V_0 = [0\ -0.5]^T\,, \quad B = 40\,, \quad \alpha = 400\,, \\ &\eta(x) = \frac{3}{\pi {r_p}^6}\left(\max\left\{0, {r_p}^2 - \|x\|^2\right\}\right)^2, \qquad r_p = 0.5\,. \end{aligned} \tag{3.3}$$

and the initial datum

$$P_0 = \begin{bmatrix} 0 \\ -0.8 \end{bmatrix}, \quad V_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \rho_0(x,y) = \chi_{[-0.2,0.2]}(x)\,\chi_{[-0.2,-0.1]}(y)\,. \tag{3.4}$$

The result is in Figure 1. In the case considered, apparently, the predator succeeds in splitting the support of the preys. This may follow from the rise of discontinuities in the prey density ρ, which in turn leads to discontinuities in the preys' speed.

3.2 Shepherd Dogs

On the plane, consider a herd of, say, sheeps controlled by N shepherd dogs. Then, ρ is the density of sheeps and $p \equiv (p_1, \ldots, p_N)$ is the vector of the positions of the dogs, so that each p_i is in $\mathbb{R}^2$. We assume that initially the sheeps are distributed around, say, the origin and tend to disperse moving radially with a speed directed by $\vec{\boldsymbol{v}}_{\boldsymbol{r}}(x)$. The duty of the dogs is to prevent this dispersion and they pursue this goal moving around sheeps or, more precisely, with a speed φ orthogonal to the gradient of the sheeps' average density. The sheeps modify their speed escaping from the dogs with a repulsive speed $\vec{\boldsymbol{v}}_{\boldsymbol{d}}(x,p) = \sum_{i=1}^{N} \vec{\boldsymbol{v}}(x - p_i)$, where $\vec{\boldsymbol{v}}$ behaves qualitatively as in Figure 3. Finally, the speed of the sheeps is then given by $v(\rho)\left(\vec{\boldsymbol{v}}_{\boldsymbol{r}}(x) + \sum_{i=1}^{N} \vec{\boldsymbol{v}}(x - p_i)\right)$ where v is maximal at the density zero and vanishes at the maximal density R. This last fact means that the sheeps can not move when their density is maximal.

Lemma 3.2 *Let $N \in \mathbb{N}$, $d = 2$, $n = 2N$, $m = 2N$ and fix a positive R. Assume $v \in \mathbf{C^2}([0,R];\mathbb{R})$, $\vec{\boldsymbol{v}}_{\boldsymbol{r}} \in \mathbf{C^2}(\mathbb{R}^2;\mathbb{R}^2)$ $\vec{\boldsymbol{v}} \in \mathbf{C^2}(\mathbb{R}^2;\mathbb{R}^2)$, $\eta \in \mathbf{C^2_c}(\mathbb{R}^2,\mathbb{R})$ with $\int_{\mathbb{R}^2} \eta \, \mathrm{d}x = 1$. Assume that $v(R) = 0$. Define*

$$\begin{aligned} f(t,x,\rho,p) &= \rho\, v(\rho) \left(\vec{\boldsymbol{v}}_{\boldsymbol{r}}(x) + \sum_{i=1}^{N} \vec{\boldsymbol{v}}(x - p_i) \right), \\ \varphi(t,p,a) &= V_d \, \frac{a^\perp}{\sqrt{1 + \|a\|^2}}\,, \\ A\rho &= \rho * \nabla\eta\,. \end{aligned} \tag{3.5}$$

Then, this setting fits in the framework of Corollary 2.3 as soon as $\bar\rho$ vanishes a.e. outside a compact set.

Here, $a \equiv (a_1, \ldots, a_N)$ is a vector in $(\mathbb{R}^2)^N$ and we set $a^\perp \equiv (a_1^\perp, \ldots, a_N^\perp)$, where for any $x, y \in \mathbb{R}$, we have $\begin{bmatrix} x \\ y \end{bmatrix}^\perp = \begin{bmatrix} y \\ -x \end{bmatrix}$.

In connection with the present shepherd dog model (1.1)–(3.5), several control problems can be stated and the existence of an optimal control follows from Theorem 2.2. For instance, let $K \subset \mathbb{R}^2$ be the compact set from which sheeps should not escape. Then, one may look for the best dogs' speed choice $V_d = V_d(t)$ that keeps as many sheeps as possible within K throughout a given time interval $[0,T]$. The existence of such an optimal choice is ensured by the next proposition.

Proposition 3.3 *Let $N \in \mathbb{N} \setminus \{0\}$ be the number of dogs, $T_{\max}$ be finite and fix a compact set $K \subset \mathbb{R}^2$. Fix the set of the admissible dogs' speed choices*

$$\mathcal{S} = \left\{ V_d \in \mathbf{W^{1,\infty}}([0,T_{\max}];\mathbb{R}^+) \colon \|V_d\|_{\mathbf{W^{1,\infty}}} \le C \right\},$$

where C is an upper bound for the dogs' speed and acceleration. Given an initial sheeps' distribution $\bar\rho \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R}^2;[0,R])$ with $\operatorname{spt}\bar\rho \subseteq K$, the time average of the total number of sheeps kept in K when the dogs have speed V_d is

$$\mathcal{J}(\bar\rho,\varphi) = \int_0^{T_{\max}} \int_K \rho(t,x) \, \mathrm{d}x \, \mathrm{d}t\,,$$

$\rho = \rho(t,x)$ being the solution to (1.1)–(3.5). Then, for any such $\bar\rho$ there exists an optimal speed choice V_d that maximizes $\mathcal{J}$ over $\mathcal{S}$.

The proof relies on a direct application of Weierstraß and Ascoli–Arzelà theorems.

Suitably modifying the choices (3.5) of f and φ, one may pass to various other problems. For instance, dogs may be asked to steer the sheeps towards a given area. On the other hand, in view of the protesters–policemen setting, one can also look for the best strategy of sheeps that let them dodge the dogs and escape from a given compact set.

Numerical Example: To fix a specific situation, we choose $N = 2$ and the following functions in (1.1):

$$\begin{aligned} v(\rho) &= V_{\max}\left(1-\frac{\rho}{R}\right), && V_{\max}=1\,, \quad R=1\,,\\ \vec{\boldsymbol{v}}(x) &= \frac{\alpha}{\sqrt{\ell}}\, e^{-\|x\|^2/\ell}\, x\,, && \alpha = 20\,, \quad \ell = 0.2\,,\\ \vec{\boldsymbol{v}}_{\boldsymbol{r}}(x) &= \frac{\beta\, x}{1+\|x\|^2}\,, && \beta = 1\,,\\ \eta(x) &= \frac{3}{\pi {r_p}^6}\left(\max\left\{0, {r_p}^2 - \|x\|^2\right\}\right)^2, && r_p = 1\,,\\ V_d &= 100\,. \end{aligned} \tag{3.6}$$

At time zero, sheeps are uniformly distributed at the maximal density $R = 1$ in the circumference centered at $(0, 0)$ with radius 0.2. Dogs start moving from (0.7, 0) and (−0.7, 0) Graphs

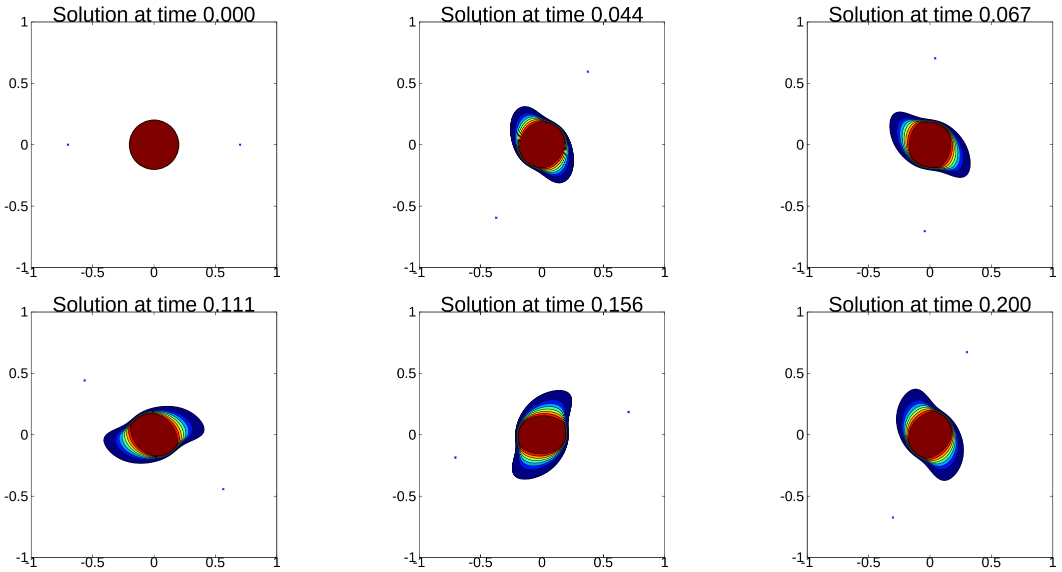

Figure 2: Solution to (1.1)–(3.5)–(3.6) at times $t = 0$, $t = 0.044$, $t = 0.067$, $t = 0.111$, $t = 0.156$, $t = 0.200$. Sheeps are initially uniformly distributed at the maximal density $R = 1$ in the circumference centered at $(0, 0)$ with radius 0.2. Dogs start moving from (0.7, 0) and (−0.7, 0), they succeed in confining the dispersion of the sheeps, at least for the tie interval considered.

of the corresponding solution are in Figure 2.

3.3 The Magic Piper

As a last toy application we consider the situation of a leader interacting with a group of followers. This case can be illustrated for example by the fairy tale of the Pied Pier [7, n. 246]. To lure rats away, the city of Hamelin (now Hamel) hires a rat-catcher who, playing his magic pipe, attracts all mice out of the city. In this case, $\rho = \rho(t, x)$ is the mice density and $p = p(t)$ is the position of the piper. Rats move with a speed $v(\rho)\, \vec{\boldsymbol{v}}(p - x)$, with the scalar v and the vector $\vec{\boldsymbol{v}}$ having the qualitative behavior in Figure 3. More precisely, at density 0 mice have the fastest speed while at density R their speed vanishes, see Figure 3,

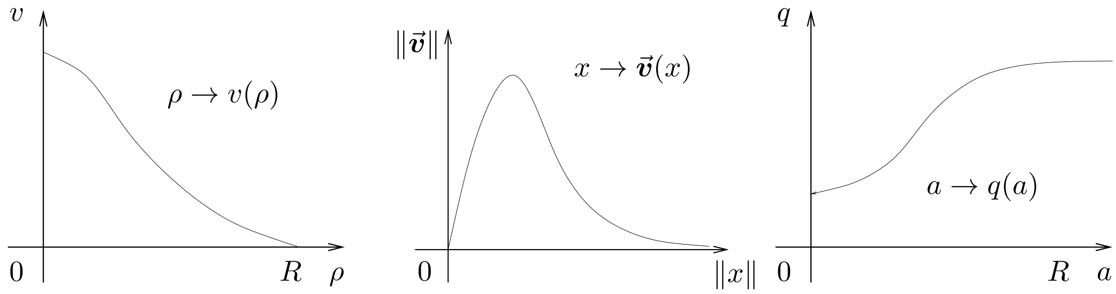

Figure 3: Left, v is assumed $\mathbf{C^2}$ and decreasing. Center, $\vec{\boldsymbol{v}}$ describes the attraction felt by the mice towards the piper. Right, q accounts for the acceleration of the piper when surrounded by a high mice density.

left. The term $\vec{\boldsymbol{v}}$ accounts for the attraction of the mice towards the piper, see Figure 3, center, and (3.8). According to this choice, when the piper is too far, mice are not attracted by the music. The magic musician has a speed $q(\rho * \eta)\,\vec{\boldsymbol{\psi}}(t)$, with q as in Figure 3, right, see also (3.8). The role of the map q is to allow the piper to move faster when the average density of mice around him is higher. On the contrary, when only few rats are near to him, he slows down. This choice is in order to avoid some of the mice remaining too far behind, where they may not hear the music.

Lemma 3.4 *Let $d = 2$, $n = 2$, $m = 1$ and fix a positive R. Assume $v \in \mathbf{C^2}([0,R];\mathbb{R})$, $\vec{\boldsymbol{v}} \in \mathbf{C^2}(\mathbb{R}^2;\mathbb{R}^2)$, $q \in \mathbf{W^{1,\infty}}([0,R];\mathbb{R})$, $\vec{\boldsymbol{\psi}} \in \mathbf{W^{1,\infty}}(\mathbb{R}^+;\mathbb{R}^2)$, $\eta \in \mathbf{C^2_c}(\mathbb{R}^2,\mathbb{R})$ with $\int_{\mathbb{R}^2}\eta\,\mathrm{d}x = 1$. Assume that $v(R) = 0$. Define*

$$f(t,x,\rho,p) = \rho\, v(\rho)\,\vec{\boldsymbol{v}}(p-x)\,, \qquad \varphi(t,p,a) = q(a)\,\vec{\boldsymbol{\psi}}(t)\,, \qquad A\rho = \rho * \eta\,. \tag{3.7}$$

Then, this setting fits in the framework of Corollary 2.3 as soon as $\bar p$ vanishes a.e. outside a compact set.

The proof is immediate and, hence, omitted.

Several optimization problems can now be stated with reference to (1.1)–(3.7)–(3.8). Referring to the situation [7, n. 246], a first natural question is the following. Let the compact set K be the area of the city and fix a finite positive time $T_{\max}$. Then, find the initial position $\bar p$ and the trajectory $\vec{\boldsymbol{\psi}}$ of the piper so that the amount of mice left in the city at time $T_{\max}$ is minimal. In other words, we want to minimize the functional

$$(\bar p, \vec{\boldsymbol{\psi}}) \mapsto \int_K \left(\rho(\bar p, \vec{\boldsymbol{\psi}})\right)(T_{\max}, x)\,\mathrm{d}x$$

over a compact set of initial positions $\bar p$ and over all strategies $\vec{\psi}$ with finite speed and acceleration. Here, $\rho(\bar p, \vec{\boldsymbol{\psi}})$ is the ρ–component of the solution to (1.1)–(3.7)–(3.8). The existence of such an optimal strategy for the piper follows from Theorem 2.2 via a standard application of Weierstraß Theorem.

Proposition 3.5 *Let $T_{\max}$ be finite. Denote by $K \subset \mathbb{R}^2$ the compact Hamelin urban area. Define the set of the possible piper's strategies*

$$\mathcal{S} = \left\{(\bar p, \vec{\boldsymbol{\psi}}) \in K \times \mathbf{W^{1,\infty}}(I;\mathbb{R}^2) \colon \left\|\vec{\boldsymbol{\psi}}\right\|_{\mathbf{W^{1,\infty}}} \le 1\right\}$$

and call $\mathcal{J}\colon \mathcal{S} \mapsto \mathbb{R}$ the total amount of mice in Hamelin at time $T_{\max}$, i.e.

$$\mathcal{J}(\bar{p}, \vec{\psi}) = \int_K \left(\rho(\bar{p}, \vec{\psi})\right)(T_{\max}, x)\, \mathrm{d}x \ ,$$

where $\rho(\bar{p}, \vec{\psi})$ is the solution to (1.1)–(3.7)–(3.8). Then, there exists an optimal trajectory $(\bar{p}_*, \vec{\psi}_*) \in \mathcal{S}$ such that $\mathcal{J}(\bar{p}_*, \vec{\psi}_*) = \min_{\mathcal{S}} \mathcal{J}(\bar{p}, \vec{\psi})$.

Thanks to the stability estimates in Theorem 2.2, the proof of this proposition directly follows from Ascoli–Arzelà Theorem that allows to prove the compactness of $\mathcal{S}$.

Numerical Example: To fix a specific situation, we choose the following functions in (1.1):

$$\begin{array}{rcllll} v(\rho) & = & V_{\max}\left(1 - \frac{\rho}{R}\right), & V_{\max} = 9\,, & R = 1\,, & \\ \vec{v}(x) & = & x\, e^{-\|x\|^2}\,, & & & \\ q(a) & = & v_p + (V_p - v_p)\,\frac{a}{R}\,, & V_p = 7\,, & v_p = 1\,, & \\ \vec{\psi}(t) & = & \left[\begin{array}{c} \cos\omega t \\ -\sin\omega t \end{array}\right], & \omega = 1\,, & & (3.8) \\ \eta(x) & = & \frac{3}{\pi {r_p}^6}\left(\max\left\{0, {r_p}^2 - \|x\|^2\right\}\right)^2, & r_p = 0.15\,. & & \end{array}$$

At time $t = 0$, we assume that rats are uniformly distributed with density $R = 1$ in the rectangle $[-0.5, 0] \times [0.35, 0.85]$. The piper starts moving at the point $(-1, 0.5)$.

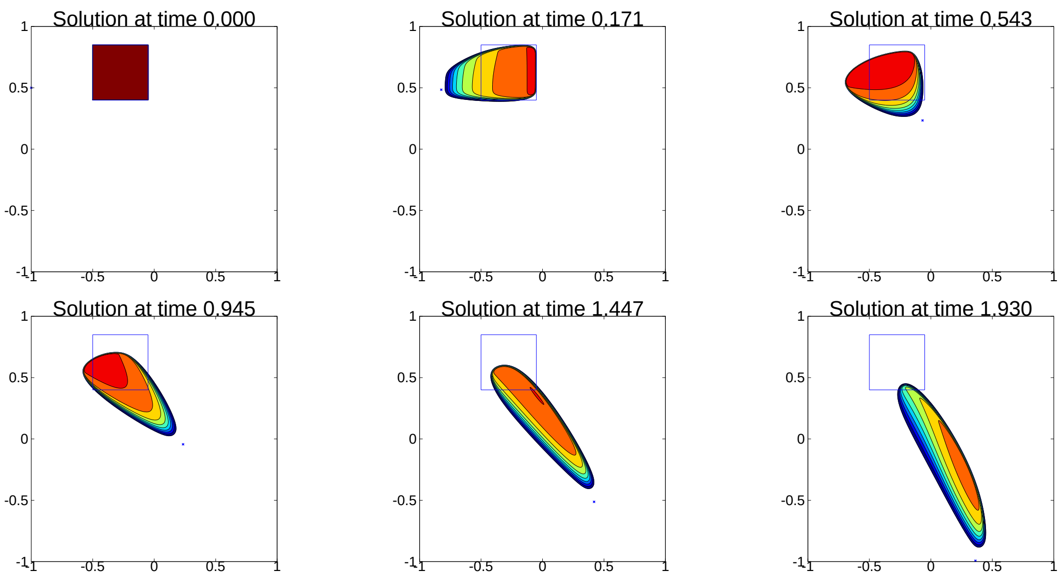

Figure 4: The pied piper and the rats, at times 0, when $p = (0, 0.5)$; 0.171, 0.543, 0.945, 1.447 and 1.930, when the rats almost completely left the rectangle and $p = (0.366, -0.983)$.

4 Technical details

Throughout this section we let $W_d = \int_0^{\pi/2} (\cos\theta)^d\, \mathrm{d}\theta$. We state and prove below the Grönwall–type lemma used in the sequel.

Lemma 4.1 *Let the functions* $\alpha \in \mathbf{C^0}(I;\mathbb{R})$*,* $\beta \in \mathbf{W^{1,1}}(I;\mathbb{R})$*,* $\gamma \in \mathbf{C^0}(I;\mathbb{R}^+)$*,* $\Delta \in \mathbf{C^0}(I;\mathbb{R})$ *be such that*

$$\Delta(t) \leq \alpha(t)\left(\beta(t) + \int_0^t \gamma(\tau)\,\Delta(\tau)\;\mathrm{d}\tau\right).$$

Then, for all $t \in I$*,*

$$\Delta(t) \quad\leq\quad \alpha(t)\left[\beta(0)\exp\left(\int_0^t \alpha(\tau)\,\gamma(\tau)\,\mathrm{d}\tau\right) + \int_0^t \beta'(\tau)\exp\left(\int_\tau^t \alpha(s)\,\gamma(s)\;\mathrm{d}s\right)\mathrm{d}\tau\right].$$

Proof. Using the following straightforward computations, we have:

$$\begin{aligned}\gamma(t)\Delta(t) \quad&\leq\quad \alpha(t)\,\beta(t)\,\gamma(t) + \alpha(t)\,\gamma(t)\int_0^t \gamma(\tau)\Delta(\tau)\,\mathrm{d}\tau\;,\\ \left(e^{-\int_0^t \alpha(\tau)\gamma(\tau)\mathrm{d}\tau}\int_0^t \gamma(\tau)\Delta(\tau)\,\mathrm{d}\tau\right)' \quad&\leq\quad \alpha(t)\,\beta(t)\,\gamma(t)\,e^{-\int_0^t \alpha(\tau)\gamma(\tau)\mathrm{d}\tau}.\end{aligned}$$

Then, by integration we obtain

$$\int_0^t \gamma(\tau)\Delta(\tau)\,\mathrm{d}\tau \quad\leq\quad \int_0^t \alpha(t)\,\beta(t)\,\gamma(t)\,e^{\int_\tau^t \alpha(s)\gamma(s)\mathrm{d}s}\,\mathrm{d}\tau\;.$$

Consequently, we have

$$\Delta(t) \quad\leq\quad \alpha(t)\left[\beta(t) + \int_0^t e^{\int_\tau^t \alpha(s)\gamma(s)\mathrm{d}s}\alpha(\tau)\beta(\tau)\gamma(\tau)\,\mathrm{d}\tau\right].$$

Integrating by part the last integral, we have finally the desired estimate. □

Below, we describe independently some properties of the conservation law in Lemma 4.2 and some properties of the ordinary differential equation in Lemma 4.3.

Lemma 4.2 *Let **(f)** hold. Choose any* $\bar\rho \in (\mathbf{L^1} \cap \mathbf{L^\infty} \cap \mathbf{BV})(\mathbb{R}^d;[0,R])$*. Fix a function* $\pi \in \mathbf{C^0}(I;\mathbb{R}^n)$*. Then, the conservation law*

$$\left\{\begin{array}{l}\partial_t\rho + \mathrm{div}_x f\left(t,x,\rho,\pi(t)\right) = 0\\ \rho(0,x) = \bar\rho(x)\end{array}\right. \tag{4.1}$$

admits a unique solution $\rho \in \mathbf{C^0}\left(I;\mathbf{L^1}(\mathbb{R}^d,[0,R])\right)$*. For all* $t \in I$*, introduce the compact set* $K_t = B_n(0,\|\pi\|_{\mathbf{C^0}([0,t])})$*, denote* $\Omega_t = [0,t]\times\mathbb{R}^d\times[0,R]\times K_t$ *and define*

$$\kappa_t = (2d+1)\left\|\nabla_x\partial_\rho f\right\|_{\mathbf{L}^\infty(\Omega_t)}. \tag{4.2}$$

Then, the following **BV** *estimate holds: for all* $t \in I$

$$\mathrm{TV}\left(\rho(t)\right) \leq \left(\mathrm{TV}(\bar\rho) + dW_d\,t\int_{\mathbb{R}^d}\left\|\nabla_x\,\mathrm{div}_x\,f(\cdot,x,\cdot,\cdot)\right\|_{\mathbf{L}^\infty([0,t]\times[0,R]\times K_t)}\mathrm{d}x\right)e^{\kappa_t t}. \tag{4.3}$$

*Moreover, there exists a function $\mathcal{C} \in \mathbf{C^0}(I;\mathbb{R}^+)$ such that, letting ρ_1, ρ_2 be the solutions to (1.1) corresponding to the initial data $\bar\rho_1$, $\bar\rho_2$ and to the equation defined by $\pi_1, \pi_2 \in \mathbf{C^0}(I;\mathbb{R}^n)$ and by f_1, f_2, satisfying **(f)**, the following estimate holds:*

$$\begin{aligned}\left\|(\rho_1-\rho_2)(t)\right\|_{\mathbf{L^1}} \leq \; & \left\|\bar\rho_1-\bar\rho_2\right\|_{\mathbf{L^1}} + t\,\mathcal{C}(t)\Bigg[\left\|\pi_1-\pi_2\right\|_{\mathbf{L^\infty}([0,t])} + \left\|\partial_\rho(f_1-f_2)\right\|_{\mathbf{L^\infty}(\Omega_t)} \\ & + \left\|\operatorname{div}(f_1-f_2)\right\|_{\mathbf{L^1}(\mathbb{R}^d)\times\mathbf{L^\infty}([0,t]\times[0,R]\times K_t)}\Bigg]\end{aligned} \tag{4.4}$$

where $\mathcal{C}(t)$ depends on $\mathrm{TV}(\bar\rho_1)$, $\left\|\nabla_x\partial_\rho f_1\right\|_{\mathbf{L^\infty}(\Omega_t)}$, $\left\|\nabla_x \operatorname{div}_x f_1\right\|_{\mathbf{L^1}(\mathbb{R}^d)\times\mathbf{L^\infty}([0,t]\times[0,R]\times K_t)}$ and $\left\|\nabla_p\partial_\rho f_2\right\|_{\mathbf{L^\infty}(\Omega_t)}$, $\left\|\operatorname{div}_x\nabla_p f_2\right\|_{\mathbf{L^1}(\mathbb{R}^d)\times\mathbf{L^\infty}([0,t]\times[0,R]\times K_t)}$, t.

An explicit expression of $\mathcal{C}(t)$ is provided in (4.5).

Proof of Lemma 4.2. This proof consists in applying to the scalar conservation law $\partial_t\rho + \operatorname{div}_x f^*(t,x,\rho) = 0$ with flux $f^*(t,x,\rho) = f\left(t,x,\rho,p(t)\right)$ first the classical Kružkov result [9, Theorem 5] and then the stability estimates in [5, 11].

To apply Kružkov Theorem, it is sufficient to verify condition **(H1)** in [5, Theorem 2.5] or the slightly weakened form **(H0*)-(H1*)** in [11]. Note that: f^* is $\mathbf{C^2}$ in x and ρ by **(f.1)**, and is $\mathbf{C^0}$ in t by the regularity of π. This regularity is sufficient in the proof of [5, Theorem 2.5], see also [9, Remark 4 in § 5]. Moreover, for any $t \in I$

$$\begin{array}{lll}\textbf{(f.3)} & \Rightarrow & \partial_\rho f^* \in \mathbf{L^\infty}([0,t]\times\mathbb{R}^d\times[0,R];\mathbb{R}^d) \text{ and } \operatorname{div}_x f^* \in \mathbf{L^\infty}([0,t]\times\mathbb{R}^d\times[0,R];\mathbb{R}), \\ \textbf{(f.4)} & \Rightarrow & \partial_\rho \operatorname{div}_x f^* \in \mathbf{L^\infty}([0,t]\times\mathbb{R}^d\times[0,R];\mathbb{R}).\end{array}$$

Kružkov Theorem can then be applied on any interval $[0,t]$.

By **(f.2)**, the constant functions $\check\rho(t,x) \equiv 0$ and $\hat\rho(t,x) \equiv R$ solve (4.1), independently from π. Then, by the Maximum Principle [9, Theorem 3], we have that any solution ρ to (4.1) satisfies $\rho(t,x) \in [0,R]$ for a.e. $(t,x) \in I\times\mathbb{R}^d$ and for all $\pi \in \mathbf{C^0}(I;\mathbb{R}^n)$.

To prove the $\mathbf{L^1}$ continuity in time and the TV bound, we apply [5, Theorem 2.5] in the weaker form [11, Theorem 2.2]. To this aim, we verify also **(H2)** therein on $[0,t]$, for any $t \in I$. By **(f.4)** and the continuity of π, $\nabla_x\partial_\rho f^* \in \mathbf{L^\infty}([0,t]\times\mathbb{R}^d\times[0,R];\mathbb{R}^{d\times d})$. Note also that, by **(f.5)**, $\int_0^t\int_{\mathbb{R}^d}\left\|\nabla_x \operatorname{div}_x f^*(\tau,x,\rho)\right\|_{\mathbf{L^\infty}} \mathrm{d}x\,\mathrm{d}\tau < +\infty$, with an upper bound that depends on π.

We denote below $\Omega_t = [0,t]\times\mathbb{R}^d\times[0,R]\times K_t$ where K_t is as above. By [11, Theorem 2.2] or [5, Theorem 2.5] we obtain the estimate

$$\mathrm{TV}\left(\rho(t)\right) \leq \mathrm{TV}(\bar\rho)e^{\kappa_t t} + dW_d\int_0^t e^{\kappa_t(t-\tau)}\int_{\mathbb{R}^d}\left\|\nabla_x \operatorname{div}_x f\left(\tau,x,\cdot,\pi(\tau)\right)\right\|_{\mathbf{L^\infty}([0,R])}\mathrm{d}x\;\mathrm{d}\tau$$

where $\kappa_t = (2d+1)\left\|\nabla_x\partial_\rho f\right\|_{\mathbf{L^\infty}(\Omega_t)}$. This implies (4.3).

The $\mathbf{L^1}$–continuity in time of ρ follows from [5, Remark 2.4], thanks to **(f.6)** and to the bound on the total variation, see also [4, Proof of Lemma 5.3].

To estimate the dependence of the solution from the initial datum, we check the hypotheses **(H3)** in [11] or [5] and apply [11, Theorem 2.6] or [5, Theorem 2.6].

Let f_1, f_2 satisfy **(f.1)**, …, **(f.5)**. Assume that π_1, π_2 are in $\mathbf{C^0}([0,t],\mathbb{R}^n)$. Let f_1^* and f_2^* be the corresponding compositions. With obvious notation, define $K = K_t^1 \cup K_t^2$ and compute

$$\sup_{\tau\in[0,t],x\in\mathbb{R}^d,\rho\in[0,R]}\left|\partial_\rho f_1^*\left(\tau,x,\rho,\pi_1(\tau)\right) - \partial_\rho f_2^*\left(\tau,x,\rho,\pi_2(\tau)\right)\right|$$

$$\leq \quad \left\|\partial_\rho f_1 - \partial_\rho f_2\right\|_{\mathbf{L}^\infty(\Omega_t)} + \left\|\partial_\rho \nabla_p f_2\right\|_{\mathbf{L}^\infty(\Omega_t)} \|\pi_1 - \pi_2\|_{\mathbf{L}^\infty([0,t])}$$

which is bounded by **(f.3)** and **(f.7)**.

To complete **(H3)**, it remains only to estimate the quantity

$$\begin{aligned}
& \int_0^t \int_{\mathbb{R}^d} \left\|\operatorname{div}_x \Big(f_1\left(\tau, x, \cdot, \pi_1(\tau)\right) - f_2\left(\tau, x, \cdot, \pi_2(\tau)\right)\Big)\right\|_{\mathbf{L}^\infty([0,R];\mathbb{R})} \mathrm{d}x \, \mathrm{d}\tau \\
\leq & \int_0^t \int_{\mathbb{R}^d} \left\|\operatorname{div}_x (f_1 - f_2)\left(\tau, x, \cdot, \pi_1(\tau)\right)\right\|_{\mathbf{L}^\infty([0,R])} \mathrm{d}x \, \mathrm{d}\tau \\
& + \int_0^t \int_{\mathbb{R}^d} \left\|\nabla_p \operatorname{div}_x f_2(x)\right\|_{\mathbf{L}^\infty} \left\|\pi_1(\tau) - \pi_2(\tau)\right\| \mathrm{d}x \, \mathrm{d}\tau
\end{aligned}$$

which is bounded thanks to **(f.6)** and **(f.7)**. Now, we compare ρ_1 and ρ_2, obtaining

$$\begin{aligned}
& \left\|(\rho_1 - \rho_2)(t)\right\|_{\mathbf{L}^1} \\
\leq & \|\bar\rho_1 - \bar\rho_2\|_{\mathbf{L}^1} \\
& + \left[\frac{e^{\kappa_t t} - 1}{\kappa_t} \mathrm{TV}(\bar\rho) + dW_d \int_0^t \frac{e^{\kappa_t (t-\tau)} - 1}{\kappa_t} \int_{\mathbb{R}^d} \left\|\nabla_x \operatorname{div}_x f_1\left(\tau, x, \cdot, \pi_1(\tau)\right)\right\|_{\mathbf{L}^\infty([0,R])} \mathrm{d}x \, \mathrm{d}\tau\right] \\
& \quad \times \left(\left\|\partial_\rho f_1 - \partial_\rho f_2\right\|_{\mathbf{L}^\infty(\Omega_t)} + \left\|\partial_\rho \nabla_p f_2\right\|_{\mathbf{L}^\infty(\Omega_t)} \|\pi_1 - \pi_2\|_{\mathbf{L}^\infty([0,t])}\right) \\
& + \int_0^t \int_{\mathbb{R}^d} \left(\left\|\operatorname{div}(f_1 - f_2)\left(\tau, x, \cdot, \pi_1(\tau)\right)\right\|_{\mathbf{L}^\infty([0,R])}\right. \\
& \qquad \left. + \left\|\nabla_p \operatorname{div}_x f_2(\tau, x, \cdot, \cdot)\right\|_{\mathbf{L}^\infty([0,R]\times K_t)} \left\|\pi_1(\tau) - \pi_2(\tau)\right\|\right) \mathrm{d}x \, \mathrm{d}\tau
\end{aligned} \tag{4.5}$$

which gives the final estimate. □

The following lemma concerns the estimates on the ordinary differential equation.

Lemma 4.3 *Let* **(φ)** *and* **(A)** *hold. Choose an initial datum* $\bar p \in \mathbb{R}^n$ *and fix a function* $r \in \mathbf{C^0}\left(I; \mathbf{L^1}(\mathbb{R}^d; [0,R])\right)$*. Then, the ordinary differential equation*

$$\left\{\begin{array}{l}
\dot p = \varphi\Big(t, p, A\left(r(t)\right)(p)\Big)\,, \\
p(0) = \bar p\,,
\end{array}\right. \tag{4.6}$$

admits a unique solution $p \in \mathbf{W^{1,\infty}_{loc}}(I; \mathbb{R}^n)$*. The following bound holds:*

$$\left\|p(t)\right\| \leq \left(\|\bar p\| + 1\right) e^{\int_0^t C_\varphi(\tau) \mathrm{d}\tau} - 1\,. \tag{4.7}$$

Given two initial conditions $\bar p_1, \bar p_2 \in \mathbb{R}^n$*, two functions* $r_1, r_2 \in \mathbf{C^0}\left(I; \mathbf{L^1}(\mathbb{R}^d; [0,R])\right)$*, two speed laws* φ_1, φ_2 *satisfying* **(φ)** *and two averaging operators* A_1, A_2 *satisfying* **(A)***, define*

$$F(t) = \Big(1 + C_A \|r_1\|_{\mathbf{L}^\infty([0,t];\mathbf{L^1})}\Big) \int_0^t C_\varphi(\tau)\, \mathrm{d}\tau\ . \tag{4.8}$$

Then,

$$\begin{aligned} &\left\|(p_1-p_2)(t)\right\| \\ &\le e^{F(t)}\|\bar p_1-\bar p_2\| + \int_0^t e^{F(t)-F(\tau)}\left\|\varphi_1(\tau)-\varphi_2(\tau)\right\|_{\mathbf{L}^\infty}\mathrm{d}\tau \\ &\quad+\int_0^t e^{F(t)-F(\tau)}C_\varphi(\tau)\left(C_A\left\|(r_1-r_2)(\tau)\right\|_{\mathbf{L}^1}+\|A_1-A_2\|_{\mathcal{L}(\mathbf{L}^1,\mathbf{W}^{1,\infty})}\left\|r_2(\tau)\right\|_{\mathbf{L}^1}\right)\mathrm{d}\tau\,. \end{aligned} \tag{4.9}$$

Proof of Lemma 4.3. By **($\boldsymbol{\varphi}$)**, we may apply [6, theorems 1 and 2, Chap. 1] to (4.6) and get the local in time existence and uniqueness of the solution. The bound (4.7) follows from a standard application of Grönwall Lemma and ensures that the solution can be extended to the whole interval I. Assume for simplicity that φ_1 and φ_2 satisfy **($\boldsymbol{\varphi}$)** with the same function C_φ. Using the representation formula $p_i=\bar p_i+\int_0^t\varphi_i\left(\tau,p_i(\tau),A\left(r_i(\tau)\right)\left(p_i(\tau)\right)\right)\mathrm{d}\tau$, we get

$$\begin{aligned} &\left\|(p_1-p_2)(t)\right\| \\ \le\ & \|\bar p_1-\bar p_2\|+\int_0^t\left\|\varphi_1\left(\tau,p_1(\tau),A_1\left(r_1(\tau)\right)\left(p_1(\tau)\right)\right)-\varphi_2\left(\tau,p_2(\tau),A_2\left(r_2(\tau)\right)\left(p_2(\tau)\right)\right)\right\|\mathrm{d}\tau \\ \le\ & \|\bar p_1-\bar p_2\|+\int_0^t\left\|(\varphi_1-\varphi_2)(\tau,p_1(\tau),A_1(r_1(\tau))(p_1(\tau)))\right\|\mathrm{d}\tau \\ &\quad+\int_0^t C_\varphi(\tau)\left(\left\|(p_1-p_2)(\tau)\right\|+\left\|A_1\left(r_1(\tau)\right)\left(p_1(\tau)\right)-A_2\left(r_2(\tau)\right)\left(p_2(\tau)\right)\right\|\right)\mathrm{d}\tau \\ \le\ & \|\bar p_1-\bar p_2\|+\int_0^t C_\varphi(\tau)\left(1+\left\|\nabla_p A_1(r_1)\right\|_{\mathbf{L}^\infty}\right)\left\|(p_1-p_2)(\tau)\right\|\mathrm{d}\tau \\ &\quad+\int_0^t C_\varphi(\tau)\left(\|A_1\|_{\mathcal{L}(\mathbf{L}^1,\mathbf{W}^{1,\infty})}\left\|(r_1-r_2)(\tau)\right\|_{\mathbf{L}^1}+\|A_1-A_2\|_{\mathcal{L}(\mathbf{L}^1,\mathbf{W}^{1,\infty})}\left\|r_2(\tau)\right\|_{\mathbf{L}^1}\right)\mathrm{d}\tau \\ &+\int_0^t\left\|(\varphi_1-\varphi_2)(t,\cdot,\cdot)\right\|_{\mathbf{L}^\infty}d\tau\,. \end{aligned}$$

An application of Lemma 4.1 with

$$\begin{aligned} \Delta(t)=&\|\bar p_1-\bar p_2\|\,,\\ \alpha(t)=&1\,,\\ \beta(t)=&\|\bar p_1-\bar p_2\|+\int_0^t\left\|(\varphi_1-\varphi_2)(\tau,\cdot,\cdot)\right\|_{\mathbf{L}^\infty}\mathrm{d}\tau\ ,\\ \gamma(t)=&C_\varphi(t)\left(1+\|A_1\|_{\mathcal{L}(\mathbf{L}^1,\mathbf{W}^{1,\infty})}\|r_1\|_{\mathbf{L}^1}\right)\\ &+\int_0^t C_\varphi(\tau)\left[\|A_1\|_{\mathcal{L}(\mathbf{L}^1,\mathbf{W}^{1,\infty})}\left\|(r_1-r_2)(\tau)\right\|_{\mathbf{L}^1}+\|A_1-A_2\|_{\mathcal{L}(\mathbf{L}^1,\mathbf{W}^{1,\infty})}\left\|r_2(\tau)\right\|_{\mathbf{L}^1}\right]\mathrm{d}\tau\,. \end{aligned}$$

completes the proof of (4.9). □

Proof of Theorem 2.2. The proof is divided in several steps.

1. Local Existence. Here we rely on an application of Banach Fixed Point Theorem. Fix first the initial data $\bar\rho\in(\mathbf{L^1}\cap\mathbf{BV})(\mathbb{R}^d;[0,R])$ and $\bar p\in\mathbb{R}^n$. Choose a positive $\hat T\in I$ and, motivated by (4.7), call

$$\delta=\left(\|\bar p\|+1\right)e^{\int_0^{\hat T}C_\varphi(\tau)\mathrm{d}\tau}-1\,.$$

For any positive $\mathcal{R}$, with $\int_{\mathbb{R}^d} \bar\rho \,\mathrm{d}x \le \mathcal{R}$, and for any $T \in \left]0, \hat T\right]$, define the complete metric spaces and the distance

$$\begin{aligned}
X_\rho &= \left\{\rho \in \mathbf{C^0}\left([0,T]; \mathbf{L^1}(\mathbb{R}^d; [0,R])\right) : \sup_{t\in[0,T]} \int_{\mathbb{R}^d} \rho(t,x)\,\mathrm{d}x \le \mathcal{R}\right\},\\
X &= X_\rho \times \mathbf{C^0}\left([0,T]; B_n(0,\delta)\right),\\
d\left((\rho_1,p_1);(\rho_2,p_2)\right) &= \sup_{t\in[0,T]} \left\|\rho_1(t)-\rho_2(t)\right\|_{\mathbf{L^1}} + \sup_{t\in[0,T]} \left\|p_1(t)-p_2(t)\right\|.
\end{aligned}$$

Define the map $\mathcal{T}\colon X \to X$ by $\mathcal{T}(r,\pi) = (\rho,p)$ if and only if ρ and p solve the problems

$$\left\{\begin{array}{l}\partial_t\rho + \mathrm{div}_x f\left(t,x,\rho,\pi(t)\right) = 0\\ \rho(0,x) = \bar\rho(x)\end{array}\right. \quad\text{and}\quad \left\{\begin{array}{l}\dot p = \varphi\left(t,p,A\left(r(t)\right)(p)\right)\\ p(0) = \bar p\,.\end{array}\right. \tag{4.10}$$

Note that both problems admit a unique solution, by lemmas 4.2 and 4.3. Moreover, by the conservative form of the former problem in (4.10), $\int_{\mathbb{R}^d}\rho(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^d}\bar\rho(x)\,\mathrm{d}x \le \mathcal{R}$, so that $\mathcal{T}$ is well defined. Moreover, Lemma 4.3 shows that the solution to the latter problem in (4.10) is in $\mathbf{W^{1,\infty}}\left([0,T]; B_n(0,\delta)\right) \subset \mathbf{C^0}\left([0,T]; B_n(0,\delta)\right)$.

To prove that $\mathcal{T}$ is a contraction, fix (r_1,π_1) and (r_2,π_2) and call $(\rho_i,p_i) = \mathcal{T}(r_i,\pi_i)$. Then, define $K_{\hat T} = B_n(0,\delta)$ and apply Lemma 4.2 with $t = T$. Note that $K_T \subseteq K_{\hat T}$. The former problem in (4.10) is then solvable in $\mathbf{C^0}\left([0,T]; \mathbf{L^1}(\mathbb{R}^d;[0,R])\right)$ and (4.4) yields

$$\sup_{t\in[0,T]} \left\|\rho_1(t)-\rho_2(t)\right\|_{\mathbf{L^1}} \le T\,\mathcal{C}(\hat T) \sup_{t\in[0,T]} \left\|\pi_1(t)-\pi_2(t)\right\|.$$

Apply now (4.9)

$$\sup_{t\in[0,T]} \left\|p_1(t)-p_2(t)\right\| \le C_A \int_0^T C_\varphi(\tau)\, e^{F(T)-F(\tau)}\,\mathrm{d}\tau \sup_{t\in[0,T]} \left\|r_1(t)-r_2(t)\right\|_{\mathbf{L^1}},$$

where F is defined as in (4.8) and can here be bounded as

$$F(t) \le (1 + C_A\mathcal{R}) \int_0^t C_\varphi(\tau)\,\mathrm{d}\tau\,. \tag{4.11}$$

Hence, $d\left(\mathcal{T}(\rho_1,p_1),\mathcal{T}(\rho_2,p_2)\right) \le \max\left\{T\,\mathcal{C}(\hat T), C_A(e^{F(T)}-1)\right\} d\left((\rho_1,p_1),(\rho_2,p_2)\right)$. Choose now a sufficiently small T so that $\mathcal{T}$ is a contraction. Then, its unique fixed point is the unique solution to (1.1) defined on the time interval $[0,T]$.

2. Global Uniqueness: Let now (ρ_1,p_1) and (ρ_2,p_2) be two solutions to the same problem (1.1) and defined at least on a common time interval $[0,\check T] \subseteq I$. Define

$$T^* = \sup\left\{T \in [0,\check T]\colon (\rho_1,p_1)(t) = (\rho_2,p_2)(t) \text{ for all } t\in[0,T]\right\}.$$

By the uniqueness of the fixed point, $(\rho_1,p_1)(t) = (\rho_2,p_2)(t)$ for all $t\in[0,T]$, so that the set in the right hand side above is not empty. Repeat Step 1 with initial datum $(\bar\rho^*,\bar p^*) = (\rho_1,p_1)(T^*) = (\rho_2,p_2)(T^*)$, which is possible since p is bounded on $[0,T^*]$ and $\mathrm{TV}(\bar\rho^*)$ is bounded, by (4.3). Thus, we obtain that $(\rho_1, p_1)(t) = (\rho_2, p_2)(t)$ also on a right neighborhood of T^*. This contradicts the maximality of T^*, unless $T^* = \check{T}$.

3. Global Existence: Define now

$$T_* = \sup\left\{T \in I \colon \exists \text{ a solution to (1.1) defined on } [0,T]\right\}$$

and assume that $T_* < +\infty$. By (4.7), p is bounded on $[0, T_*[$ and since

$$\left\|p(t_2) - p(t_1)\right\| \le \left| \int_{t_1}^{t_2} C_\varphi(\tau) \left(1 + \left\|p(\tau)\right\|\right) d\tau \right| \le \left(1 + \sup_{t\in[0,T_*]} \left\|p(t)\right\|\right) \left| \int_{t_1}^{t_2} C_\varphi(\tau)\, d\tau \right|,$$

p is also uniformly continuous. Hence the limit $p_* = \lim_{t\to T_*^-} p(t)$ exists and is finite.

Apply now Lemma 4.2 on $[0, T_*]$, obtaining that the solution ρ to (4.1) is defined on all $[0, T_*]$ and, together with p, also solves (1.1). Now, we repeat Step 1 with initial datum $(\bar\rho_*, \bar p_*) = (\bar\rho, \bar p)(T_*)$, which is possible thanks to (4.3). In turn, this allows to extend $(\bar\rho, \bar p)$ to a right neighborhood of T_*. This contradicts the maximality of T_*, unless $T_* = T_{\max}$.

4. Stability Estimates: Fix $t > 0$ and let $\tau \in [0, t]$. Let $\mathcal{R} \ge \max\left\{\int_{\mathbb{R}^d} \bar\rho_1 \,\mathrm{d}x\,, \int_{\mathbb{R}^d} \bar\rho_2 \,\mathrm{d}x\right\}$. Then, by (4.4) and (4.9), the solutions to (2.1) satisfy

$$\begin{aligned}
& \left\|(\rho_1-\rho_2)(t)\right\|_{\mathbf{L^1}} \\
\le\; & \|\bar\rho_1 - \bar\rho_2\|_{\mathbf{L^1}} + t\,\mathcal{C}(t) \Big[\|p_1 - p_2\|_{\mathbf{L}^\infty([0,t])} + \left\|\partial_\rho (f_1 - f_2)\right\|_{\mathbf{L}^\infty(\Omega_t)} \\
& \qquad + \left\|\mathrm{div}\,(f_1 - f_2)\right\|_{\mathbf{L^1}(\mathbb{R}^d)\times\mathbf{L}^\infty([0,t]\times[0,R]\times K_t)} \Big] , \\
& \left\|(p_1-p_2)(t)\right\| \\
\le\; & e^{F(t)} \|\bar p_1 - \bar p_2\| + \int_0^t e^{F(t)-F(\tau)} \left\|(\varphi_1 - \varphi_2)(\tau,\cdot,\cdot)\right\|_{\mathbf{L}^\infty} \mathrm{d}\tau \\
& + \int_0^t e^{F(t)-F(\tau)} C_\varphi(\tau) \left(C_A \left\|(\rho_1-\rho_2)(\tau)\right\|_{\mathbf{L^1}} + \mathcal{R} \|A_1 - A_2\|_{\mathcal{L}(\mathbf{L^1}, \mathbf{W^{1,\infty}})} \right) \mathrm{d}\tau \;,
\end{aligned}$$

with $\mathcal{C}$ as in Lemma 4.2, F as in (4.11), $K_t = B(0,\delta_t)$ and $\delta_t = \left(\|\bar p\| + 1\right) e^{\int_0^t C_\varphi(\tau)\mathrm{d}\tau} - 1$. Insert now the former estimate in the latter one and apply Lemma 4.1 with

$$\begin{aligned}
\Delta &= \left\|(p_1 - p_2)(t)\right\| , \\
\alpha(t) &= e^{F(t)} , \\
\beta(t) &= \|\bar p_1 - \bar p_2\| + \frac{\mathcal{R}}{1 + C_A \mathcal{R}} \left(1 - e^{-F(t)}\right) \|A_1 - A_2\|_{\mathcal{L}(\mathbf{L^1}, \mathbf{W^{1,\infty}})} \\
& \quad + \int_0^t \left\|(\varphi_1 - \varphi_2)(\tau,\cdot,\cdot)\right\|_{\mathbf{L}^\infty} e^{-F(\tau)} \,\mathrm{d}\tau + C_A \int_0^t e^{-F(\tau)} C_\varphi(\tau) \|\bar\rho_1 - \bar\rho_2\|_{\mathbf{L^1}} \,\mathrm{d}\tau \\
& \quad + C_A \int_0^t \tau\, \mathcal{C}(\tau)\, C_\varphi(\tau)\, e^{-F(\tau)} \\
& \qquad \times \left(\left\|\partial_\rho (f_1 - f_2)\right\|_{\mathbf{L}^\infty(\Omega_\tau)} + \left\|\mathrm{div}\,(f_1 - f_2)\right\|_{\mathbf{L^1}(\mathbb{R}^d)\times\mathbf{L}^\infty([0,\tau]\times[0,R]\times K_\tau)} \right) , \\
\gamma(t) &= C_A\, t\, C_\varphi(t)\, \mathcal{C}(t) e^{F(t)} ,
\end{aligned}$$

obtaining, with $\mathcal{H}(\tau,t) = \exp \int_\tau^t C_\varphi(s)\left(1 + C_A\,\mathcal{R} + C_A\,s\,\mathcal{C}(s)\right) \mathrm{d}s$,

$$
\begin{aligned}
\|p_1 - p_2\| \leq & \left(\exp\left(F(t) + C_A \int_0^t \tau C_\varphi(\tau)\mathcal{C}(\tau)\,\mathrm{d}\tau\right)\right) \|\bar p_1 - \bar p_2\| \\
& + \left(\int_0^t \mathcal{H}(\tau,t)\,\mathrm{d}\tau\right) \|\varphi_1 - \varphi_2\|_{\mathbf{L}^\infty([0,t]\times K_t \times [0,C_A]} \\
& + \left(\mathcal{R}\int_0^t C_\varphi(\tau)\mathcal{H}(\tau,t)\,\mathrm{d}\tau\right) \|A_1 - A_2\|_{\mathcal{L}(\mathbf{L^1},\mathbf{W^{1,\infty}})} \\
& + \left(C_A \int_0^t C_\varphi(\tau)\,\mathcal{H}(\tau,t)\,\mathrm{d}\tau\right) \|\bar\rho_1 - \bar\rho_2\|_{\mathbf{L^1}} \\
& + \left(C_A \int_0^t \tau C_\varphi(\tau)\mathcal{C}(\tau)\mathcal{H}(\tau,t)\,\mathrm{d}\tau\right) \\
& \quad \times \left[\left\|\partial_\rho(f_1 - f_2)\right\|_{\mathbf{L}^\infty([0,R]\times\mathbb{R}^d\times K_t)} + \left\|\mathrm{div}_x\,(f_1 - f_2)\right\|_{\mathbf{L^1}(\mathbb{R}^d)\times\mathbf{L}^\infty([0,R]\times K_t)}\right].
\end{aligned}
$$

Then, we immediately get the other bound

$$
\begin{aligned}
& \|\rho_1 - \rho_2\|_{\mathbf{L^1}} \\
\leq\ & \|\bar\rho_1 - \bar\rho_2\| \left(1 + t\mathcal{C}(t)\exp\left(F(t) + C_A \int_0^t \tau C_\varphi(\tau)\mathcal{C}(\tau)\,\mathrm{d}\tau\right)\right) \\
& + \left(t\mathcal{C}(t)\int_0^t \mathcal{H}(\tau,t)\,\mathrm{d}\tau\right) \|\varphi_1 - \varphi_2\|_{\mathbf{L}^\infty([0,t]\times K_t\times[0,C_A]} \\
& + \left(\mathcal{R}t\mathcal{C}(t)\int_0^t C_\varphi(\tau)\mathcal{H}(\tau,t)\,\mathrm{d}\tau\right) \|A_1 - A_2\|_{\mathcal{L}(\mathbf{L^1},\mathbf{W^{1,\infty}})} \\
& + C_A t\mathcal{C}(t)\exp\left(F(t) + C_A\left(1 + \mathrm{TV}(\bar\rho_1)\right)\int_0^t \tau C_\varphi(\tau)\mathcal{C}(\tau)\,\mathrm{d}\tau\right) \|\bar p_1 - \bar p_2\|_{\mathbf{L^1}} \\
& + t\mathcal{C}(t)\left(1 + C_A\int_0^t \tau C_\varphi(\tau)\mathcal{C}(\tau)\mathcal{H}(\tau,t)\,\mathrm{d}\tau\right) \\
& \quad \times \left(\left\|\partial_\rho(f_1 - f_2)\right\|_{\mathbf{L}^\infty([0,R]\times\mathbb{R}^d\times K_t)} + \left\|\mathrm{div}_x\,(f_1 - f_2)\right\|_{\mathbf{L^1}(\mathbb{R}^d)\times\mathbf{L}^\infty([0,R]\times K_t)}\right)
\end{aligned}
$$

completing the proof. □

Now, we want to prove corollary 2.3. A first step is the following consequence of Kružkov Theorem [9, Theorem 5].

Proposition 4.4 *Let $T > 0$. Consider the conservation law*

$$
\left\{
\begin{array}{l}
\partial_t \rho + \mathrm{div}_x\,\bar f(t,x,\rho) = 0 \\
\rho(t,0) = \bar\rho
\end{array}
\right.
\tag{4.12}
$$

with $\bar f \in \mathbf{C^0}([0,T]\times\mathbb{R}^d\times\mathbb{R};\mathbb{R}^d)$; $\partial_\rho \bar f$, $\partial_\rho\nabla_x \bar f$ and $\nabla_x^2 \bar f$ continuous wherever defined; $\partial_\rho \bar f$, $\mathrm{div}_x\,\bar f \in \mathbf{L}^\infty([0,T]\times\mathbb{R}^d\times[-H,H])$ for all $H > 0$. Assume that $\bar\rho \in (\mathbf{L^1}\cap\mathbf{L}^\infty)(\mathbb{R}^d;\mathbb{R})$

is such that $\bar\rho(x) = 0$ *for a.e.* $x \in \mathbb{R}^d \setminus B_d(0,\ell)$ *for a given* $\ell > 0$. *Moreover,* $\bar f(t,x,0) = 0$ *for all* $t \in [0,T]$ *and* $x \in \mathbb{R}^d$. *Call* ρ *the Kružkov solution to* (4.12) *and let* $K = \sup_{t\in[0,T]} \left\| \rho(t) \right\|_{\mathbf{L}^\infty(\mathbb{R}^d)}$. *Then, for all* $t \in [0,T]$, $\rho(t,x) = 0$ *for a.e.* $x \in \mathbb{R}^d \setminus B_d(0,\ell+Vt)$, *where* $V = \left\| \partial_\rho \bar f \right\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^d\times[-K,K])}$.

Above, $\bar f$ is assumed to satisfy the usual Kružkov conditions, see [11, **(H1)**], or [5, 9]. The proof essentially relies on [9, Theorem 1].

Proof of Proposition 4.4. Choose an $x \in \mathbb{R}^d \setminus B_d(0,\ell+Vt)$. Let $\delta > 0$ be such that $B_d(x,\delta) \cap B_d(0,\ell+Vt) = \emptyset$, so that $B_d(x,\delta+Vt) \cap B_d(0,\ell) = \emptyset$. Applying [9, Theorem 1], with $u = \rho$ and $v = 0$, we have that

$$\int_{B_d(x,\delta)} \left|\rho(t,x)\right| \mathrm{d}x \le \int_{B_d(x,\delta+Vt)} \left|\bar\rho(x)\right| \mathrm{d}x = 0$$

hence $\rho(t)$ vanishes a.e. outside $B_d(0,\ell+Vt)$. □

Proof of Corollary 2.3. Fix a positive $T \in I$. Let ℓ be such that $\bar\rho$ vanishes outside $B_d(0,\ell)$. Let $\chi \in \mathbf{C_c^\infty}(\mathbb{R},[0,1])$ be such that $\chi(x) = 1$ for all $x \in B_d(0,\ell+VT)$. Define the convolution in the space variable $f^* = f * \chi$, so that f^* has compact support in x. Then, thanks also to the *a priori* bound (4.7), f^* satisfies **(f)** on $[0,T]$. Hence to the problem

$$\begin{cases} \partial_t \rho + \mathrm{div}_x f^*\left(t,x,\rho,p(t)\right) = 0\,, \\ \dot p = \varphi\Big(t,p,A\left(\rho(t)\right)(p)\Big)\,, \\ \rho(0,x) = \bar\rho(x)\,, \\ p(0) = \bar p\,. \end{cases}$$

Theorem 2.2 can be applied, yielding on all $[0,T]$ the existence and uniqueness of a solution (ρ,p) in the sense of Definition 2.1. Let now $\bar f(t,x,\rho) = f^*\left(t,x,\rho,p(t)\right)$. Then, ρ is a Kružkov solution to (4.12) and by Proposition 4.4 its support is contained in $B_d(0,\ell+VT)$, for all $t \in [0,T]$. Therefore, on the same time interval, by the definition of f^*, (ρ,p) is the unique solution also to (1.1), always according to Definition 2.1. The rest of the proof follows. □

References

[1] R. Borsche, R. M. Colombo, and M. Garavello. On the coupling of systems of hyperbolic conservation laws with ordinary differential equations. *Nonlinearity*, To appear.

[2] A. Bressan and C. De Lellis. Existence of optimal strategies for a fire confinement problem. *Comm. Pure Appl. Math.*, 62(6):789–830, 2009.

[3] V. Capasso, A. Micheletti, and D. Morale. Stochastic geometric models, and related statistical issues in tumour-induced angiogenesis. *Math. Biosci.*, 214(1-2):20–31, 2008.

[4] R. M. Colombo, M. Herty, and M. Mercier. Control of the continuity equation with a non-local flow. *To appear on ESAIM Control Optim. Calc. Var.*, 2010.

[5] R. M. Colombo, M. Mercier, and M. D. Rosini. Stability and total variation estimates on general scalar balance laws. *Commun. Math. Sci.*, 7(1):37–65, 2009.

[6] A. F. Filippov. *Differential equations with discontinuous righthand sides.* Kluwer Academic Publishers Group, Dordrecht, 1988. Translated from the Russian.

[7] J. Grimm and W. Grimm. *Deutsche Sagen*. Nicolaische Verlagsbichhandlung, Berlin, second edition edition, 1865.

[8] W. D. Hoff, M. A. van der Horst, C. B. Nudel, and K. J. Hellingwerf. *Prokaryotic Phototaxis*, volume 571, chapter 2, pages 25–49. Springer, 2009.

[9] S. N. Kružkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81 (123):228–255, 1970.

[10] C. Lattanzio, A. Maurizi, and B. Piccoli. Moving bottlenecks in car traffic flow: a pde-ode coupled model. Preprint, 2010.

[11] M. Lécureux-Mercier. Improved stability estimates on general scalar balance laws. To appear on Journal of Hyperbolic Differential Equations, 2011.

[12] R. J. LeVeque. *Finite volume methods for hyperbolic problems*. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2002.

[13] D. Serre. Chute libre d'un solide dans un fluide visqueux incompressible. Existence. *Japan J. Appl. Math.*, 4(1):99–110, 1987.

[14] J. L. Vázquez and E. Zuazua. Large time behavior for a simplified 1D model of fluid-solid interaction. *Comm. Partial Differential Equations*, 28(9-10):1705–1738, 2003.

[15] G. B. Witman. Chlamydomonas phototaxis. *Trends in Cell Biology*, 3(11):403 – 408, 1993.