A priori estimates and analytical construction of shock waves in the gas dynamics

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Abstract

In this article we derive $\mathscr{C}^1$-a priori estimates on the Riemann invariants of the Euler compressible equations in the case of cylindrical or spherical symmetry. These estimates allow then to construct shock waves with a time of existence proportional to the distance to the origin at the initial time.
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1 Introduction

We are interested in the Euler compressible system in the isentropic case:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0\,, \\ \partial_t u + (u \cdot \nabla) u + \frac{p'(\rho)}{\rho} \nabla \rho = 0\,. \end{cases} \tag{1.1}$$

In the case of cylindrical ($d = 2$) or spherical ($d = 3$) symmetry, the system (1.1) can be written

$$\begin{cases} \partial_t \rho + \partial_r(\rho u) = \frac{-(d-1)\rho u}{r}\,, \\ \partial_t u + u \partial_r u + \frac{p'(\rho)}{\rho} \partial_r \rho = 0\,. \end{cases} \tag{1.2}$$

The case $d = 1$ corresponds to the one-dimensional case.

Very often the authors only consider the case of a perfect polytropic gas satisfying the sate law $pv = \Re T$. However, in numerous phenomenon such as cavitation or sonoluminescence [2, 4] or considering a dusty gas [7, 9, 19, 24, 25], it seems more adapted to consider at least a Van der Waals gas satisfying $p(v - b) = \Re T$. Our goal here is to construct shock wave solutions in the isentropic spherical or cylindrical case. Thanks to a priori estimates on the Riemann invariant, we obtain a lower bound on the time of existence of these solutions, proportional to the position at initial time of the discontinuity.

The Euler compressible equations have already been widely studied. Concerning regular solutions, general classical criteria on hyperbolic systems (Leray [12], Gårding [5], Kato [10]) provide us local in time existence of smooth solutions for the Cauchy problem. However, the time of existence can be very small: several results by T. C. Sideris [22, 23], T. Makino, S. Ukai & S. Kawashima [16], J.-Y. Chemin [3] provide explosion's criteria. We also know that some regular solutions can be global in time: besides the stationary solutions, under some expansivity hypothesis (see T. T. Li [13], D. Serre [21] or M. Grassin [6], M. Lécureux-Mercier [11]).

Concerning piecewise regular solutions, a result by A. Majda [15] states that we can associate a piecewise regular solution to a given piecewise initial data satisfying some compatibility conditions. But the time of existence of these solutions can be once again very small.

In this paper, we want to construct shock waves with a long time of existence for an isentropic Van der Waals gas in the particular case of spherical or cylindrical symmetry. In order to do that, we use a method that Li Ta Tsien [13] employed in order to construct 1D shock waves for an isentropic gas. This method is inspired from the scalar one-dimensional case $\partial_t u + \partial_x(f(u)) = 0$, in which it is possible to obtain a shock wave solution just glueing two regular solutions along a line of discontinuity satisfying the Rankine-Hugoniot shock condition

$$U = \frac{f(u^+) - f(u^-)}{u^+ - u^-},$$

where u^+ is the limit of u at the discontinuity from the right and u^- is the limit of u at the discontinuity from the left. This provides us an ODE that the line of discontinuity has to satisfy, being defined as $\frac{\mathrm{d}x}{\mathrm{d}t} = U$. Then, we have just to check that under suitable conditions the Lax entropy conditions meaning the characteristic curves are entering the shock (see Figure 1) are satisfied. We would like to apply this strategy to the isentropic spherical Euler

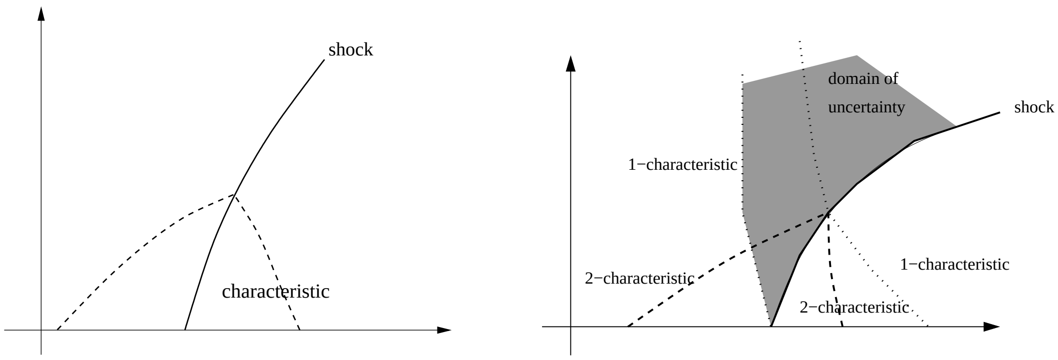

Figure 1: Shock curve and characteristics in the scalar case (left) and in the case of a system of two equations (right). the domain of uncertainty is in gray on the right picture.

equation. However, in the case of a system of two equations in one space dimension, the strategy above is no longer possible since now the Rankine-hugoniot conditions provide us two equations. Consequently, we have not only an equation for the behavior of the shock, but also a compatibility condition along the shock. Graphically, we see that, for a 2-shock, if the 2-characteristics are still entering the shock, the 1-characteristics are entering from the right and exiting from the left (see Figure 1), which brings us an uncertainty zone between the shock and the 1-characteristic exiting from the foot initial position of the shock to the left. In order to construct a shock wave solution from two regular solutions, we have to study the existence of a smooth solution in this angular domain between the shock and the 1-characteristic. Li Ta Tsien [13] already studied this subject in the one-dimensional case. In particular, he obtained local in time existence of a smooth solution for the angular problem.

The goal of this paper is to prove the following

Theorem 1.1. *Let $R_0 > 0$. Let us consider a Bethe-Weyl gas satisfying $1 < \mathscr{G} < 2$.*

Assume there exists two regular solutions of (1.2) (ρ^-, u^-) and (ρ^+, u^+) whose times of existence are respectively T^- and T^+, such that the compatibility condition (5.5) are satisfied at time $t = 0$, in $r = R_0$. Assume furthermore that the Riemann invariants of the system, denoted w_1 and w_2, satisfy

$$\min_{r \leqslant R_0} w_{1,0}^-(r) > 0\,, \qquad w_{1,0}^-(R_0) + \inf_{r \geqslant R_0} w_{1,0}^+(r) > 0\,, \qquad w_{1,0}^-(R_0) > \sup_{r \geqslant R_0} w_{2,0}^+\,.$$

and that there exists $C_0 > 0$ such that, for all (t, r):

$$\left|\partial_r w_2^-(t,r)\right| \leqslant \frac{C_0}{r}\,, \qquad \partial_r w_1^+(t,r) \geqslant \frac{-C_0}{r}\,, \qquad \partial_r w_2^+(t,r) \leqslant \frac{C_0}{r}\,.$$

Then there exists a shock wave solution of (1.2) with initial conditions

$$\begin{aligned} &\text{for } r \leqslant R_0\,, \qquad && u_0 = u_0^-\,, \qquad && \rho_0 = \rho_0^-\,, \\ &\text{for } r \geqslant R_0\,, \qquad && u_0 = u_0^+\,, \qquad && \rho_0 = \rho_0^+\,, \end{aligned}$$

and its time of existence is bounded below in the following way

$$T_{ex} \geqslant \min(T^-, T^+, C R_0)\,,$$

where C depends on the initial conditions.

The definition of the Riemann invariants w_1 and w_2 is given in (3.1) and the definition of a Bethe-Weyl fluid is given in definition 2.2. A condition on the initial conditions is proposed in Proposition 5.7 to validate the condition "$\rho^- > \rho^+$ along $\mathcal{K}$".

In Theorem 5.6 we construct a shock wave solution whose time of existence is bounded below by a quantity proportional to the initial radius of dicontinuity. To prove this theorem, we first have to estimate the time of existence of a smooth solutions in an angular domain whose boundaries are chosen in a way to be a 1-characteristic on the left and a shock on the right (see Figure 1) ; the time of existence is then obtained by deriving $\mathscr{C}^1$ estimates on the solution of this boundary problem.

As we consider the isentropic spherical or cylindrical case, we can diagonalize the system (3.2) and make estimates along the characteristics. However, these estimates are quite intricate due to the presence of source terms in the equations. Considering in the same way a diagonalized system on the derivatives $\partial_r \rho$ and $\partial_r u$, we also obtain estimates on the derivatives.

This paper is structured as followed: in Section 2, we describe the thermodynamical quantities and their properties. In Section 3, we compute a priori estimates in $\mathscr{C}^0$ along the characteristics. In Section 4, we compute a priori estimates in $\mathscr{C}^1$ along the characteristics. In Section 5, we introduce the shock conditions, the angular problem and we prove the main results of this article: in Proposition 5.5 we give an estimate of the time of existence of a smooth solution in an angular domain and in Theorem 5.6 we finally construct a shock wave. In Sections A and B, we finally give some details about a useful lemma on ODEs and about explicit computations in the cases of a perfect gas and of a Van der Waals gas.

2 Thermodynamics

2.1 Fundamental relationships

Definition 2.1. We consider a fluid, whose internal energy is a regular function of its specific volume[1] $v = 1/\rho$ and of its specific entropy s. We say that the gas is entitled with a *complete* state law, or energy law $e = e(v, s)$.

For a gas entitled with a complete state law, the fundamental thermodynamic principle is then

$$\mathrm{d}e = -p\mathrm{d}v + T\mathrm{d}s \tag{2.1}$$

where p is the pressure and T the temperature of the gas. Consequently, the pressure p and the temperature T can be defined as

$$p = -\left.\frac{\partial e}{\partial v}\right|_s, \qquad T = \left.\frac{\partial e}{\partial s}\right|_v, \tag{2.2}$$

where the notation $|$ precises the variable maintained constant in the partial derivation.

The higher order derivatives of e have also an important role; we introduce the following adimensional quantities:

$$\gamma = -\frac{v}{p}\left.\frac{\partial p}{\partial v}\right|_s, \qquad \Gamma = -\frac{v}{T}\left.\frac{\partial T}{\partial v}\right|_s, \qquad \delta = \frac{pv}{T^2}\left.\frac{\partial T}{\partial s}\right|_v, \qquad \mathscr{G} = -\frac{v}{2}\frac{\left.\frac{\partial^3 e}{\partial v^3}\right|_s}{\left.\frac{\partial^2 e}{\partial v^2}\right|_s}. \tag{2.3}$$

The coefficient γ is called the *adiabatic exponent*, and Γ is the *Grüneisen coefficient*. The quantities γ, δ, Γ and $\mathscr{G}$ characterise the geometrical properties of the isentropic curves in the (v, p) plane (see [17]). They can be expressed in function of e through the relationships:

$$\gamma = \frac{v}{p}\frac{\partial^2 e}{\partial v^2}, \qquad \Gamma = -\frac{v}{T}\frac{\partial^2 e}{\partial s \partial v}, \qquad \delta = \frac{pv}{T^2}\frac{\partial^2 e}{\partial s^2}.$$

We also introduce the *calorific capacity at constant volume* c_v and the *calorific capacity at constant pressure* c_p by

$$c_v = \left.\frac{\partial e}{\partial T}\right|_v = \frac{T}{\left.\frac{\partial^2 e}{\partial s^2}\right|_v}, \qquad c_p = T\left.\frac{\partial s}{\partial T}\right|_p. \tag{2.4}$$

These two quantities are linked with $\frac{pv}{T}$ and with γ, δ, Γ through

$$\delta c_v = \frac{pv}{T}, \qquad c_p = \frac{pv}{T}\frac{\gamma}{\gamma\delta - \Gamma^2}. \tag{2.5}$$

The quantity $\gamma_* = \frac{c_p}{c_v}$ can besides be expressed as $\gamma_* = \frac{\gamma\delta}{\gamma\delta - \Gamma^2}$. It is not equal to γ in the general case, but for an ideal gas we have $\delta = \Gamma = \gamma - 1$, and consequently $\gamma_* = \gamma$.

[1]specific is a synonym of massic

2.2 Thermodynamical constraints.

It is very natural to assume that the massic volume v is positive. We assume furthermore that the pressure p and the temperature T are positive, which imposes that e is a function increasing in T and decreasing in v.

A classical thermodynamical hypothesis requires furthermore e to be a convex function of s and v, which means:

$$\gamma\delta - \Gamma^2 \geqslant 0\,, \qquad \delta \geqslant 0\,, \qquad \gamma \geqslant 0\,.$$

Furthermore, we require usually $\Gamma > 0$ and $\mathscr{G} > 0$. The condition $\Gamma > 0$ is not thermodynamically required but is satisfied for many gases and ensures that the isentropes do not cross each other in the (v,p) plan. The condition $\mathscr{G} > 0$ means that the isentropes are strictly convex in the (v,p) plan.

Definition 2.2. We call *Bethe-Weyl fluid* any fluid endowed with a complete state law e bounded below such that

- the pressure and the temperature defined by (2.2) are positive,
- the coefficients γ, δ, Γ and $\mathscr{G}$ defined by (2.3) satisfy :

$$\gamma > 0\,, \qquad \gamma\delta \geqslant \Gamma^2\,, \qquad \Gamma > 0\,, \qquad \mathscr{G} > 0\,, \tag{2.6}$$

- there exists a maximal density $\rho_{max} \in]0,+\infty]$ such that $\lim\limits_{\rho \to \rho_{max}} p(\rho,s) = +\infty$.

The condition $\gamma \geqslant 0$ means that p increases with the density $\rho = 1/v$, which allows us to define the *adiabatic sound speed* by

$$c = \sqrt{\left.\frac{\partial p}{\partial \rho}\right|_s} = \sqrt{\gamma \frac{p}{\rho}}\,. \tag{2.7}$$

Then, we show that $\mathscr{G}$ can be expressed in function of ρ and c through the expression

$$\mathscr{G} = \frac{1}{c} \left.\frac{\partial(\rho c)}{\partial \rho}\right|_s\,.$$

2.3 Van der Waals Gas.

Definition 2.3. A gas is said to follow the *Van der Waals* law, if there exists a constant $\mathfrak{R}$ such that it satisfies the following pressure law:

$$p\,(v-b) = \mathfrak{R}T\,, \tag{2.8}$$

where v is the massic volume and b is the *covolume*, representing the compressibility limit of the fluid, due to the volume of the molecules. The constant $\mathfrak{R} = 8.314\,\mathrm{J.K^{-1}.mol^{-1}}$ is called the perfect gas constant.

In the case $b = 0$, we obtain the *perfect gas* law.

The state law (2.8) is a particular case of the state law $p = \frac{\mathfrak{R}T}{V-b} - \frac{a}{V^2}$, that authorises the change of phase when T goes under a threshold $T_c = \frac{8a}{27b\mathfrak{R}}$ (see [18]).

In order to obtain expressions for the quantities e, p, γ, Γ... in function of v and s, we use the fundamental relationship (2.1). This equation gives us the PDE: $\partial_v e + \frac{\mathfrak{R}}{v-b}\partial_s e = 0$. Thus, we introduce new variables $w = (v-b)^{-\mathfrak{R}}$, $\sigma = (v-b)^{-\mathfrak{R}}\exp(s)$ and $\hat{e}(w,\sigma) = e(v,s)$. We obtain $\partial_w \hat{e} = 0$, so that $e = \mathcal{E}((v-b)^{-\mathfrak{R}}\exp(s))$ for any regular function $\mathcal{E}$.

If we assume furthermore that c_v is constant, thanks to the definition of c_v and (2.1), we get that $\frac{\partial^2 e}{\partial s^2}\Big|_v = \frac{1}{c_v}\frac{\partial e}{\partial s}\Big|_v$, hence $\sigma\mathcal{E}'' = (\frac{1}{c_v} - 1)\mathcal{E}'$ and $\mathcal{E}(\sigma) = C\sigma^{1/c_v}$ which leads to:

$$e = (v-b)^{-\frac{\mathfrak{R}}{c_v}}\exp(\frac{s}{c_v}), \qquad p = \frac{\mathfrak{R}}{c_v}\frac{e}{v-b}.$$

After some computations we finally obtain

$$\gamma = \gamma_0\frac{v}{v-b}, \qquad \Gamma = \delta = (\gamma_0 - 1)\frac{v}{v-b}, \qquad \mathscr{G} = \frac{\gamma_0+1}{2}\frac{v}{v-b},$$

where

$$\gamma_0 := \frac{\mathfrak{R}}{c_v} + 1. \tag{2.9}$$

The conditions of Section 2.2 are then satisfied for $\gamma_0 > 1$.

In the following, we only consider Van der Waals fluids with constant and strictly positive calorific capacity c_v:

$$c_v > 0, \tag{2.10}$$

which implies $\gamma_0 > 1$ and $\mathscr{G} > 1$.

3 A priori $\mathscr{C}^0$ estimates along the characteristics

First we want to obtain $\mathscr{C}^0$ estimates on regular solutions of (1.2), ρ and u. To do that, we use the Riemann invariants of the system and we make computations along the characteristics.

3.1 Change of variables

Lemma 3.1. *Let us introduce the Riemann invariants*

$$w_1 = u - H(\rho), \qquad w_2 = u + H(\rho), \tag{3.1}$$

where $H(\rho)$ is a primitive of $\rho \mapsto \frac{c(\rho)}{\rho}$. Then, the system (1.2) reduces to

$$\begin{cases} \partial_t w_1 + \lambda_1(w)\partial_r w_1 = f(r,w), \\ \partial_t w_2 + \lambda_2(w)\partial_r w_2 = -f(r,w), \end{cases} \tag{3.2}$$

where $f(r,w) = \frac{(d-1)uc}{r}$, $\lambda_1 = u - c(\rho)$, $\lambda_2 = u + c(\rho)$.

Proof. Direct computation. □

Let us give below some properties of the new unknown H, w_1, w_2.

Lemma 3.2. *Let us consider a Bethe-Weyl gas. We assume that $1 < \mathscr{G} < 2$. Then we have $H \geqslant c$ and in particular $u - H \leqslant u - c$, that is to say $w_1 \leqslant \lambda_1$.*

In particular, $w_1 = u - H \geqslant 0$ implies $\lambda_1 = u - c \geqslant 0$.

Remark 3.3. In the case of a perfect polytropic gas, we have $\mathscr{G} = \frac{\gamma_0+1}{2}$ and the condition $1 < \mathscr{G} < 2$ is equivalent to $1 < \gamma_0 < 3$, which is a natural hypothesis on γ_0.

Proof. Note first that $c'(\rho) = H'(\rho)(\mathscr{G} - 1) \geqslant 0$. Then $H'(\rho) - c'(\rho) = H'(\rho)(2 - \mathscr{G}) = \frac{c(\rho)}{\rho}(2 - \mathscr{G}) \geqslant 0$. Hence, integrating on $[0, \rho]$, we obtain $H(\rho) \geqslant c(\rho)$. □

Lemma 3.4. *Let $\rho^+ > 0$. Let us define, for $\rho \geqslant \rho^+$,*

$$F(\rho, \rho^+) := \left(p - p^+\right)\left(\frac{1}{\rho^+} - \frac{1}{\rho}\right). \tag{3.3}$$

Then, for all $\rho \geqslant \rho^+$, we have $H(\rho) - H(\rho^+) \leqslant \sqrt{F(\rho, \rho^+)}$.

Proof. Let $\rho \geqslant \rho^+$. Let us derivate $\sqrt{F(\rho, \rho^+)} - H(\rho)$ with respect to ρ. We obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\rho}(\sqrt{F(\rho,\rho^+)} - H(\rho)) &= \frac{1}{2\sqrt{F(\rho,\rho^+)}}\left(c^2(\frac{1}{\rho^+} - \frac{1}{\rho}) + \frac{1}{\rho^2}(p - p^+)\right) - \frac{c}{\rho} \\
&= \frac{c}{2\rho\sqrt{F(\rho,\rho^+)}}\left(c\rho(\frac{1}{\rho^+} - \frac{1}{\rho}) + \frac{1}{c\rho}(p - p^+) - 2\sqrt{F(\rho,\rho^+)}\right) \\
&= \frac{c}{2\rho\sqrt{F(\rho,\rho^+)}}\left(\sqrt{c\rho(\frac{1}{\rho^+} - \frac{1}{\rho})} - \sqrt{\frac{1}{c\rho}(p - p^+)}\right)^2 \\
&\geqslant 0\,.
\end{aligned}$$

Noting that $F(\rho^+, \rho^+) = 0$ and integrating on $[\rho^+, \rho]$, we obtain $\sqrt{F(\rho, \rho^+)} - H(\rho) \geqslant -H(\rho^+)$, which is the expected result. □

Lemma 3.5. *Let $u^+, \rho^-, \rho^+ \in \mathbb{R}$. Let us assume $\rho^- \geqslant \rho^+ > 0$ and define $u^- := u^+ + \sqrt{F(\rho^-, \rho^+)}$, with F defined as in (3.3). Then $u^- - H(\rho^-) \geqslant u^+ - H(\rho^+)$, that is to say $w_1^- \geqslant w_1^+$.*

In particular, $w_1^+ = u^+ - H(\rho^+) \geqslant 0$ implies $w_1^- = u^- - H(\rho^-) \geqslant 0$.

Proof. This is a direct consequence of Lemma 3.4. Indeed, the inequality $\sqrt{F(\rho^-, \rho^+)} \geqslant H^- - H^+$ implies

$$u^- - H(\rho^-) = u^+ + \sqrt{F(\rho^-, \rho^+)} - H(\rho^-) \geqslant u^+ + H(\rho^-) - H(\rho^+) - H(\rho^-) = u^+ - H(\rho^+)\,.$$

□

3.2 $\mathscr{C}^0$ estimate on w_1 and w_2

Relying on computations along the characteristics, we now obtain estimates in $\mathbf{L}^\infty$ for w_1 and w_2, the Riemann invariants associated to a regular solution of (1.2).

Lemma 3.6. *Let us consider a Bethe-Weyl gas, satisfying $1 < \mathscr{G} < 2$. Let $w = (w_1, w_2)$ be a regular solution of (3.2). Let X_1 and X_2 be the characteristics defined by*

$$\begin{aligned} \frac{\mathrm{d}X_1}{\mathrm{d}t} &= \lambda_1(w(t, X_1(t)))\,, & \frac{\mathrm{d}X_2}{\mathrm{d}t} &= \lambda_2(w(t, X_1(t)))\,, \\ X_1(0) &= r_1\,, & X_2(0) &= r_2\,. \end{aligned} \tag{3.4}$$

Let us assume that for all $r > 0$, $(w_2 - w_1)(0, r) \geqslant 0$ and $w_1(0, r) > 0$. Then, we get: $X_1' > 0$, $X_2' > 0$ and

$$w_1(0, X_1(0; t, r)) \leqslant w_1(t, r) \leqslant w_2(t, r) \leqslant w_2(0, X_2(0; t, r))\,. \tag{3.5}$$

Furthermore, for all $t \geqslant 0$ we also have the estimates:

$$w_1(0, r_1) \leqslant w_1(t, X_1(t)) \leqslant w_1(0, r_1) + \left\|w_2(0, \cdot)\right\|^2_{\mathbf{L}^\infty} \int_0^t \frac{d-1}{4X_1(\tau)}\,\mathrm{d}\tau\ , \tag{3.6}$$

$$\frac{w_2(0,r_2)}{1+w_2(0,r_2)\int_0^t \frac{(d-1)}{4X_2(\tau)}\mathrm{d}\tau} \leqslant w_2(t, X_2(t)) \leqslant w_2(0, r_2)\,. \tag{3.7}$$

Proof. First, let us introduce χ be the solution of the ODE:

$$\frac{\mathrm{d}\chi}{\mathrm{d}t} = u(t, \chi)\,, \qquad \chi(t_0) = r_0\,.$$

Note that the solution of this ODE exists at least in finite time. Some trouble could appear only if χ meets the line of equation $r = 0$. In the same way, the characteristics X_1 and X_2 satisfying (3.4) are defined at least in finite time. Let $r_0, r_1, r_2 > 0$ be fixed. Let us denote $T > 0$ a time such that χ, X_1, X_2 are defined on $[0, T]$.

Positivity of the density: Let us prove first that ρ remains positive because of the first equation of (1.2). By the definition of χ, for any $t \in [0, T]$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\chi^{d-1}\rho(t, \chi)\exp\Big(\int_{t_0}^t \partial_r u\,\mathrm{d}s\Big)\right) = 0\,.$$

Consequently, the densitity remains positive and in particular, the inequality $w_2 \geqslant w_1$ is satisfied for all time $t \in [0, T]$ since it is true at time $t = 0$.

w_1 is increasing along the characteristic: At least locally in time, since $w_1(0, r_1) > 0$ we have $w_1(t, X_1(t)) \geqslant 0$. We want to prove that w_1 remains positive along the characteristic. Assume there is a time at which $w_1(t, X_1(t)) = 0$. Let us denote $t_0 \leqslant T$ the first time at which $w_1(t_0, X_1(t_0)) = 0$. Thanks to the previous result on the positivity of the density, we obtain $w_2(t, X_1(t)) \geqslant w_1(t, X_1(t)) \geqslant 0$ on $[0, t_0]$. Then on $[0, t_0]$ we have $u = \frac{w_2+w_1}{2} \geqslant 0$ and $f(r, w) = \frac{(d-1)uc}{r} \geqslant 0$ which implies

$$\frac{\mathrm{d}w_1\,(t, X_1(t))}{\mathrm{d}t} \geqslant 0\,.$$

Integrating, we get $w_1(t_0, X_1(t_0)) \geqslant w_1(0, r_1) > 0$, which is in contradiction with the hypothesis. Finally, w_1 is strictly positive for all $t \in [0, T]$ and increasing along the first characteristic.

Upper bound on w_2: Along the second characteristic we get, since $u \geqslant w_1 \geqslant 0$,

$$\frac{\mathrm{d}w_2(t, X_2(t))}{\mathrm{d}t} = -f(X_2, w(t, X_2(t))) \leqslant 0\,.$$

Integrating we obtain $w_2(t, X_2(t)) \leqslant w_2(0, r_2)$, which provides us with an upper bound.

Lower bound on w_2: Thanks to Lemma 3.2 we know that $c \leqslant H$. Hence, we obtain

$$\frac{\mathrm{d}w_2(t, X_2(t))}{\mathrm{d}t} = \frac{-(d-1)}{X_2(t)} uc \geqslant \frac{-(d-1)}{X_2(t)} uH = \frac{-(d-1)}{4X_2(t)}(w_2^2 - w_1^2) \geqslant \frac{-(d-1)}{4X_2(t)} w_2^2\,.$$

Consequently, we have $-\frac{1}{w_2^2(t,X_2(t))}\frac{\mathrm{d}w_2(t,X_2(t))}{\mathrm{d}t} \leqslant \frac{d-1}{4X_2(t)}$ and integrating we finally obtain

$$\frac{1}{w_2(t, X_2(t))} \leqslant \frac{1}{w_2(0, r_2)} + \int_0^t \frac{d-1}{4X_2(\tau)}\,\mathrm{d}\tau\ .$$

Since $w_2 \geqslant 0$, we can invert this relation, and obtain the desired lower bound on w_2.

Upper bound on w_1: Similarly for w_1 we get

$$\frac{\mathrm{d}w_1(t, X_1(t))}{\mathrm{d}t} = \frac{(d-1)}{X_1(t)} uc \leqslant \frac{d-1}{X_1(t)} uH = \frac{(d-1)}{4X_1(t)}(w_2^2 - w_1^2) \leqslant \frac{(d-1)}{4X_1(t)} \|w_2\|^2_{\mathbf{L}^\infty}\,.$$

Hence, as announced,

$$w_1(t, X_1(t)) \leqslant w_1(0, r_1) + \int_0^t \frac{(d-1)}{X_1(\tau)} \|w_2\|^2_{\mathbf{L}^\infty}\,\mathrm{d}\tau\ .$$

Time of existence: Note now that $\frac{\mathrm{d}\chi}{\mathrm{d}t} = u(t, \chi(t)) \geqslant 0$ implies that this curve never meets the origin and is defined on $\mathbb{R}^+$. Similarly, $\frac{\mathrm{d}X_2}{\mathrm{d}t} = (u + c)(t, X_2(t)) \geqslant 0$ implies the 2-characteristics are going away from the origin and are defined for all $t \in \mathbb{R}^+$.

Concerning the first characteristic, Lemma 3.2 gives us that $\frac{\mathrm{d}X_1}{\mathrm{d}t} = (u - c)(t, X_1(t)) \geqslant (u - H)(t, X_1(t)) = w_1(t, X_1(t)) \geqslant 0$. Consequently, the 1-characteristics are going away from the origin and are defined for all $t \in \mathbb{R}^+$. □

Let us modily slightly the hypotheses in order to obtain a better result, but now the time of validity is finite:

Proposition 3.7. *Let us consider a Bethe-Weyl gas, satisfying $1 < \mathscr{G} < 2$. Let $w = (w_1, w_2)$ be a regular solution of (3.2). Let X_1 and X_2 be the characteristics defined by (3.4), crossing in (T, R). Let us assume that for all $r > 0$, $(w_2 - w_1)(0, r) \geqslant 0$ and $\min(w_{1,0}) + \min(w_{2,0}) > 0$. Let T_0 be defined by $T_0 = +\infty$ if $\min(w_{1,0}) \geqslant 0$ and*

$$T_0 = \frac{-4r_2}{(d-1)\min_{r_1}(w_{1,0}(r_1))\min_{r_2}(w_{2,0}(r_2))}\left(\min_{r_1}(w_{1,0}(r_1)) + \min_{r_2}(w_{2,0}(r_2))\right),$$

if $\min(w_{1,0}) < 0$. *Then, if* $T \leqslant T_0$,

$$w_1(0,r_1) \leqslant w_1(T,R) \leqslant w_2(T,R) \leqslant w_2(0,r_2)\,. \tag{3.8}$$

Furthermore, for all $t \in [0,T_0]$,

$$w_1(0,r_1) \leqslant w_1(t,X_1(t)) \leqslant w_1(0,r_1) + \left\|w_2(0,\cdot)\right\|^2_{\mathbf{L}^\infty} \int_0^t \tfrac{d-1}{4X_1(\tau)}\,\mathrm{d}\tau\ , \tag{3.9}$$

$$\frac{w_2(0,r_2)}{1+\frac{w_2(0,r_2)}{r_2}\frac{(d-1)t}{4}} \leqslant w_2(t,X_2(t)) \leqslant w_2(0,r_2)\,. \tag{3.10}$$

Proof. As in the preceding proposition, we know that if the density if positive at the initial time, then it is positive for any positive time. Hence, $w_2(t,r) \geqslant w_1(t,r)$ for all $(t,r) \in \mathbb{R}^2_+$.

By hypothesis $u(0,r) \geqslant \frac{1}{2}(\min(w_{1,0}) + \min(w2,0)) > 0$. Consequently, u is positive at least on a small time interval $[0,t_0]$. On this time interval, we obtain $\frac{\mathrm{d}w_1(t,x_1(t))}{\mathrm{d}t} \geqslant 0$ and $\frac{\mathrm{d}w_2(t,x_2(t))}{\mathrm{d}t} \leqslant 0$. Hence for $t \in [0,t_0]$, $w(t,X_1(t)) \geqslant w_1(0,r_1)$ and $w_2(t,X_2(t)) \leqslant w_2(0,r_2)$.

Furthermore, as $u \geqslant 0$, $\frac{\mathrm{d}w_2(t,X_2(t))}{\mathrm{d}t} \geqslant \frac{-(d-1)}{4X_2(t)}w_2^2$ and $\frac{\mathrm{d}w_1(t,X_1(t))}{\mathrm{d}t} \leqslant \frac{(d-1)}{4X_1(t)}w_2^2$. Hence, the same estimates as before hold true:

$$\frac{1}{w_2(t,X_2(t))} \leqslant \frac{1}{w_2(0,r_2)} + \int_0^t \frac{(d-1)}{4X_2(\tau)}\,\mathrm{d}\tau$$

Since, for $t \in [0,t_0]$, $\lambda_2 = u + H \geqslant 0$, X_2 is increasing and as $w_2 \geqslant u \geqslant 0$, we can invert the relation, obtaining

$$w_2(t,X_2(t)) \geqslant \frac{w_2(0,r_2)}{1 + w_2(0,r_2)\frac{(d-1)t}{4r_2}}\,.$$

Let us use these estimate to find a lower bound for u. If $T \leqslant t_0$ then

$$u(T,R) = \frac{1}{2}(w_1+w_2)(T,R) \geqslant \frac{1}{2}\left(w_1(0,r_1) + \frac{w_2(0,r_2)}{1 + w_2(0,r_2)\frac{(d-1)T}{4r_2}}\right)\,.$$

If $w_1(0,r_1) \geqslant 0$, then u is positive for all time and we recover the result of Lemma 3.6. If $w_1(0,r_1) < 0$ and $w_1(0,r_1) + w_2(0,r_2) > 0$, then u is positive if

$$T \leqslant \frac{-4r_2}{(d-1)w_1(0,r_1)w_2(0,r_2)}(w_1(0,r_1) + w_2(0,r_2))\,,$$

which provides us a lower bound for t_0. □

Remark 3.8. In Lemma 3.6 and Proposition 3.7, we are integrating along the characterists on the time interval $[0,t]$. Note that we could obtain similar result integrating on any time interval $[\beta,t]$ with $\beta \in [0,t]$.

4 A priori $\mathscr{C}^1$ estimates along the characteristics

We want now to obtain estimates in $\mathbf{L}^\infty$ on $\partial_r w_1$ and $\partial_r w_2$ where $w = (w_1,w_2)$ is a regular solution of (3.2). We apply the same strategy as in the previous section. That is to say, we want to have a diagonal form for the system of equation on $\partial_r w_1$ and $\partial_r w_2$, obtained by derivating with respect to r the system (3.2), and then make computations along the characteristics. As the obtained system in $\partial_r w_1$ and $\partial_r w_2$ is not diagonal, we have to introduce new variables v_1 and v_2 as described below.

4.1 Change of variable

Lemma 4.1. *Let us define*

$$v_1 = e^h(\partial_r w_1 + \Phi)\,, \qquad\qquad v_2 = e^k(\partial_r w_2 + \Psi)\,, \tag{4.1}$$

where h, k, Φ and Ψ are such that:

$$\partial_2 h = \frac{\partial_2\lambda_1}{\lambda_1-\lambda_2}\,, \qquad\qquad \partial_1 k = \frac{\partial_1\lambda_2}{\lambda_2-\lambda_1}\,, \tag{4.2}$$

$$\partial_2(e^h\Phi) = \frac{-e^h\partial_2 f}{\lambda_1-\lambda_2}\,, \qquad\qquad \partial_1(e^k\Psi) = \frac{e^k\partial_1 f}{\lambda_2-\lambda_1}\,. \tag{4.3}$$

Then v_1 and v_2 satisfy the equations

$$\left\{\begin{array}{rcl} \partial_t v_1 + \lambda_1(w)\partial_r v_1 & = & a_0(r,w)v_1^2 + a_1(r,w)v_1 + a_2(r,w)\,, \\ \partial_t v_2 + \lambda_2(w)\partial_r v_2 & = & b_0(r,w)v_2^2 + b_1(r,w)v_2 + b_2(r,w)\,. \end{array}\right. \tag{4.4}$$

where

$$\begin{array}{rcl} a_0(r,w) & = & -e^{-h}\partial_1\lambda_1\,, \\ a_1(r,w) & = & \partial_1 f + 2\Phi\partial_1\lambda_1 + (\partial_1 h - \partial_2 h)f\,, \\ a_2(r,w) & = & e^h\left(\partial_r f - \Phi\partial_1 f - \Phi^2\partial_1\lambda_1 + (\partial_1\Phi - \partial_2\Phi)f + \lambda_1\partial_r\Phi\right)\,, \\ b_0(r,w) & = & -e^{-k}\partial_2\lambda_2\,, \\ b_1(r,w) & = & -\partial_2 f + 2\partial_2\lambda_2\Psi + (\partial_1 k - \partial_2 k)f\,, \\ b_2(r,w) & = & e^k\left(-\partial_r f + \partial_2 f\Psi - \partial_2\lambda_2\Psi^2 + \lambda_2\partial_r\Psi + (\partial_1\Psi - \partial_2\Psi)f\right)\,. \end{array}$$

Remark 4.2. Note that, for any gas law, we have $\partial_1\lambda_1 = \partial_2\lambda_2 = \frac{\mathscr{G}}{2} \geqslant 0$. Hence, we have $a_0 \leqslant 0$ and $b_0 \leqslant 0$.

Proof. On the first hand, derivating v_1 with respect to t and r, we get

$$\begin{aligned} \partial_t v_1 =& e^h\left(\partial_t\partial_r w_1 + \partial_1\Phi\partial_t w_1 + \partial_2\Phi\partial_t w_2\right) + e^h(\partial_1 h\partial_t w_1 + \partial_2 h\partial_t w_2)(\partial_r w_1 + \Phi)\,, \\ \partial_r v_1 =& e^h(\partial_r^2 w_1 + \partial_1\Phi\partial_r w_1 + \partial_2\Phi\partial_r w_2 + \partial_r\Phi) + e^h(\partial_1 h\partial_r w_1 + \partial_2 h\partial_r w_2)(\partial_r w_1 + \Phi)\,. \end{aligned}$$

Hence

$$\begin{aligned} & e^{-h}(\partial_t v_1 + \lambda_1\partial_r v_1) \\ & = \partial_t\partial_r w_1 + \lambda_1\partial_r^2 w_1 + \lambda_1\partial_r\Phi + \partial_1\Phi(\partial_t w_1 + \lambda_1\partial_r w_1) + \partial_2\Phi(\partial_t w_2 + \lambda_1\partial_r w_2) \\ & \quad + (\partial_r w_1 + \Phi)(\partial_1 h(\partial_t w_1 + \lambda_1\partial_r w_1) + \partial_2 h(\partial_t w_2 + \lambda_1\partial_r w_2))\,. \end{aligned}$$

Note that $\partial_t w_1 + \lambda_1\partial_r w_1 = f$, and $\partial_t w_2 + \lambda_1\partial_r w_2 = -f + (\lambda_1 - \lambda_2)\partial_r w_2$ so that

$$\begin{aligned} & e^{-h}(\partial_t v_1 + \lambda_1\partial_r v_1) \\ & = \partial_t\partial_r w_1 + \lambda_1\partial_r^2 w_1 + \lambda_1\partial_r\Phi + f\partial_1\Phi + \partial_2\Phi(-f + (\lambda_1 - \lambda_2)\partial_r w_2) \\ & \quad + (\partial_r w_1 + \Phi)(f\partial_1 h + \partial_2 h(-f + (\lambda_1 - \lambda_2)\partial_r w_2))\,. \end{aligned} \tag{4.5}$$

On the other hand, derivating with respect to time the equation in w_1, we obtain:

$$\begin{aligned}
&\partial_t\partial_r w_1 + \lambda_1\partial_r^2 w_1 \\
&= \partial_r f + \partial_1 f \partial_r w_1 + \partial_2 f \partial_r w_2 - \partial_1\lambda_1(\partial_r w_1)^2 - \partial_2\lambda_1\partial_r w_2 \partial_r w_1 \\
&= \partial_r f + \partial_1 f(e^{-h}v_1 - \Phi) + \partial_2 f\partial_r w_2 - \partial_1\lambda_1(e^{-h}v_1 - \Phi)^2 - \partial_2\lambda_1\partial_r w_2(e^{-h}v_1 - \Phi) \\
&= -e^{-2h}v_1^2\partial_1\lambda_1 + e^{-h}v_1(\partial_1 f + 2\Phi\partial_1\lambda_1) + (\partial_r f - \Phi\partial_1 f - \Phi^2\partial_1\lambda_1) \\
&\quad + \partial_r w_2(\partial_2 f + \Phi\partial_2\lambda_1) - \partial_2\lambda_1 e^{-h}v_1\partial_r w_2\,.
\end{aligned}$$

Replacing $\partial_t\partial r w_1 + \lambda_1\partial_1^2 w_1$ by its expression in (4.5), we get

$$\begin{aligned}
&e^{-h}(\partial_t v_1 + \lambda_1\partial_r v_1) \\
&= -e^{-2h}v_1^2\partial_1\lambda_1 + e^{-h}v_1(\partial_1 f + 2\Phi\partial_1\lambda_1) + (\partial_r f - \Phi\partial_1 f - \Phi^2\partial_1\lambda_1) \\
&\quad + \partial_r w_2(\partial_2 f + \Phi\partial_2\lambda_1) - \partial_2\lambda_1 e^{-h}v_1\partial_r w_2 + \lambda_1\partial_r\Phi + f(\partial_1\Phi - \partial_2\Phi) + \partial_2\Phi(\lambda_1 - \lambda_2)\partial_r w_2 \\
&\quad + e^{-h}v_1 f(\partial_1 h - \partial_2 h) + \partial_2 h(\lambda_1 - \lambda_2)e^{-h}v_1\partial_r w_2 \\
&= -e^{-2h}v_1^2\partial_1\lambda_1 + e^{-h}v_1(\partial_1 f + 2\Phi\partial_1\lambda_1 + f(\partial_1 h - \partial_2 h)) \\
&\quad + (\partial_r f - \Phi\partial_1 f - \Phi^2\partial_1\lambda_1 + f(\partial_1\Phi - \partial_2\Phi) + \lambda_1\partial_r\Phi) \\
&\quad + \partial_r w_2(\partial_2 f + \Phi\partial_2\lambda_1 + \partial_2\Phi(\lambda_1 - \lambda_2)) + (\partial_2 h(\lambda_1 - \lambda_2) - \partial_2\lambda_1)e^{-h}v_1\partial_r w_2\,.
\end{aligned}$$

With our choice for h and Φ, the last line above vanishes.

In the same way, we have for v_2

$$\begin{aligned}
\partial_t v_2 =& e^k(\partial_t\partial_r w_2 + \partial_1\Psi\partial_t w_1 + \partial_2\Psi\partial_t w_2) + e^k(\partial_1 k\partial_t w_1 + \partial_2 k\partial_t w_2)(\partial_r w_2 + \Psi)\,, \\
\partial_r v_2 =& e^k(\partial_r^2 w_2 + \partial_1\Psi\partial_r w_1 + \partial_2\Psi\partial_r w_2 + \partial_r\Psi) + e^k(\partial_1 k\partial_r w_1 + \partial_2 k\partial_r w_2)(\partial_r w_2 + \Psi)\,.
\end{aligned}$$

Hence

$$\begin{aligned}
&e^{-k}(\partial_t v_2 + \lambda_2\partial_r v_2) \\
&= \partial_t\partial_r w_2 + \lambda_2\partial_r^2 w_2 + \lambda_2\partial_r\Psi + \partial_1\Psi(\partial_t w_1 + \lambda_2\partial_r w_1) + \partial_2\Psi(\partial_t w_2 + \lambda_2\partial_r w_2) \\
&\quad + (\partial_r w_2 + \Psi)(\partial_1 k(\partial_t w_1 + \lambda_2\partial_r w_1) + \partial_2 k(\partial_t w_2 + \lambda_2\partial_r w_2))\,.
\end{aligned}$$

Note that $\partial_t w_1 + \lambda_2\partial_r w_1 = f + (\lambda_2 - \lambda_1)\partial_r w_1$, and $\partial_t w_2 + \lambda_2\partial_r w_2 = -f$ so that

$$\begin{aligned}
&e^{-k}(\partial_t v_2 + \lambda_2\partial_r v_2) \\
&= \partial_t\partial_r w_2 + \lambda_2\partial_r^2 w_2 + \lambda_2\partial_r\Psi + (f + (\lambda_2 - \lambda_1)\partial_r w_1)\partial_1\Psi - f\partial_2\Psi \\
&\quad + (\partial_r w_2 + \Psi)((f + (\lambda_2 - \lambda_1)\partial_r w_1)\partial_1 k - f\partial_2 k)\,.
\end{aligned}$$

Derivating with respect to time the equation in w_2, we obtain:

$$\begin{aligned}
&\partial_t\partial_r w_2 + \lambda_2\partial_r^2 w_2 \\
&= -\partial_r f - \partial_1 f\partial_r w_1 - \partial_2 f\partial_r w_2 - \partial_1\lambda_2\partial_r w_1\partial_r w_2 - \partial_2\lambda_2(\partial_r w_2)^2 \\
&= -\partial_r f - \partial_1 f\partial_r w_1 - \partial_2 f(e^{-k}v_2 - \Psi) - \partial_1\lambda_2(e^{-k}v_2 - \Psi)\partial_r w_1 - \partial_2\lambda_2(e^{-k}v_2 - \Psi)^2 \\
&= -e^{-2k}v_2^2\partial_2\lambda_2 + e^{-k}v_2(-\partial_2 f + 2\Psi\partial_2\lambda_2) + (-\partial_r f + \Psi\partial_2 f - \Psi^2\partial_2\lambda_2) \\
&\quad + \partial_r w_1(-\partial_1 f + \Psi\partial_1\lambda_2) - \partial_1\lambda_2 e^{-k}v_2\partial_r w_1\,.
\end{aligned}$$

Replacing, we get

$$
\begin{aligned}
&e^{-k}(\partial_t v_2 + \lambda_2 \partial_r v_2) \\
&= -e^{-2k} v_2^2 \partial_2 \lambda_2 + e^{-k} v_2(-\partial_2 f + 2\Psi \partial_2 \lambda_2) + (-\partial_r f + \Psi \partial_2 f - \Psi^2 \partial_2 \lambda_2) \\
&\quad + \partial_r w_1(-\partial_1 f + \Psi \partial_1 \lambda_2) - \partial_1 \lambda_2 e^{-k} v_2 \partial_r w_1 + \lambda_2 \partial_r \Psi + (f + (\lambda_2 - \lambda_1)\partial_r w_1)\partial_1 \Psi - f \partial_2 \Psi \\
&\quad + e^{-k} v_2((f + (\lambda_2 - \lambda_1)\partial_r w_1)\partial_1 k - f \partial_2 k) \\
&= -e^{-2k} v_2^2 \partial_2 \lambda_2 + e^{-k} v_2(-\partial_2 f + 2\Psi \partial_2 \lambda_2 + f(\partial_1 k - \partial_2 k)) \\
&\quad + (-\partial_r f + \Psi \partial_2 f - \Psi^2 \partial_2 \lambda_2 + \lambda_2 \partial_r \Psi + f(\partial_1 \Psi - \partial_2 \Psi)) \\
&\quad + \partial_r w_1(-\partial_1 f + \Psi \partial_1 \lambda_2 + (\lambda_2 - \lambda_1)\partial_1 \Psi) + e^{-k} v_2 \partial_r w_1((\lambda_2 - \lambda_1)\partial_1 k - \partial_1 \lambda_2)\,.
\end{aligned}
$$

With our choice for k and Ψ, the last line above vanishes. □

Similarly as above, we want to derive a diagonal system on rv_1 and rv_2.

Corollary 4.3. *With the notation of Lemma 4.1, we have for all $\ell \in \mathbb{N}$,*

$$
\begin{cases}
\partial_t(r^\ell v_1) + \lambda_1(w)\partial_r(r^\ell v_1) = \frac{a_0(r,w)}{r^\ell}(r^\ell v_1)^2 + (a_1(r,w) + \frac{\ell\lambda_1}{r})(r^\ell v_1) + r^\ell a_2(r,w)\,, \\
\partial_t(r^\ell v_2) + \lambda_2(w)\partial_r(r^\ell v_2) = \frac{b_0(r,w)}{r^\ell}(r^\ell v_2)^2 + (b_1(r,w) + \frac{\ell\lambda_2}{r})(r^\ell v_2) + r^\ell b_2(r,w)\,.
\end{cases}
\tag{4.6}
$$

In the following we give some properties of the coefficients $a_0, a_1, a_2, b_0, b_1, b_2$:

Lemma 4.4. *There exist smooth functions $\bar{a}_i : \mathbb{R}^2 \to \mathbb{R}$ and $\bar{b}_i : \mathbb{R}^2 \to \mathbb{R}$ such that, for $i \in \{1,2\}$ the coefficients a_i and b_i defined as in Lemma 4.1 can be written $a_i(r,w) = \frac{\bar{a}_i(w)}{r^i}$ and $b_i(r,w) = \frac{\bar{b}_i(w)}{r^i}$. In particular, $\bar{a}_i$ and $\bar{b}_i$ are not depending on r.*

Similarly, there exist smooth functions $\varphi : \mathbb{R}^2 \to \mathbb{R}$ and $\psi : \mathbb{R}^2 \to \mathbb{R}$ such that the coefficients Φ and Ψ can be written $\Phi(r,w) = \frac{\varphi(w)}{r}$ and $\Psi(r,w) = \frac{\psi(w)}{r^i}$. In particular φ and ψ are not depending on r.

Proof. We have $\lambda_1 = u - c$, and $u = \frac{w_1+w_2}{2}$, $H = \frac{w_2-w_1}{2}$. Hence $\partial_2\lambda_1 = \frac{1}{2} - \frac{c'}{2H'}$ where $\partial_2\lambda_1$ stands for $\frac{\partial\lambda_1(w_1,w_2)}{\partial w_2}$, and

$$
\partial_2 h = \frac{H''}{4(H')^2}\,.
$$

Finally $h = \frac{1}{2}\ln(H')$ satisfies the equation. In the same way, we obtain $k = h = \frac{1}{2}\ln(H')$.

Using $H'(\rho) = \frac{c(\rho)}{\rho}$ and $\mathscr{G} = \frac{1}{c(\rho)}(c(\rho)\rho)'$, we note that $c' = H'(\mathscr{G} - 1)$ and that $H'' = \frac{H'}{\rho}(\mathscr{G} - 2)$. Then

$$
\partial_2(e^h\Phi) = \frac{\sqrt{H'}}{2c}\frac{d-1}{r}\left(\frac{c}{2} + u\frac{c'}{2H'}\right) = \frac{d-1}{2r}\left(\frac{1}{2}\sqrt{H'} + u\frac{c'}{2c\sqrt{H'}}\right)\,.
$$

Let us define g, A and B such that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\rho}(2\sqrt{H'(\rho)}g(\rho)) = \frac{1}{\rho}\sqrt{H'(\rho)}\,, \qquad A = 1 + g\,, \\
&\frac{\mathrm{d}}{\mathrm{d}\rho}(2\sqrt{H'(\rho)}B(\rho)) = H'(\rho)\sqrt{H'(\rho)}(1 + 2g(\rho))\,.
\end{aligned}
$$

Then, noting that, for any function f of u and ρ we have

$$\partial_2 f = \partial_2 u\, \partial_u f + \partial_2 \rho\, \partial_\rho f = \frac{1}{2}\partial_u f + \frac{1}{2H'(\rho)}\partial_\rho f\,.$$

Hence, we can check that $\partial_2(e^h\Phi) = \frac{d-1}{2r}\partial_2\left(2\sqrt{H'}uA - 2\sqrt{H'}B\right)$ and we can choose

$$\Phi = \frac{d-1}{r}\left(uA - B\right)\,.$$

In the same way, we have

$$\Psi = \frac{d-1}{r}\left(uA + B\right)\,.$$

Note that

$$\partial_\rho A = \frac{1}{2\rho}\left(1 - g(\mathscr{G} - 2)\right)\,, \qquad \partial_\rho B = \frac{1}{2\rho}\left(c(1+2g) - B(\mathscr{G} - 2)\right)\,.$$

Let us now compute the expression of a_0, a_1, a_2:

$$\begin{aligned} a_0 &= -e^{-h}\partial_1\lambda_1 = -\frac{1}{\sqrt{H'}}\left(\frac{1}{2} + \frac{c'}{2H'}\right) = \frac{-1}{\sqrt{H'}}\frac{\mathscr{G}}{2} \leqslant 0\,,\\ a_1 &= \frac{d-1}{2r}(c - u(\mathscr{G}-1)) + \mathscr{G}\frac{d-1}{r}\left(uA - B\right) - \frac{d-1}{2r}u(\mathscr{G}-2)\\ &= \frac{d-1}{r}\left[\frac{c}{2} - B\mathscr{G} + \frac{u}{2}\left(3 + 2\mathscr{G}g\right)\right]\,, \end{aligned}$$

and

$$\begin{aligned} a_2 = &\frac{(d-1)\sqrt{H'}}{r^2}\Big[- uc - \frac{d-1}{2}\Big((uA-B)(c-u(\mathscr{G}-1)) + \mathscr{G}(uA-B)^2\Big)\\ &- \frac{d-1}{2}u(u(1-(\mathscr{G}-2)(A-1)) + (B(\mathscr{G}-2) - c(2A-1)))\\ &- (u-c)(uA-B)\Big]\\ = &\frac{(d-1)\sqrt{H'}}{r^2}\Big[- uc - (u^2A - u(B+cA) + cB)\\ &\quad - \frac{d-1}{2}\Big(u(Ac + B(\mathscr{G}-1)) - Bc - u^2A(\mathscr{G}-1) + \mathscr{G}(u^2A^2 - 2uAB + B^2)\\ &\quad + u^2(1-(\mathscr{G}-2)(A-1)) + u(B(\mathscr{G}-2) - c(2A-1))\Big)\Big] \end{aligned}$$

Let us now consider the expression of b_0, b_1, b_2 as given in (4.8).

$$\begin{aligned} b_0 &= a_0 = \frac{-\mathscr{G}}{2\sqrt{H'}}\,,\\ b_1 &= \frac{d-1}{2r}\left(u(2\mathscr{G}(A-1)+3) + 2\mathscr{G}B - c\right)\,,\\ b_2 &= \frac{(d-1)\sqrt{H'}}{r^2}\Big[- u^2A - u(c(A-1)+B) - cB + \frac{d-1}{2}\Big(u^2(-\mathscr{G}A^2 + (2\mathscr{G}-3)A + 1 - \mathscr{G})\\ &\qquad + u(B(2\mathscr{G} - 3 - 2\mathscr{G}A) - c(A-1)) + Bc - \mathscr{G}B^2\Big)\Big]\,. \end{aligned}$$

□

4.2 $\mathscr{C}^1$ estimate on w_1 and w_2

We derive now $\mathscr{C}^1$-estimates for w_1 and w_2. First we derive an upper bound on $\partial_r w_1$ and $\partial_r w_2$.

Lemma 4.5. *Let $\alpha, R > 0$. Let $t \mapsto X_1(t)$ and $t \mapsto X_2(t)$ be the characteristics defined as in (3.4) and passing through (α, R). Let $T > \alpha$. We assume that $X_1(t), X_2(t)$ are well-defined on $[\alpha, T]$. Let furthermore v_1, v_2 be defined as in (4.1). Then, as long as v_1 and v_2 are well-defined, we have*

$$\begin{aligned} v_1(t, X_1(t))e^{-\int_\alpha^t a_1(s,X_1(s))\mathrm{d}s} &\leqslant v_1(\alpha, X_1(\alpha)) + \int_\alpha^t a_2(s, X_1(s))e^{\int_s^t a_1(\tau,X_1(\tau))\mathrm{d}\tau}\,, \\ v_2(t, X_2(t))e^{-\int_\alpha^t b_1(s,X_2(s))\mathrm{d}s} &\leqslant v_2(\alpha, X_2(\alpha)) + \int_\alpha^t b_2(s, X_2(s))e^{\int_s^t b_1(\tau,X_2(\tau))\mathrm{d}\tau}\,. \end{aligned}$$

More generally, for any $\ell \in \mathbb{N}$,

$$\begin{aligned} & X_1(t)^\ell v_1(t, X_1(t)) \\ \leqslant\; & X_1(\alpha)^\ell v_1(\alpha, X_1(\alpha))e^{\int_\alpha^t(\frac{\ell\lambda_1}{X_1(s)}+a_1(s,X_1(s)))\mathrm{d}s} + \int_\alpha^t X_1(s)^\ell a_2(s, X_1(s))e^{\int_s^t(\frac{\ell\lambda_1}{X_1(\tau)}+a_1(\tau,X_1(\tau)))\mathrm{d}\tau}\,, \\ & X_2(t)^\ell v_2(t, X_2(t)) \\ \leqslant\; & X_2(\alpha)^\ell v_2(\alpha, X_2(\alpha))e^{\int_\alpha^t(\frac{\ell\lambda_2}{X_2(s)}+b_1(s,X_2(s)))\mathrm{d}s} + \int_\alpha^t X_2^\ell(s) b_2(s, X_2(s))e^{\int_s^t(\frac{\ell\lambda_2}{X_2(\tau)}+b_1(\tau,X_2(\tau)))\mathrm{d}\tau}\,. \end{aligned}$$

Proof. Let us consider v_1, the argument being similar for v_2. Let us denote $y_1(t) = v_1(t, X_1(t))$. The coefficient a_0 being negative, we have by Lemma 4.1 $y_1'(t) \leqslant a_1 y_1 + a_2$. Hence,

$$(y_1 e^{-\int_\alpha^t a_1(s,X_1(s))\mathrm{d}s})' \leqslant a_2(t, X_1(t))e^{-\int_\alpha^t a_1(s,X_1(s))\mathrm{d}s}\,.$$

Integrating we obtain the desired estimate. The second set of estimates is obtained in the same way considering the system (4.6) instead of (4.4). □

We now derive a lower bound on v_1 and v_2. First we consider the case in which the initial condition $v_1(\alpha, R)$ is positive.

Lemma 4.6. *Let $\alpha, R > 0$. Let $t \mapsto X_1(t)$ and $t \mapsto X_2(t)$ be the characteristics defined as in (3.4) and passing through (α, R). Let $T > \alpha$. We assume that $X_1(t), X_2(t)$ are well-defined on $[\alpha, T]$. Let us denote $\bar{a}_1 = ra_1$, $\bar{a}_2 = r^2 a_2$, $\bar{b}_1 = rb_1$, $\bar{b}_2 = r^2 b_2$ as in Lemma 4.4, and*

$$A_0 = \max_{[0,T]} \left|a_0(t, X_1(t))\right|, \quad A_1 = \max_{[0,T]} \left|\bar{a}_1(t, X_1(t))\right|, \quad A_2 = \max_{[0,T]} \left|\bar{a}_2(t, X_1(t))\right|, \tag{4.7}$$

$$B_0 = \max_{[0,T]} \left|b_0(t, X_2(t))\right|, \quad B_1 = \max_{[0,T]} \left|\bar{b}_1(t, X_2(t))\right|, \quad B_2 = \max_{[0,T]} \left|\bar{b}_2(t, X_2(t))\right|. \tag{4.8}$$

We also denote, for all $\zeta_0, \zeta_1, \zeta_2$ positive, $x(\zeta_0, \zeta_1, \zeta_2)$ the unique positive solution of

$$Q(r) = \frac{\zeta_1}{\sqrt{\zeta_0 \zeta_2}}\,, \qquad \textit{where } Q(r) := re^r\,. \tag{4.9}$$

Furthermore, we denote

$$K_a(\theta) = \int_\alpha^\theta \big|a_2(t, X_1(t))\big|\,\mathrm{d}t \exp\big(\int_\alpha^\theta \big|a_1(t, X_1(t))\big|\,\mathrm{d}t\big)\,, \tag{4.10}$$

$$K_b(\theta) = \int_\alpha^\theta \big|b_2(t, X_2(t))\big|\,\mathrm{d}t \exp\big(\int_\alpha^\theta \big|b_1(t, X_2(t))\big|\,\mathrm{d}t\big)\,, \tag{4.11}$$

Then:

1. *If $v_1(\alpha, R) \geqslant 0$, then for all $t \in \Big[\alpha, \alpha + \frac{x(A_0,A_1,A_2)}{A_1} \min_{[0,T]} X_1(t)\Big]$, we have*

$$\frac{-K_a(t)}{1 - K_a(t)\int_\alpha^t \big|a_0(\tau, X_1(\tau))\big| e^{\int_\alpha^\tau |a_1(u,X_1(u))|\mathrm{d}u}\,\mathrm{d}\tau} \leqslant v_1(t, X_1(t))e^{-\int_\alpha^t a_1(s,X_1(s))\mathrm{d}s} \leqslant v_1(\alpha, X_1(\alpha)) + K_a(t)\,.$$

2. *If $v_2(\alpha, R) \geqslant 0$, then for all $t \in \Big[\alpha, \alpha + \frac{x(B_0,B_1,B_2)}{B_1} \min_{[0,T]} X_2(t)\Big]$, we have*

$$\frac{-K_b(t)}{1 - K_b(t)\int_\alpha^t \big|b_0(\tau, X_2(\tau))\big| e^{\int_\alpha^\tau |b_1(u,X_2(u))|\mathrm{d}u}\,\mathrm{d}\tau} \leqslant v_2(t, X_2(t))e^{-\int_\alpha^t b_1(s,X_2(s))\mathrm{d}s} \leqslant v_2(\alpha, X_2(\alpha)) + K_b(t)\,.$$

Remark 4.7. The quantities A_i and B_i, for $i \in \{0, 1, 2\}$, are not depending directly on r but only on $\|w_1\|_{\mathbf{L}^\infty}$ and $\|w_2\|_{\mathbf{L}^\infty}$. In the case $\frac{\mathrm{d}X_2}{\mathrm{d}t} > 0$ and $\alpha = 0$, we obtain a lower bound for the time of existence of $\partial_r w_2$

$$T_{ex}(\partial_r w_2) \geqslant \frac{x(B_0, B_1, B_2)}{B_1} R\,.$$

Remark 4.8. The same lemma can be applied to (4.6) with

$$\tilde{a}_0 = \frac{a_0}{r^\ell}\,, \qquad \tilde{a}_1 = a_1 + \frac{\ell\lambda_1}{r}\,, \qquad \tilde{a}_2 = r^\ell a_2\,,$$
$$\tilde{b}_0 = \frac{b_0}{r^\ell}\,, \qquad \tilde{b}_1 = b_1 + \frac{\ell\lambda_2}{r}\,, \qquad \tilde{b}_2 = r^\ell b_2\,.$$

The time of validity of the estimates remains similar, replacing A_1 by $\tilde{A}_1 = \max_{[0,T]}(\bar{a}_1 + \lambda_1)(t, X_1(t)))$ and B_1 by $\tilde{B}_1 = \max_{[0,T]}(\bar{b}_1 + \lambda_2)(t, X_2(t)))$, with A_0, B_0, A_2, B_2 unchanged.

Proof. First note that the upper bounds come directly from Lemma 4.5.

Let us consider the equation on v_2. Without a lot of changes we can adapt the following to v_1. Let us denote $y_2(t) = v_2(t, X_2(t))$ where v_2 is defined in in (4.1). The equation on v_2 in (4.4) writes, with $y_2(t) = v_2(t, X_2(t))$,

$$y_2' = b_0 y_2^2 + b_1 y_2 + b_2\,, \tag{4.12}$$

where $b_0 \leqslant 0$.

According to a Lemma A.2 (see also Hörmander [8]), if we have $y_2(\alpha) \geqslant 0$ and, for $\theta > \alpha$

$$\int_\alpha^\theta \big|b_2(t, X_2(t))\big|\,\mathrm{d}t \int_\alpha^\theta \big|b_0(t, X_2(t))\big|\,\mathrm{d}t \exp\big(2\int_\alpha^\theta \big|b_1(t, X_2(t))\big|\,\mathrm{d}t\big) < 1\,, \tag{4.13}$$

then the equation (4.12) with initial condition $y_2(\alpha) = v_2(\alpha, X_2(\alpha))$ in $t = \alpha$ admits a solution on $[\alpha, \theta]$ and, with K_b as in (4.11), we have the estimate

$$\begin{aligned}&\frac{-K_b(\theta)}{1 - K_b(\theta)\int_\alpha^\theta \left|b_0(t, X_2(t))\right| \mathrm{d}t \exp\left(\int_\alpha^\theta \left|b_1(t, X_2(t))\right| \mathrm{d}t\right)}\\ \leqslant & y_2(\theta)e^{-\int_\alpha^\theta b_1(s,X_1(s))\mathrm{d}s} \leqslant v_2(\alpha, X_2(\alpha)) + K_b(\theta)\,.\end{aligned}$$

Note, thanks to Lemma 4.4, that we have

$$b_0(r,w) = \bar{b}_0(w)\,, \qquad b_1(r,w) = \frac{\bar{b}_1(w)}{r}\,, \qquad b_2(r,w) = \frac{\bar{b}_2(w)}{r^2}\,.$$

As we are not considering the 1-D case, b_0, b_2 are not constantly zero, see the expression of b_0, b_2 in Lemma 4.1. Since

$$\begin{aligned}&\int_\alpha^\theta \left|b_2(t, X_2(t))\right| \mathrm{d}t \int_\alpha^\theta \left|b_0(t, X_2(t))\right| \mathrm{d}t \exp\left(2\int_\alpha^\theta \left|b_1(t, X_2(t))\right| \mathrm{d}t\right)\\ \leqslant & \frac{B_0 B_2}{B_1^2}\left[\frac{B_1}{\min_{[\alpha,\theta]}(X_2(t))}(\theta-\alpha)\exp\left[\frac{B_1}{\min_{[\alpha,\theta]}(X_2(t))}(\theta-\alpha)\right]\right]^2\\ = & \frac{B_0 B_2}{B_1^2} Q\left(\frac{B_1}{\min_{[\alpha,\theta]}(X_2(t))}(\theta-\alpha)\right)^2\,,\end{aligned}$$

the condition (4.13) is satisfied for all $\theta \leqslant T$ such that $Q\left(\frac{B_1}{\min_{[\alpha,\theta]}(X_2(t))}(\theta-\alpha)\right) < \frac{B_1}{\sqrt{B_0 B_2}}$. Hence, it is sufficient to ask that θ satisfies:

$$\theta \leqslant \alpha + \min_{[\alpha,\theta]}(X_2(t))\frac{x(B_0, B_1, B_2)}{B_1}\,.$$

□

Let us now derive a lower bound on v_1 and v_2 in the case the initial condition is negative.

Lemma 4.9. *Let $\alpha, R > 0$. With the same notations as introduced in Lemma 4.6, we obtain:*

1. *If $v_1(\alpha, R) \leqslant 0$, then v_1 is well-defined on every interval $[\alpha, \theta]$ such that*

$$\theta \leqslant \alpha + \frac{\min_{t\in[\alpha,\theta]} X_1(t)}{A_1} Q^{-1}(\Theta_+)\,,$$

where

$$\Theta_+ = \frac{A_1}{\sqrt{A_0 A_2}} \frac{1}{\left(1 + \frac{\sqrt{A_0}}{\sqrt{A_2}}\left(\left|v_1(\alpha,R)\right| \min_{t\in[\alpha,\theta]} X_1(t)\right)\right)}\,, \qquad (4.14)$$

and we have the estimate

$$\begin{aligned}&\frac{-\left(\left|v_1(\alpha,R)\right| + K_a\right)}{1 - \left(\left|v_1(\alpha,R)\right| + K_a\right)\int_\alpha^\theta \left|a_0(t, X_1(t))\right| e^{\int_\alpha^t \left|a_1(u,X_1(u))\right| \mathrm{d}u}\, \mathrm{d}t}\\ &\qquad\leqslant v_1(t, X_1(t))e^{\int_\alpha^t a_1(s,X_1(s))\mathrm{d}s} \leqslant K_a + v_1(\alpha, R)\,.\end{aligned}$$

2. If $v_2(\alpha,R)\leqslant 0$, then v_2 is well-defined on every interval $[\alpha,\theta]$ such that

$$\theta \leqslant \alpha + \frac{\min_{t\in[\alpha,\theta]} X_2(t)}{B_1} Q^{-1}\left(\Xi_+\right) ,$$

where

$$\Xi_+ = \frac{B_1}{\sqrt{B_0 B_2}} \frac{1}{\left(1+\frac{\sqrt{B_0}}{\sqrt{B_2}}(\left|v_2(\alpha)\right| \min_{t\in[\alpha,\theta]} X_2(t))\right)} , \tag{4.15}$$

and we have the estimate

$$\frac{-(\left|v_2(\alpha,R)\right| + K_b)}{1-(\left|v_2(\alpha,R)\right|+K_b)\int_\alpha^\theta \left|b_0(t,X_2(t))\right| e^{\int_\alpha^t |b_1(u,X_2(u))|\mathrm{d}u}\,\mathrm{d}t}$$
$$\leqslant v_2(t,X_2(t)) e^{\int_\alpha^t b_1(s,X_2(s))\mathrm{d}s} \leqslant K_b + v_2(\alpha,R)\,.$$

Proof. First note that the upper bounds come directly from Lemma 4.5.

Let us consider v_1. The same computations apply to v_2 after small changes. Let us denote $z_1(t) = -v_1(t,X_1(t))$. According to Lemma 4.1, z_1 satisfies the ODE:

$$z_1' = -a_0 z_1^2 + a_1 z_1 - a_2 ,$$

where $-a_0 \geqslant 0$ (see Lemma 4.4). Let us introduce K_a as in (4.10). To apply Hörmander Lemma (see Lemma A.2), conditions (A.2) and (A.3) have to be satisfied. Note that, since $-a_0 \geqslant 0$, condition (A.2) implies condition (A.3) so it is sufficient to see what is a sufficient condition allowing (A.2) to be satisfied. Condition (A.2) is equivalent to

$$\left(z_1(\alpha)+K_a\right)\int_\alpha^\theta \left|a_0(t,X_1(t))\right| \mathrm{d}t\, e^{\int_\alpha^\theta |a_1(t,X_1(t))|\mathrm{d}t} - 1 < 0\,.$$

It is sufficient to have

$$z_1(\alpha) A_0 (\theta-\alpha) e^{\frac{A_1}{\min_t X_1(t)}(\theta-\alpha)} + \frac{A_0 A_2}{\min_t X_1(t)^2}(\theta-\alpha)^2 e^{\frac{2A_1}{\min_t X_1(t)}(\theta-\alpha)} - 1 < 0\,.$$

Let us denote $\Theta = \frac{A_1}{\min_{t\in[\alpha,\theta]} X_1(t)}(\theta-\alpha) e^{\frac{A_1}{\min_{t\in[\alpha,\theta]} X_1(t)}(\theta-\alpha)} = Q(\frac{A_1}{\min_{t\in[\alpha,\theta]} X_1(t)}(\theta-\alpha))$, where Q is defined as in (4.9). Then to satisfy condition (A.2) it is sufficient to have

$$z_1(\alpha)\frac{A_0}{A_1}\min_{t\in[\alpha,\theta]}\{X_1(t)\}\Theta + \frac{A_0 A_2}{A_1^2}\Theta^2 - 1 < 0\,.$$

Let $\Delta = \left(z_1(\alpha)\frac{A_0}{A_1}\min_t\{X_1(t)\}\right)^2 + 4\frac{A_0 A_2}{A_1^2}$ be the discriminant of this equation in Θ. This equation admits two distinct roots, one positive and one negative. Let us denote ξ_+ the positive root. Then we have

$$\begin{aligned}
\xi_+ &= \frac{A_1 z_1(\alpha)\min_{t\in[\alpha,\theta]} X_1(t)}{2A_2}\left(-1+\sqrt{1+\frac{4A_2}{z_1(\alpha)^2 A_0 \min_{t\in[\alpha,\theta]} X_1(t)^2}}\right)\\
&= \frac{A_1}{\sqrt{A_0 A_2}} \frac{1}{\sqrt{1+\frac{A_0}{4A_2}(z_1(\alpha)\min_{t\in[\alpha,\theta]} X_1(t))^2} + \frac{\sqrt{A_0}}{2\sqrt{A_2}}(z_1(\alpha)\min_{t\in[\alpha,\theta]} X_1(t))}\\
&\geqslant \frac{A_1}{\sqrt{A_0 A_2}} \frac{1}{1+\frac{\sqrt{A_0}}{\sqrt{A_2}}(z_1(\alpha)\min_{t\in[\alpha,\theta]} X_1(t))} =: \Theta_+\,.
\end{aligned}$$

Then (A.2) is satisfied if $0 \leqslant \Theta \leqslant \Theta_+$. Hence, the application Q defined in (4.9) being strictly increasing on $\mathbb{R}_+$, it is sufficient to have

$$\theta \leqslant \alpha + \frac{\min_{t\in[\alpha,\theta]} X_1(t)}{A_1} Q^{-1}\left(\Theta_+\right) .$$

□

5 Construction of a shock wave

5.1 Rankine-Hugoniot conditions

The Rankine-Hugoniot conditions (cf. [1, p. 312]) appear when we consider weak and piecewise smooth solution for first order systems. For the Euler equations (1.1), these conditions are written:

$$\begin{cases} -U[\rho] + [\rho(u\cdot\nu)] = 0\,, \\ -U[\rho u] + [\rho(u\cdot\nu)u + p\nu] = 0\,, \end{cases}$$

through a discontinuity with normal vector ν and with normal speed U. The usual notation $[\cdot]$ stands for the jump between the two limit values at the both sides of the discontinuity. We denote u^+ the limit of u from the right and u^- the limit of u from the left. In the spherical case, these conditions become

$$\begin{cases} -U[\rho] + [\rho u] = 0\,, \\ -U[\rho u] + [\rho u^2 + p] = 0\,. \end{cases} \tag{5.1}$$

A weak solution containing a discontinuity is called *shock* when U differs from the speed of the fluid on the both sides of the discontinuity. Note that the first condition of (5.1) gives us $\rho^+(U-u^+) = \rho^-(U-u^-)$ so that $U-u^+$ and $U-u^-$ have the same sign. If we assume $U - u^\pm \geqslant 0$ then the shock moves from the left to the right.

Let $W^\pm = U - u^\pm$. With some classical computations, we get that (5.1) is equivalent to

$$j := \rho^+(U-u^+) = \rho^-(U-u^-)\,, \qquad [p+\rho W^2] = 0\,. \tag{5.2}$$

Hence, we get

$$u^+ - u^- = j\left(\frac{1}{\rho^-} - \frac{1}{\rho^+}\right), \qquad p^+ - p^- = j^2\left(\frac{1}{\rho^-} - \frac{1}{\rho^+}\right).$$

Finally, we obtain $(u^+ - u^-)^2 = (p^+ - p^-)\left(\frac{1}{\rho^-} - \frac{1}{\rho^+}\right) \geqslant 0$. Let us recall furthermore the Lax entropy conditions for a 2-shock (see [20]):

$$\rho^- > \rho^+\,, \qquad p^- > p^+\,, \qquad u^- > u^+\,, \tag{5.3}$$

$$\lambda_1(w^-) < U(w^+,w^-) < \lambda_2(w^-)\,, \qquad U(w^+) \geqslant \lambda_2(w^+)\,. \tag{5.4}$$

Finally, the jump conditions at the shock are

$$U = \frac{\rho^+u^+ - \rho^-u^-}{\rho^+ - \rho^-}\,, \qquad u^- - u^+ = \sqrt{(p^+-p^-)\left(\frac{1}{\rho^-} - \frac{1}{\rho^+}\right)}\,, \qquad \rho^- > \rho^+\,. \tag{5.5}$$

Proposition 5.1. *For a Bethe-Weyl gas, the Rankine-Hugoniot and Lax shock conditions can be reduced to (5.5).*

Proof. By (5.5), we have immediatly $u^- > u^+$ and $p^- > p^+$ since $\frac{\partial p}{\partial \rho} = c^2 \geqslant 0$. Hence (5.3) is satisfied.

Let us prove (5.4). First, note that

$$U - u^- = \frac{\rho^+ \sqrt{F(\rho^-, \rho^+)}}{\rho^- - \rho^+},$$

where F is defined as in (3.3). Hence $U - u^- \geqslant 0 \geqslant -c^-$ and $U \geqslant u^- - c^- = \lambda_1^-$.

Let us compute now $U - u^- - c^-$:

$$\begin{aligned} U - u^- - c^- &= \frac{\rho^+}{\rho^- - \rho^+} \sqrt{\rho^- c^- \left(\frac{1}{\rho^+} - \frac{1}{\rho^-}\right)} \left(\sqrt{\frac{1}{\rho^- c^-}(p^- - p^+)} - \sqrt{\rho^- c^- \left(\frac{1}{\rho^+} - \frac{1}{\rho^-}\right)} \right) \\ &= \frac{\rho^+}{\rho^- - \rho^+} \sqrt{\rho^- c^- \left(\frac{1}{\rho^+} - \frac{1}{\rho^-}\right)} \frac{\frac{1}{\rho^- c^-}\left(p^- - p^+ - (\rho^- c^-)^2 \left(\frac{1}{\rho^+} - \frac{1}{\rho^-}\right)\right)}{\left(\sqrt{\frac{1}{\rho^- c^-}(p^- - p^+)} + \sqrt{\rho^- c^- \left(\frac{1}{\rho^+} - \frac{1}{\rho^-}\right)}\right)} \end{aligned}$$

Let us denote $\xi(\rho^-, \rho^+) = p^- - p^+ - (\rho^- c^-)^2 \left(\frac{1}{\rho^+} - \frac{1}{\rho^-}\right)$. We have, for $\rho \geqslant \rho^+$

$$\frac{\partial \xi(\rho, \rho^+)}{\partial \rho} = -2\rho c^2 \mathscr{G} \left(\frac{1}{\rho^+} - \frac{1}{\rho}\right) \leqslant 0.$$

Integrating on $[\rho^+, \rho^-]$, we obtain $\xi(\rho^-, \rho^+) \leqslant \xi(\rho^+, \rho^+) = 0$. Finally, we have $U - u^- - c^- \leqslant 0$.

A similar computation prove that $U - u^+ - c^+ \geqslant 0$. □

Let us invert the second of the jump conditions (5.5), in order to express it as a condition on w_1^-, depending on w_2^-, w_1^+, w_2^+.

Lemma 5.2. *We assume there exists $(w_{1,0}^-, w_{2,0}^-)$ and $(w_{1,0}^+, w_{2,0}^+)$ such that $\mathcal{F}(w_0^-, w_0^+) = 0$. Then, there exists $g \in \mathscr{C}^1(\mathbb{R}^3, \mathbb{R})$ such that the second condition of (5.5) is equivalent to the compatibility condition*

$$w_1^- = g(w_2^-, w_1^+, w_2^+), \tag{5.6}$$

as long as the condition $\rho^- > \rho^+$ is satisfied. That is to say, as long as $g(w_2^-, w^+) < w_2^- - w_2^+ + w_1^+$.

Proof. We can write (5.5) as the following:

$$\mathcal{F}(w^-, w^+) := u^- - u^+ - \sqrt{F(\rho^-, \rho^+)} = 0, \tag{5.7}$$

with $F(\rho, \rho^+) = (\frac{1}{\rho} - \frac{1}{\rho^+})(p(\rho^+) - p(\rho))$ as in (3.3). Since by hypothesis $\rho^+ < \rho^-$, we have

$$\begin{aligned} \frac{\partial \mathcal{F}}{\partial w_1^-} &= \frac{1}{4\sqrt{F}} \left(\sqrt{\rho^- c^- (\frac{1}{\rho^+} - \frac{1}{\rho^-})} + \sqrt{\frac{1}{\rho^- c^-}(p^- - p^+)} \right)^2 \\ &> 0. \end{aligned} \tag{5.8}$$

According to the hypotheses $\mathcal{F}(w^-(0,R_0),w^+(0,R_0)) = 0$, that is to say that the jump condition given by the second equation of (5.5) is satisfied at time $t=0$. By the implicit function Theorem, there exist (locally) a unique function $g(w_2^-,w^+)$ such that

$$\mathcal{F}(g(w_2^-,w^+),w_2^-,w^+) = 0\,.$$

Furthermore

$$\frac{\partial \mathcal{F}}{\partial w_2^-} = \frac{-1}{4\sqrt{F}}\left(\sqrt{\rho^- c^-(\frac{1}{\rho^+}-\frac{1}{\rho^-})} - \sqrt{\frac{1}{\rho^- c^-}(p^- - p^+)}\right)^2 \leqslant 0\,,$$

$$\frac{\partial \mathcal{F}}{\partial w_1^+} = \frac{-1}{4\sqrt{F}}\left(\sqrt{\rho^+ c^+(\frac{1}{\rho^+}-\frac{1}{\rho^-})} + \sqrt{\frac{1}{\rho^+ c^+}(p^- - p^+)}\right)^2 \leqslant 0\,,$$

$$\frac{\partial \mathcal{F}}{\partial w_2^+} = \frac{1}{4\sqrt{F}}\left(\sqrt{\rho^+ c^+(\frac{1}{\rho^+}-\frac{1}{\rho^-})} - \sqrt{\frac{1}{\rho^+ c^+}(p^- - p^+)}\right)^2 \geqslant 0\,,$$

and we can see that the sign of the partial derivatives remain constant. As we have $(\frac{\partial g}{\partial w_2^-},\frac{\partial g}{\partial w_1^+},\frac{\partial g}{\partial w_2^+}) = \frac{-1}{\partial_{w_1^-}\mathcal{F}}(\partial_{w_2^-}\mathcal{F},\partial_{1+}\mathcal{F},\partial_{2+}\mathcal{F})$ is well defined as long as $\rho^- > \rho^+$. Hence, the implicit function is defined as long as the condition $\rho^- > \rho^+$ is satisfied. □

Remark 5.3. Note that the condition $\rho^- > \rho^+$ on $\mathcal{K}$ is equivalent to $w_2^- > w_2^+$ on $\mathcal{K}$. Indeed, as H is a strictly increasing function of ρ, $\rho^- > \rho^+$ is equivalent to $H^- > H^+$, that is to say $w_2^- - w_1^- > w_2^+ - w_1^+$. On $\mathcal{K}$, $w_1^- = g(w_2^-,w^+)$. Hence, $\rho^- > \rho^+$ is equivalent to $g(w_2^-,w_1^+,w_2^+) < w_2^- - w_2^+ + w_1^+$.

Let us denote $h(w_2^-,w_1^+,w_2^+) = w_2^- - w_2^+ + w_1^+$. As $\partial_{w_1^-}\mathcal{F} > 0$, then $g(w_2^-,w^+) < h(w_2^-,w^+)$ is equivalent to $\mathcal{F}(g(w_2^-,w^+),w_2^-,w^+) < \mathcal{F}(h(w_2^-,w^+),w_2^-,w^+)$. We note that

$$u(h,w_2^-) = \frac{1}{2}(h+w_2^-) = \frac{1}{2}(2w_2^- - (w_2^+ - w_1^+)) = w_2^- - H(w_1^+,w_2^+)$$

and

$$\rho(h,w_2^-) = H^{-1}\left(\frac{1}{2}(w_2^- - h)\right) = H^{-1}\left(\frac{1}{2}(w_2^+ - w_1^+)\right) = \rho(w_1^+,w_2^+)\,.$$

Hence we obtain that $\mathcal{F}(g,w_2^-,w^+) < \mathcal{F}(h,w_2^-,w^+)$ is equivalent to

$$0 < (w_2^- - H^+) - u^+ = w_2^- - (u^+ + H^+) = w_2^- - w_2^+\,.$$

5.2 Angular domain

We want now to construct a solution by solving three problems : two classical problems with initial conditions obtained by prolongating the initial conditions on the right and on the left of R_0; and an angular problem with boundary conditions given in (5.6)–(5.9). According to T. T. Li & W. C. Yu [14, Chap. 3], this last problem admits a local in time solution. In order to obtain an estimate on the time of existence we make a priori estimates on the solution.

Let us denote D_- the domain in the (x,t)-plan which is bounded on the right by the curve $\mathcal{C}_1$ defined in (5.9) and by the lines of equations $t=0$, $t=T_*$ (see Figure 2). In the same way, we denote D_+ the domain in the (x,t)-plan which is bounded on the left by the curve $\mathcal{K}$ defined in (5.9) and by the lines of equations $t=0$, $t=T_*$. The domain D_0 is the domain in the (x,t)-plan which is bounded on the left by the curve $\mathcal{C}_1$ defined in (5.9) and on the right by the curve $\mathcal{K}$ defined in (5.9).

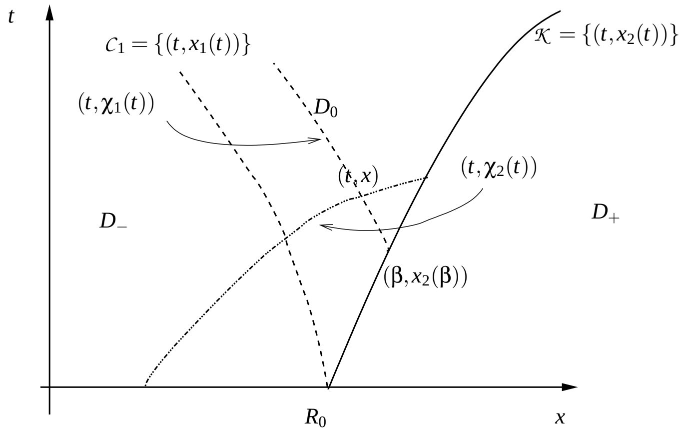

Figure 2: Angular Domain and some related curves.

More precisely, let $R_0>0$. Let us define the free boundary domain D (see Figure 2) where the boundaries $\mathcal{C}_1=\{(t,x_1(t))\,,\ t\geqslant 0\}$ and $\mathcal{K}=\{(t,x_2(t))\,,\ t\geqslant 0\}$ are respectively the curves defined by

$$\frac{\mathrm{d}x_1}{\mathrm{d}t}=\lambda_1(w(t,x_1(t))),\quad x_1(0)=R_0\,,\quad \frac{\mathrm{d}x_2}{\mathrm{d}t}=U(t,w(t,x_2(t))),\quad x_2(0)=R_0\,. \tag{5.9}$$

where U is the speed of a 2-shock linking $w^-(t,x_2(t))$ to $w^+(t,x_2(t))$ and is defined by the first equation of (5.5).

So that the angular problem is well-posed, we need to add some boundary conditions on $\mathcal{C}_1$ and $\mathcal{K}$. Defining g thanks to Lemma 5.2, we have the following boundary conditions:

$$w_2(t,x_1(t))=\omega(t)\,,\qquad w_1(t,x_2(t))=g(t,x_2(t),w_2(t,x_2(t)))\,, \tag{5.10}$$

where ω is chosen so that w is $\mathscr{C}^1$ through $\mathcal{C}_1$. The definition of g ensures that the compatibility condition given by the second equation of (5.5) is satisfied along $\mathcal{K}$.

Lemma 5.4. *Let us assume that, along the shock $\mathcal{K}$, the conditions (5.5) are satisfied. Then along the shock $\mathcal{K}$, we have*

$$\begin{aligned}\partial_r w_1^-(t,x_2(t))=&\frac{1}{U-\lambda_1^-}\Big[(U-\lambda_2^-)\partial_{w_2^-}g\,\partial_r w_2^- -(\partial_{w_2^-}g+1)f^-\\ &+(U-\lambda_1^+)\partial_{w_1^+}g\,\partial_r w_1^+ +(U-\lambda_2^+)\partial_{w_2^+}g\,\partial_r w_2^+ +(\partial_{w_1^+}g-\partial_{w_2^+}g)f^+\Big]\,.\end{aligned}$$

Proof. Derivating $w_1^-(t,x_2(t))$ we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}w_1^-(t,x_2(t))\\ =&\partial_{w_2^-}g((U-\lambda_2^-)\partial_r w_2^- - f^-)+\partial_{w_1^+}g((U-\lambda_1^+)\partial_r w_1^+ + f^+)+\partial_{w_2^+}g((U-\lambda_2^+)\partial_r w_2^+ - f^+)\end{aligned}$$

but also $\frac{\mathrm{d}}{\mathrm{d}t}w_1^-(t,x_2(t))=\partial_t w_1^- + U\partial_r w_1^- = (U-\lambda_1^-)\partial_r w_1^- + f^-$. □

Proposition 5.5. *Let $R_0 > 0$. Consider a Bethe-Weyl gas satisfying $1 < \mathscr{G} < 2$.*

Let us consider the free boundary problem consisting in the system (1.2) in D with the following boundary conditions:

$$w_2(t, x_1(t)) = \omega(t) \geqslant 0\,, \qquad \text{on } \mathcal{C}_1\,; \qquad w_1(t, x_2(t)) = G(t, w_2) > 0\,, \qquad \text{on } \mathcal{K}\,,$$

with $\mathcal{C}_1$, $\mathcal{K}$, defined as above.

Assume that the compatibility conditions (5.5) are satisfied at time $t = 0$ and that there exists $C_0 > 0$ such that, for any $M, T > 0$:

$$\text{along } \mathcal{C}_1\,, \qquad \left|\partial_r w_2(t, x_1(t))\right| = \left|\frac{\omega' + f}{\lambda_1 - \lambda_2}(t, x_1(t))\right| \leqslant \frac{C_0}{x_1(t)}\,,$$

$$\text{along } \mathcal{K}\,, \qquad \partial_t G(t, x_2(t)) \geqslant \frac{-C_0}{x_2(t)}\,, \qquad \sup_{t\in[0,T]} \left|\partial_2 G(t, x_2(t))\right| \leqslant C_0\,.$$

Then, assuming that there exists $\delta > 0$ such that $\rho^- - \rho^+ \geqslant \delta > 0$ along $\mathcal{K}$, there exists a $\mathscr{C}^1$ solution of (1.2) in D whose time of existence is bounded below in the following way

$$T_{ex} \geqslant C R_0\,,$$

where C depends on the boundary conditions.

Proof. Let $(t, r) \in D$. The 2-characteristic going through (t, r) originates in D_- and crosses $\mathcal{C}_1$ in $(\beta, x_1(\beta))$; meanwhile the 1-characteristic going through (t, r) originates in $(\alpha, x_2(\alpha)) \in \mathcal{K}$.

As $w_1 > 0$ along $\mathcal{K}$, we can apply Lemma 3.6 on D_0 to obtain a $\mathscr{C}^0$ estimate of w_1 and w_2. We obtain

$$0 \leqslant w_1(\alpha, x_2(\alpha)) \leqslant w_1(t, r) \leqslant w_2(t, r) \leqslant w_2(\beta, x_1(\beta)) \leqslant \|\omega\|_{\mathbf{L}^\infty}\,.$$

Next, we want to obtain $\mathscr{C}^1$ estimates for w_1 and w_2. First, we derive an $\mathbf{L}^\infty$ estimate for $\partial_r w_2$. Let us remind that $v_2 = e^h(\partial_r w_2 + \Psi)$ with $\Psi(r, w) = \frac{\psi(w)}{r}$.

- In the case $(\partial_r w_2 + \Psi)(\beta, x_1(\beta)) \geqslant 0$, then we have $v_2(\beta, x_1(\beta)) \geqslant 0$ and we can apply Lemma 4.6 on D to obtain an $\mathbf{L}^\infty$-estimate of $\partial_r w_2$ in D. Since $X_2' \geqslant 0$, we obtain a lower bound for the time of existence of $\partial_r w_2$:

$$T_{ex} \geqslant x_1(\beta)\frac{x(B_0, B_1, B_2)}{B_1} \geqslant R_0 \frac{x(B_0, B_1, B_2)}{B_1}\,,$$

  where B_0, B_1, B_2 are defined as in (4.8).

- In the case $v_2(\beta, x_1(\beta)) \leqslant 0$, as $X_2' \geqslant 0$, by Lemma 4.9 we obtain the estimate

$$T_{ex} \geqslant \frac{R_0}{B_1} Q^{-1}\left(\Xi_+\right)$$

  where B_1 is defined as in (4.8) and $\Xi_+ = \frac{B_1}{\sqrt{B_0 B_2}} \frac{1}{1 + \frac{\sqrt{B_0}}{\sqrt{B_2}}\left(\left|v_2(\beta, x_1(\beta))\right| x_1(\beta)\right)}$.

  Since $\Psi(w, r) = \frac{\psi(w)}{r} \leqslant \frac{\|\psi\|_{\mathbf{L}^\infty}}{r}$ and, by hypothesis $\left|\partial_r w_2(\beta, x_1(\beta))\right| \leqslant \frac{C_0}{x_1(\beta)}$, we have

$$\left|v_2(\beta, x_1(\beta))\right| x_1(\beta) \leqslant \|\psi\|_{\mathbf{L}^\infty} + C_0 := C_1\,,$$

and we finally have

$$T_{ex} \geqslant \frac{R_0}{B_1} Q^{-1} \left(\frac{B_1}{\sqrt{B_0 B_2}} \frac{1}{1 + \frac{\sqrt{B_0}}{\sqrt{B_2}} C_1} \right) .$$

Let us now find an estimate in $\mathbf{L}^\infty$ for $\partial_r w_1$. We want to proceed in the same way. First we prove that there exists C such that $\partial_r w_1 + \Phi \geqslant \frac{-C}{r}$ along $\mathcal{K}$. derivating the boundary condition, we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} w_1(t, x_2(t)) &= \partial_t w_1 + U \partial_r w_1 \\ &= f + (U - \lambda_1) \partial_r w_1 \\ &= \partial_t G + \partial_2 G \left((U - \lambda_2) \partial_r w_2 - f \right) . \end{aligned}$$

Hence $\partial_r w_1 = \frac{1}{U - \lambda_1} \left[\partial_t G - f + \partial_2 G \left((U - \lambda_2) \partial_r w_2 - f \right) \right]$ By hypothesis we have such a lower bound on $\partial_t G$. We can find a similar lower bound on f, whose expression is known. By hypothesis $U - \lambda_2^- \leqslant 0$, $U - \lambda_1^- > 0$ and they are depending only on (w_1, w_2). Thanks to Lemma 4.5, we obtain the following upper bound on $v_2 = e^h(\partial_r w_2 + \Psi)$, taking $i = 1$ and reminding that, by Lemma 4.4, $b_i(r, w) = \bar{b}_i(w)/r^i$

$$v_2(t, X_2(t)) e^{-\int_\alpha^t \frac{(\lambda_2 + \bar{b}_1)(s, X_2(s))}{X_2(s)} \mathrm{d}s} \leqslant \frac{X_2(\alpha) v_2(\alpha, X_1(\alpha))}{X_2(t)} + \frac{1}{X_2(t)} \int_\alpha^t \frac{\bar{b}_2}{X_2(s)} e^{-\int_\alpha^s \frac{(\lambda_2 + \bar{b}_1)(\tau, X_2(\tau))}{X_2(\tau)} \mathrm{d}\tau} \mathrm{d}s .$$

Reminding that $X_1' \geqslant 0$ and $X_2' \geqslant 0$, if $(\alpha, X_2(\alpha)) \in \mathcal{C}_1$, we have, for any $s \geqslant \alpha$, $1/X_2(s) \leqslant 1/X_2(\alpha) \leqslant 1/R_0$. Using furthermore the hypothesis $\left| r(\partial_r w_2 + \Psi) \right| \leqslant C_0$ along $\mathcal{C}_1$, we obtain

$$v_2(t, X_2(t)) \leqslant \left(\frac{C_0}{X_2(t)} + \frac{1}{X_2(t)} (t - \alpha) \frac{B_2}{R_0} \right) e^{(t - \alpha)(\sup(\lambda_2) + B_1)/R_0} ,$$

which provide us the estimate $v_2(t, x_2(t)) \leqslant \frac{1}{x_2(t)} C e^{\frac{C(t-\alpha)}{R_0}}$. We obtain the desired estimate taking a time t bounded.

- If $(\partial_r w_1 + \Phi)(\alpha, x_2(\alpha)) \geqslant 0$, then we have $v_1(\alpha, x_2(\alpha)) \geqslant 0$ and we can apply Lemma 4.6 on D to obtain estimate of $\partial_r w_1$ in $\mathbf{L}^\infty$ in D. Since $X_1' \geqslant 0$, we obtain a lower bound for the time of existence of $\partial_r w_1$:

$$T_{ex} \geqslant x_2(\alpha) \frac{x(A_0, A_1, A_2)}{A_1} \geqslant R_0 \frac{x(A_0, A_1, A_2)}{A_1} ,$$

  where A_0, A_1, A_2 are defined as in (4.7) and $x(A_0, A_1, A_2)$ is defined in Lemma 4.6.

- If $v_1(\alpha, x_2(\alpha)) \leqslant 0$, as $X_1' \geqslant 0$, by Lemma 4.9 we obtain the estimate

$$T_{ex} \geqslant \frac{R_0}{A_1} Q^{-1} (\Theta_+)$$

  where A_1 is defined as in (4.7) and $\Theta_+ = \frac{A_1}{\sqrt{A_0 A_2}} \frac{1}{1 + \frac{\sqrt{A_0}}{\sqrt{A_2}} (\left| v_1(\alpha, x_2(\alpha)) \right| x_2(\alpha))}$. Since $\Phi(w, r) = \frac{\varphi(w)}{r} \leqslant \frac{C_1}{r}$ by Lemma 4.4 and $\left| \partial_r w_1(\alpha, x_2(\alpha)) \right| \leqslant \frac{C_0}{x_2(\alpha)}$, we have

$$\left| v_1(\alpha, x_2(\alpha)) \right| x_2(\alpha) \leqslant C_1 + C_0 := C_2 ,$$

and we finally have

$$T_{ex} \geqslant \frac{R_0}{A_1} Q^{-1} \left(\frac{A_1}{\sqrt{A_0 A_2}} \frac{1}{1 + \frac{\sqrt{A_0}}{\sqrt{A_2}} C_2} \right).$$

□

We are now able to construct a piecewise solution:

Theorem 5.6. *Let $R_0 > 0$. Let us consider a Bethe-Weyl gas satisfying $1 < \mathscr{G} < 2$.*

Assume there exists two regular solutions of (1.2) (ρ^-, u^-) and (ρ^+, u^+) whose times of existence are respectively T^- and T^+, such that the compatibility condition (5.5) are satisfied at time $t = 0$, in $r = R_0$. Assume furthermore that

$$\min_{r \leqslant R_0} w_{1,0}^-(r) > 0 , \qquad w_{1,0}^-(R_0) + \inf_{r \geqslant R_0} w_{1,0}^+(r) > 0 .$$

and that there exists $C_0 > 0$ such that:

$$\left| \partial_r w_2^-(t,r) \right| \leqslant \frac{C_0}{r} , \qquad \partial_r w_1^+(t,r) \geqslant \frac{-C_0}{r} , \qquad \partial_r w_2^+(t,r) \leqslant \frac{C_0}{r} .$$

Then, as long as $\rho^- > \rho^+$ along $\mathcal{K}$, there exists a shock wave solution of (1.2) with initial conditions

$$\begin{aligned} &\text{for } r \leqslant R_0 , && u_0 = u_0^- , && \rho_0 = \rho_0^- , \\ &\text{for } r \geqslant R_0 , && u_0 = u_0^+ , && \rho_0 = \rho_0^+ , \end{aligned}$$

and its time of existence is bounded below in the following way

$$T_{ex} \geqslant \min(T^-, T^+, CR_0) ,$$

where C depends on the initial conditions.

For example, we can use the result obtained in [11] in order to have $T^+ = +\infty$ or $T^- = +\infty$. For (ρ^+, u^+), a stationary solution could also work.

Proof. Let D_-, D_0, D_+ be defined as before. Let us define $G(t, w_2) := g(w_2, w^+)$, where g has been defined in Lemma 5.2. Using the estimate of Lemma 3.5 in D_+, we obtain along $\mathcal{K}$: $w_1^- = g(w_2^-, w^+) \geqslant w_1^+ \geqslant w_{1,0}^+ \geqslant \inf_{r \geqslant R_0} w_{1,0}^+$. Besides, $u > 0$ in D_-, in particular $u^-(0, R_0) > 0$, and by continuity of u^- in $D_- \cup D_0$, at least for a small time, u^- is positive in D_0. Note $[0, t_0]$ the time interval in which u^- is positive in D_0. For $(t, r) \in D_0$ such that $t \leqslant t_0$. Let us denote β the time at which the 2-characteristic going through (t, r) cross $\mathcal{C}_1$.

$$u^-(t,r) \geqslant \frac{1}{2}(w_1^- + w_2^-)(t, x_2(t)) \geqslant \frac{1}{2}\big(\min_{r \geqslant R_0}(w_{1,0}^+) + \frac{w_2(\beta, x_1(\beta))}{1 + w_2(\beta, x_1(\beta) \frac{(d-1)(t-\beta)}{4x_1(\beta)})}\big).$$

If $w_{1,0}^+ \geqslant 0$, $u^- \geqslant 0$ in all D_0. Otherwise, if $w_{1,0}^+ < 0$ then u^- is positive as long as

$$t - \beta \leqslant \frac{4x_1(\beta)}{(d-1)} \left(\frac{1}{\min(w_{1,0}^+)} + \frac{1}{w_2(\beta, x_1(\beta))} \right).$$

Besides, as $(\beta, x_1(\beta)) \in D_-$, we have $w_2(\beta, x_1(\beta)) \geqslant w_1(\beta, x_1(\beta)) \geqslant w_1(0, R_0)$. Furthermore, $x_1' \geqslant 0$ then $x_1(\beta) \geqslant R_0$. Then we obtain that u^- is positive if

$$t \leqslant \frac{4R_0}{(d-1)} \left(\frac{1}{\min(w_{1,0}^+)} + \frac{1}{w_1^-(0, R_0)} \right).$$

Hence, we can applying Proposition 3.7 to $(t, r) \in D_0$, we find a time of validity for the $\mathscr{C}^0$-estimates proportional to R_0.

Besides, the expression of $\partial_r w_1^-$ along $\mathcal{K}$ in Lemma 5.4 and the hypotheses on $\partial_r w_1^+$, $\partial_r w_2^+$ allow us to check that the hypotheses of Proposition 5.5. Hence, we have a regular solution in the angular domain satisfying the boundary condition $w_2(t, x_1(t)) = w_2^-(t, x_1(t))$ and $w_1(t, x_2(t)) = g(w_2, w^+)$.

□

We determine now a set of hypotheses on the initial conditions so that the hypotheses are satisfied. The condition $|\partial_r w_2| \leqslant \frac{C}{r}$ along $\mathcal{C}_1$ can not be computed with this method. Consequently, the construction of a regular solution in D_- has to be obtained by another result on regular solution (see for example D. Serre [21] or M. Grassin [6] or M. Lécureux-Mercier [11]).

Proposition 5.7. *In the same context as in Theorem 5.6, we assume that $w_{1,0}^-(R_0) + \min_{r \geqslant R_0} w_{1,0}^+ > 0$ and $w_{1,0}^-(R_0) > \max_{r \geqslant R_0} w_{2,0}^+$. Then, the estimates of Proposition 3.7 are available in D_0 for a time proportional to R_0 and furthermore, along $\mathcal{K}$, $\rho^+ < \rho^-$.*

Proof. Let $t \in \mathbb{R}_+$ we denote β the time at which the 2-characteristic going through $(t, x_2(t))$ cross $\mathcal{C}_1$. By Proposition 3.7, we have

$$\begin{aligned} w_2^-(t, x_2(t)) \geqslant & \frac{w_2(\beta, x_1(\beta))}{1 + (t-\beta)\frac{(d-1)}{4x_1(\beta)} w_2(\beta, x_1(\beta))} \\ \geqslant & \frac{1}{\frac{1}{w_{1,0}^-(R_0)} + (t-\beta)\frac{(d-1)}{4R_0}}. \end{aligned}$$

Besides, $w_2^+(t, x_2(t)) \leqslant \max(w_{2,0}^+)$. By remark 5.3, the condition $\rho^+ < \rho^-$ on $\mathcal{K}$ is equivalent to $w_2^- > w_2^+$. Hence it is sufficient to have

$$\frac{1}{\frac{1}{w_{1,0}^-(R_0)} + (t-\beta)\frac{(d-1)}{4R_0}} \geqslant \max(w_{2,0}^+)$$

Finally it is sufficient to have $w_{1,0}^-(R_0) > \max(w_{2,0}^+)$ and $t \leqslant \frac{4R_0}{(d-1)} \frac{w_1^-(0,R_0) - \max w_{2,0}^+}{\max w_{2,0}^+}$. □

A Time of existence for ODE

Lemma A.1 (Maximum principle for ODE.). *Let $a, b : [0, T] \to \mathbb{R}$ be continuous applications. Assume that $w, z : [0, T] \to \mathbb{R}$ are continuous applications such that $z(0) \geqslant w(0)$ and $w' = aw^2 + b$, $z' \geqslant az^2 + b$ in $[0, T]$. Then $z(t) \geqslant w(t)$ for all $t \in [0, T]$.*

Proof. By hypothesis,

$$(z-w)'(t) = z' - (a(t)w(t)^2 + b(t)) \geqslant a(t)(z(t)^2 - w(t)^2),$$

that is to say $(z-w)'(t) \geqslant a(t)(z(t)-w(t))(z(t)+w(t))$. It follows that,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((z-w)(t)e^{-\int_0^t a(s)(z+w)(s)\mathrm{d}s}\right) \geqslant 0\,.$$

Hence $(z-w)(t)e^{-\int_0^t a(s)(z+w)(s)\mathrm{d}s} \geqslant (z-w)(0) \geqslant 0$, for all $t \in [0,T]$. □

Lemma A.2. *Let $T > 0$ and $a_0, a_1, a_2 \in \mathscr{C}^0([0,T];\mathbb{R})$. Let $a_0^+ = \max(a_0, 0)$ and K be defined by*

$$K = \int_0^T \left|a_2(t)\right| \mathrm{d}t \exp\left(\int_0^T \left|a_1(t)\right| \mathrm{d}t\right)\,. \tag{A.1}$$

If $y_0 \geqslant 0$ and

$$\frac{1}{y_0+K} > \int_0^T a_0^+(t)\,\mathrm{d}t \exp\left(\int_0^T \left|a_1(t)\right| \mathrm{d}t\right)\,, \tag{A.2}$$

$$\frac{1}{K} > \int_0^T \left|a_0(t)\right| \mathrm{d}t \exp\left(\int_0^T \left|a_1(t)\right| \mathrm{d}t\right)\,, \tag{A.3}$$

then the maximal solution of the Cauchy problem

$$y' = a_0(t)y^2 + a_1(t)y + a_2(t)\,, \qquad y(0) = y_0\,, \tag{A.4}$$

is defined at least on $[0,T]$ and satisfies

$$\frac{1}{y(T)} > \frac{1}{y_0+K} - \int_0^T \left|a_0^+(t)\right| \mathrm{d}t \exp\left(\int_0^T \left|a_1(t)\right| \mathrm{d}t\right)\,, \qquad \text{if } y(T) \geqslant 0\,, \tag{A.5}$$

$$\frac{1}{\left|y(T)\right|} > \frac{1}{K} - \int_0^T \left|a_0(t)\right| \mathrm{d}t \exp\left(\int_0^T \left|a_1(t)\right| \mathrm{d}t\right)\,, \qquad \text{if } y(T) < 0\,. \tag{A.6}$$

Proof. First, denoting $\tilde{y} = \exp\left(-\int_0^t a_1(s)\,\mathrm{d}s\right) y$, $\tilde{a}_0 = \exp\left(\int_0^t a_1(s)\,\mathrm{d}s\right) a_0$ and $\tilde{a}_2 = \exp\left(-\int_0^t a_1(s)\,\mathrm{d}s\right) a_2$, we see that the equation (A.4) becomes the ordinary differential equation

$$\tilde{y}' = \tilde{a}_0(t)\tilde{y}^2 + \tilde{a}_2(t)\,. \tag{A.7}$$

and $\tilde{y}(0) = y(0)$. We can thus assume without loss of generality that $a_1 \equiv 0$.

Let us introduce the increasing function v defined by $v(t) = \int_0^t \left|\tilde{a}_2(s)\right| \mathrm{d}s$. Let z be the maximal solution of the Cauchy problem

$$z' = \tilde{a}_0^+(t)(z+K)^2\,, \qquad z(0) = y_0\,.$$

Then z is increasing and since $y_0 + K > 0$, we have

$$\frac{1}{z(t)+K} = \frac{1}{y_0+K} - \int_0^t \tilde{a}_0^+(s)\,\mathrm{d}s\;.$$

Note that the right hand side does not vanish for all $t \in [0,T]$, thanks to (A.2). Besides, we have $(z+v)(0) = y_0$ and

$$\begin{aligned}(z+v)' &= \tilde{a}_0^+(t)(z+K)^2 + |\tilde{a}_2| \\ &\geqslant \tilde{a}_0^+(t)(z+v)^2 + \tilde{a}_2 \\ &\geqslant \tilde{a}_0(t)(z+v)^2 + \tilde{a}_2\,.\end{aligned}$$

Consequently, according to Lemma A.1, we have $y(t) \leqslant (z+v)(t) \leqslant z(t)+K$ for all $t \in [0,T]$ if y exists. In particular, as long as $y(t) > 0$, we have

$$\frac{1}{y(t)} \geqslant \frac{1}{z(t)+K} = \frac{1}{y_0+K} - \int_0^t \tilde{a}_0^+(s)\,\mathrm{d}s = \frac{1}{y_0+K} - \int_0^t a_0^+(\tau) e^{\int_0^\tau a_1(s)\mathrm{d}s}\,\mathrm{d}\tau\ ,$$

hence

$$\frac{1}{y(t)} \geqslant \frac{1}{y_0+K} - \int_0^t a_0^+(\tau)\,\mathrm{d}\tau\ e^{\int_0^t |a_1(s)|\mathrm{d}s}\,. \tag{A.8}$$

Assume now that y vanishes and changes its sign in $t_0 \in [0,T]$. We can apply the same procedure as above to $Y = -y$, replacing y_0 by $Y(t_0) = Y_0 = 0$ and beginning at time t_0. The application Y is then solution of $Y' = A_0 Y^2 + a_1 Y + A_2$, where $A_0 = -a_0$, $A_2 = -a_2$. Denoting $A_0^+ = \max(-a_0, 0)$, we get that, for all $t \geqslant t_0$ such that $Y(t) > 0$

$$\frac{1}{Y(t)} \geqslant \frac{1}{K} - \int_{t_0}^t A_0^+(\tau)\,\mathrm{d}\tau\ e^{\int_0^t |a_1(s)|\mathrm{d}s}\,.$$

Consequently, for all t such that $y(t) < 0$

$$\frac{1}{\big|y(t)\big|} \geqslant \frac{1}{K} - \int_0^t \big|a_0(\tau)\big|\,\mathrm{d}\tau\ e^{\int_0^t |a_1(s)|\mathrm{d}s}\,. \tag{A.9}$$

Finally, the inequalities (A.8)–(A.9) give us some bounds on y for all time, y being positive or negative. Hence, y can not tend to $\pm\infty$ and exists up to time T. Indeed, we proved above that there exists a function $\varphi \in \mathscr{C}^0([0,T],\mathbb{R})$ such that, if y is solution of (A.4), then $\big|y(t)\big| \leqslant \varphi(t)$ for all $t \in [0,T]$. Let us denote T_* the maximal time of existence of y. If we assume $T_* < T$, then we obtain that y is bounded on $[0,T_*[$ by $\max_{[0,T]}\varphi$, which contradicts the fact that y has to go out of all compact set when $t \to T_* < \infty$. □

B Explicit expression of the coefficients

We use here the same notation as in Lemma 4.4.

Remark B.1. Note that, in the case $\rho \mapsto \frac{\sqrt{H'(\rho)}}{\rho}$ is integrable at $+\infty$, we have

$$A = 1 + g = \frac{1}{2\sqrt{H'}} \int_{+\infty}^{\rho} \frac{1}{u}\sqrt{H'(u)}(\mathscr{G}-1)\,\mathrm{d}u \leqslant 0\,.$$

Hence $1+2g \leqslant 0$ and we have also $B \leqslant 0$. Besides,

$$(1+g)H - B = \frac{1}{2\sqrt{H'}} \int_0^{\rho} \left(H'\sqrt{H'} + \frac{H\sqrt{H'}}{\rho}(\mathscr{G}-1)\right) \geqslant 0\,.$$

Hence, if $w_2 = u + H \geqslant 0$, then $\Psi \leqslant \frac{(d-1)}{r}(1+g)(u+H) \leqslant 0$

Remark B.2. In the case $d = 3$, we obtain

$$a_2 = \frac{2\sqrt{H'}}{r^2}\Big[- u^2 \left(\mathscr{G}A^2 - 2A(\mathscr{G}-2) + \mathscr{G} - 1\right) + u\left(2(A-1)(\mathscr{G}B + c) + 4B\right) - \mathscr{G}B^2\Big]$$
$$= \frac{2\sqrt{H'}}{r^2}\Big[- \mathscr{G}(u(A-1+\frac{2}{\mathscr{G}}) - B)^2 - 4u^2(\frac{3}{4} - \frac{1}{\mathscr{G}}) + 2uc(A-1)\Big] ,$$

and

$$b_2 = \frac{2\sqrt{H'}}{r^2}\Big[- u^2 \left(\mathscr{G}A^2 - 2A(\mathscr{G}-2) + \mathscr{G} - 1\right) + u\left(2B(\mathscr{G} - 2 - \mathscr{G}A) - 2c(A-1)\right) - \mathscr{G}B^2\Big]$$

In the case $d = 2$, we obtain

$$a_2 = \frac{\sqrt{H'}}{2r^2}\Big[- \mathscr{G}\left(u(A-1+\frac{5}{2\mathscr{G}}) - B\right)^2 - \frac{25}{4}u^2\left(\frac{16}{25} - \frac{1}{\mathscr{G}}\right) + 3uc(A-1)\Big] .$$

B.1 Perfect gas.

For a perfect gas, we have: $c(\rho) = \sqrt{\gamma_0(\gamma_0 - 1)}\,(\rho)^{\frac{\gamma_0 - 1}{2}}$, thus

$$H = 2\sqrt{\frac{\gamma_0}{\gamma_0 - 1}}\,(\rho)^{\frac{\gamma_0 - 1}{2}} .$$

This implies, denoting $\nu = \frac{\gamma_0+1}{\gamma_0-1} > 1$

$$\rho = (\frac{\gamma_0 - 1}{4\gamma_0})^{\frac{1}{\gamma_0 - 1}} H^{\nu - 1} ,$$
$$c(\rho) = \frac{\gamma_0 - 1}{2} H ,$$
$$H' = \frac{\gamma_0 - 1}{2}(\frac{4\gamma_0}{\gamma_0 - 1})^{\frac{1}{\gamma_0 - 1}} H^{2-\nu}$$

Consequently, noting that $\frac{\gamma_0 - 1}{2} = \frac{1}{\nu - 1}$, $u = \frac{w_1 + w_2}{2}$ and $H = \frac{w_2 - w_1}{2}$, we get

$$\lambda_1 = \frac{w_1 + w_2}{2} - \frac{1}{\nu - 1}\left(\frac{w_2 - w_1}{2}\right) ,$$
$$\lambda_2 = \frac{w_1 + w_2}{2} + \frac{1}{\nu - 1}\left(\frac{w_2 - w_1}{2}\right) ,$$
$$\partial_1\lambda_1 = \partial_2\lambda_2 = \frac{\mathscr{G}}{2} = \frac{\nu}{2(\nu - 1)} > 0 ,$$
$$\partial_2\lambda_1 = \partial_1\lambda_2 = \frac{(\nu - 2)}{2(\nu - 1)} \geqslant 0 .$$

Then $h = k = \ln\left(H^{(2-\nu)/2}\right)$ and

$$
\begin{aligned}
g &= \frac{\nu-1}{(2-\nu)} \leqslant 0\,,\\
B &= \frac{\nu}{(2-\nu)(4-\nu)}H\,,\\
a_0 &= \frac{-\nu}{2(\nu-1)^{1/2}(2\nu+2)^{\frac{\nu-1}{2}}}H^{\nu-1}\,,\\
a_1 &= \frac{d-1}{(2-\nu)r}\Big[3u + \frac{2(4-3\nu)H}{(\nu-1)(4-\nu)}\Big]\,,\\
a_2 &= \frac{d-1}{(2-\nu)r^2}\Big[-u^2 + \frac{4uH}{(4-\nu)} - \frac{\nu H^2}{(\nu-1)(4-\nu)}\\
&\quad + \frac{d-1}{2}\left(\frac{-2u^2}{(2-\nu)} - \frac{8uH}{(2-\nu)(4-\nu)} + \Big(\frac{\nu H^2}{(\nu-1)(4-\nu)} - \frac{\nu^3}{(\nu-1)(2-\nu)(4-\nu)^2}\Big)H^2\right)\Big]
\end{aligned}
$$

B.2 Van der Waals gas.

For a Van der Waals gas, we have: $c(\rho) = \frac{1}{1-b\rho}\sqrt{\gamma_0(\gamma_0-1)}\left(\frac{\rho}{1-b\rho}\right)^{\frac{\gamma_0-1}{2}}$, thus

$$
H = 2\sqrt{\frac{\gamma_0}{\gamma_0-1}}\left(\frac{\rho}{1-b\rho}\right)^{\frac{\gamma_0-1}{2}}\,.
$$

This implies, denoting $\tilde{b} = b(\frac{\gamma_0-1}{4\gamma_0})^{\frac{1}{\gamma_0-1}}$ and $\nu = \frac{\gamma_0+1}{\gamma_0-1} > 1$

$$
\begin{aligned}
\rho &= \frac{(\frac{\gamma_0-1}{4\gamma_0})^{\frac{1}{\gamma_0-1}}H^{\nu-1}}{1+\tilde{b}H^{\nu-1}}\,,\\
c(\rho) &= \frac{\gamma_0-1}{2}(1+\tilde{b}H^{\nu-1})H\,,\\
H' &= \frac{\gamma_0-1}{2}(\frac{4\gamma_0}{\gamma_0-1})^{\frac{1}{\gamma_0-1}}(1+\tilde{b}H^{\nu-1})^2H^{2-\nu}
\end{aligned}
$$

Consequently, noting that $\frac{\gamma_0-1}{2} = \frac{1}{\nu-1}$, $u = \frac{w_1+w_2}{2}$ and $H = \frac{w_2-w_1}{2}$, we get

$$
\begin{aligned}
\lambda_1 &= \frac{w_1+w_2}{2} - \frac{1}{\nu-1}\left(\frac{w_2-w_1}{2} + \tilde{b}(\frac{w_2-w_1}{2})^{\nu}\right)\,,\\
\lambda_2 &= \frac{w_1+w_2}{2} + \frac{1}{\nu-1}\left(\frac{w_2-w_1}{2} + \tilde{b}(\frac{w_2-w_1}{2})^{\nu}\right)\,,\\
\partial_1\lambda_1 = \partial_2\lambda_2 &= \frac{\mathscr{G}}{2} = \frac{\nu}{2(\nu-1)}(1+\tilde{b}H^{\nu-1}) > 0\,,\\
\partial_2\lambda_1 = \partial_1\lambda_2 &= \frac{1}{2(\nu-1)}\left((\nu-2) - \tilde{b}\nu H^{\nu-1}\right)\,.
\end{aligned}
$$

Then $h = k = \ln\left(H^{(2-\nu)/2}(1+\tilde{b}H^{\nu-1})\right)$ and

$$
\begin{aligned}
g &= \frac{\nu-1}{(2-\nu)(1+\tilde{b}H^{\nu-1})}\,, \\
B &= \frac{1}{(2-\nu)(4-\nu)}\frac{H}{1+\tilde{b}H^{\nu-1}}\left(1+\frac{(2-\nu)(4-\nu)}{2+\nu}\tilde{b}H^{\nu-1}\right)\,, \\
a_0 &= \frac{-\nu}{2(\nu-1)^{1/2}(2\nu+2)^{\frac{\nu-1}{2}}}H^{\nu-1}\,.
\end{aligned}
$$

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