$\mathbf{L}^1$ Stability for scalar balance laws; Application to pedestrian traffic.

Magali Mercier
Université de Lyon, Université Lyon 1, Institut Camille Jordan;
43 blvd. du 11 Novembre 1918, 69622 Villeurbanne Cedex;
mercier@math.univ-lyon1.fr

Abstract

We present here a stability result for the solutions of scalar balance laws. The estimates we obtained are then used to study the continuity equation with a non-local flow, which appears for example in a new model of pedestrian traffic.

1 Introduction

We consider the Cauchy problem for scalar balance laws of the form $\partial_t u + Div f(t,x,u) = F(t,x,u)$, which often appear in physics. Thanks to Kružkov's theorem [8, Thm 1 & 5] we know that this kind of equation admits a unique weak entropy solution and we can describe the dependence on the initial condition of the solution.

In the first part, we describe the dependence of the solution with respect to the flow f and the source F. Some cases were already studied: for example Lucier [9] or Bouchut & Perthame [2] have considered the case in which the flow depends only on u and in which there is no source. We treat here the general case, which includes the preceding results. These results come from a collaboration with R. Colombo and M. Rosini and are more precisely described in [5].

The second part is devoted to the study of the continuity equation with a non-local flow. Using estimates of the first part, we show not only that this model admits a unique weak entropy solution, but also that the linearized equation admits a weak entropy solution. Furthermore, the non-linear local semi-group obtained by solving the initial value problem is Gâteaux-differentiable with respect to the initial condition and the Gâteaux-derivative is the solution of the linearized equation. This fact allows us to characterize the minima or maxima of a given cost functional depending on the initial condition. This is of interest in pedestrian traffic if for example we want to minimize the time of exit out of a room, avoiding high density in the crowd. These results come from a collaboration with R. Colombo and M. Herty and are presented in [4].

2 $\mathbf{L}^1$ Stability for scalar balance laws

Below, for a vector valued function $f = f(x,u)$ with $u = u(x)$, $\mathrm{Div} f$ stands for the total divergence whereas $\mathrm{div}\, f$ and ∇f denote the partial divergence and gradient with respect to the x variable. Moreover, ∂_u and ∂_t are the usual partial derivatives with respect to the variables u and t. Hence, if $u \in \mathscr{C}^1(\mathbb{R}^N, \mathbb{R})$, we have $\mathrm{Div} f = \mathrm{div}\, f + \partial_u f \cdot \nabla u$.

We study here Cauchy problems for scalar balance laws:

$$\begin{cases} \partial_t u + \mathrm{Div} f(t,x,u) = F(t,x,u) & (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^N \\ u(0,x) = u_0(x) \in \mathbf{L}^1 \cap \mathbf{L}^\infty & x \in \mathbb{R}^N \,, \end{cases} \tag{2.1}$$

where $f \in \mathscr{C}^2([0,T] \times \mathbb{R}^N \times \mathbb{R}; \mathbb{R}^N)$ is the flow and $F \in \mathscr{C}^1([0,T] \times \mathbb{R}^N \times \mathbb{R}; \mathbb{R})$ is the source. The properties of this kind of equation have already been intensively investigated, see for example [7, 8]. Here we want to describe the dependence of the solutions with respect to flow and source.

2.1 Previous Results

Let us first recall the Kružkov Theorem [8, Theorem 5]:

Theorem 2.1 (Kružkov)**.** *Let $T > 0$. For any $A \geqslant 0$, we denote $\Omega_A = [0,T] \times \mathbb{R}^N \times [-A, A]$. Under the conditions $f \in \mathscr{C}^2$, $F \in \mathscr{C}^1$ and:*

$$(\mathbf{K}) \ \ \forall A > 0 \ : \partial_u f \,,\ \partial_u (F - \mathrm{div}\, f) \,,\ F - \mathrm{div}\, f \text{ are bounded on } \Omega_A$$

there exists a unique weak entropy solution $u \in \mathbf{L}^\infty([0,T]; \mathbf{L}^1(\mathbb{R}^N; \mathbb{R}))$ of (2.1) right-continuous in time.

Let $v_0 \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^N; \mathbb{R})$. Let u be the solution associated to the initial condition u_0 and v be the solution associated to the initial condition v_0. Let M be such that $M \geqslant \sup(\|u\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^N;\mathbb{R})}, \|v\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^N;\mathbb{R})})$. Then, for all $t \in [0,T]$, with $\gamma = \|\partial_u F\|_{\mathbf{L}^\infty(\Omega_M)}$, we have

$$\|(u-v)(t)\|_{\mathbf{L}^1} \leqslant e^{\gamma t} \|u_0 - v_0\|_{\mathbf{L}^1} \,.$$

We also know some other results concerning the dependence of the solution with respect to flow and source. The following was first proved by Lucier [9], and later improved by Bouchut & Perthame [2]. Their results are about the homogeneous conservation laws: the flow depends only on u and there is no source. More precisely, if $f, g : \mathbb{R} \to \mathbb{R}^N$ are globally lipschitz, then for all $u_0, v_0 \in \mathbf{L}^1 \cap \mathbf{L}^\infty(\mathbb{R}^N; \mathbb{R})$ initial conditions for

$$\partial_t u + \mathrm{Div} f(u) = 0 \,, \qquad \partial_t v + \mathrm{Div} g(v) = 0 \,.$$

with furthermore $v_0 \in \mathbf{BV}(\mathbb{R}^N; \mathbb{R})$, we have for all $t \geqslant 0$,

$$\|(u-v)(t)\|_{\mathbf{L}^1} \leqslant \|u_0 - v_0\|_{\mathbf{L}^1} + C\, t\, \mathrm{TV}\,(v_0)\, \mathbf{Lip}\,(f-g) \,.$$

A flow depending also on x was considered by Chen & Karlsen [3], in the special case $f(x,u) = \lambda(x)l(u)$. There, under appropriate hypotheses, with $f(t,x,u) = \lambda(x)\, l(u)$, $g(t,x,v) = \mu(x)\, m(v)$, and without source ($F = G = 0$), they obtained the estimate:

$$\begin{aligned}\|(u-v)(t)\|_{\mathbf{L}^1} \leqslant \|u_0 - v_0\|_{\mathbf{L}^1} + C_1\, t\, (&\|\lambda-\mu\|_{\mathbf{L}^\infty} + \|\lambda-\mu\|_{\mathbf{W}^{1,1}} \\ &+ \|l-m\|_{\mathbf{L}^\infty} + \|l-m\|_{\mathbf{W}^{1,\infty}})\end{aligned}$$

where $C_1 = C \sup_{[0,T]} (\mathrm{TV}\,(u(t)), \mathrm{TV}\,(v(t)))$. However, this general settings contains the Cauchy problem: $\partial_t u + \partial_x(\cos x) = 0$ with $u_0 = 0$, is $u(t,x) = t \sin x$ for which $\mathrm{TV}\,(u_0) = 0$ and $\mathrm{TV}\,(u(t)) = +\infty$ for any $t > 0$. Hence, the coefficient C_1 is also $+\infty$. This fact motivated us for searching first an estimate on the total variation in the case the flow and source depend on the three variables t, x and u.

2.2 Estimate on the Total Variation

Let us recall here the definition of total variation.

Definition 2.2. For $u \in \mathbf{L}^1_{loc}(\mathbb{R}^N;\mathbb{R})$ we denote the total variation of u:

$$\mathrm{TV}\,(u) = \sup\Big\{ \int_{\mathbb{R}^N} u \,\mathrm{div}\, \Psi\,; \quad \Psi \in \mathscr{C}^1_c(\mathbb{R}^N;\mathbb{R}^N)\,, \quad \|\Psi\|_{\mathbf{L}^\infty} \leqslant 1 \Big\}\,.$$

Then we introduce the space of function with bounded variation $\mathbf{BV}(\mathbb{R}^N;\mathbb{R}) = \big\{ u \in \mathbf{L}^1_{loc}; \mathrm{TV}\,(u) < \infty \big\}$.

When f and F depend only on u we already know that $u_0 \in \mathbf{L}^\infty \cap \mathbf{BV}$ implies that for all $t \geqslant 0$, $u(t) \in \mathbf{L}^\infty \cap \mathbf{BV}$, with the same notation as in Theorem 2.1, we have $\mathrm{TV}\,(u(t)) \leqslant \mathrm{TV}\,(u_0) e^{\gamma t}$, where $\gamma = \|\partial_u F\|_{\mathbf{L}^\infty(\Omega_M)}$.

Now we give a more general estimate on the total variation. Let $W_N = \int_0^{\pi/2} (\cos\theta)^N \,\mathrm{d}\theta$, $\Omega = \Omega_\infty$ and:

$$(\mathbf{H1}) \;:\; \begin{cases} f \in \mathscr{C}^2(\Omega;\mathbb{R}^N)\,, & F \in \mathscr{C}^1(\Omega;\mathbb{R})\,, \\ \nabla\partial_u f \in \mathbf{L}^\infty(\Omega;\mathbb{R}^{N\times N})\,, & \partial_t \,\mathrm{div}\, f \in \mathbf{L}^\infty(\Omega;\mathbb{R})\,, \\ \partial_t\partial_u f \in \mathbf{L}^\infty(\Omega;\mathbb{R}^N)\,, & \partial_t F \in \mathbf{L}^\infty(\Omega;\mathbb{R})\,. \\ \displaystyle\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N} \|\nabla(F - \mathrm{div}\, f)(t,x,\cdot)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^N)} \,\mathrm{d}x\,\mathrm{d}t < +\infty & \end{cases}$$

Theorem 2.3 (see Theorem 2.5 in [5])**.** *Assume (f,F) satisfies **(K)** and **(H1)**. Let $u_0 \in (\mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^N;\mathbb{R})$, $M = \|u\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^N)}$, $\kappa_0 = (2N+1)\,\|\nabla\partial_u f\|_{\mathbf{L}^\infty(\Omega_M)} + \|\partial_u F\|_{\mathbf{L}^\infty(\Omega_M)}$. Then for all $t \in [0,T]$, $u(t) \in (\mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^N;\mathbb{R})$ and*

$$\begin{aligned}\mathrm{TV}\,(u(t)) \leqslant{} & \mathrm{TV}\,(u_0) e^{\kappa_0 t} \\ & + N W_N \int_0^t e^{\kappa_0(t-\tau)} \int_{\mathbb{R}^N} \|\nabla(F - \mathrm{div}\, f)(\tau,x,\cdot)\|_{\mathbf{L}^\infty([-M,M])} \,\mathrm{d}x\,\mathrm{d}\tau\,.\end{aligned}$$

Remark 2.4. In some cases, we recover known estimates. When f depends only on u and $F = 0$, we have a result similar to the one that was already known : $\mathrm{TV}\,(u(t)) \leqslant \mathrm{TV}\,(u_0)$.

When f, F do not depend on u, we have in fact an ODE $\partial_t u = (F - \mathrm{div}\, f)(t,x)$. The solution is consequently $u(t,x) = u_0(x) + \int_0^t (F - \mathrm{div}\, f)(\tau, x)\,\mathrm{d}\tau$. Meanwhile, the bound above reduces to $\mathrm{TV}\,(u(t)) \leqslant \mathrm{TV}\,(u_0) + NW_N \int_0^t \int_{\mathbb{R}^N} |(F - \mathrm{div}\, f)(\tau,x)|\,\mathrm{d}\tau$ which is essentially what we expected.

Remark 2.5. The set of hypotheses **(H1)** is in fact very strong. We expect it can be relaxed to

$$(\mathbf{H1}*) : \begin{cases} f \in \mathscr{C}^0(\Omega;\mathbb{R}^N)\,, \qquad F \in \mathscr{C}^0(\Omega;\mathbb{R})\,, \text{ and } \forall\, A > 0 : \\ \nabla\partial_u f \in \mathbf{L}^\infty(\Omega_A;\mathbb{R}^{N\times N})\,, \quad \partial_u F \in \mathbf{L}^\infty(\Omega_A;\mathbb{R})\,, \\ \displaystyle\int_0^T \int_{\mathbb{R}^N} \|\nabla(F - \mathrm{div}\, f)(t,x,\cdot)\|_{\mathbf{L}^\infty([-A,A];\mathbb{R}^N)} < \infty\,, \end{cases} \tag{2.2}$$

which is useful for example in [4]. This is a work in progress [11].

2.3 $\mathbf{L}^1$ Stability of the solution

Now, we can study the dependence of the solution with respect to flow and source. Let us introduce the set of hypotheses:

$$(\mathbf{H2}) \; : \begin{cases} f \in \mathscr{C}^1(\Omega;\mathbb{R}^N)\,, \qquad F \in \mathscr{C}^0(\Omega;\mathbb{R})\,, \\ \partial_u F \in \mathbf{L}^\infty(\Omega;\mathbb{R})\,, \qquad \partial_u f \in \mathbf{L}^\infty(\Omega;\mathbb{R}^N)\,, \\ \displaystyle\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N} \|(F - \mathrm{div}\, f)(t,x,\cdot)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\,\mathrm{d}x\,\mathrm{d}t < +\infty\,. \end{cases}$$

Theorem 2.6 (see Theorem 2.6 in [5])**.** *Assume $(f,F),(g,G)$ satisfy **(K)**, (f,F) satisfies **(H1)** and $(f-g, F-G)$ satisfies **(H2)**. Let $u_0, v_0 \in (\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^N;\mathbb{R})$. We denote*

$$\kappa = 2N\,\|\nabla\partial_u f\|_{\mathbf{L}^\infty(\Omega_M)} + \|\partial_u F\|_{\mathbf{L}^\infty(\Omega_M)} + \|\partial_u(F-G)\|_{\mathbf{L}^\infty(\Omega_M)}\,.$$

Let u and v be the solutions associated to (f,F) and (g,G) respectively and with initial conditions u_0 and v_0. Then for all $t \in [0,T]$:

$$\begin{aligned} \|(u-v)(t)\|_{\mathbf{L}^1} \leqslant\; & e^{\kappa t}\,\|u_0 - v_0\|_{\mathbf{L}^1} + \frac{e^{\kappa_0 t} - e^{\kappa t}}{\kappa_0 - \kappa}\,\mathrm{TV}\,(u_0)\,\|\partial_u(f-g)\|_{\mathbf{L}^\infty} \\ & + \int_0^t \frac{e^{\kappa_0(t-\tau)} - e^{\kappa(t-\tau)}}{\kappa_0 - \kappa}\int_{\mathbb{R}^N} \|\nabla(F - \mathrm{div}\, f)(\tau,x,\cdot)\|_{\mathbf{L}^\infty([-M,M])}\,\mathrm{d}x\mathrm{d}\tau \\ & \quad \times NW_N\,\|\partial_u(f-g)\|_{\mathbf{L}^\infty} \\ & + \int_0^t e^{\kappa(t-\tau)}\int_{\mathbb{R}^N} \|((F-G) - \mathrm{div}\,(f-g))(\tau,x,\cdot)\|_{\mathbf{L}^\infty([-M,M])}\,\mathrm{d}x\mathrm{d}\tau\,. \end{aligned}$$

Remark 2.7. As in Remark 2.4, we recover known estimates in some particular cases

1. f, g depend only on u, $F = G = 0$,
2. f, g, F, G do not depend on u.

Remark 2.8. As in Remark 2.5, we think the set of hypotheses **(H2)** can be slightly weakened (see [11]) into

$$(\mathbf{H2}*) \ : \begin{cases} \forall\, A > 0\,,\ \partial_u F \in \mathbf{L}^\infty(\Omega_A; \mathbb{R}^N)\,, \\ \displaystyle\int_{\mathbb{R}_+^*} \int_{\mathbb{R}^N} \left\| (F - \operatorname{div} f)(t, x, \cdot) \right\|_{\mathbf{L}^\infty([-A,A];\mathbb{R})} \, \mathrm{d}x \, \mathrm{d}t < +\infty\,. \end{cases}$$

Furthermore, κ can be replaced by $\kappa^* = \left\| \partial_u G \right\|_{\mathbf{L}^\infty(\Omega_U;\mathbb{R})}$ where $U = \sup(\left\| u \right\|_{\mathbf{L}^\infty}, \left\| v \right\|_{\mathbf{L}^\infty})$.

3 The continuity equation with a non-local flow

This section is a short version of [4]: we study the continuity equation:

$$\partial_t u + \operatorname{Div}(u\, V(x, u(t))) = 0\,, \qquad u(0, \cdot) = u_0 \in \mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV}\,, \quad (3.1)$$

where $V : \mathbb{R}^N \times \mathbf{L}^1(\mathbb{R}^N; \mathbb{R}) \to \mathscr{C}^2(\mathbb{R}^N; \mathbb{R})$ is a non-local averaging functional, for example, if $\varphi : \mathbb{R} \to \mathbb{R}$ is a regular function:

1. $V(u) = \varphi\left(\int_{\mathbb{R}} u \, \mathrm{d}x\right)$ for a supply-chain [1],
2. $V(x, u) = \varphi(\eta *_x u) w(x)$ for pedestrian traffic. This model follows several other macroscopic models [6, 10, 12].

Our goals are: first, prove existence and uniqueness of a weak entropy solution, second find the extrema of a cost functional depending on the initial condition. The second point leads us to differentiate the semi-group in the Gâteaux sense with respect to initial conditions.

3.1 Existence and uniqueness of a solution

Let us introduce the following sets of hypotheses:
(V1) There exists $C \in \mathbf{L}^\infty_{loc}(\mathbb{R}_+; \mathbb{R}_+)$ such that for all $u \in \mathbf{L}^1(\mathbb{R}^N; \mathbb{R})$

$$\begin{aligned} V(u) &\in \mathbf{L}^\infty\,, & \left\| \nabla V(u) \right\|_{\mathbf{L}^\infty} &\leqslant C(\left\| u \right\|_{\mathbf{L}^\infty})\,, \\ \left\| \nabla V(u) \right\|_{\mathbf{L}^1} &\leqslant C(\left\| u \right\|_{\mathbf{L}^\infty})\,, & \left\| \nabla^2 V(u) \right\|_{\mathbf{L}^1} &\leqslant C(\left\| u \right\|_{\mathbf{L}^\infty})\,, \end{aligned}$$

and or all $u_1, u_2 \in \mathbf{L}^1(\mathbb{R}^N; \mathbb{R})$

$$\begin{aligned} \left\| V(u_1) - V(u_2) \right\|_{\mathbf{L}^\infty} &\leqslant C(\left\| u_1 \right\|_{\mathbf{L}^\infty}) \left\| u_1 - u_2 \right\|_{\mathbf{L}^1}\ , \\ \left\| \nabla(V(u_1) - V(u_2)) \right\|_{\mathbf{L}^1} &\leqslant C(\left\| u_1 \right\|_{\mathbf{L}^\infty}) \left\| u_1 - u_2 \right\|_{\mathbf{L}^1}\ . \end{aligned}$$

(V2) There exists a positive function $C \in \mathbf{L}^\infty_{loc}(\mathbb{R}_+;\mathbb{R}_+)$ such that $\left\|\nabla^2 V(u)\right\|_{\mathbf{L}^\infty} \leqslant C(\|u\|_{\mathbf{L}^\infty})$ and $\left\|\nabla^3 V(u)\right\|_{\mathbf{L}^\infty} \leqslant C(\|u\|_{\mathbf{L}^\infty})$.

Theorem 3.1 (see Theorem 2.2 in [4])**.** *Let $u_0 \in \mathbf{L}^\infty \cap \mathbf{L}^1 \cap \mathbf{BV}$. If V satisfies **(V1)**, then there exists a time $T_{ex} > 0$ and a unique entropy solution $u \in \mathscr{C}^0([0,T_{ex}[;\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})$ to (3.1) and we denote $S_t u_0 = u(t,\cdot)$. Besides, we have*

$$T_{ex} \geqslant \sup\Big\{ \sum_n \frac{\ln(\alpha_{n+1}/\alpha_n)}{C(\alpha_{n+1})} \, ; \, (\alpha_n)_n \text{ strict. increasing}, \, \alpha_0 = \|u_0\|_{\mathbf{L}^\infty} \Big\},$$

*where the function C is the one appearing in **(V1)**.*

*If furthermore, V satisfies **(V2)** then*

$$u_0 \in \mathbf{W}^{2,1} \cap \mathbf{L}^\infty \Rightarrow \forall t \in [0,T_{ex}[\,, \quad u(t) \in \mathbf{W}^{2,1}\,.$$

Idea of the proof: We introduce the space $X_\alpha = \mathbf{L}^1 \cap \mathbf{BV}(\mathbb{R}^N;[0,\alpha])$ and the application $\mathcal{Q}$ that associates to $w \in \mathcal{X}_\beta = \mathscr{C}^0([0,T[,X_\beta)$ the solution $u \in \mathcal{X}_\beta$ of the Cauchy problem

$$\partial_t u + \mathrm{Div}(uV(w)) = 0\,, \quad u(0,\cdot) = u_0 \in X_\alpha\,.$$

For $w_1, w_2 \in \mathcal{X}_\beta$, we obtain, thanks to the estimate of Theorem 2.6:

$$\|\mathcal{Q}(w_1) - \mathcal{Q}(w_2)\|_{\mathbf{L}^\infty([0,T[,\mathbf{L}^1)} \leqslant f(T)\,\|w_1 - w_2\|_{\mathbf{L}^\infty([0,T[,\mathbf{L}^1)}\,,$$

where f is increasing, $f(0) = 0$ and $f \to \infty$ when $T \to \infty$. Then we apply the Banach Fixed Point Theorem.

3.2 Gâteaux derivative of the semi-group

Let us recall the standard (local) situation: the semi-group generated by a conservation law is in general lipschitz continuous and *not* differentiable. Here, the non-local property gives us more regularity and we are able to differentiate the semi-group in the Gâteaux sense. Let us first recall the definition of Gâteaux differentiability.

Definition 3.2. The application $S : \mathbf{L}^1(\mathbb{R}^N;\mathbb{R}) \to \mathbf{L}^1(\mathbb{R}^N;\mathbb{R})$ is said to be $\mathbf{L}^1$ *Gâteaux differentiable in $u_0 \in \mathbf{L}^1$ in the direction $r_0 \in \mathbf{L}^1$* if there exists a linear continuous application $DS(u_0) : \mathbf{L}^1 \to \mathbf{L}^1$ such that

$$\left\| \frac{S(u_0 + hr_0) - S(u_0)}{h} - DS(u_0)(r_0) \right\|_{\mathbf{L}^1} \to 0 \qquad \text{when } h \to 0$$

Formally, we expect the Gâteaux derivative of the semi-group to be the solution of the linearized problem:

$$\partial_t r + \mathrm{Div}(rV(u) + uDV(u)(r)) = 0\,, \quad r(0,\cdot) = r_0\,.$$

In order to give sense to this equation, we have first to require the differentiability of V. Let us introduce stronger hypotheses:
(V3) $V : \mathbf{L}^1 \to \mathscr{C}^2$ is differentiable and there exists $C \in \mathbf{L}^\infty_{loc}$ such that for all $u, r \in \mathbf{L}^1$:

$$\|V(u+r) - V(u) - DV(u)(r)\|_{\mathbf{W}^{2,\infty}} \leqslant C\left(\|u\|_{\mathbf{L}^\infty} + \|u+r\|_{\mathbf{L}^\infty}\right)\|r\|^2_{\mathbf{L}^1},$$
$$\|DV(u)(r)\|_{\mathbf{W}^{2,\infty}} \leqslant C(\|u\|_{\mathbf{L}^\infty})\|r\|_{\mathbf{L}^1}.$$

(V4) There exists $C \in \mathbf{L}^\infty_{loc}(\mathbb{R}_+;\mathbb{R}_+)$ such that for all $u, \tilde{u}, r \in \mathbf{L}^1$:

$$\|\mathrm{div}\,(V(\tilde{u}) - V(u) - DV(u)(\tilde{u}-u))\|_{\mathbf{L}^1} \leqslant C(\|\tilde{u}\|_{\mathbf{L}^\infty} + \|u\|_{\mathbf{L}^\infty})\|\tilde{u}-u\|^2_{\mathbf{L}^1}$$
$$\|\mathrm{div}\,(DV(u)(r))\|_{\mathbf{L}^1} \leqslant C(\|u\|_{\mathbf{L}^\infty})\|r\|_{\mathbf{L}^1}.$$

We show that the linearized problem admits a unique entropy solution:

Theorem 3.3 (see Proposition 2.8 in [4])**.** *Assume that V satisfies **(V1), (V3)**. Let $u \in \mathscr{C}^0([0,T_{ex}[;\mathbf{W}^{1,\infty}\cap\mathbf{W}^{1,1})$, $r_0 \in (\mathbf{L}^1\cap\mathbf{L}^\infty)(\mathbb{R}^N;\mathbb{R})$. Then the linearized Cauchy problem*

$$\partial_t r + \mathrm{Div}(rV(u) + uDV(u)(r)) = 0\,, \qquad \text{with } r(0,x) = r_0 \tag{3.2}$$

admits a unique entropy solution $r \in \mathscr{C}^0([0,T_{ex}[;\mathbf{L}^1(\mathbb{R}^N;\mathbb{R}))$ and we denote $\Sigma^u_t r_0 = r(t,\cdot)$.

*If furthermore V satisfies **(V2)**, and $r_0 \in \mathbf{W}^{1,1}$, then $\forall t \in [0,T_{ex}[$, $r(t) \in \mathbf{W}^{1,1}$.*

Now, we can prove that the solution of the linearized equation is really the derivative of the semi-group.

Theorem 3.4 (see Theorem 2.10 in [4])**.** *Assume that V satisfies **(V1), (V2), (V3), (V4)**. Let $u_0 \in \mathbf{W}^{1,\infty}\cap\mathbf{W}^{2,1}$, $r_0 \in \mathbf{W}^{1,1}\cap\mathbf{L}^\infty$ and let T_{ex} be the time of existence for the initial problem given by Theorem 3.1.*

Then, for all $t \in [0,T_{ex}[$ the local semi-group of the pedestrian traffic problem is $\mathbf{L}^1$ Gâteaux differentiable in the direction r_0 and

$$DS_t(u_0)(r_0) = \Sigma^{S_t u_0}_t r_0\,.$$

Idea of the proof: Let u, u_h be the solutions of the Cauchy problem $\partial_t u + \mathrm{Div}(uV(u)) = 0$ with initial conditions $u_0, u_0 + hr_0$. Let r be the solution of the linearized equation (3.2), with $r(0) = r_0$. We define then $z_h = u + hr$ that satisfies $z_h(0) = u_0 + hr_0$ and

$$\partial_t z_h + \mathrm{Div}\,(z_h(V(u) + hDV(u)(r))) = h^2\mathrm{Div}(rDV(u)(r))\,.$$

Next, we use Theorem 2.6 to compare u_h and z_h. We obtain

$$\|u_h - z_h\|_{\mathbf{L}^\infty([0,T[,\mathbf{L}^1)} \leqslant F(T)\left[\|u_h - u\|^2_{\mathbf{L}^\infty(\mathbf{L}^1)} + \|u_h - z_h\|_{\mathbf{L}^\infty(\mathbf{L}^1)}\right]$$
$$+\, h^2 C(\beta)Te^{C(\beta)T}\|r\|_{\mathbf{L}^\infty(\mathbf{W}^{1,1})}\|r\|_{\mathbf{L}^\infty(\mathbf{L}^1)}\,,$$

where F is increasing and $F(0) = 0$. After a good choice of T, we can divide by h, make $h \to 0$ and conclude.

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