Riemann problem for traffic flow on a roundabout

Magali Mercier

ICJ, Lyon

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Modelization.

Model

- LWR model on each open segment: let $v$ be a given (decreasing) speed law, then the total density $r$ verifies

$$\partial_t r + \partial_x (v(r)) = 0;$$

- Special boundary conditions:
  - bounds on the flows of exiting and entering vehicles,
  - conservation of the flow of the vehicles staying on the road.
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The one-T road.

Figure: Road with an entry and an exit in $x = 0$.

$\rho_1$: density of vehicles that go straight,
$\rho_2$: density of vehicles that are to turn right,
$\rho_3$: density of vehicles that are entering the road.
The one T road.

Equations:

We write the mass conservation of the vehicles with the same speed law \( v \) (multi-class extension of the LWR model):

\[
\begin{align*}
\partial_t \rho_1 + \partial_x (\rho_1 v(\rho_1 + \rho_2)) &= 0 \quad \text{for } x < 0, \\
\partial_t \rho_2 + \partial_x (\rho_2 v(\rho_1 + \rho_2)) &= 0 \\
\partial_t \rho_1 + \partial_x (\rho_1 v(\rho_1 + \rho_3)) &= 0 \quad \text{for } x > 0, \\
\partial_t \rho_3 + \partial_x (\rho_3 v(\rho_1 + \rho_3)) &= 0
\end{align*}
\]

We also add piecewise constant initial data (Riemann problem):

\[
\begin{align*}
\rho_1(0, x) &= \rho_1^- \quad \text{for } x < 0 \\
\rho_1(0, x) &= \rho_1^+ \quad \text{for } x > 0 \\
\rho_2(0, x) &= \rho_2^- \quad \text{for } x < 0 \\
\rho_3(0, x) &= \rho_3^- \quad \text{for } x > 0,
\end{align*}
\]
Special boundary conditions.

The boundary conditions give bounds on the flows of the vehicles:

\[
\begin{align*}
\rho_1 v(\rho_1 + \rho_2)(t,0-) &= \rho_1 v(\rho_1 + \rho_3)(t,0+) \max, \\
\rho_2 v(\rho_1 + \rho_2)(t,0-) &\leq o(t) \max, \\
\rho_3 v(\rho_1 + \rho_3)(t,0+) &\leq i(t) \max, \\
\end{align*}
\]  

"max" means here that the flows of \( \rho_1 \), \( \rho_2 \) and \( \rho_3 \) are maximised. We also add a priority rule:

- we maximise first the flows of \( \rho_1 \) and \( \rho_2 \)
- then the flow of \( \rho_3 \).

We can obtain similar result with the other rule.
The Riemann problem for the one T road admits a unique weak entropy solution.

**Theorem**

Under the hypotheses

- \((V)\) : the speed law \(v\) is \(C^{0,1}\), decreasing and vanishes in 1.
- \((R)\) : the flow \(q(r) = rv(r)\) is strictly concave and attains its maximum in \(r_c\).

the Riemann problem for the one T road admits a unique weak entropy solution.
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Figure: Solution.
$\rho_{i,j}$ : density of the vehicles which enter in $i$ and exit in $j$.

(P) we know the numbers $p_{i,j}$ that represent the proportion of vehicles entering in $i$ that are to exit in $j$.

(T) the vehicles are not authorized to do more than one turn!

This means: $\rho_{i,k}(x^+_k) = 0$, for $i \neq k$, $\rho_{k,j}(x^-_k) = 0$, for $j \neq k$. 

Magali Mercier
We have:

$$\forall i \in [0, n], \forall j \in [1, n+1], \quad \partial_t \rho_{i,j} + \partial_x \left( \rho_{i,j} v \left( \sum_{l,m} \rho_{l,m} \right) \right) = 0 \quad (4)$$

with boundary conditions in $x_k$:

$$\begin{align*}
\text{for } i,j \neq k, \quad & \rho_{i,j} v \left( \sum_{l,m} \rho_{l,m} \right) (t, x_k^-) = \rho_{i,j} v \left( \sum_{l,m} \rho_{l,m} \right) (t, x_k^+) \quad \text{max}, \\
\sum_{0 \leq i \leq n} \rho_{i,k} v \left( \sum_{l,m} \rho_{l,m} \right) (t, x_k^-) & \leq o_k(t) \quad \text{max}, \\
\sum_{1 \leq j \leq n+1} \rho_{k,j} v \left( \sum_{l,m} \rho_{l,m} \right) (t, x_k^+) & \leq i_k(t) \quad \text{max},
\end{align*}$$

the flows being maximized first in $x_k^-$ and then in $x_k^+$ because of the priority rule.
Theorem

Under the hypotheses $(V)$, $(F)$ and $(P)$, there exists $T > 0$ such that the Riemann problem for the $n$-T road admits a unique weak entropy solution for $t \in [0, T]$.

Furthermore, we can give a lower bound for the time of existence: let $L = \min(x_{k+1} - x_k) > 0$, then $T \geq \frac{L}{2V}$.

Figure: Solution with $n$ points of entry and exit.
Discontinuity points.

We have some points of discontinuity for the Riemann solver:
- when $o, \rho_2^- \to 0$;
- when $r^+ = \rho_1^+ + \rho_3^+ \to 1$ and $\rho_1^- \to 0$.

For example, in the case: $o = 0$, $i$ large, $\rho_1^- < r_c$, $r^+ \geq r_c$, we obtain
Invariant sets.

We have some invariant sets.

On $S$ the Riemann solver for the considered problem is not continuous. However, it is continuous on some subset: for $o \in [\varepsilon, 1]$ with $\varepsilon > 0$ and on $\rho \in T_{0,b}$, with $b < 1$ and $T_{0,b}$ invariant, the solution is obtained continuously.

Figure: Invariant set.
We consider here a multi-class extension of the LWR model. The solution of the Riemann problem is obtained by following the Hugoniot loci.

- The 1-waves are shocks or rarefaction waves;
- the 2-waves are contact discontinuities.
We call $N(\rho^-)$ the set of states attainable in $x = 0$ by the left and $P(\rho^+)$ the set of states attainable in $x = 0$ by the right.

Figure: Left-Riemann problem (top), right-riemann problem (bottom).
Let $d$ be a function such that $d(r) = q_c$ if $r \leq r_c$ and $d(r) = q(r)$ if $r \geq r_c$. Then, we introduce:

$$
\mathcal{N}(\rho_1^-, \rho_2^-) = \mathcal{N}(\rho_1^-, \rho_2^-) \cap \{(\rho_1, \rho_2), \rho_2 v(\rho_1 + \rho_2) \leq o\}
\cap \{(\rho_1, \rho_2), \rho_1 v(\rho_1 + \rho_2) \leq d(r^+)\},
$$

$$
\mathcal{P}(\rho_1^+, \rho_3^+) = \mathcal{P}(\rho_1^+, \rho_3^+) \cap \{(\rho_1, \rho_3), \rho_1 v(\rho_1 + \rho_3) = M\}
\cap \{(\rho_1, \rho_3), \rho_3 v(\rho_1 + \rho_3) \leq i\}.
$$

We maximise the flow of $\rho_2$ on $\mathcal{N}(\rho^-)$ and then the flow of $\rho_3$ on $\mathcal{P}(\rho^+)$. 

![Diagram illustrating the authorized set and the flow maximization.](image-url)
Conclusion:

- Colombo, Rinaldo M. and Goatin, Paola, A well posed conservation law with a variable unilateral constraint, J. Differential Equations, 2007,
- Serre, D., Systèmes de lois de conservation. I, Diderot Editeur, Paris, 1996,