Improved stability estimates on general scalar balance laws

Magali Lécureux-Mercier

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Abstract

Consider the general scalar balance law $\partial_t u + \text{Div} f(t,x,u) = F(t,x,u)$ in several space dimensions. The aim of this note is to improve the results of Colombo, Mercier, Rosini who gave an estimate of the dependence of the solutions from the flow f and from the source F. The improvements are twofold: first the expression of the coefficients in these estimates are more precise; second, we eliminate some regularity hypotheses thus extending significantly the applicability of our estimates.

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1 Introduction

We consider the Cauchy problem for the general scalar balance law

$$\begin{cases} \partial_t u + \text{Div} f(t,x,u) = F(t,x,u) & (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^N \\ u(0,x) = u_0(x) & x \in \mathbb{R}^N . \end{cases} \tag{1.1}$$

This kind of equation has already been intensively studied: a fundamental result is the one of S. N. Kružkov [12, Theorem 1 & 5], stating the existence and uniqueness of a weak entropy solution for an initial data $u_0 \in \mathbf{L}^\infty(\mathbb{R}^N, \mathbb{R})$. In addition, Kružkov describes the dependence of the solutions with respect to the initial condition: if u_0 and v_0 are two initial data, then the associated entropy solutions u and v satisfy

$$\left\|(u-v)(t)\right\|_{\mathbf{L}^1} \leqslant e^{\gamma t} \|u_0 - v_0\|_{\mathbf{L}^1}, \qquad \text{with } \gamma = \|\partial_u F\|_{\mathbf{L}^\infty} . \tag{1.2}$$

A huge literature on this subject is available in the special case the flow f depends only on u and not on the variables t and x and there is no source $F = 0$ (see for example [3, 10, 14, 15]).

We are interested here in the dependence of the solution with respect to flow f and source F in the case these functions depend on the three variables t, x and u.

This dependence with respect to flow and source has already been investigated: this question was first addressed from the point of view of numerical analysis by B. Lucier [13] who studied the case of an homogeneous flow $(f(u))$, without source term $(F = 0)$. More recently F. Bouchut & B. Perthame [2] improved this result, always in the case of an homogeneous flow and without source. G.-Q. Chen & K. Karlsen [4] also studied this dependence, for a flow depending also on x, but the estimate they obtained was depending on an a priori (unknown) bound on $\text{TV}(u(t))$.

The purpose of the present paper is to improve the recent result of R. Colombo, M. Mercier & M. Rosini [8], which provided an estimate of the total variation in the general case (with flow and source depending on the three variables t, x and u) and of the $\mathbf{L}^1$ distance between solutions. In particular, this estimate can be compared to the one of Kružkov (1.2) which gives a bound on the $\mathbf{L}^1$ distance between solutions with different initial data (but with same flow and source). The estimates (1.2) and [8, Theorem 2.6] look similar but in [8], the coefficient γ given by Kružkov in

(1.2) is replaced by $\kappa = 2N\|\nabla\partial_u f\|_{\mathbf{L}^\infty} + \|\partial_u F\|_{\mathbf{L}^\infty}$. Consequently, we do not recover (1.2) from [8] in the case $F = 0$ (because $\gamma = 0$ whereas $\kappa = 2N\|\nabla\partial_u f\|_{\mathbf{L}^\infty} \neq 0$ a priori).

In the same setting as in [8, 12], we provide here an estimate on the total variation of the solution to (1.1), and on the dependence of the solutions to (1.1) on the flow f and on the source F, with better hypotheses and coefficients than in [8]. The advances are twofold. Firstly, we relax hypotheses, and thus widely extend the usability of our results. More precisely, we require here less regularity in time than in [8], which is very useful for applications (see [6, 7]). Furthermore, we recover the same estimate as Kružkov when we consider the dependence toward initial conditions only.

This note is organized as follows. In Section 2 we state the main results and compare them to those in [8]. In Section 3, we give some tools on functions with bounded variations; in Sections 4 and 5 we prove Theorems 2.2 and 2.5; finally Section 6 contains some technical lemmas used in the preceding sections.

2 Main results

We shall use the notations $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{R}_+^* = (0, +\infty)$. Below, N is a positive integer, $\Omega = \mathbb{R}_+^* \times \mathbb{R}^N \times \mathbb{R}$; for any positive T, U we denote $\Omega_T^U = [0, T] \times \mathbb{R}^N \times [-U, U]$; $B(x, r)$ stands for the ball in $\mathbb{R}^N$ with center $x \in \mathbb{R}^N$ and radius $r > 0$ and $\mathrm{Supp}(u)$ stands for the support of u. The volume of the unit ball $B(0, 1)$ is ω_N. For notational simplicity, we set $\omega_0 = 1$. The following induction formula gives ω_N in terms of the Wallis integral W_N:

$$\frac{\omega_N}{\omega_{N-1}} = 2\,W_N \qquad \text{where} \qquad W_N = \int_0^{\pi/2} (\cos\theta)^N \,\mathrm{d}\theta\,. \tag{2.1}$$

In the present work, $\mathbf{1}_A$ is the characteristic function of the set A, and δ_t is the Dirac measure centered at t. Besides, for a vector valued function $f = f(x, u)$ with $u = u(x)$, $\mathrm{Div} f$ stands for the total divergence. On the other hand, $\mathrm{div}\, f$, respectively ∇f, denotes the partial divergence, respectively gradient, with respect to the space variables. Moreover, ∂_u and ∂_t are the usual partial derivatives. Thus, $\mathrm{Div} f = \mathrm{div}\, f + \partial_u f \cdot \nabla u$.

The following sets of assumptions on f and F will be of use below.

$$(\mathbf{H1}^*) \quad \begin{cases} f \in \mathscr{C}^0(\Omega; \mathbb{R}^N)\,, \qquad F \in \mathscr{C}^0(\Omega; \mathbb{R})\,, \\ f, F \text{ have continuous derivatives } \partial_u f\,,\ \partial_u \nabla f\,,\ \nabla^2 f\,,\ \partial_u F\,,\ \nabla F\,; \\ \text{for all } U,\, T > 0, \qquad \partial_u f \in \mathbf{L}^\infty(\Omega_T^U; \mathbb{R}^N)\,, \\ F - \mathrm{div}\, f \in \mathbf{L}^\infty(\Omega_T^U; \mathbb{R})\,, \qquad \partial_u(F - \mathrm{div}\, f) \in \mathbf{L}^\infty(\Omega_T^U; \mathbb{R})\,. \end{cases} \tag{2.2}$$

$$(\mathbf{H2}^*) \quad \begin{cases} \text{for all } U,\ T > 0\,, \qquad \nabla\partial_u f \in \mathbf{L}^\infty(\Omega_T^U; \mathbb{R}^{N\times N})\,, \qquad \partial_u F \in \mathbf{L}^\infty(\Omega_T^U; \mathbb{R})\,, \\ \displaystyle\int_0^T \int_{\mathbb{R}^N} \left\| \nabla(F - \mathrm{div}\, f)(t, x, \cdot) \right\|_{\mathbf{L}^\infty([-U,U]; \mathbb{R}^N)} \mathrm{d}x \ \mathrm{d}t < \infty\,. \end{cases} \tag{2.3}$$

$$(\mathbf{H3}^*) \quad \begin{cases} \text{for all } U,\, T > 0 \qquad \partial_u f \in \mathbf{L}^\infty(\Omega_T^U; \mathbb{R}^N)\,, \qquad \partial_u F \in \mathbf{L}^\infty(\Omega_T^U; \mathbb{R})\,, \\ \displaystyle\int_0^T \int_{\mathbb{R}^N} \left\| (F - \mathrm{div}\, f)(t, x, \cdot) \right\|_{\mathbf{L}^\infty([-U,U]; \mathbb{R})} \mathrm{d}x\, \mathrm{d}t < +\infty\,. \end{cases} \tag{2.4}$$

Comparing these sets of hypotheses to **(H1)**,**(H2)** and **(H3)** in [8], we note that

- no derivatives in time are now needed;
- the $\mathbf{L}^\infty$ norm are now taken on the domain $\Omega_T^U = [0, T] \times \mathbb{R}^N \times [-U, U]$ which is smaller than $\Omega = \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}$, which was the domain considered in [8].

Let us recall the fundamental theorem

Theorem 2.1 (Kružkov [12])**.** *Assume* $(\mathbf{H1}^*)$ *hold. Then, for any* $u_0 \in \mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R})$*, there exists a unique weak entropy solution* u *to (1.1) in* $\mathbf{L}^\infty\left(\mathbb{R}_+;\mathbf{L}^1_{loc}(\mathbb{R}^N;\mathbb{R})\right)$ *continuous from the right. Moreover, if a sequence* $u_0^n \in \mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R})$ *converges to* u_0 *in* $\mathbf{L}^1_{loc}$*, then for all* $t>0$ *the corresponding solutions* $u^n(t)$ *converge to* $u(t)$ *in* $\mathbf{L}^1_{loc}$.

2.1 Estimate on the Total Variation

We give here a result similar to the one obtained by Colombo, Mercier and Rosini [8, Theorem 2.5], but under weaker assumptions.

Theorem 2.2. *Assume that* $(\mathbf{H1}^*)$ *and* $(\mathbf{H2}^*)$ *hold. Let* $u_0 \in (\mathbf{L}^\infty \cap \mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}^N;\mathbb{R})$*. Then, the weak entropy solution* u *of (1.1) satisfies* $u(t) \in \mathbf{BV}(\mathbb{R}^N;\mathbb{R})$ *for all* $t>0$*. Let* T_0 *be real positive. Let us denote* $\mathcal{U} = \|u\|_{\mathbf{L}^\infty([0,T_0]\times\mathbb{R}^N)}$, $U_t = \sup_{y\in\mathbb{R}^N}\left|u(t,y)\right|$, $\mathcal{S}_{T_0}(u) = \bigcup_{t\in[0,T_0]}\mathrm{Supp}(u(t))$ *and*

$$\begin{aligned}\Sigma^u_{T_0} &= [0,T_0]\times\mathcal{S}_{T_0}(u)\times[-\mathcal{U},\mathcal{U}]\,, &(2.5)\\ \kappa_0^* &= (2N+1)\,\|\nabla\,\partial_u f\|_{\mathbf{L}^\infty(\Sigma^u_{T_0};\mathbb{R}^{N\times N})} + \|\partial_u F\|_{\mathbf{L}^\infty(\Sigma^u_{T_0};\mathbb{R})} &(2.6)\end{aligned}$$

then for all $T\in[0,T_0]$*, with* W_N *as in (2.1),*

$$\mathrm{TV}\left(u(T)\right) \leqslant \mathrm{TV}\left(u_0\right)e^{\kappa_0^* T} + NW_N\int_0^T e^{\kappa_0^*(T-t)}\int_{\mathbb{R}^N}\left\|\nabla(F-\mathrm{div}\,f)(t,x,\cdot)\right\|_{\mathbf{L}^\infty([-U_t,U_t];\mathbb{R})}\mathrm{d}x\,\mathrm{d}t\,. \quad (2.7)$$

Remark 2.3. Note that, with $c = \|\partial_u f\|_{\mathbf{L}^\infty(\Omega^{\mathcal{U}}_{T_0})}$, we have $\mathrm{Supp}\,u(t)\subset\mathrm{Supp}\,u_0 + B(0,ct)$. Consequently,

$$\mathcal{S}_{T_0}(u)\subset\mathrm{Supp}\,u_0 + B(0,c\,T_0)\,.$$

We can note here several improvements with respect to [8, Theorem 2.5]. First, as we already noted, the set of hypotheses is weaker since we do not require f to be $\mathscr{C}^2$ and F to be $\mathscr{C}^1$ with respect to the time variable: they only have to be continuous in time, which is useful in applications, see for example [6].

A second improvement stands in the $\mathbf{L}^\infty$ norms, that are taken on smaller domains than in [8].

Last, the expression of the coefficient κ_0^* that does not content any longer the constant NW_N. Indeed, in [8, Theorem 2.5] it was given by

$$\kappa_0 = NW_N\left((2N+1)\|\nabla\partial_u f\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{N\times N})} + \|\partial_u F\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})}\right)$$

Besides, it does not seem possible to erase the coefficient NW_N completely from the expression (2.7), except in the case F and f do not depend on u, see Remark 4.1.

An important corollary of this theorem is that we have now a criterium for having solution continuous in time instead of continuous from the right. This is the analogous of [10, Theorem 4.3.1] for general flows and sources. We use here the same notations as in Theorem 2.2.

Corollary 2.4. *Assume that* (f,F) *satisfy* $(\mathbf{H1}^*)$, $(\mathbf{H2}^*)$ *and* $(\mathbf{H3}^*)$*. Let* $u_0 \in (\mathbf{L}^\infty\cap\mathbf{L}^1\cap\mathbf{BV})(\mathbb{R}^N;\mathbb{R})$ *and let* u *be the weak entropy solution of (1.1). Then* $u\in\mathscr{C}^0([0,T],\mathbf{L}^1(\mathbb{R}^N;\mathbb{R}))$ *and for any* $s,t\in[0,T]$ *we have the estimate*

$$\begin{aligned}\left\|u(t)-u(s)\right\|_{\mathbf{L}^1} \leqslant\ & \left|\int_s^t\int_{\mathbb{R}^N}\left\|(F-\mathrm{div}\,f)(\tau,x,\cdot)\right\|_{\mathbf{L}^\infty([-\mathcal{U},\mathcal{U}];\mathbb{R})}\mathrm{d}x\right| &(2.8)\\ & + |s-t|\|\partial_u f\|_{\mathbf{L}^\infty(\Sigma^u_T)}\sup_{\tau\in[0,T]}\mathrm{TV}\left(u(\tau)\right).\end{aligned}$$

If furthermore, for $T_0>0$*, instead of **(H3*)**, the condition*

$$\sup_{t\in[0,T_0]}\int_{\mathbb{R}^N}\left\|(F-\mathrm{div}\,f)(t,x,\cdot)\right\|_{\mathbf{L}^\infty([-\mathcal{U},\mathcal{U}];\mathbb{R})}\mathrm{d}x\,\mathrm{d}t<\infty$$

holds, then the application $t\in[0,T_0]\to u(t,\cdot)\in\mathbf{L}^1(\mathbb{R}^N,\mathbb{R})$ *is Lipschitz.*

2.2 Stability of Solutions with Respect to Flow and Source

We want now to estimate the difference $u - v$, where

- u is the solution of (1.1) with flow f, source F and initial condition u_0,
- v is the solution of (1.1) with flow g, source G and initial condition v_0.

We search for an estimate of $u - v$ in term of $f - g$, $F - G$ and $u_0 - v_0$.

F. Bouchut & B. Perthame in [2] obtained such an estimate in the particular case f, g depend only on u and $F = G = 0$. The following result is an improvement of the result of R. Colombo, M. Mercier and M. Rosini [8, Theorem 2.6], in which we gave a similar result under stronger assumptions and with a coefficient κ^* that was not compatible with the result of Kružkov (1.2).

Theorem 2.5. *Let (f, F), (g, G) satisfy $(\mathbf{H1}^*)$, (f, F) satisfy $(\mathbf{H2}^*)$ and $(f - g, F - G)$ satisfy $(\mathbf{H3}^*)$. Let $u_0, v_0 \in \mathbf{L}^\infty \cap \mathbf{L}^1 \cap \mathbf{BV}(\mathbb{R}^N; \mathbb{R})$. Let $T > 0$ and let us denote*

$$\begin{aligned} \mathcal{V} &= \max(\|u\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^N)}, \|v\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^N)}) \,, \\ V_t &= \sup_{y\in\mathbb{R}^N}(\left|u(t,y)\right|, \left|v(t,y)\right|) \,, \\ \mathcal{S}_T(u,v) &= \bigcup_{t\in[0,T]} \left(\operatorname{Supp} u(t) \cup \operatorname{Supp} v(t)\right) \,, \\ \Sigma_T^{u,v} &= [0,T] \times \mathcal{S}_T(u,v) \times [-\mathcal{V}, \mathcal{V}] \,. \end{aligned} \tag{2.9}$$

Furthermore, we define κ_0^*, U_t, Σ_T^u as in (2.6) and

$$\kappa^* = \|\partial_u F\|_{\mathbf{L}^\infty(\Sigma_T^{u,v};\mathbb{R})} + \left\|\partial_u \operatorname{div}(g - f)\right\|_{\mathbf{L}^\infty(\Sigma_{T_0}^{u,v})} \,, \qquad M = \|\partial_u g\|_{\mathbf{L}^\infty(\Omega_T^{\mathcal{V}};\mathbb{R}^N)} \,. \tag{2.10}$$

Then, for any $R > 0$ and $x_0 \in \mathbb{R}^N$, the following estimate holds:

$$\begin{aligned} \int_{\|x-x_0\|\leq R} \left|u(T,x) - v(T,x)\right| \mathrm{d}x \;&\leqslant\; e^{\kappa^* T} \int_{\|x-x_0\|\leqslant R+MT} \left|u_0(x) - v_0(x)\right| \mathrm{d}x \\ &+ \frac{e^{\kappa_0^* T} - e^{\kappa^* T}}{\kappa_0^* - \kappa^*} \operatorname{TV}(u_0) \left\|\partial_u(f-g)\right\|_{\mathbf{L}^\infty(\Sigma_T^u;\mathbb{R}^N)} \\ &+ NW_N \left(\int_0^T \frac{e^{\kappa_0^*(T-t)} - e^{\kappa^*(T-t)}}{\kappa_0^* - \kappa^*} \int_{\mathbb{R}^N} \left\|\nabla(F - \operatorname{div} f)(t,x,\cdot)\right\|_{\mathbf{L}^\infty([-U_t,U_t])} \mathrm{d}x \, \mathrm{d}t \right) \\ &\quad \times \left\|\partial_u(f-g)\right\|_{\mathbf{L}^\infty(\Sigma_T^u;\mathbb{R}^N)} \\ &+ \int_0^T e^{\kappa^*(T-t)} \int_{\|x-x_0\|\leqslant R+M(T-t)} \left\| \left((F-G) - \operatorname{div}(f-g)\right)(t,x,\cdot) \right\|_{\mathbf{L}^\infty([-V_t,V_t])} \mathrm{d}x \, \mathrm{d}t \,. \end{aligned}$$

This theorem is a direct consequence of lemma 5.1.

Remark 2.6. Note as above that, with $c' = \max(\|\partial_u f\|_{\mathbf{L}^\infty(\Omega_{T_0}^{\mathcal{U}})}, \|\partial_u g\|_{\mathbf{L}^\infty(\Omega_{T_0}^{\mathcal{V}})})$, we have $\operatorname{Supp} u(t) \subset \operatorname{Supp} u_0 + B(0, c'\,t)$ and $\operatorname{Supp} v(t) \subset \operatorname{Supp} v_0 + B(0, c'\,t)$. Consequently,

$$\mathcal{S}_T(u,v) \subset (\operatorname{Supp} u_0 \cup \operatorname{Supp} v_0) + B(0, c'\,T) \,.$$

Remark 2.7. As above, we can note some improvements with respect to [8, Theorem 2.6]:

- The hypotheses are weaker: no derivative in time is needed for f and F.
- The $\mathbf{L}^\infty$ norms are taken on smaller domains.
- The coefficient κ^* is better than the κ given in [8, Theorem 2.6] by

$$\kappa = 2N\|\nabla\partial_u f\|_{\mathbf{L}^\infty(\Omega;\mathbb{R}^{N\times N})} + \|\partial_u F\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} + \left\|\partial_u(F - G)\right\|_{\mathbf{L}^\infty(\Omega;\mathbb{R})} \,.$$

Indeed, κ^* coincides with γ in the case $f = g$ and consequently we recover the previous Kružkov's result (1.2), which was not the case with κ.

Remark 2.8. Note that, if $\kappa_0^* \geqslant \kappa^*$ then

$$\frac{e^{\kappa_0^* t} - e^{\kappa^* t}}{\kappa_0^* - \kappa^*} \leqslant e^{\kappa^* t} \int_0^t e^{(\kappa_0^* - \kappa^*)\tau} \, \mathrm{d}\tau \leqslant t e^{\kappa_0^* t} \, .$$

As the expression is symmetric, we can conclude in the general case that, denoting $\kappa_1 = \max(\kappa_0^*, \kappa^*)$, we have $\frac{e^{\kappa_0^* t} - e^{\kappa^* t}}{\kappa_0^* - \kappa^*} \leqslant t e^{\kappa_1 t}$. Let us assume that $\kappa_0^* \geqslant \kappa^*$; then the estimate of Theorem 2.5 can be rewritten

$$\begin{aligned}
\int_{\|x-x_0\| \leq R} \left| u(T,x) - v(T,x) \right| \mathrm{d}x \ &\leqslant \ e^{\kappa^* T} \int_{\|x-x_0\| \leqslant R+MT} \left| u_0(x) - v_0(x) \right| \mathrm{d}x \\
+ \ & T \, \mathrm{TV}\,(u(T)) \left\| \partial_u (f-g) \right\|_{\mathbf{L}^\infty(\Sigma_T^u; \mathbb{R}^N)} \\
+ \ & e^{\kappa^* T} \int_0^T \int_{\|x-x_0\| \leqslant R+M(T-t)} \left\| \big((F-G) - \mathrm{div}\,(f-g) \big) (t,x,\cdot) \right\|_{\mathbf{L}^\infty([-V_t, V_t])} \mathrm{d}x \, \mathrm{d}t \, .
\end{aligned}$$

Another consequence of Lemma 5.1 is the following proposition.

Proposition 2.9. *Let (f,F), (g,G) satisfy $(\mathbf{H1}^*)$, (f,F) satisfy $(\mathbf{H2}^*)$ and $(f-g, F-G)$ satisfy $(\mathbf{H3}^*)$. Let $u_0, v_0 \in \mathbf{L}^\infty \cap \mathbf{L}^1 \cap \mathbf{BV}(\mathbb{R}^N; \mathbb{R})$. Let $T > 0$. Then, using the same notation as in (2.9)–(2.10), for any $R > 0$ and $x_0 \in \mathbb{R}^N$, the following estimate holds:*

$$\begin{aligned}
\int_{\|x-x_0\| \leq R} \left| u(T,x) - v(T,x) \right| \mathrm{d}x \ &\leqslant \ e^{\kappa^* T} \int_{\|x-x_0\| \leqslant R+MT} \left| u_0(x) - v_0(x) \right| \mathrm{d}x \\
+ \ & \left[\mathrm{TV}\,(u_0) + N W_N \int_0^T e^{-\kappa_0^* t} \int_{\mathbb{R}^N} \left\| \nabla (F - \mathrm{div}\, f)(t,x,\cdot) \right\|_{\mathbf{L}^\infty([-U_t, U_t])} \mathrm{d}x \, \mathrm{d}t \right] \\
& \times \frac{\kappa_0^* e^{\kappa_0^* t} - \kappa^* e^{\kappa^* t}}{\kappa_0^* - \kappa^*} \int_0^T \left\| \partial_u (f-g)(t) \right\|_{\mathbf{L}^\infty(\mathcal{S}_T \times [-V_t, V_t])} \mathrm{d}t \\
+ \ & e^{\kappa^* T} \int_0^T \int_{\|x-x_0\| \leqslant R+M(T-t)} \left\| \big((F-G) - \mathrm{div}\,(f-g) \big) (t,x,\cdot) \right\|_{\mathbf{L}^\infty([-V_t, V_t])} \mathrm{d}x \, \mathrm{d}t \, .
\end{aligned}$$

This proposition is useful in [5], where we studied the equation

$$\partial_t u + \mathrm{div}\,(u(1-u) w(u *_x \eta)) = 0 \, , \qquad\qquad u(0,\cdot) = \bar{u} \, ,$$

and in particular, the stability with respect to η. The use of proposition 2.9 allows then to apply Gronwall lemma and gives us the following stability result. We assume here that we have existence and uniqueness of weak entropy solutions, as obtained in [5].

Proposition 2.10. *Let $w \in \mathbf{Lip}\,(\mathbb{R}, \mathbb{R})$ be such that $w' \in \mathbf{W}^{1,\infty}(\mathbb{R}, \mathbb{R})$, $\eta_1, \eta_2 \in \mathbf{W}^{2,1} \cap \mathbf{W}^{1,\infty}(\mathbb{R}^N, \mathbb{R})$, $\bar{u} \in \mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV}(\mathbb{R}^N, [0,1])$. Let $u_1, u_2 \in \mathscr{C}^0(\mathbb{R}_+, \mathbf{L}^1(\mathbb{R}^N, [0,1]))$ be weak entropy solutions to the Cauchy problems (for $i = 1, 2$):*

$$\partial_t u_i + \mathrm{div}\,(u_i(1-u_i) w(u_i *_x \eta_i)) = 0 \, , \qquad\qquad u_i(0,\cdot) = \bar{u} \, .$$

Then, we have the stability estimate:

$$\left\| (u_1 - u_2)(t) \right\|_{\mathbf{L}^1} \leqslant C(t) \|\eta_1 - \eta_2\|_{\mathbf{W}^{1,1}} \, ,$$

where C depends on $\|\bar{u}\|_{\mathbf{L}^1}, \|u_1\|_{\mathbf{L}^\infty([0,T] \times \mathbb{R}^N)}, \|u_2\|_{\mathbf{L}^\infty([0,T] \times \mathbb{R}^N)}$ and on various norms on η and w.

Proof. Applying Theorem 2.9, we obtain

$$\left\| u_1(T) - u_2(T) \right\|_{\mathbf{L}^1} \leqslant a(T) + b(T) \int_0^T \left\| u_1 - u_2(t) \right\|_{\mathbf{L}^1} \mathrm{d}t$$

where a and b are regular and increasing functions of T. Applying Gronwall Lemma, we obtain the desired estimate. □

3 Tools on functions with bounded variation

Recall the following theorem (see [1, Theorem 3.9 and Remark 3.10]):

Theorem 3.1. *Let $u \in \mathbf{L}^1_{loc}(\mathbb{R}^N;\mathbb{R})$. Then $u \in \mathbf{BV}(\mathbb{R}^N;\mathbb{R})$ if and only if there exists a sequence (u_n) in $\mathscr{C}^\infty(\mathbb{R}^N;\mathbb{R})$ converging to u in $\mathbf{L}^1_{loc}$ and satisfying*

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} \left\|\nabla u_n(x)\right\| \mathrm{d}x = L \quad \text{with} \quad L < \infty\,.$$

Moreover, $\mathrm{TV}\,(u)$ is the smallest constant L for which there exists a sequence as above.

Let us also recall the following property of any function $u \in \mathbf{BV}(\mathbb{R}^N;\mathbb{R})$:

$$\int_{\mathbb{R}^N} \left|u(x) - u(x-z)\right| \mathrm{d}x \leqslant \|z\|\ \mathrm{TV}\,(u) \qquad \text{for all } z \in \mathbb{R}^N. \tag{3.1}$$

For a proof, see [1, Remark 3.25].

Now, in a similar way as J. Dávila [11], we prove the following proposition, which is an improvement of [8, Proposition 4.3]. Indeed, in [8, Proposition 4.3], the equality (3.3) is valid only for $u \in \mathscr{C}^1$. In the present proposition we extend this result to all $u \in \mathbf{BV}$.

Proposition 3.2. *Let $\rho_1 \in \mathscr{C}^\infty_c(\mathbb{R},\mathbb{R}_+)$ with $\operatorname{Supp}\rho_1 \subset [-1,1]$. Let $u \in \mathbf{L}^1_{loc}(\mathbb{R}^N;\mathbb{R})$. For all $\lambda > 0$, we introduce ρ_λ such that $\rho_\lambda(x) = \frac{1}{\lambda^N}\rho_1\left(\frac{\|x\|}{\lambda}\right)$. Assume that there exists a constant $\widetilde{C}$ such that for all λ, R positive,*

$$\frac{1}{\lambda}\int_{\mathbb{R}^N}\int_{B(x_0,R)} \left|u(x) - u(x-z)\right| \rho_\lambda(z)\,\mathrm{d}x\,\mathrm{d}z \leqslant \widetilde{C}\,. \tag{3.2}$$

Then $u \in \mathbf{BV}(\mathbb{R}^N;\mathbb{R})$ and

$$\mathrm{TV}\,(u) = \frac{1}{C_1}\lim_{\lambda\to0}\frac{1}{\lambda}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left|u(x) - u(x-z)\right| \rho_\lambda(z)\,\mathrm{d}x\,\mathrm{d}z\,, \tag{3.3}$$

where

$$C_1 = \int_{\mathbb{R}^N} |x_1|\,\rho_1\left(\|x\|\right)\,\mathrm{d}x\,. \tag{3.4}$$

Proof.

Note that the first part of the proof is the same as the first part of the proof of [8, Proposition 4.3]. We introduce a regularisation of u: $u_h = u * \mu_h$, with $\mu_h(x) = \mu_1\left(\|x\|/h\right)/h^N$, where μ_1 is defined as in (6.1). We note that $u_h \in \mathscr{C}^\infty(\mathbb{R}^N;\mathbb{R})$ and that u_h tends to u in $\mathbf{L}^1_{loc}$ when $h \to 0$. Furthermore, for R and h positive, by change of variables we get

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{B(x_0,R-h)} \left|\int_0^1 \nabla u_h(x-\lambda s z)\cdot z\,\mathrm{d}s\right| \rho_1(\|z\|)\,\mathrm{d}x\,\mathrm{d}z \\
=\;& \frac{1}{\lambda}\int_{\mathbb{R}^N}\int_{B(x_0,R-h)} \left|u_h(x) - u_h(x-\lambda z)\right| \rho_1(\|z\|)\,\mathrm{d}x\,\mathrm{d}z \\
\leqslant\;& \frac{1}{\lambda}\int_{\mathbb{R}^N}\int_{B(x_0,R)} \left|u(x) - u(x-\lambda z)\right| \rho_1(\|z\|)\,\mathrm{d}x\,\mathrm{d}z \\
\leqslant\;& \widetilde{C}\,.
\end{aligned}$$

Making $R \to \infty$ and using the Dominated Convergence Theorem when $\lambda \to 0$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left|\nabla u_h(x)\cdot z\right| \rho_1(\|z\|)\,\mathrm{d}x\,\mathrm{d}z \\
\leqslant\;& \liminf_{\lambda\to0}\frac{1}{\lambda}\int_{\mathbb{R}^N}\int_{B(x_0,R)} \left|u(x) - u(x-\lambda z)\right| \rho_1(\|z\|)\,\mathrm{d}x\,\mathrm{d}z\,.
\end{aligned}$$

Remark that for fixed $x \in \mathbb{R}^N$, when $\nabla u_h(x) \neq 0$, the scalar product $\nabla u_h(x) \cdot z$ is positive (respectively, negative) when z is in a half-space, say H_x^+ (respectively, H_x^-). We can write $z = \alpha \frac{\nabla u_h(x)}{\|\nabla u_h(x)\|} + w$, with $\alpha \in \mathbb{R}$ and w in the hyperplane $H_x^o = \nabla u_h(x)^\perp$. Hence

$$\begin{aligned}
\int_{\mathbb{R}^N} \left|\nabla u_h(x) \cdot z\right| \mu_1(\|z\|) \, \mathrm{d}z &= \int_{H_x^+} \nabla u_h(x) \cdot z \, \mu_1(\|z\|) \, \mathrm{d}z + \int_{H_x^-} \nabla u_h(x) \cdot (-z) \, \mu_1(\|z\|) \, \mathrm{d}z \\
&= 2 \int_{H_x^+} \nabla u_h(x) \cdot z \, \mu_1(\|z\|) \, \mathrm{d}z \\
&= 2 \int_{\mathbb{R}_+^*} \int_{H_x^o} \alpha \left\|\nabla u_h(x)\right\| \mu_1(\sqrt{\alpha^2 + \|w\|^2}) \, \mathrm{d}w \, \mathrm{d}\alpha \\
&= \int_{\mathbb{R}} \int_{H_x^o} |\alpha| \left\|\nabla u_h(x)\right\| \mu_1(\sqrt{\alpha^2 + \|w\|^2}) \, \mathrm{d}w \, \mathrm{d}\alpha \\
&= \left\|\nabla u_h(x)\right\| \int_{\mathbb{R}^N} |z_1| \, \mu_1(\|z\|) \, \mathrm{d}z \, .
\end{aligned}$$

So we obtain

$$\mathrm{TV}\,(u) \leqslant \frac{1}{C_1} \liminf_{\lambda \to 0} \frac{1}{\lambda} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left|u(x) - u(x-z)\right| \rho_\lambda(z) \, \mathrm{d}x \, \mathrm{d}z \leqslant \frac{\widetilde{C}}{C_1} \, . \tag{3.5}$$

Now, let (u_n) be a sequence of functions in $\mathscr{C}^\infty(\mathbb{R}^N, \mathbb{R})$ converging to u in $\mathbf{L}^1_{loc}$ and such that $\int_{\mathbb{R}^N} \left\|\nabla u_n(x)\right\| \mathrm{d}x$ converges to $\mathrm{TV}\,(u)$ when $n \to \infty$. Then, doing the same computation as above, we obtain

$$\begin{aligned}
& \frac{1}{\lambda} \int_{\mathbb{R}^N} \int_{B(x_0,R)} \left|u_n(x) - u_n(x - \lambda z)\right| \rho_1(\|z\|) \, \mathrm{d}x \, \mathrm{d}z \\
\leqslant & \int_{\mathbb{R}^N} \int_{B(x_0,R)} \int_0^1 \left|\nabla u_n(x - \lambda s z) \cdot z\right| \rho_1(\|z\|) \, \mathrm{d}s \, \mathrm{d}x \, \mathrm{d}z \\
\leqslant & \int_{\mathbb{R}^N} \int_0^1 \int_{B(x_0,R+\lambda)} \left|\nabla u_n(x') \cdot z\right| \rho_1(\|z\|) \, \mathrm{d}x' \, \mathrm{d}s \, \mathrm{d}z \\
= & \int_{B(x_0,R+\lambda)} \left\|\nabla u_n(x)\right\| C_1 \, \mathrm{d}x \\
\leqslant & \; C_1 \, \mathrm{TV}\,(u_n, B(x_0, R+\lambda)) \, .
\end{aligned}$$

Taking $R \to \infty$ and then $n \to \infty$, we have consequently

$$\frac{1}{\lambda} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left|u(x) - u(x - \lambda z)\right| \rho_1(\|z\|) \, \mathrm{d}x \, \mathrm{d}z \leqslant C_1 \, \mathrm{TV}\,(u) \, .$$

Then, we take the supremum limit when λ goes to 0. We obtain

$$\limsup_{\lambda \to 0} \frac{1}{\lambda} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left|u(x) - u(x-z)\right| \rho_\lambda(z) \, \mathrm{d}x \, \mathrm{d}z \leqslant C_1 \, \mathrm{TV}\,(u) \, . \tag{3.6}$$

We conclude the proof by reassembling (3.5) and (3.6). □

4 Proof of the Total Variation estimate

The following proof is quite similar to the one of [8, Theorem 2.5]. The differences come from the use of Proposition 3.2 instead of [8, Proposition 4.3] and from avoiding the derivatives in time to appear. In order to be clear, we rewrite here most of the steps of the proof. In particular, the beginning of the proof is similar to [8, proof of Theorem 2.5] up to (4.10).

Proof of Theorem 2.2. First, we assume that $u_0 \in \mathscr{C}^1(\mathbb{R}^N;\mathbb{R})$. The general case will be considered only at the end of this proof.

By Kružkov Theorem [12, Theorem 5 & Section 5 Remark 4], the set of hypotheses **(H1*)** gives us existence and uniqueness of a weak entropy solution for any initial condition $u_0 \in \mathbf{L}^\infty \cap \mathbf{L}^1(\mathbb{R}^N;\mathbb{R})$. Let u be the weak entropy solution to (1.1) associated to $u_0 \in (\mathbf{L}^\infty \cap \mathbf{L}^1 \cap \mathbf{BV})(\mathbb{R}^N;\mathbb{R})$. Let us denote $u = u(t,x)$ and $v = u(s,y)$ for $(t,x),(s,y) \in \mathbb{R}_+^* \times \mathbb{R}^N$. Then, for all $k,l \in \mathbb{R}$ and for all test functions $\varphi = \varphi(t,x,s,y)$ in $\mathscr{C}_c^1\left((\mathbb{R}_+^* \times \mathbb{R}^N)^2;\mathbb{R}_+\right)$, we have

$$\begin{aligned}\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\Big[(u-k)\,\partial_t\varphi + \big(f(t,x,u)-f(t,x,k)\big)\,\nabla_x\varphi + \big(F(t,x,u)-\operatorname{div} f(t,x,k)\big)\,\varphi\Big]&\\ \times\operatorname{sign}(u-k)\,\mathrm{d}x\,\mathrm{d}t \quad &\geqslant 0\end{aligned} \tag{4.1}$$

for all $(s,y) \in \mathbb{R}_+^* \times \mathbb{R}^N$, and

$$\begin{aligned}\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\Big[(v-l)\,\partial_s\varphi + \big(f(s,y,v)-f(s,y,l)\big)\,\nabla_y\varphi + (F(s,y,v)-\operatorname{div} f(s,y,l))\varphi\Big]&\\ \times\operatorname{sign}(v-l)\,\mathrm{d}y\,\mathrm{d}s \quad &\geqslant 0\end{aligned} \tag{4.2}$$

for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}^N$. Let $\Phi \in \mathscr{C}_c^\infty(\mathbb{R}_+^* \times \mathbb{R}^N;\mathbb{R}_+)$, $\Psi \in \mathscr{C}_c^\infty(\mathbb{R} \times \mathbb{R}^N;\mathbb{R}_+)$ and set

$$\varphi(t,x,s,y) = \Phi(t,x)\,\Psi(t-s,x-y)\,. \tag{4.3}$$

Observe that $\partial_t\varphi + \partial_s\varphi = \Psi\,\partial_t\Phi$, $\nabla_x\varphi = \Psi\,\nabla_x\Phi + \Phi\,\nabla_x\Psi$, $\nabla_y\varphi = -\Phi\,\nabla_x\Psi$. Choose $k = v(s,y)$ in (4.1) and integrate with respect to (s,y). Analogously, take $l = u(t,x)$ in (4.2) and integrate with respect to (t,x). Summing the obtained inequalities, we obtain

$$\begin{aligned}\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\operatorname{sign}(u-v)\Bigg[&(u-v)\,\Psi\,\partial_t\Phi + \big(f(t,x,u)-f(t,x,v)\big)\cdot(\nabla\Phi)\,\Psi\\ &+\big(f(s,y,v)-f(s,y,u)-f(t,x,v)+f(t,x,u)\big)\cdot(\nabla\Psi)\,\Phi\\ +\big(F(t,x,u)-F(s,y,v)+\operatorname{div} f(s,y,u)-\operatorname{div} f(t,x,v)\big)\,\varphi\Bigg]&\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s \quad \geqslant 0.\end{aligned} \tag{4.4}$$

Introduce a family of functions $\{Y_\vartheta\}_{\vartheta>0}$ such that for any $\vartheta > 0$:

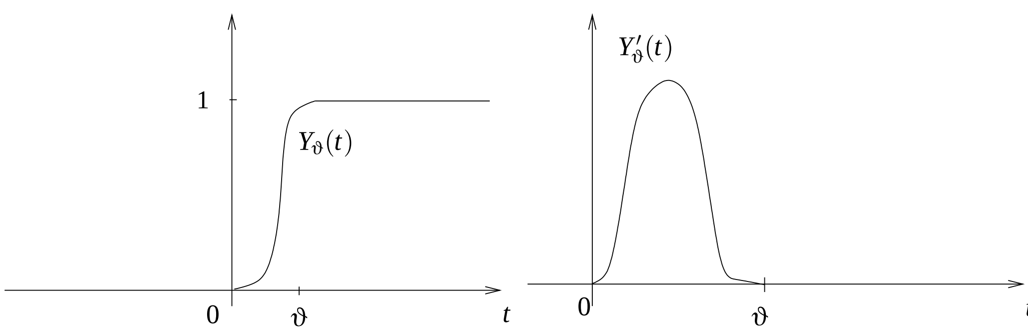

Figure 1: Graphs of Y_ϑ, left, and of Y'_ϑ, right.

$$\begin{aligned}&Y_\vartheta(t) = \int_{-\infty}^t Y'_\vartheta(s)\,\mathrm{d}s\,, && Y'_\vartheta(t) = \frac{1}{\vartheta}\,Y'\left(\frac{t}{\vartheta}\right), && Y' \in \mathscr{C}_c^\infty(\mathbb{R};\mathbb{R})\,,\\ &\operatorname{Supp}(Y') \subset\,]0,1[\;, && Y' \geqslant 0\,, && \int_{\mathbb{R}} Y'(s)\,\mathrm{d}s = 1\,.\end{aligned} \tag{4.5}$$

Let $T_0 > 0$, $\mathcal{U} = \|u\|_{\mathbf{L}^\infty([0,T_0]\times\mathbb{R}^N;\mathbb{R})}$ and $M = \|\partial_u f\|_{\mathbf{L}^\infty(\Omega^{\mathcal{U}}_{T_0};\mathbb{R}^N)}$ which is bounded by **(H1*)**. Let us also define, for $\varepsilon, \theta, R > 0$, $x_0 \in \mathbb{R}^N$, (see Figure 2):

$$\chi(t) = Y_\varepsilon(t) - Y_\varepsilon(t-T) \quad \text{and} \quad \psi(t,x) = 1 - Y_\theta\left(\|x-x_0\| - R - M(T_0-t)\right) \geqslant 0, \tag{4.6}$$

where we also need the compatibility conditions $T_0 \geqslant T$ and $M\varepsilon \leqslant R + M(T_0-T)$.

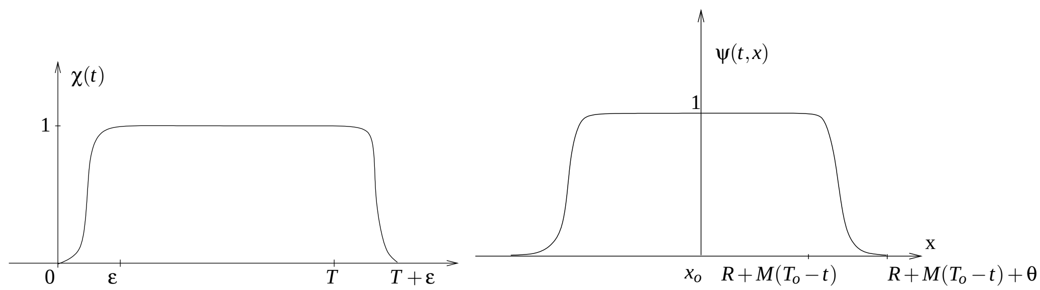

Figure 2: Graphs of χ, left, and of ψ, right.

Observe that $\chi \to \mathbf{1}_{[0,T]}$ and $\chi' \to \delta_0 - \delta_T$ as ε tends to 0. On χ and ψ we use the bounds

$$\chi \leqslant \mathbf{1}_{[0,T+\varepsilon]} \quad \text{and} \quad \mathbf{1}_{B(x_0,R+M(T_0-t))} \leqslant \psi \leqslant \mathbf{1}_{B(x_0,R+M(T_0-t)+\theta)}\,.$$

In (4.4), choose $\Phi(t,x) = \chi(t)\,\psi(t,x)$. With this choice, we have

$$\partial_t\Phi = \chi'\,\psi - M\,\chi\,Y'_\theta \quad \text{and} \quad \nabla\Phi = -\chi\,Y'_\theta\,\frac{x-x_0}{\|x-x_0\|}\,. \tag{4.7}$$

Setting $B(t,x,u,v) = |u-v|M + \operatorname{sign}(u-v)\left(f(t,x,u) - f(t,x,v)\right)\cdot\dfrac{x-x_0}{\|x-x_0\|}$, the first line in (4.4) becomes

$$\begin{aligned}
&\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\Big[(u-v)\Psi\,\partial_t\Phi + \big(f(t,x,u)-f(t,x,v)\big)\cdot(\nabla\Phi)\,\Psi\Big]\operatorname{sign}(u-v)\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\\
=\;&\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\left(|u-v|\,\chi'\,\psi - B(t,x,u,v)\chi\,Y'_\theta\right)\,\Psi\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\\
\leqslant\;&\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}|u-v|\,\chi'\,\psi\,\Psi\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\,,
\end{aligned}$$

since $B(t,x,u,v)$ is positive for all $(t,x,u,v)\in\Omega\times\mathbb{R}$. Due to the above estimate and to (4.4), we have

$$\begin{aligned}
\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\;&\Big[(u-v)\,\chi'\,\psi\,\Psi\\
&+\big(f(s,y,v)-f(s,y,u)-f(t,x,v)+f(t,x,u)\big)\cdot(\nabla\Psi)\,\Phi\\
&+\big(F(t,x,u)-F(s,y,v)-\operatorname{div} f(t,x,v)+\operatorname{div} f(s,y,u)\big)\,\varphi\Big]\\
&\times\operatorname{sign}(u-v)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s \qquad\qquad \geqslant\; 0.
\end{aligned}$$

Now, we aim at bounds for each term of this sum. Introduce the following notations:

$$
\begin{aligned}
I &= \int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N} |u-v|\,\chi'\,\psi\,\Psi\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\,,\\
J_x &= \int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N} \big(f(t,y,v)-f(t,y,u)+f(t,x,u)-f(t,x,v)\big)\cdot(\nabla\Psi)\;\Phi\\
&\qquad\qquad \times \operatorname{sign}(u-v)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\,,\\
J_t &= \int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N} \big(f(s,y,v)-f(s,y,u)+f(t,y,u)-f(t,y,v)\big)\cdot(\nabla\Psi)\;\Phi\\
&\qquad\qquad \times \operatorname{sign}(u-v)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\,,\\
L_1 &= \int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N} \big[\operatorname{div} f(t,x,v)-\operatorname{div} f(t,x,u)+F(t,y,v)-F(t,y,u)\big]\\
&\qquad\qquad \varphi\operatorname{sign}(u-v)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s \qquad (4.8)\\
L_2 &= \int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N} \left[\int_0^1 \nabla(F-\operatorname{div} f)\big(t,rx+(1-r)y,u\big)\cdot(x-y)\,\mathrm{d}r\right]\varphi\\
&\qquad\qquad \times\operatorname{sign}(u-v)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\,. \qquad (4.9)\\
L_t &= \int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N} \big(F(t,y,v)-F(s,y,v)-\operatorname{div} f(t,y,u)+\operatorname{div} f(s,y,u)\big)\;\varphi\\
&\qquad\qquad \times \operatorname{sign}(u-v)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\,.
\end{aligned}
$$

Then, the above inequality is rewritten as $I+J_x+J_t+L_1+L_2+L_t \geqslant 0$. Choose $\Psi(t,x)=\nu(t)\,\mu(x)$ where, for $\eta,\lambda>0$, $\mu\in\mathscr{C}_c^\infty(\mathbb{R}_+;\mathbb{R}_+)$ satisfies (6.1)–(6.2) and

$$
\nu(t)=\frac{1}{\eta}\,\nu_1\left(\frac{t}{\eta}\right),\quad \int_{\mathbb{R}}\nu_1(s)\,\mathrm{d}s=1\,,\quad \nu_1\in\mathscr{C}_c^\infty(\mathbb{R};\mathbb{R}_+)\,,\quad \operatorname{supp}(\nu_1)\subset\,]-1,0[\;. \tag{4.10}
$$

Now, we want to estimate separately I, J_x, J_t, L_1, L_2 and L_t. Note first that if $x,y\in\mathbb{R}^N\setminus\{\bigcup_{t\in[0,T_0]}\operatorname{Supp} u(t)\}$, the integrand in J_x and L_1 vanishes, so denoting

$$
\mathcal{S}_T(u)=\bigcup_{t\in[0,T_0]}\operatorname{Supp} u(t)\,, \tag{4.11}
$$

the space of integration of J_x and L_1 is in fact $\mathbb{R}_+\times\mathcal{S}_T(u)\times\mathbb{R}_+\times\mathcal{S}_T(u)$. The main differences with respect to the proof of [8, Theorem 2.5] are the following:

- The $\mathbf{L}^\infty$ norm that we took on $\mathbb{R}_+\times\mathbb{R}^N\times\mathbb{R}$, are now taken on $\Sigma_{T_0}^u=[0,T_0]\times\mathcal{S}_T(u)\times[-\mathcal{U},\mathcal{U}]$, where $\mathcal{U}=\sup(\left\|u(t)\right\|_{\mathbf{L}^\infty(\mathbb{R}^N)}, t\in[0,T_0])$.
- For J_t and L_t, by Dominated Convergence Theorem, we get when $\eta\to 0$

$$
\lim_{\eta\to 0} J_t=\lim_{\eta\to 0} L_t=0\,, \tag{4.12}
$$

  which avoids the use of time derivatives.
- The $\mathbf{L}^\infty$ norm of u in L_2 is now taken on $[-U_t,U_t]$ where $U_t=\left\|u(t)\right\|_{\mathbf{L}^\infty(\mathbb{R}^N)}$.

We do not rewrite the estimates on I, J_x, L_1, L_2, that are the same as in [8, Theorem 2.5], up to the space in the $\mathbf{L}^\infty$ norm. See remark 4.1 for precisions on the estimate of L_2.

Letting $\varepsilon, \eta, \theta \to 0$ we get

$$\begin{aligned}
\limsup_{\varepsilon,\eta,\theta\to 0} I &= \int_{\mathbb{R}^N}\int_{\|x-x_0\|\leqslant R+MT_0} \left|u(0,x)-u(0,y)\right| \mu(x-y)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad -\int_{\mathbb{R}^N}\int_{\|x-x_0\|\leqslant R+M(T_0-T)} \left|u(T,x)-u(T,y)\right| \mu(x-y)\,\mathrm{d}x\,\mathrm{d}y\,, \\
\limsup_{\varepsilon,\eta,\theta\to 0} J_x &\leqslant \|\nabla\partial_u f\|_{\mathbf{L}^\infty(\Sigma^u_{T_0})} \int_0^T\int_{\mathbb{R}^N}\int_{B(x_0,R+M(T_0-t))} \|x-y\|\left|u(t,x)-u(t,y)\right| \\
&\qquad \times \left\|\nabla\mu(x-y)\right\|\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,, \\
\limsup_{\varepsilon,\eta,\theta\to 0} J_t &= 0\,, \\
\limsup_{\varepsilon,\eta,\theta\to 0} L_1 &\leqslant \left(N\|\nabla\partial_u f\|_{\mathbf{L}^\infty(\Sigma^u_{T_0})} + \|\partial_u F\|_{\mathbf{L}^\infty(\Sigma^u_{T_0})}\right) \\
&\qquad \times \int_0^T\int_{\mathbb{R}^N}\int_{\|x-x_0\|\leqslant R+M(T_0-t)} \left|u(t,x)-u(t,y)\right|\mu(x-y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,, \\
\limsup_{\varepsilon,\eta,\theta\to 0} L_2 &= \lambda M_1 \int_0^T\int_{\mathbb{R}^N} \left\|\nabla(F-\operatorname{div} f)(t,y,\cdot)\right\|_{\mathbf{L}^\infty([-U_t,U_t])}\,\mathrm{d}y\,\mathrm{d}t\,, \\
\limsup_{\varepsilon,\eta,\theta\to 0} L_t &= 0\,,
\end{aligned}$$

where

$$M_1 = \int_{\mathbb{R}^N} \|x\|\,\mu_1\left(\|x\|\right)\,\mathrm{d}x\,. \tag{4.13}$$

Above, the right hand sides are bounded thanks to **(H2*)**.

Collating all the obtained results and using the equality, $\left\|\nabla\mu(x)\right\| = -\frac{1}{\lambda^{N+1}}\mu_1'\left(\frac{\|x\|}{\lambda}\right)$

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\|x-x_0\|\leqslant R+M(T_0-T)} \left|u(T,x)-u(T,y)\right| \frac{1}{\lambda^N}\mu_1\left(\frac{\|x-y\|}{\lambda}\right)\mathrm{d}x\,\mathrm{d}y \\
\leqslant\;& \int_{\mathbb{R}^N}\int_{\|x-x_0\|\leqslant R+MT_0} \left|u(0,x)-u(0,y)\right| \frac{1}{\lambda^N}\mu_1\left(\frac{\|x-y\|}{\lambda}\right)\mathrm{d}x\,\mathrm{d}y \\
&-\|\nabla\partial_u f\|_{\mathbf{L}^\infty(\Sigma^u_{T_0})}\int_0^T\int_{\mathbb{R}^N}\int_{\|x-x_0\|\leqslant R+M(T_0-t)} \left|u(t,x)-u(t,y)\right| \\
&\qquad \times\frac{1}{\lambda^{N+1}}\,\mu_1'\left(\frac{\|x-y\|}{\lambda}\right)\|x-y\|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&+\left(N\|\nabla\partial_u f\|_{\mathbf{L}^\infty(\Sigma^u_{T_0})} + \|\partial_u F\|_{\mathbf{L}^\infty(\Sigma^u_{T_0})}\right)\int_0^T\int_{\mathbb{R}^N}\int_{\|x-x_0\|\leqslant R+M(T_0-t)} \left|u(t,x)-u(t,y)\right| \\
&\qquad \times\frac{1}{\lambda^N}\mu_1\left(\frac{\|x-y\|}{\lambda}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&+\lambda M_1\int_0^T\int_{\mathbb{R}^N}\left\|\nabla(F-\operatorname{div} f)(t,y,\cdot)\right\|_{\mathbf{L}^\infty([-U_t,U_t])}\,\mathrm{d}y\,\mathrm{d}t\,.
\end{aligned} \tag{4.14}$$

If $\|\nabla\partial_u f\|_{\mathbf{L}^\infty(\Sigma^u_{T_0})} = \|\partial_u F\|_{\mathbf{L}^\infty(\Sigma^u_{T_0})} = 0$ and under the present assumption that $u_0 \in \mathscr{C}^1(\mathbb{R}^N;\mathbb{R})$, using Proposition 3.2, (3.4) and (4.13), we directly obtain that

$$\mathrm{TV}\left(u(T)\right) \leqslant \mathrm{TV}\left(u_0\right) + \frac{M_1}{C_1}\int_0^T\int_{\mathbb{R}^N}\left\|\nabla(F-\operatorname{div} f)(t,y,\cdot)\right\|_{\mathbf{L}^\infty([-U_t,U_t])}\mathrm{d}y\,\mathrm{d}t\,. \tag{4.15}$$

The same procedure at the end of this proof allows to extend (4.15) to more general initial data, providing an estimate of $\mathrm{TV}\left(u(t)\right)$ in the situation studied in [2].

Now, it remains to treat the case when $\|\nabla\partial_u f\|_{\mathbf{L}^\infty(\Sigma^u_{T_0})} \neq 0$. As in [8, Theorem 2.5], a direct use of Gronwall lemma is not possible, but we can first obtain an estimate of the function:

$$\mathcal{F}(T,\lambda) = \int_0^T \int_{\mathbb{R}^N} \int_{\|x-x_0\|\leqslant R+M(T_0-t)} \left|u(t,x)-u(t,x-z)\right| \frac{1}{\lambda^N}\,\mu_1\left(\frac{\|z\|}{\lambda}\right) \mathrm{d}x\,\mathrm{d}z\,\mathrm{d}t\,.$$

Indeed, we get that if T is such that

$$T < \frac{1}{(1+2N)\|\nabla\partial_u f\|_{\mathbf{L}^\infty(\Sigma^u_{T_0})} + \|\partial_u F\|_{\mathbf{L}^\infty(\Sigma^u_{T_0})}}\,,$$

then we obtain, with $\alpha = \left(2N\|\nabla\partial_u f\|_{\mathbf{L}^\infty} + \|\partial_u F\|_{\mathbf{L}^\infty} - \frac{1}{T}\right)\left(\|\nabla\partial_u f\|_{\mathbf{L}^\infty(\Sigma^u_{T_0})}\right)^{-1} < -1$,

$$\frac{1}{\lambda}\mathcal{F}(T',\lambda) \leqslant \frac{1}{-\alpha-1}\left(M_1\,\mathrm{TV}\,(u_0) + C(T')\right)\frac{1}{\|\nabla\partial_u f\|_{\mathbf{L}^\infty(\Sigma^u_{T_0})}}\,. \tag{4.16}$$

Furthermore, by (6.1) and (6.2) there exists a constant $Q>0$ such that for all $z\in\mathbb{R}^N$

$$-\,\mu_1'(\|z\|) \leqslant Q\,\mu_1\left(\frac{\|z\|}{2}\right)\,. \tag{4.17}$$

Divide both sides in (4.14) by λ, rewrite them using (4.16), (4.17), apply (3.1) and obtain

$$\begin{aligned}
&\frac{1}{\lambda}\int_{\mathbb{R}^N}\int_{\|x-x_0\|\leqslant R+M(T_0-T)} \left|u(T,x)-u(T,y)\right| \frac{1}{\lambda^N}\mu_1\left(\frac{\|x-y\|}{\lambda}\right)\mathrm{d}x\,\mathrm{d}y\\
\leqslant\;& M_1\,\mathrm{TV}\,(u_0) + \frac{\mathcal{F}(T,\lambda)}{\lambda}\left(2N\|\nabla\partial_u f\|_{\mathbf{L}^\infty(\Sigma^u_{T_0})} + \|\partial_u F\|_{\mathbf{L}^\infty(\Sigma^u_{T_0})}\right)\\
&+\frac{\mathcal{F}(T,2\lambda)}{2\lambda}2^{N+2}\,Q\,\|\nabla\partial_u f\|_{\mathbf{L}^\infty(\Sigma^u_{T_0})}\\
&+M_1\int_0^T\int_{\mathbb{R}^N}\left\|\nabla(F-\mathrm{div}\,f)(t,y,\cdot)\right\|_{\mathbf{L}^\infty([-U_t,U_t])}\mathrm{d}y\,\mathrm{d}t\,.
\end{aligned}$$

An application of (4.16) yields an estimate of the type

$$\frac{1}{\lambda}\int_{\mathbb{R}^N}\int_{B(x_0,R+M(T_0-T))}\left|u(T,x)-u(T,x-z)\right|\mu(z)\,\mathrm{d}x\,\mathrm{d}z \leqslant \check{C}\,, \tag{4.18}$$

where the positive constant $\check{C}$ is independent from R and λ. Applying Proposition 3.2 we obtain that $u(t)\in\mathbf{BV}(\mathbb{R}^N;\mathbb{R})$ for $t\in[0,2\,T_1[$, where

$$T_1 = \frac{1}{2\left((1+2N)\|\nabla\partial_u f\|_{\mathbf{L}^\infty(\Sigma^u_{T_0})} + \|\partial_u F\|_{\mathbf{L}^\infty(\Sigma^u_{T_0})}\right)}\,. \tag{4.19}$$

The next step is to obtain a general estimate of the TV norm. The starting point is (4.14). Recall the definitions (4.13) of M_1 and (4.19) of T_1. Moreover, by integration by part we obtain

$$\int_{\mathbb{R}^N}|z_1|\|z\|\,\mu_1'(\|z\|)\,\mathrm{d}z = -(N+1)\,C_1\,.$$

The following step is *not* similar to [8, proof of theorem 2.5]: we divide both terms in (4.14) by λ, apply (3.3) on the first, second and third terms in the right hand side, with $\rho_1=\mu_1\geqslant 0$ in the second and third case, and with $\rho_1 = -\mu_1' \geqslant 0$ in the second case. We obtain for all $T \in [0, T_1]$ with $T_1 < T_0$

$$\begin{aligned}\mathrm{TV}\left(u(T)\right) \quad &\leqslant \quad \mathrm{TV}\left(u_0\right) + \left((2N+1)\|\nabla\partial_u f\|_{\mathbf{L}^\infty(\Sigma^u_{T_0})} + \|\partial_u F\|_{\mathbf{L}^\infty(\Sigma^u_{T_0})}\right)\int_0^T \mathrm{TV}\left(u(t)\right)\,\mathrm{d}t\\ &\quad + \frac{M_1}{C_1}\int_0^T\int_{\mathbb{R}^N}\left\|\nabla(F-\operatorname{div} f)(t,x,\cdot)\right\|_{\mathbf{L}^\infty([-U_t,U_t])}\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}$$

Next, an application of the Gronwall Lemma shows that $\mathrm{TV}\left(u(t)\right)$ is bounded on $[0, T_1]$

$$\mathrm{TV}\left(u(T)\right) \leqslant e^{\kappa_0^* T}\ \mathrm{TV}\left(u_0\right) + \frac{M_1}{C_1}\int_0^T e^{\kappa_0^*(T-t)}\int_{\mathbb{R}^N}\left\|\nabla(F-\operatorname{div} f)(t,x,\cdot)\right\|_{\mathbf{L}^\infty([-U_t,U_t])}\,\mathrm{d}x\,\mathrm{d}t \tag{4.20}$$

for $T \in [0, T_1]$, M_1, C_1 as in (4.13), (3.4) and $\kappa_0^* = (2N+1)\|\nabla\partial_u f\|_{\mathbf{L}^\infty(\Sigma^u_{T_0})} + \|\partial_u F\|_{\mathbf{L}^\infty(\Sigma^u_{T_0})}$.

Now, it remains only to relax assumption on the regularity of u_0 and to note that the bound (4.20) is additive in time. These steps are the same as in [8, Theorem 2.5], so we do not write them. □

Remark 4.1. The constant NW_N in front of $\int_0^T\int_{\mathbb{R}^N}\left\|\nabla(F-\mathrm{div} f)(t,x,\cdot)\right\|_{\mathbf{L}^\infty([-U_t,U_t])}\,\mathrm{d}x\ \mathrm{d}t$ in Theorem 2.2 comes from the estimate of the term L_2 defined by (4.9).

We have indeed

$$\begin{aligned} L_2 \quad &\leqslant \quad \int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\int_0^1\left|\nabla(F-\mathrm{div} f)(t,x-\lambda(1-r)z,u)\cdot(\lambda z)\right|\\ &\qquad\qquad \times\chi\psi\mu_1(\|z\|)\nu\mathrm{d}r\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}z\,\mathrm{d}s\\ &\leqslant \quad \lambda\int_0^{T+\varepsilon}\int_{B(x_0,R+M(T_0-t)+\theta)}\int_{\mathbb{R}^N}\int_0^1\left|\nabla(F-\mathrm{div} f)(t,x-\lambda(1-r)z,u)\cdot(z)\right|\\ &\qquad\qquad \times\mu_1(\|z\|)\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant \quad \lambda\int_0^{T+\varepsilon}\int_0^1\int_{B(x_0,R+M(T_0-t)+\theta+\lambda)}\int_{\mathbb{R}^N}\left|\nabla(F-\mathrm{div} f)(t,x',u(t,x'+\lambda(1-r)z))\cdot z\right|\\ &\qquad\qquad \times\mu_1(\|z\|)\mathrm{d}z\,\mathrm{d}x'\,\mathrm{d}r\,\mathrm{d}t\end{aligned}$$

If $F-\operatorname{div} f$ does not depend on u, then, with the same computations as in the proof of Proposition 3.2, considering $z\mapsto\nabla(F-\mathrm{div} f)(t,x')\cdot z$ as a linear application, we get:

$$L_2 \quad \leqslant \quad \lambda\int_0^{T+\varepsilon}\int_{B(x_0,R+M(T_0-t)+\theta+\lambda)}\left|\nabla(F-\mathrm{div} f)(t,x')\right|\mathrm{d}x'\,\mathrm{d}t\int_{\mathbb{R}^N}|z_1|\mu_1(\|z\|)\mathrm{d}z\,,$$

which allows us to get rid of the constant NW_N into the bound of L_2.

However, in the general case, because of the dependence of u in z, we are led to take the supremum of $u(t)$. We obtain the following:

$$L_2 \leqslant \lambda\int_0^{T+\varepsilon}\int_{B(x_0,R+M(T_0-t)+\theta+\lambda)}\int_{\mathbb{R}^N}\sup_{y\in\mathbb{R}^N}\left|\nabla(F-\mathrm{div} f)(t,x',u(t,y))\cdot z\right|\mu_1(\|z\|)\mathrm{d}z\mathrm{d}x'\mathrm{d}t.$$

We can no longer do the same computations as in the proof of Proposition 3.2. Indeed, it is not allowed to permute sup and $\int_{\mathbb{R}^N}$, consequently, if we want to isolate the variable z from the other variables, we use the Cauchy-Schwartz inequality to obtain:

$$\begin{aligned} L_2 \leqslant &\lambda\int_0^{T+\varepsilon}\int_{B(x_0,R+M(T_0-t)+\theta+\lambda)}\sup_{y\in\mathbb{R}^N}\left\|\nabla(F-\mathrm{div} f)(t,x',u(t,y))\right\|\,\mathrm{d}x'\,\mathrm{d}t\\ &\times\int_{\mathbb{R}^N}\|z\|\mu_1(\|z\|)\mathrm{d}z\,.\end{aligned}$$

The constant NW_N appears here when we divide by $C_1 = \int_{\mathbb{R}^N} |z_1|\mu_1(\|z\|)\mathrm{d}z$, since, by Lemma 6.1, $\frac{1}{C_1}\int_{\mathbb{R}^N} \|z\|\mu_1(\|z\|)\mathrm{d}z = NW_N$.

In the general case, we were consequently not able, using this method, to erase the constant NW_N on the right hand side of (2.7).

Proof of Corollary 2.4. This is the same argument as in [9, Theorem 4.3.1], the flow and the source depending here on the three variables t, x and u.

The weak entropy solution u of (1.1) is also a weak solution. Consequently, for any $\varphi \in \mathscr{C}_c^\infty([0,T]\times\mathbb{R}^N,\mathbb{R})$ such that $|\varphi| \leqslant 1$, for any $t \in [0,T]$, we have

$$\begin{aligned}&\int_t^T \int_{\mathbb{R}^N} \big(u\partial_t\varphi + f(\tau,x,u)\cdot\nabla\varphi\big)\,\mathrm{d}x\,\mathrm{d}\tau + \int_{\mathbb{R}^N} u(t,x)\varphi(t,x)\,\mathrm{d}x\\ =&-\int_t^T\int_{\mathbb{R}^N} F(\tau,x,u)\varphi(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau\ .\end{aligned}$$

Let $s,t \in [0,T]$. Then, with $\varphi(t,x) = \psi(x)$, we obtain

$$\begin{aligned}&\int_s^t \int_{\mathbb{R}^N} f(\tau,x,u)\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}\tau + \int_{\mathbb{R}^N} (u(s,x)-u(t,x))\psi(x)\,\mathrm{d}x\\ =&-\int_s^t\int_{\mathbb{R}^N} F(\tau,x,u)\psi(x)\,\mathrm{d}x\,\mathrm{d}\tau\ .\end{aligned}$$

That is to say

$$\begin{aligned}&\int_{\mathbb{R}^N} (u(s,x)-u(t,x))\psi(x)\,\mathrm{d}x\\ =&-\int_s^t\int_{\mathbb{R}^N} (F(\tau,x,u) - \operatorname{div} f(\tau,x,u))\psi(x)\,\mathrm{d}x\,\mathrm{d}\tau\\ &-\int_s^t\int_{\mathbb{R}^N} (\operatorname{div} f(\tau,x,u)\psi(x) + f(\tau,x,u)\cdot\nabla\psi)\,\mathrm{d}x\,\mathrm{d}\tau\ .\end{aligned}$$

By a regularization process, we prove that

$$\begin{aligned}&\left|\int_s^t\int_{\mathbb{R}^N} (\operatorname{div} f(\tau,x,u)\psi(x) + f(\tau,x,u)\cdot\nabla\psi)\,\mathrm{d}x\,\mathrm{d}\tau\right|\\ \leqslant&|s-t|\|\partial_u f\|_{\mathbf{L}^\infty(\Sigma_{T^u})}\sup_{[0,T]}\mathrm{TV}\,(u(t))\,.\end{aligned}$$

Taking the supremum over all $\psi \in \mathscr{C}_c^\infty(\mathbb{R}^N,\mathbb{R})$ such that $|\psi| \leqslant 1$, we obtain

$$\begin{aligned}\big\|u(t)-u(s)\big\|_{\mathbf{L}^1(\mathbb{R}^N)} \leqslant& \left|\int_s^t\int_{\mathbb{R}^N} \big\|F - \operatorname{div} f(\tau,x,\cdot)\big\|_{\mathbf{L}^\infty([-U_\tau,U_\tau])}\,\mathrm{d}x\,\mathrm{d}\tau\right|\\ &+|s-t|\|\partial_u f\|_{\mathbf{L}^\infty(\Sigma_T^u])}\sup_{[0,T]}\mathrm{TV}\,(u(t))\,.\end{aligned}$$

□

5 Proof of the stability estimates

We give now the proof of Theorems 2.5 and 2.9. We prove first prove the following lemma.

Lemma 5.1. *Let (f, F), (g, G) satisfy* $(\mathbf{H1}^*)$*,* (f, F) *satisfy* $(\mathbf{H2}^*)$ *and* $(f-g, F-G)$ *satisfy* $(\mathbf{H3}^*)$*. Let $u_0, v_0 \in \mathbf{L}^\infty \cap \mathbf{L}^1 \cap \mathbf{BV}(\mathbb{R}^N;\mathbb{R})$. We denote u and v the solutions associated respectively to the initial conditions u_0 and v_0. Let $T > 0$. Then, using the same notation as in (2.9)–(2.10), for any $R > 0$ and $x_0 \in \mathbb{R}^N$, the following estimate holds:*

$$
\begin{aligned}
&\int_{B(x_0,R+M(T_0-T))} \left|u(T,x) - v(T,x)\right| \mathrm{d}x \\
\leqslant\ & \int_{B(x_0,R+MT_0)} \left|u(0,x) - v(0,x)\right| \mathrm{d}x \\
&+ (\|\partial_u F\|_{\mathbf{L}^\infty(\Sigma^{u,v}_{T_0})} + \left\|\partial_u \operatorname{div}(g-f)\right\|_{\mathbf{L}^\infty(\Sigma^{u,v}_{T_0})}) \int_0^T \int_{B(x_0,R+M(T_0-t))} \left|v(t,x) - u(t,x)\right| \mathrm{d}x\,\mathrm{d}t \\
&+ \left[\int_0^T \left\|\partial_u (f-g)(t)\right\|_{\mathbf{L}^\infty(\mathcal{S}_{T_0}(u)\times[-U_t,U_t])} \mathrm{TV}\,(u(t))\,\mathrm{d}t \right. \\
&\qquad \left. + \int_0^T \int_{B(x_0,R+M(T_0-t))} \left\| \big((F-G) - \operatorname{div}(f-g)\big)(t,y,\cdot) \right\|_{\mathbf{L}^\infty([-V_t,V_t])} \mathrm{d}y\,\mathrm{d}t \right].
\end{aligned}
$$

The beginning of this proof is similar, up to (5.4), to the proof of Theorem 2.6 in [8]. We rewrite it in order to be complete and clear.

Proof of Lemma 5.1.

Let $\Phi \in \mathscr{C}_c^\infty(\mathbb{R}_+^* \times \mathbb{R}^N;\mathbb{R}_+)$, $\Psi \in \mathscr{C}_c^\infty(\mathbb{R} \times \mathbb{R}^N;\mathbb{R}_+)$, and set $\varphi(t,x,s,y) = \Phi(t,x)\Psi(t-s,x-y)$ as in (4.3).

By Kružkov Theorem [12, Theorem 5 & Section 5, Remark 4], the set of hypotheses **(H1*)** gives us existence and uniqueness of a weak entropy solution for any initial condition in $\mathbf{L}^\infty \cap \mathbf{L}^1(\mathbb{R}^N;\mathbb{R})$. Let u be the Kružkov solution associated to u_0 and v be the Kružkov solution associated to v_0. By definition of Kružkov weak entropy solution, we have for all $l \in \mathbb{R}$, for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}^N$

$$
\begin{aligned}
\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N} \Big[(u-l)\,\partial_s\varphi + \big(f(s,y,u) - f(s,y,l)\big)\cdot\nabla_y\varphi + \big(F(s,y,u) - \operatorname{div} f(s,y,l)\big)\,\varphi\Big] \\
\times \operatorname{sign}(u-l)\,\mathrm{d}y\,\mathrm{d}s \geqslant 0
\end{aligned}
\tag{5.1}
$$

and for all $k \in \mathbb{R}$, for all $(s,y) \in \mathbb{R}_+^* \times \mathbb{R}^N$

$$
\begin{aligned}
\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N} \Big[(v-k)\,\partial_t\varphi + \big(g(t,x,v) - g(t,x,k)\big)\cdot\nabla_x\varphi + \big(G(t,x,v) - \operatorname{div} g(t,x,k)\big)\,\varphi\Big] \\
\times \operatorname{sign}(v-k)\,\mathrm{d}x\,\mathrm{d}t \geqslant 0.
\end{aligned}
\tag{5.2}
$$

Choose $k = u(s,y)$ in (5.2) and integrate with respect to (s,y). Analogously, take $l = v(t,x)$ in (5.1) and integrate with respect to (t,x). By summing the obtained equations, we get, denoting $u = u(s,y)$ and $v = v(t,x)$:

$$
\begin{aligned}
\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N} \quad & \Big[(u-v)\Psi\partial_t\Phi + \big(g(t,x,u) - g(t,x,v)\big)\cdot(\nabla\Phi)\,\Psi \\
& + \big(g(t,x,u) - g(t,x,v) - f(s,y,u) + f(s,y,v)\big)\cdot(\nabla\Psi)\Phi \\
& + \big(F(s,y,u) - G(t,x,v) + \operatorname{div} g(t,x,u) - \operatorname{div} f(s,y,v)\big)\,\varphi\Big] \\
& \times \operatorname{sign}(u-v)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s \qquad \geqslant 0\,.
\end{aligned}
\tag{5.3}
$$

We introduce a family of functions $\{Y_\vartheta\}_{\vartheta>0}$ as in (4.5). Let $T_0 > 0$ and denote $M = \|\partial_u g\|_{\mathbf{L}^\infty(\Omega^{\mathcal{V}}_{T_0};\mathbb{R}^N)}$ with $\mathcal{V} = \max(\|u\|_{\mathbf{L}^\infty([0,T_0]\times\mathbb{R}^N)}, \|v\|_{\mathbf{L}^\infty([0,T_0]\times\mathbb{R}^N)})$. We also define χ, ψ as in (4.6), for $\varepsilon, \theta, R > 0$, $x_0 \in \mathbb{R}^N$ (see also Figure 2). Note that with these choices, equalities (4.7) still hold. Note that here the definition of the test function φ is essentially the same as in the preceding proof; the only change is the definition of the constant M, which is now defined with reference to g. We also introduce as above the function $B(t,x,u,v) = M|u-v| + \mathrm{sign}(u-v)\left(g(t,x,u)-g(t,x,v)\right)\cdot \dfrac{x-x_0}{\|x-x_0\|}$ that is positive for all $(t,x,u,v)\in\Omega\times\mathbb{R}$, and we have:

$$\begin{aligned}
&\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N} \left[(u-v)\partial_t\Phi + \left(g(t,x,u)-g(t,x,v)\right)\cdot\nabla\Phi\right]\Psi\,\mathrm{sign}(u-v)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\\
\leqslant&\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N} \left[|u-v|\chi'\psi - B(t,x,u,v)\chi Y_\theta'\right]\Psi\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\\
\leqslant&\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N} |u-v|\,\chi'\,\psi\,\Psi\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\,.
\end{aligned}$$

Due to the above estimate and (5.3), we obtain

$$\begin{aligned}
\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N} &\Big[(u-v)\chi'\psi\Psi\\
&+\left(g(t,x,u)-g(t,x,v)-f(s,y,u)+f(s,y,v)\right)\cdot(\nabla\Psi)\Phi\\
&+\left(F(s,y,u)-G(t,x,v)+\mathrm{div}\,g(t,x,u)-\mathrm{div}\,f(s,y,v)\right)\varphi\Big]\\
&\times\mathrm{sign}(u-v)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s \qquad\geqslant\quad 0\,,
\end{aligned}$$

i.e. $I+J_x+J_t+K+L_x+L_t\geqslant 0$, where

$$I = \int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N} |u-v|\chi'\psi\Psi\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\,, \tag{5.4}$$

$$\begin{aligned}
J_x = \int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N} &\Big[\left(g(t,x,u)-g(t,x,v)+g(t,y,v)-g(t,y,u)\right)\cdot(\nabla\Psi)\Phi\\
&+(\mathrm{div}\,g(t,x,u)-\mathrm{div}\,g(t,x,v))\varphi\Big]\mathrm{sign}(u-v)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\,,
\end{aligned}\tag{5.5}$$

$$\begin{aligned}
J_t = \int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N} &\Big[\left(f(s,y,v)-f(s,y,u)+f(t,y,u)-f(t,y,v)\right)\cdot(\nabla\Psi)\Phi\\
&+(\mathrm{div}\,f(t,y,v)-\mathrm{div}\,f(s,y,v))\Big]\times\mathrm{sign}(u-v)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\,,
\end{aligned}$$

$$\begin{aligned}
K = \int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N} &\left((g-f)(t,y,u)-(g-f)(t,y,v)\right)\cdot(\nabla\Psi)\Phi\\
&\times\mathrm{sign}(u-v)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\,,
\end{aligned}\tag{5.6}$$

$$\begin{aligned}
L_x = \int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N} &\left(F(t,y,u)-G(t,x,v)+\mathrm{div}\,g(t,x,v)-\mathrm{div}\,f(t,y,v)\right)\varphi\\
&\times\mathrm{sign}(u-v)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\,,\\
L_t = \int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N} &\left(F(s,y,u)-F(t,y,u)\right)\varphi\\
&\times\mathrm{sign}(u-v)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\,.
\end{aligned}\tag{5.7}$$

Now, we choose $\Psi(t,x)=\nu(t)\,\mu(x)$ as in (4.10), (6.1), (6.2). Let us estimate each of these integrals separately.

a) Estimate on I. The estimate on I is the same as in the proof of [8, Theorem 2.6]: thanks to Lemma 6.2, we obtain

$$\begin{aligned}\limsup_{\varepsilon,\eta,\lambda\to 0} I \quad \leqslant & \int_{\|x-x_0\|\leqslant R+MT_0+\theta} \left|u(0,x)-v(0,x)\right| \mathrm{d}x \\ & - \int_{\|x-x_0\|\leqslant R+M(T_0-T)} \left|u(T,x)-v(T,x)\right| \mathrm{d}x\,.\end{aligned} \tag{5.8}$$

b) Estimate on J_x. For J_x, we derive a new estimate with respect to [8, Theorem 2.6]. Indeed, as g is $\mathscr{C}^2$ in space, we can use the following Taylor expansion:

$$\begin{aligned}g(t,y,v) =& g(t,x,v) + \nabla g(t,x,v)\cdot(y-x) \\ &+ \int_0^1 (1-r)\nabla^2 g(t,ry+(1-r)x,v)\,\mathrm{d}r\cdot(y-x)^2\,, \\ g(t,y,u) =& g(t,x,u) + \nabla g(t,y,u)\cdot(y-x) \\ &+ \int_0^1 (1-r)\nabla^2 g(t,ry+(1-r)x,u)\,\mathrm{d}r\cdot(y-x)^2\,.\end{aligned}$$

Besides, we note that

$$\begin{aligned} & \big(\nabla g(t,x,v)\cdot(y-x)\big)\cdot\nabla\mu(x-y) - \operatorname{div} g(t,x,v)\,\mu(x-y) \\ = \;& \sum_{i,j}\partial_j g_i(t,x,v)\,(y_j-x_j)\,\partial_i\mu(x-y) - \sum_i \partial_i g_i(t,x,v)\,\mu(x-y) \\ = \;& -\sum_{i,j}\partial_j g_i(t,x,v)\,\partial_i\left(z_j\mu(z)\right)|_{z=x-y} \\ = \;& -\nabla g(t,x,v)\cdot\nabla((x-y)\mu(x-y))\end{aligned}$$

In the same way, we have

$$\begin{aligned} & \big(\nabla g(t,x,u)\cdot(x-y)\big)\,\nabla\mu(x-y) + \operatorname{div} g(t,x,u)\mu(x-y) \\ = \;& \nabla g(t,x,u)\cdot\nabla((x-y)\mu)\end{aligned}$$

so that finally

$$\begin{aligned} & \big(g(t,y,v)-g(t,x,v)+g(t,x,u)-g(t,y,u)\big)\,\nabla\mu + \big(\operatorname{div} g(t,x,u)-\operatorname{div} g(t,x,v)\big)\,\mu(x-y) \\ & = \big(\nabla g(t,x,u)-\nabla g(t,x,v)\big)\cdot\nabla((x-y)\mu) \\ & \quad + \left[\int_0^1 (1-r)\Big(\nabla^2 g(t,ry+(1-r)x,u) - \nabla^2 g(t,ry+(1-r)x,v)\Big)\,\mathrm{d}r\cdot(x-y)^2\right]\cdot\nabla\mu\end{aligned}$$

After a change of variable, we obtain

$$\begin{aligned} & \lim_{\varepsilon,\eta,\theta\to 0} J_x \\ & = \int_0^T\int_{B(x_0,R+M(T_0-t))}\int_{\mathbb{R}^N}\Big\{\Big(\nabla g(t,x,u(t,x-\lambda z)) - \nabla g(t,x,v(t,x))\Big) \\ & \qquad\qquad \cdot\nabla(z\mu_1(\|z\|))\operatorname{sign}(u-v) \\ & \qquad + \lambda\left[\int_0^1 (1-r)\Big(\nabla^2 g(t,ry+(1-r)x,u) - \nabla^2 g(t,ry+(1-r)x,v)\Big)\,\mathrm{d}r\cdot z^2\right] \\ & \qquad\qquad \cdot\frac{z}{\|z\|}\mu_1'(\|z\|)\,\operatorname{sign}(u-v)\Big\}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}t\;.\end{aligned}$$

When λ goes to 0, we obtain by the Dominated Convergence Theorem

$$\lim_{\varepsilon,\eta,\theta,\lambda\to 0} J_x = \int_0^T \int_{B(x_0,R+M(T_0-t))} \big(\nabla g(t,x,u(t,x)) - \nabla g(t,x,v(t,x))\big)\ \mathrm{sign}(u-v)\,\mathrm{d}x\,\mathrm{d}t \cdot \int_{\mathbb{R}^N} \nabla(z\mu_1(\|z\|))\,\mathrm{d}z\ .$$

As $\int_{\mathbb{R}^N} \nabla(z\mu_1(\|z\|))\,\mathrm{d}z = 0$, we finally get

$$\lim_{\varepsilon,\eta,\theta,\lambda\to 0} J_x = 0\,. \tag{5.9}$$

c) Estimates of J_t and L_t. For J_t and L_t, we avoid now the use of the derivatives in time thanks to an application of the Dominated Convergence Theorem. We obtain

$$\lim_{\varepsilon,\eta,\theta,\lambda\to 0} J_t = \lim_{\varepsilon,\eta,\theta,\lambda\to 0} L_t = 0\,. \tag{5.10}$$

d) Estimate of L_x. For L_x, we have

$$L_x \leqslant \int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}_+^*}\int_{\mathbb{R}^N} \Big[(F-G-\mathrm{div}\,(f-g))(t,y,v) + (F(t,y,u)-F(t,y,v)) + \int_0^1 \nabla G(t,ry+(1-r)x,v)\cdot(y-x)\,\mathrm{d}r\Big]\varphi\,\mathrm{d}y\ \mathrm{d}s\ \mathrm{d}x\ \mathrm{d}t\ .$$

Note that $F(t,y,u) - F(t,y,v) = \int_v^u \partial_u F(t,y,w)\,\mathrm{d}w$ vanishes for $y \in \mathbb{R}^N \setminus \mathcal{S}_T(u,v)$. Consequently, with $\mathcal{V} = \sup_{t\in[0,T_0]}(\left\|u(t)\right\|_{\mathbf{L}^\infty(\mathbb{R}^N)}, \left\|v(t)\right\|_{\mathbf{L}^\infty(\mathbb{R}^N)})$ and

$$\Sigma_{T_0}^{u,v} = [0,T_0]\times\mathcal{S}_T(u,v)\times[-\mathcal{V},\mathcal{V}]\,, \tag{5.11}$$

we obtain

$$\begin{aligned}\lim_{\varepsilon,\eta,\theta,\lambda\to 0} L_x \leqslant{} & \int_0^T \int_{B(x_0,R+M(T_0-t))} \left\|(F-G)(t,x,\cdot) - \mathrm{div}\,(f-g)(t,x,\cdot)\right\|_{\mathbf{L}^\infty([-U_t,U_t])}\mathrm{d}x\,\mathrm{d}t \\ & + \|\partial_u F\|_{\mathbf{L}^\infty(\Sigma_{T_0}^{u,v})} \int_0^T \int_{B(x_0,R+M(T_0-t))} \left|u(t,x)-v(t,x)\right|\mathrm{d}x\,\mathrm{d}t\ .\end{aligned} \tag{5.12}$$

e) Estimate of K. In order to estimate K as given in (5.6), we follow the same procedure as in [8, Theorem 2.6]: let us introduce a regularisation of the y dependent functions. In fact, let $\rho_\alpha(z) = \frac{1}{\alpha}\rho\left(\frac{z}{\alpha}\right)$ and $\sigma_\beta(y) = \frac{1}{\beta^N}\sigma\left(\frac{y}{\beta}\right)$, where $\rho \in \mathscr{C}_c^\infty(\mathbb{R};\mathbb{R}_+)$ and $\sigma \in \mathscr{C}_c^\infty(\mathbb{R}^N;\mathbb{R}_+)$ are such that $\|\rho\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})} = \|\sigma\|_{\mathbf{L}^1(\mathbb{R}^N;\mathbb{R})} = 1$ and $\mathrm{Supp}(\rho) \subseteq \,]-1,1[$, $\mathrm{Supp}(\sigma)\subseteq B(0,1)$. Then, we introduce

$$\begin{array}{rclcrcl} P(w) & = & (g-f)(t,y,w)\,, & \quad & s_\alpha & = & \mathrm{sign} *_u \rho_\alpha\,,\\ \Upsilon_\alpha^i(w) & = & s_\alpha(w-v)\,\big(P_i(w)-P_i(v)\big)\,, & & u_\beta & = & \sigma_\beta *_y u\,,\\ \Upsilon^i(w) & = & \mathrm{sign}(w-v)\,\big(P_i(w)-P_i(v)\big)\,, & & & & \end{array}$$

so that we obtain

$$\begin{aligned} & \langle \Upsilon_\alpha^i(u_\beta) - \Upsilon_\alpha^i(u),\, \partial_{y_i}\varphi\rangle \\ = {} & \int_{\mathbb{R}^N} \Big[\big(s_\alpha(u-v)P_i(u) - s_\alpha(u_\beta-v)P_i(u_\beta)\big) + \big(s_\alpha(u-v) - s_\alpha(u_\beta-v)\big)\,P_i(v)\Big]\,\partial_{y_i}\varphi\,\mathrm{d}y \\ = {} & \int_{\mathbb{R}^N}\int_u^{u_\beta} \big(\partial_U(s_\alpha(U-v)P_i(U)) - \partial_U s_\alpha(U-v)P_i(v)\big)\,\partial_{y_i}\varphi\,\mathrm{d}y \\ = {} & \int_{\mathbb{R}^N}\int_u^{u_\beta} \big(s_\alpha'(U-v)(P_i(U)-P_i(v)) + s_\alpha(U-v)P_i'(U)\big)\,\partial_{y_i}\varphi\,\mathrm{d}y\ . \end{aligned}$$

Now, we use the relation $s'_\alpha(U) = \frac{2}{\alpha}\rho\left(\frac{U}{\alpha}\right)$ to obtain

$$\begin{aligned}\left|\langle\Upsilon^i_\alpha(u_\beta) - \Upsilon^i_\alpha(u),\, \partial_{y_i}\varphi\rangle\right| \;\leqslant\; & \int_{\mathbb{R}^N}\int_{\mathbb{R}} 2\rho\,(z)\left|P_i(v+\alpha z) - P_i(v)\right|\partial_{y_i}\varphi\,\mathrm{d}z\;\mathrm{d}y \\ & + \int_{\mathbb{R}^N}\int_u^{u_\beta} \left|P'_i(U)\right|\partial_{y_i}\varphi\,\mathrm{d}U\,\mathrm{d}y\,.\end{aligned}$$

When α tends to 0, using the Dominated Convergence Theorem we obtain

$$\left|\langle\Upsilon^i(u_\beta) - \Upsilon^i(u), \partial_{y_i}\varphi\rangle\right| \;\leqslant\; \int_{\mathbb{R}^N} \left|u - u_\beta\right| \left\|P'_i\right\|_{\mathbf{L}^\infty}\partial_{y_i}\varphi\,\mathrm{d}y\,.$$

Applying the Dominated Convergence Theorem again, we see that

$$\begin{aligned}\lim_{\beta\to0}\lim_{\alpha\to0}\langle\Upsilon^i_\alpha(u_\beta),\, \partial_{y_i}\varphi\rangle &= \langle\Upsilon^i(u),\, \partial_{y_i}\varphi\rangle\,,\\ \lim_{\beta\to0}\lim_{\alpha\to0}\langle\Upsilon_\alpha(u_\beta),\, \nabla_y\varphi\rangle &= \langle\Upsilon(u),\, \nabla_y\varphi\rangle\,.\end{aligned}$$

Consequently, it is sufficient to find a bound independent of α and β on $K_{\alpha,\beta}$, where

$$K_{\alpha,\beta} = -\int_{\mathbb{R}^*_+}\int_{\mathbb{R}^N}\int_{\mathbb{R}^*_+}\int_{\mathbb{R}^N} \Upsilon_\alpha(u_\beta)\cdot\nabla_y\varphi\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\,.$$

Integrating by parts, we obtain

$$\begin{aligned}K_{\alpha,\beta} =& \int_{\mathbb{R}^*_+}\int_{\mathbb{R}^N}\int_{\mathbb{R}^*_+}\int_{\mathbb{R}^N} \mathrm{Div}_y\,\Upsilon_\alpha(u_\beta)\,\varphi\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\\ =& \int_{\mathbb{R}^*_+}\int_{\mathbb{R}^N}\int_{\mathbb{R}^*_+}\int_{\mathbb{R}^N} \partial_u s_\alpha(u_\beta - v)\nabla u_\beta\cdot\big((g-f)(t,y,u_\beta) - (g-f)(t,y,v)\big)\,\varphi\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\\ &+ \int_{\mathbb{R}^*_+}\int_{\mathbb{R}^N}\int_{\mathbb{R}^*_+}\int_{\mathbb{R}^N} s_\alpha(u_\beta - v)\,\big(\partial_u(g-f)(t,y,u_\beta)\cdot\nabla u_\beta\big)\,\varphi\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\\ &+ \int_{\mathbb{R}^*_+}\int_{\mathbb{R}^N}\int_{\mathbb{R}^*_+}\int_{\mathbb{R}^N} s_\alpha(u_\beta - v)\,\big(\mathrm{div}\,(g-f)(t,y,u_\beta) - \mathrm{div}\,(g-f)(t,y,v)\big)\,\varphi\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\\ =& K_1 + K_2 + K_3\,.\end{aligned}$$

We now search for a bound for each term of the sum above.

- For K_1, recall that $\partial_u s_\alpha(u) = \frac{2}{\alpha}\rho\left(\frac{u}{\alpha}\right)$. Hence, by the Dominated Convergence Theorem, we get that $K_1 \to 0$ when $\alpha\to0$. Indeed,

$$\begin{aligned}&\left|\frac{2}{\alpha}\rho\left(\frac{u_\beta - v}{\alpha}\right)\nabla u_\beta\cdot\big((g-f)(t,y,u_\beta) - (g-f)(t,y,v)\big)\,\varphi\right|\\ \leqslant\;& \frac{2}{\alpha}\rho\left(\frac{u_\beta - v}{\alpha}\right)\varphi\left\|\nabla u_\beta(s,y)\right\|\int_v^{u_\beta}\left\|\partial_u(f-g)(t,y,w)\right\|\mathrm{d}w\\ \leqslant\;& 2\|\rho\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\left\|\nabla u_\beta(s,y)\right\|\left\|\partial_u(f-g)\right\|_{\mathbf{L}^\infty(\Omega_U^{T_0};\mathbb{R}^N)}\varphi \qquad \in \mathbf{L}^1\left((\mathbb{R}^*_+\times\mathbb{R}^N)^2;\mathbb{R}\right).\end{aligned}$$

- For K_2, denoting $\mathcal{D} = \{\mathcal{S}_{T_0}(u) + B(0,\beta)\}\times[-\left\|u(t)\right\|_{\mathbf{L}^\infty}, \left\|u(t)\right\|_{\mathbf{L}^\infty}]$, we get

$$\begin{aligned}K_2 \;\leqslant\;& \int_0^{T+\varepsilon+\eta}\int_{\mathbb{R}^N}\int_{\mathbb{R}_+}\left\|\partial_u(f-g)(t)\right\|_{\mathbf{L}^\infty(\mathcal{D};\mathbb{R}^N)}\left\|\nabla u_\beta(s,y)\right\|\nu(t-s)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}t\\ \leqslant\;& \int_0^{T+\varepsilon+\eta}\int_{\mathbb{R}_+}\left\|\partial_u(f-g)(t)\right\|_{\mathbf{L}^\infty(\mathcal{D};\mathbb{R}^N)}\,\mathrm{TV}\,(u_\beta(s))\,\nu(t-s)\,\mathrm{d}s\,\mathrm{d}t\,.\end{aligned}$$

  We note besides that $\mathcal{D}\to\mathcal{S}_{T_0}(u)\times[-U_t, U_t]$ when $\beta\to0$.

- For K_3, we have

$$\lim_{\alpha,\beta,\varepsilon,\eta,\lambda\to 0} K_3 \leqslant \int_0^T \int_{B(x_0,R+M(T_0-t))} \left\|\partial_u \operatorname{div}(g-f)\right\|_{\mathbf{L}^\infty(\Sigma_{T_0}^{u,v})} \left|(u-v)(t,x)\right| \mathrm{d}x\,\mathrm{d}t\,.$$

Finally, letting $\alpha,\beta\to 0$ and $\varepsilon,\eta,\lambda\to 0$, due to [1, Proposition 3.7], we obtain

$$\begin{aligned}\limsup_{\varepsilon,\eta,\lambda\to 0} K \quad \leqslant \quad & \int_0^T \left\|\partial_u(f-g)(t)\right\|_{\mathbf{L}^\infty(\mathcal{S}_{T_0}(u)\times[-U_t,U_t];\mathbb{R}^N)} \mathrm{TV}\,(u(t))\,\mathrm{d}t \\ & + \int_0^T \int_{B(x_0,R+M(T_0-t))} \left\|\partial_u \operatorname{div}(g-f)\right\|_{\mathbf{L}^\infty(\Sigma_{T_0}^{u,v})} \left|(u-v)(t,x)\right| \mathrm{d}x\,\mathrm{d}t\,.\end{aligned} \tag{5.13}$$

f) Collecting of the estimates. Now, we collate the estimates obtained in (5.8), (5.9), (5.12), and (5.13). Remark the order in which we pass to the various limits: first $\varepsilon,\eta,\theta\to 0$ and, after, $\lambda\to 0$. Therefore, we get

$$\begin{aligned} & \int_{B(x_0,R+M(T_0-T))} \left|u(T,x)-v(T,x)\right| \mathrm{d}x \\ \leqslant \quad & \int_{B(x_0,R+MT_0)} \left|u(0,x)-v(0,x)\right| \mathrm{d}x \\ & +\left(\|\partial_u F\|_{\mathbf{L}^\infty(\Sigma_{T_0}^{u,v})} + \left\|\partial_u \operatorname{div}(g-f)\right\|_{\mathbf{L}^\infty(\Sigma_{T_0}^{u,v})}\right) \int_0^T \int_{B(x_0,R+M(T_0-t))} \left|v(t,x)-u(t,x)\right| \mathrm{d}x\,\mathrm{d}t \\ & + \left[\int_0^T \left\|\partial_u(f-g)(t)\right\|_{\mathbf{L}^\infty(\mathcal{S}_{T_0}(u)\times[-U_t,U_t])} \mathrm{TV}\,(u(t))\,\mathrm{d}t \right. \\ & \quad \left. + \int_0^T \int_{B(x_0,R+M(T_0-t))} \left\|\big((F-G)-\operatorname{div}(f-g)\big)(t,y,\cdot)\right\|_{\mathbf{L}^\infty([-V_t,V_t])} \mathrm{d}y\,\mathrm{d}t\right].\end{aligned}$$

□

Remark 5.2. In the preceding proof, the main changes comparing to [8] are essentially in the bound of J_x. Furthermore, we also gain some regularity hypotheses by avoiding the use of the derivative in time.

Proof of Theorem 2.5. Thanks to Lemma 5.1, we can write

$$A'(T) \leqslant A'(0) + \kappa^*\, A(T) + R(T)\,, \tag{5.14}$$

where

$$\begin{aligned} A(T) \quad &= \quad \int_0^T \int_{B(x_0,R+M(T_0-t))} \left|v(t,x)-u(t,x)\right| \mathrm{d}x\,\mathrm{d}t\,, \\ \kappa^* \quad &= \quad \|\partial_u F\|_{\mathbf{L}^\infty(\Sigma_{T_0}^{u,v})} + \left\|\partial_u \operatorname{div}(g-f)\right\|_{\mathbf{L}^\infty(\Sigma_{T_0}^{u,v})}\,, \\ R(T) \quad &= \quad \left\|\partial_u(f-g)\right\|_{\mathbf{L}^\infty(\Sigma_{T_0}^{u})} \int_0^T \mathrm{TV}\,\big(u(t)\big)\,\mathrm{d}t \\ & \qquad + \int_0^T \int_{B(x_0,R+M(T_0-t))} \left\|\big((F-G)-\operatorname{div}(f-g)\big)(t,y,\cdot)\right\|_{\mathbf{L}^\infty([-V_t,V_t])} \mathrm{d}y\,\mathrm{d}t. \end{aligned} \tag{5.15}$$

The bound (2.7) on $\mathrm{TV}\,\big(u(t)\big)$ gives:

$$R(T) \quad \leqslant \quad \frac{e^{\kappa_0^* T}-1}{\kappa_0^*} a + \int_0^T \frac{e^{\kappa_0^*(T-t)}-1}{\kappa_0^*} b(t)\mathrm{d}t + \int_0^T c(t)\mathrm{d}t\,,$$

where κ_0^* is defined in (2.6) and

$$\begin{aligned} a &= \left\|\partial_u (f-g)\right\|_{\mathbf{L}^\infty(\Sigma^u_{T_0})} \mathrm{TV}\,(u_0)\,, \\ b(t) &= NW_N \left\|\partial_u (f-g)\right\|_{\mathbf{L}^\infty(\Sigma^u_{T_0})} \int_{\mathbb{R}^N} \left\|\nabla (F - \mathrm{div}\, f)(t,x,\cdot)\right\|_{\mathbf{L}^\infty([-U_t,U_t])} \mathrm{d}x\,, \\ c(t) &= \int_{B(x_0,R+M(T_0-t))} \left\| \big((F-G) - \mathrm{div}\,(f-g)\big)\,(t,y,\cdot)\right\|_{\mathbf{L}^\infty([-V_t,V_t])} \mathrm{d}y\,, \end{aligned}$$

since $T \leqslant T_0$. Consequently

$$A'(T) \leqslant A'(0) + \kappa^* A(T) + \left(\frac{e^{\kappa_0^* T}-1}{\kappa_0^*} a + \int_0^T \frac{e^{\kappa_0^*(T-t)}-1}{\kappa_0^*} b(t) \mathrm{d}t + \int_0^T c(t) \mathrm{d}t \right). \tag{5.16}$$

By a Gronwall type argument, we obtain

$$A'(T) \leqslant e^{\kappa^* T} A'(0) + \frac{e^{\kappa_0^* T} - e^{\kappa^* T}}{\kappa_0^* - \kappa^*}\, a + \int_0^T \frac{e^{\kappa_0^*(T-t)} - e^{\kappa^*(T-t)}}{\kappa_0^* - \kappa^*}\, b(t)\,\mathrm{d}t + \int_0^T e^{\kappa^*(T-t)} c(t)\,\mathrm{d}t\,.$$

Taking $T = T_0$, we finally obtain the result. □

Proof of Proposition 2.9. Thanks to Lemma 5.1, we can write

$$B'(T) \leqslant B'(0) + \kappa^*\, B(T) + S(T)\,, \tag{5.17}$$

where

$$\begin{aligned} B(T) &= \int_0^T \int_{B(x_0,R+M(T_0-t))} \left|v(t,x) - u(t,x)\right| \mathrm{d}x\,\mathrm{d}t\,, \\ \kappa^* &= \left\|\partial_u F\right\|_{\mathbf{L}^\infty(\Sigma^{u,v}_{T_0})} + \left\|\partial_u\, \mathrm{div}\,(g-f)\right\|_{\mathbf{L}^\infty(\Sigma^{u,v}_{T_0})}\,, \\ S(T) &= \sup_{t\in[0,T_0]} \mathrm{TV}\,\big(u(t)\big) \int_0^T \left\|\partial_u (f-g)(t)\right\|_{\mathbf{L}^\infty(\mathcal{S}_{T_0}(u)\times[-U_t,U_t])} \mathrm{d}t \\ &\quad + \int_0^T \int_{B(x_0,R+M(T_0-t))} \left\| \big((F-G) - \mathrm{div}\,(f-g)\big)\,(t,y,\cdot)\right\|_{\mathbf{L}^\infty([-V_t,V_t])} \mathrm{d}y\,\mathrm{d}t. \end{aligned} \tag{5.18}$$

The bound (2.7) on $\mathrm{TV}\,\big(u(t)\big)$ gives:

$$\begin{aligned} S(T) \leqslant\ & \left(e^{\kappa_0^* T}\, \mathrm{TV}\,(u_0) + NW_N \int_0^T e^{\kappa_0^*(T-t)} \int_{\mathbb{R}^N} \left\|\nabla (F - \mathrm{div}\, f)(t,x,\cdot)\right\|_{\mathbf{L}^\infty([-U_t,U_t])} \mathrm{d}x\,\mathrm{d}t \right) \\ & \quad \times \int_0^T \left\|\partial_u (f-g)(t)\right\|_{\mathbf{L}^\infty(\mathcal{S}_{T_0}(u)\times[-U_t,U_t])} \mathrm{d}t \\ & + \int_0^T \int_{B(x_0,R+M(T_0-t))} \left\| \big((F-G) - \mathrm{div}\,(f-g)\big)\,(t,y,\cdot)\right\|_{\mathbf{L}^\infty([-V_t,V_t])} \mathrm{d}y\,,\mathrm{d}t \end{aligned}$$

where κ_0^* is defined in (2.6). Let us denote

$$\begin{aligned} a &= \mathrm{TV}\,(u_0)\,, \\ b(t) &= NW_N e^{-\kappa_0^* t} \int_{\mathbb{R}^N} \left\|\nabla (F - \mathrm{div}\, f)(t,x,\cdot)\right\|_{\mathbf{L}^\infty([-U_t,U_t])} \mathrm{d}x\,, \\ c(t) &= \left\|\partial_u (f-g)(t)\right\|_{\mathbf{L}^\infty(\mathcal{S}_{T_0}(u)\times[-U_t,U_t])}\,, \\ d(t) &= \int_{B(x_0,R+M(T_0-t))} \left\| \big((F-G) - \mathrm{div}\,(f-g)\big)\,(t,y,\cdot)\right\|_{\mathbf{L}^\infty([-V_t,V_t])} \mathrm{d}y\,. \end{aligned}$$

Then we have

$$A'(T) \leqslant A'(0) + \kappa^* A(T) + e^{\kappa_0^* T} \left(a + \int_0^T b(t)\,\mathrm{d}t \right) \int_0^T c(t)\,\mathrm{d}t + \int_0^T d(t)\,\mathrm{d}t \,.$$

Consequently, by a Gronwall type argument, we obtain

$$B'(T) \leqslant e^{\kappa^* T} B'(0) + \frac{\kappa_0^* e^{\kappa_0^* T} - \kappa^* e^{\kappa^* T}}{\kappa_0^* - \kappa^*} \left(a + \int_0^T b(t)\,\mathrm{d}t \right) \int_0^T c(t)\,\mathrm{d}t + e^{\kappa^* T} \int_0^T d(t)\,\mathrm{d}t \,.$$

Taking $T = T_0$, we finally obtain the result. □

6 Technical tools

We give below a lemma that was used in the previous proof. Let us recall from [8] the following useful technical results:

Lemma 6.1. *Fix a function* $\mu_1 \in \mathscr{C}_c^\infty(\mathbb{R}_+;\mathbb{R}_+)$ *with*

$$\operatorname{Supp}(\mu_1) \subseteq [0,1[\,, \quad \int_{\mathbb{R}_+^*} r^{N-1} \mu_1(r)\,\mathrm{d}r = \frac{1}{N\omega_N}, \quad \mu_1' \leqslant 0, \quad \mu_1^{(n)}(0) = 0 \text{ for } n \geqslant 1. \tag{6.1}$$

Define

$$\mu(x) = \frac{1}{\lambda^N}\, \mu_1\left(\frac{\|x\|}{\lambda} \right). \tag{6.2}$$

Then, recalling that $\omega_0 = 1$,

$$\int_{\mathbb{R}^N} \mu(x)\,\mathrm{d}x = 1\,, \tag{6.3}$$

$$\int_{\mathbb{R}^N} |x_1|\, \mu_1\left(\|x\|\right)\,\mathrm{d}x = \frac{2}{N} \frac{\omega_{N-1}}{\omega_N} \int_{\mathbb{R}^N} \|x\|\, \mu_1\left(\|x\|\right)\,\mathrm{d}x\,, \tag{6.4}$$

$$\int_{\mathbb{R}^N} \|x\| \left\|\nabla\mu(x)\right\| \mathrm{d}x = -\int_{\mathbb{R}^N} \|x\|\, \mu_1'\left(\|x\|\right)\,\mathrm{d}x = N\,, \tag{6.5}$$

$$\int_{\mathbb{R}^N} \|x\|^2\, \mu_1'\left(\|x\|\right)\,\mathrm{d}x = -(N+1) \int_{\mathbb{R}^N} \|x\|\, \mu_1\left(\|x\|\right)\,\mathrm{d}x\,. \tag{6.6}$$

Lemma 6.2. *Let* I *be defined as in (5.4). Then,*

$$\begin{aligned}
\limsup_{\varepsilon \to 0} I \;\leqslant\; & \int_{\|x-x_0\| \leq R + MT_0 + \theta} \left|u(0,x) - v(0,x)\right| \mathrm{d}x \\
& - \int_{\|x-x_0\| \leq R + M(T_0 - T)} \left|u(T,x) - v(T,x)\right| \mathrm{d}x + 2 \sup_{\tau \in \{0,T\}} \mathrm{TV}\left(u(\tau)\right) \lambda \\
& + 2 \sup_{\substack{t \in \{0,T\} \\ s \in]t, t+\eta[}} \int_{\|y-x_0\| \leq R + \lambda + M(T_0 - t) + \theta} \left|u(t,y) - u(s,y)\right| \mathrm{d}y\,.
\end{aligned}$$

Proof. See [8, Lemma 5.2]. □

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