Some aspects of hyperbolic PDE : Persistence of shock waves in fluid dynamics, Traffic flow, Stability of scalar conservation laws.

M. Mercier

Institut Camille Jordan, Lyon

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The study of hyperbolic conservation laws differs depending on the type of the equation.

- System of quasi-linear equations : $\mathbf{H}^m$ theory, no access to general weak solutions, except $\mathcal{C}^1$ by pieces – Compressible fluid equations.
- 1D-case : Analysis through numerical schemes – Traffic Flow,
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- Scalar case : $\mathbf{L}^1$ theory – Pedestrian Traffic, Supply chain,
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  [3] R. M. Colombo, M. Mercier and M. D. Rosini, Stability and Total Variation estimates on general scalar balance laws, *Comm. in Math. Sc.*, 2009
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   - Local in time existence
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   - Shock waves
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3. Stability of conservation laws.
   - $\mathbf{L}^1$ stability with respect to flow and source
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We study the Cauchy problem for the Euler compressible equations:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0\,, \\ \partial_t u + (u \cdot \nabla) u + \frac{1}{\rho} \nabla p = 0\,, \\ \partial_t s + u \cdot \nabla s = 0\,. \end{cases} \tag{1}$$

where ρ, u, s are the density, the speed and the specific entropy.
Furthermore, we are given a state law $p : \rho, s \mapsto p(\rho, s)$.
Perfect polytropic gas (PPG):

$$p = (\gamma_0 - 1)\rho^{\gamma_0} \exp(s/c_v)\,,$$

with $\gamma_0 \in]1, 3]$ and especially $\gamma_0 = 5/3, 7/5$ ou $6/5$.
Van der Waals gas (VdW):

$$p = (\gamma_0 - 1)\left(\frac{\rho}{1 - b\rho}\right)^{\gamma_0} \exp(s/c_v)\,.$$

Remark: Same law as for "dutsy" gas.

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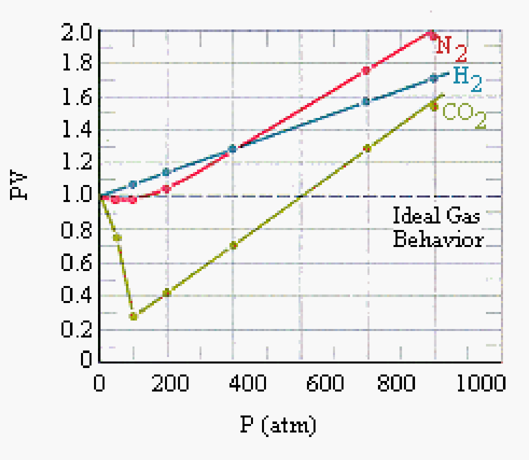
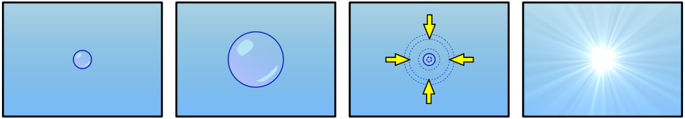
Gaz	*He*	*Ne*	H_2	CO_2	Water vapor
b (cm3/mol)	23.71	17.10	26.61	42.69	30.52

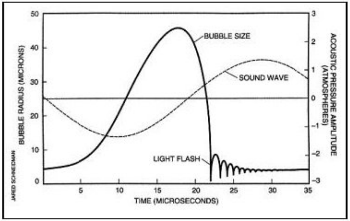
Table: van der Waals Coefficients, Data from Fishbane, et al.



Sonoluminescence

Wu & Roberts (1995) study spherical shock waves in a Van der Waals gas in order to modelise sonoluminescence.
When the bubble shrink, we think that the pression can reach 200 Mbars !





Goals

We want to find some spherically symmetric shock waves with a long time of existence in a dusty gas/VdW gas.

- Find global in time classical solutions for VdW gases;
- Build spherically symmetric shock waves by gluing two classical solutions along a line of discontinuity.

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Properties of the Euler equations

1. Hyperbolicity, if $\left.\frac{\partial p}{\partial \rho}\right|_s > 0$. Then we can define $c^2 = \left.\frac{\partial p}{\partial \rho}\right|_s$.
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   - General case,

$$\begin{cases} \partial_t p + u \cdot \nabla p + c^2 \rho \operatorname{div} u = 0\,, \\ \partial_t u + \frac{1}{\rho} \nabla p + (u \cdot \nabla) u = 0\,, \\ \partial_t s + u \cdot \nabla s = 0\,. \end{cases}$$

   - For a Van der Waals gas, I have adapted the symetrisation from Makino, Ukai & Kawashima (for a PPG). In variables (π, u, s), we have

$$\begin{cases} \partial_t \pi + u \cdot \nabla \pi + \frac{\gamma_0 - 1}{2}(1 + b\,[illegible])\pi \operatorname{div} u &= 0, \\ \partial_t u + e^{[illegible]} \frac{\gamma_0 - 1}{2}(1 + b\,[illegible])\pi \nabla \pi + (u \cdot \nabla) u &= 0, \\ \partial_t s + u \cdot \nabla s &= 0, \end{cases} \quad (2)$$

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Local in time existence results

- Classical Solutions:

Theorem

The Cauchy problem for (1) with initial conditions $(\rho_0, u_0, s_0) \in (\bar{\rho}, \bar{u}, \bar{s}) + \mathbf{H}^m(\mathbb{R}^d)$ *with* $\bar{\rho} > 0$ *and* $m > 1 + \frac{d}{2}$ *admit a classical solution* $(\rho, u, s) \in \mathcal{C}^1([0, T] \times \mathbb{R}^d)$ *with* $T > 0$.

Theorem

We consider a Van der Waals gas. The Cauchy problem (2) with initial conditions $(\pi_0, u_0, s_0) \in \mathbf{H}^m(\mathbb{R}^d)$ *with* $m > 1 + d/2$ *admit a classical solution* $(\pi, u, s) \in \mathcal{C}^1([0, T] \times \mathbb{R}^d)$ *with* $T > 0$.

- Shock waves : Majda's theorem.

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Preceding results

- Result of explosion that do not precise the time of explosion (for PPG): T. C. Sideris (1985), T. Makino, S. Ukai & S. Kawashima (1986), J.-Y. Chemin (1990);
- Almost global existence, with spherical symmetry: Q. Qiu & H. Yin (1999) (isentropic case), P. Godin (2005) (general case);
- Global in time existence. Li Ta Tsien (1981) :Isentropic 1D case; D. Serre (1997), M. Grassin (1998) : multi-D case (PPG).

Global existence for a VdW gas

Théorème (M.)

We consider a Van der Waals gas such that (π_0, u_0, s_0) *are the initial conditions associated to the system (2). There exists* $\varepsilon_0 > 0$ *such that, if* $\nu = \frac{\gamma_0+1}{\gamma_0-1}$ *and*

- $\nu \in \mathbb{N}\backslash\{0,1\}$ *or* $\nu \in [m, +\infty[$,
- $\left\|(\pi_0, s_0)\right\|_{\mathbf{H}^m} \leq \varepsilon_0$, *where* $m > 1 + d/2$, *and* π_0, s_0 *have compact support,*
- $\mathrm{D} u_0 \in \mathbf{L}^\infty$, $\mathrm{D}^2 u_0 \in \mathbf{H}^{m-1}$,
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then there exists a global in time classical solution.

Remark : the first condition can be written: $\gamma_0 \in]1, \frac{m+1}{m-1}] \cup \{\frac{n+1}{n-1} \,;\, n \in \mathbb{N}\,,\, n \geq 2\}$.

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Idea of the proof

As M. Grassin, we introduce an approximated problem

$$\partial_t \bar{u} + (\bar{u} \cdot \nabla)\bar{u} = 0\,, \qquad \bar{u}(0,x) = u_0(x)\,.$$

We consider first the isentropic case. The system writes

$$\partial_t U + \sum_j A_j(U) = B(\bar{u}, \mathrm{D}\bar{u}\,, U, \mathrm{D}U) + F(\mathrm{D}\bar{u}\,, U, \mathrm{D}U)$$

where $U = (\rho, u - \bar{u})$ and

$$F(\mathrm{D}\bar{u}\,, U, \mathrm{D}U) = -\frac{\gamma_0 - 1}{2}\,\tilde{b}\,\pi^\nu \begin{pmatrix} \mathrm{div}\, u \\ \nabla \pi \end{pmatrix}.$$

We estimate $\int_{\mathbb{R}^d} \mathrm{D}^k U \cdot \mathrm{D}^k \left(F(\mathrm{D}\bar{u}\,, U, \mathrm{D}U)\right)$, thanks to the Lemma:

Lemma

Let $f \in \mathbf{L}^\infty \cap \mathbf{H}^m$. If $\nu \in \mathbb{N}\backslash\{0,1\}$ or $\nu \in [m, +\infty[$, then $f^\nu \in \mathbf{H}^m$ and for all $\alpha \in \mathbb{N}^d$ such that $|\alpha| = k \leq m$ we have

$$\left\|\partial^\alpha (f^\nu)\right\|_{\mathbf{L}^2} \leq C \|f\|_{\mathbf{L}^\infty}^{\nu-1} \left\|\mathrm{D}^k f\right\|_{\mathbf{L}^2}.$$

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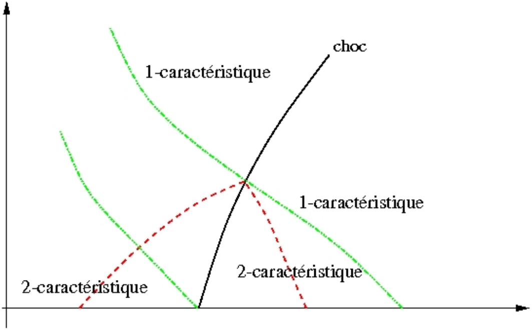
   - Existence of solutions when f, F are non-local

Preceding results

- G. Q. Chen : global in time existence of weak entropy solutions in the isentropic spherical case with $0 \leq \pi_0 \leq u_0 < \infty$ for a PPG.
- Li Ta Tsien (1994): isentropic 1D case
- P. Godin (1997): isentropic irrotationnal in an angular domaine case,

Idea for building a shock wave



Isentropic spherical case

Théorème (M.)

In the isentropic spherical case, the angular problem admit a classical solution if

- $\rho \geq 0$ *on* $\mathcal{C}$ *and* $\mathcal{K}$,
- $\partial_t g + U \partial_r g \geq 0$, *on* $\mathcal{K}$,
- $w_1 \geq 0$ *on* $\mathcal{K}$,
- $\partial_r w_2 \leq -C$ *along* $\mathcal{C}_1$,
- $\rho_0^-(R_0) > \max(\rho^+)$,

and we can estimate the time of existence. For a Van der Waals gas, we have

$$T_{ex} \geq T_* = R_0 C \,,$$

where C *depends on* $\left\| w^- \right\|_{\mathbf{L}^\infty}$

Proof

In the spherical case, the equations are:

$$\begin{cases} \partial_t w_1 + \lambda_1(w)\partial_r w_1 = f(r,w), \\ \partial_t w_2 + \lambda_2(w)\partial_r w_2 = -f(r,w), \end{cases} \tag{3}$$

where $f(r,w) = \frac{(d-1)uc}{r}$.
We estimate $w = (w_1, w_2)$ in $\mathbf{W}^{1,\infty}$ by introducing $v_2 = e^{k(w)}(\partial_r w_2 + \Phi(r,w))$, where

$$\partial_1 k = \frac{\partial_1 \lambda_2}{\lambda_2 - \lambda_1}, \qquad \partial_1(e^k \Phi) = \frac{e^k \partial_1 f}{\lambda_2 - \lambda_1}.$$

Denoting $y_2(t) = -v_2(t, \chi_2(t))$, this leads us to study the ODE

$$y_2' = a_0 y_2^2 - a_1 y_2 - a_2,$$

for some given functions a_0, a_1, a_2.

Perspectives

- What happens in the case of a focalisation ?
- Stability of the spherically symmetric solutions obtained ?
  Other methods to obtain existence of spherical shock waves ?

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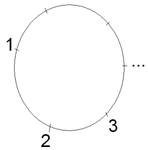
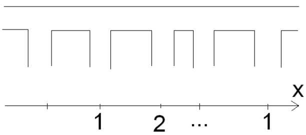
Modelization.





Model

- LWR model on each open segment: let v be a given (decreasing) speed law, then the total density r verifies

$$\partial_t r + \partial_x (r v(r)) = 0;$$

- Special boundary conditions:
  - bounds on the flows of exiting and entering vehicles,
  - conservation of the flow of the vehicles staying on the road.

Result

We write the mass conservation of the vehicles with the same speed law v (multi-class extension of the LWR model) $\partial_t \rho_i + \partial_x(\rho_i v(\sum \rho_j)) = 0$.
The boundary conditions give bounds on the flows of the vehicles:

$$\begin{array}{rcll} \rho_1 v(\rho_1+\rho_2)(t,0-) & = & \rho_1 v(\rho_1+\rho_3)(t,0+) & \max, \\ \rho_2 v(\rho_1+\rho_2)(t,0-) & \leq & o(t) & \max, \\ \rho_3 v(\rho_1+\rho_3)(t,0+) & \leq & i(t) & \max. \end{array} \tag{4}$$

Theorem (M.)

Under the hypotheses

- ***(V)*** *: the speed law v is $\mathcal{C}^{0,1}$, decreasing and vanishes in 1.*
- ***(R)*** *: the flow $q(r) = rv(r)$ is strictly concave and attains its maximum in r_c,*

the Riemann problem for the one T road admits a unique weak entropy solution.

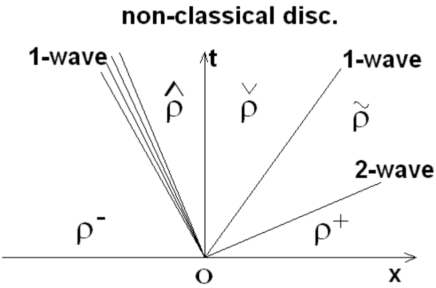


Figure: Solution.

The n T road

Theorem (M.)

*Under the hypotheses **(V)**, **(F)** and **(P)**, there exists $T > 0$ such that the Riemann problem for the n-T road admits a unique weak entropy solution for $t \in [0, T]$. Furthermore, we can give a lower bound for the time of existence: let $L = \min(x_{k+1} - x_k) > 0$, then $T \geq \frac{L}{2V}$.*

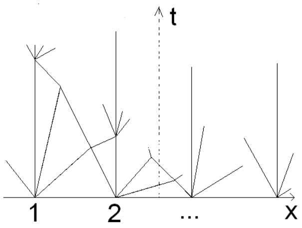
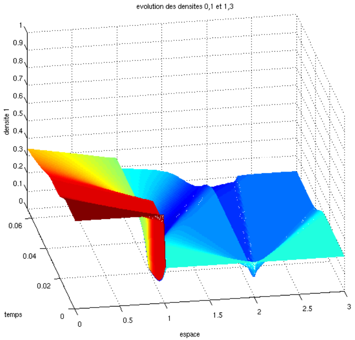
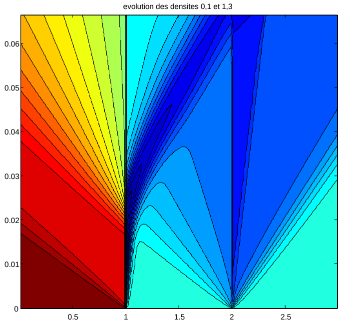


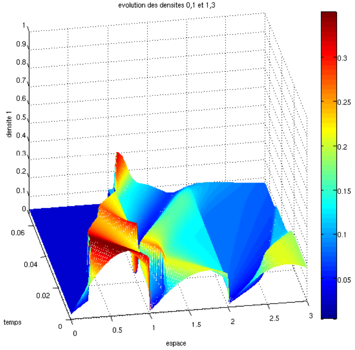
Figure: Solution with *n* points of entry and exit.

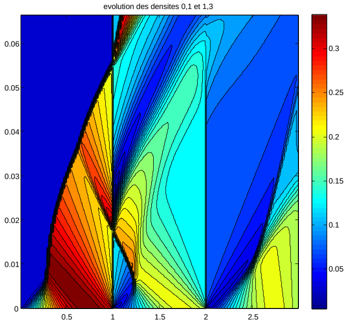
Numerical results : Riemann problem with two T



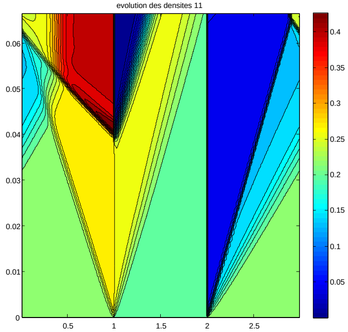


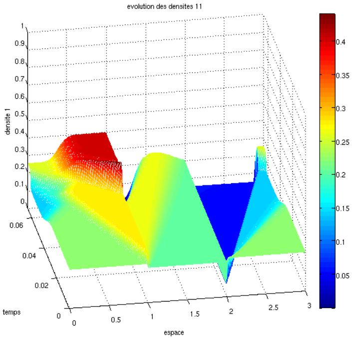
Numerical results : Cauchy problem with two T



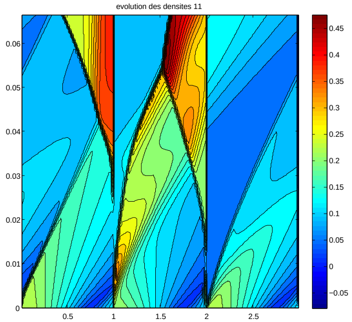


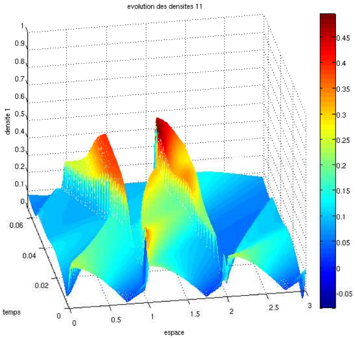
Numerical results : Riemann problem on a roundabout





Numerical results : Cauchy problem for a roundabout





Perspectives

Global in time existence for the n-T case ?
Cauchy problem for the T road ? for the n T road ?
Asymptotic behavior ?

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Introduction

Here we consider scalar balance laws

$$\begin{cases} \partial_t u + \mathrm{Div} f(t,x,u) = F(t,x,u) & (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^N \\ u(0,x) = u_0(x) \in \mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV} & x \in \mathbb{R}^N, \end{cases}$$

where $f \in \mathcal{C}^2([0,T] \times \mathbb{R}^N \times \mathbb{R}; \mathbb{R}^N)$, $F \in \mathcal{C}^1([0,T] \times \mathbb{R}^N \times \mathbb{R}; \mathbb{R})$.

- Existence and uniqueness, dependence toward initial conditions: Kružkov's Theorem
- Dependence with respect to flow and source ?

Preceding results

- Kružkov (1970): Existence and uniqueness of a weak entropy solution in $\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV}$ + dependance toward initial conditions.
  Let $v_0 \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^N;\mathbb{R})$ such that $\|v_0\|_{\mathbf{L}^\infty} \leq M_0$, then

$$\left\|(u-v)(t)\right\|_{\mathbf{L}^1} \leq e^{\gamma t}\|u_0 - v_0\|_{\mathbf{L}^1},$$

  where $\gamma = \left\|\partial_u(F - \mathrm{div} f)\right\|_{\mathbf{L}^\infty(\Omega_M)}$.

- Lucier (1986) : flow not depending on u, no source $F = G = 0$.
  If $f, g : \mathbb{R} \to \mathbb{R}^N$ are globally Lipschitz, then $\exists C > 0$ such that $\forall u_0, v_0 \in \mathbf{L}^1 \cap \mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R})$ initial conditions for

$$\partial_t u + \mathrm{Div} f(u) = 0\,, \qquad \partial_t v + \mathrm{Div} g(v) = 0\,.$$

  with furthermore $v_0 \in \mathbf{BV}(\mathbb{R}^N;\mathbb{R})$, we get $\forall t \geq 0$,

$$\left\|(u-v)(t)\right\|_{\mathbf{L}^1} \leq \|u_0 - v_0\|_{\mathbf{L}^1} + Ct\,\mathrm{TV}(v_0)\,\mathbf{Lip}\,(f-g)\,.$$

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Total Variation

Definition: For $u \in \mathbf{L}^1_{\text{loc}}(\mathbb{R}^N;\mathbb{R})$ we get

$$\begin{aligned}\mathrm{TV}(u) &= \sup\left\{\int_{\mathbb{R}^N} u \,\mathrm{div}\Psi\,;\quad \Psi \in \mathcal{C}^1_c(\mathbb{R}^N;\mathbb{R}^N)\,,\quad \|\Psi\|_{\mathbf{L}^\infty} \leq 1\right\}\,;\\ &\text{et}\\ \mathbf{BV}(\mathbb{R}^N;\mathbb{R}) &= \left\{u \in \mathbf{L}^1_{\text{loc}} ; \mathrm{TV}(u) < \infty\right\}\,.\end{aligned}$$

Remark: When f and F are depending only on u, we have

$$u_0 \in \mathbf{L}^\infty \cap \mathbf{BV} \Rightarrow \forall t \geq 0\,,\quad u(t) \in \mathbf{L}^\infty \cap \mathbf{BV}$$

and denoting $\gamma = \|\partial_u F\|_{\mathbf{L}^\infty(\Omega_M)}$,

$$\mathrm{TV}(u(t)) \leq \mathrm{TV}(u_0)e^{\gamma t}\,.$$

Estimate on total variation

Theorem (TV — Colombo, M., Rosini)

Assume (f, F) *satisfy* ***(K) + (H1)****. Let*
$\kappa_0 = NW_N \left((2N+1) \|\nabla_x \partial_u f\|_{\mathbf{L}^\infty(\Omega_M)} + \|\partial_u F\|_{\mathbf{L}^\infty(\Omega_M)} \right)$. *If* $u_0 \in (\mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^N; \mathbb{R})$, *then* $\forall t \in [0, T]$, $u(t) \in (\mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^N; \mathbb{R})$ *and*

$$\begin{aligned} \mathrm{TV}(u(t)) \leq & \mathrm{TV}(u_0) e^{\kappa_0 t} \\ & + NW_N \int_0^t e^{\kappa_0 (t-\tau)} \int_{\mathbb{R}^N} \left\| \nabla_x (F - \mathrm{div} f)(\tau, x, \cdot) \right\|_{\mathbf{L}^\infty(\mathrm{d}u)} \mathrm{d}x \, \mathrm{d}\tau \, . \end{aligned}$$

(H1) : $\int_0^T \int_{\mathbb{R}^N} \left\| \nabla_x (F - \mathrm{div} f) \right\|_{\mathbf{L}^\infty(\mathrm{d}u)} \mathrm{d}x \mathrm{d}t < \infty$ and $\nabla_x \partial_u f \in \mathbf{L}^\infty(\Omega_M)$

Remark : We have the same estimate as in known particular cases

- f, F depend only on u,
- f, F not depend on u.

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Dependence with respect to flow and source

Theorem (Flow/Source... – Colombo, M. & Rosini)

Assume that $(f, F), (g, G)$ *satisfy* ***(K)***, (f, F) *satisfy* ***(H1)*** *and* $(f - g, F - G)$ *satisfy* ***(H2)***. *Let* $u_0, v_0 \in (\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^N; \mathbb{R})$. *We denote*

$$\kappa = 2N \|\nabla_x \partial_u f\|_{\mathbf{L}^\infty(\Omega_M)} + \|\partial_u F\|_{\mathbf{L}^\infty(\Omega_M)} + \left\|\partial_u (F - G)\right\|_{\mathbf{L}^\infty(\Omega_M)}.$$

Let u *and* v *be the solutions associated respectively to the flows and sources* (f, F) *and* (g, G) *and to initial conditions* (u_0, v_0).

(H2) : $\partial_u(F - G) \in \mathbf{L}^\infty(\Omega_M)$, $\partial_u(f - g) \in \mathbf{L}^\infty(\Omega_M)$ and $\int_0^T \int_{\mathbb{R}^N} \left\|F - G - \mathrm{div}(f - g)\right\|_{\mathbf{L}^\infty(\mathrm{d}u)} \mathrm{d}x\mathrm{d}t < \infty$.

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Theorem (...Flow/Source – Colombo, M. & Rosini)

then $\forall t \in [0, T]$*:*

$$\begin{aligned}\left\|(u-v)(t)\right\|_{\mathbf{L}^1} \leq{}& e^{\kappa t}\left\|u_0 - v_0\right\|_{\mathbf{L}^1} + \frac{e^{\kappa_0 t} - e^{\kappa t}}{\kappa_0 - \kappa}\mathrm{TV}(u_0)\left\|\partial_u(f-g)\right\|_{\mathbf{L}^\infty}\\ &+ \int_0^t \frac{e^{\kappa_0(t-\tau)} - e^{\kappa(t-\tau)}}{\kappa_0 - \kappa}\int_{\mathbb{R}^N}\left\|\nabla_x(F - \mathrm{div} f)(\tau, x, \cdot)\right\|_{\mathbf{L}^\infty(\mathrm{d}u)}\mathrm{d}x\mathrm{d}\tau\\ &\quad \times N W_N \left\|\partial_u(f-g)\right\|_{\mathbf{L}^\infty}\\ &+ \int_0^t e^{\kappa(t-\tau)}\int_{\mathbb{R}^N}\left\|((F-G) - \mathrm{div}(f-g))(\tau, x, \cdot)\right\|_{\mathbf{L}^\infty(\mathrm{d}u)}\mathrm{d}x\mathrm{d}\tau\,.\end{aligned}$$

Remark : As for Theorem (TV), we find the same estimates as in some known particular cases.

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Continuity equation

$$\partial_t \rho + \mathrm{Div}(\rho V(\rho(t))) = 0\,, \qquad \rho(0,\cdot) = \rho_0 \in \mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV}\,,$$

where $V : \mathbf{L}^1(\mathbb{R}^N;\mathbb{R}) \to \mathcal{C}^2(\mathbb{R}^N;\mathbb{R})$ is a non-local fonctional (regularizing), for example, if $v : \mathbb{R} \to \mathbb{R}$ is regular:

- $V(\rho) = v\left(\int_{\mathbb{R}^N} \rho\, \mathrm{d}x\right)$ in the supply-chain model
- $V(\rho) = v(\eta *_x \rho)\vec{v}(x)$, η being a regularizing kernel, in a pedestrian trafic model.

Goal:

- Existence and uniqueness of an entropy solution ?
- Gâteaux derivative of the (non-linear) semi-group ?

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Goal:

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Existence of a solution

Theorem (Traffic — Colombo, Herty, M.)

*If V satisfy **(V1)**, then there exists $T_{ex} > 0$ and a unique weak entropy solution $u \in \mathcal{C}^0([0, T_{ex}[; \mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})$ and we denote $S_t u_0 = u(t, \cdot)$.*
We can bound by below the time of existence by

$$T_{ex} = \sup \left\{ \sum_n \frac{\ln(\alpha_{n+1}/\alpha_n)}{C(\alpha_{n+1})} \, ; \, (\alpha_n)_n \text{ strict. increasing} \, , \, \alpha_0 = \|u_0\|_{\mathbf{L}^\infty} \right\} .$$

*If furthermore V satisfy **(V2)** then*

$$u_0 \in \mathbf{W}^{2,1} \cap \mathbf{L}^\infty \Rightarrow \forall t \in [0, T_{ex}[\, , \quad u(t) \in \mathbf{W}^{2,1} \, .$$

Hypothesis

(V1) there exists $C \in \mathbf{L}^{\infty}_{\mathbf{loc}}(\mathbb{R}_+;\mathbb{R}_+)$ such that $\forall u \in \mathbf{L}^{1}(\mathbb{R}^N;\mathbb{R})$

$$V(u) \in \mathbf{L}^{\infty}, \qquad \left\|\nabla_x V(u)\right\|_{\mathbf{L}^\infty} \leq C(\|u\|_{\mathbf{L}^\infty}),$$

$$\left\|\nabla_x V(u)\right\|_{\mathbf{L}^1} \leq C(\|u\|_{\mathbf{L}^\infty}), \qquad \left\|\nabla_x^2 V(u)\right\|_{\mathbf{L}^1} \leq C(\|u\|_{\mathbf{L}^\infty}),$$

and $\forall u_1, u_2 \in \mathbf{L}^{1}(\mathbb{R}^N;\mathbb{R})$

$$\left\|V(u_1) - V(u_2)\right\|_{\mathbf{L}^\infty} \leq C(\|u_1\|_{\mathbf{L}^\infty})\|u_1 - u_2\|_{\mathbf{L}^1},$$

$$\left\|\nabla_x(V(u_1) - V(u_2))\right\|_{\mathbf{L}^1} \leq C(\|u_1\|_{\mathbf{L}^\infty})\|u_1 - u_2\|_{\mathbf{L}^1}.$$

(V2) there exists $C \in \mathbf{L}^{\infty}_{\mathbf{loc}}(\mathbb{R}_+;\mathbb{R}_+)$ such that $\left\|\nabla_x^3 V(u)\right\|_{\mathbf{L}^\infty} \leq C(\|u\|_{\mathbf{L}^\infty})$.

Idea of the proof:
Let $\beta > \alpha > 0$ and $T \leq T_* = \frac{\ln(\beta/\alpha)}{C(\beta)}$. We introduce the space $X_\alpha = \mathbf{L}^1 \cap \mathbf{BV}(\mathbb{R}^N; [0, \alpha])$ and the application $\mathcal{Q}$ that associates to $w \in \mathcal{X}_\beta = \mathcal{C}^0([0, T[, X_\beta)$ the solution $u \in \mathcal{X}_\beta$ of the problem

$$\partial_t u + \mathrm{Div}(uV(w)) = 0\,, \quad u(0, \cdot) = u_0 \in X_\alpha$$

For w_1, w_2, we get, thanks to the estimate of Theorem (Flow/Source)

$$\left\|\mathcal{Q}(w_1) - \mathcal{Q}(w_2)\right\|_{\mathbf{L}^\infty([0,T[,\mathbf{L}^1)} \leq f(T)\|w_1 - w_2\|_{\mathbf{L}^\infty([0,T[,\mathbf{L}^1)}\,,$$

where f is increasing, $f(0) = 0$ and $f \to_{T\to\infty} \infty$
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Definition : We say that the application $S : \mathbf{L}^1(\mathbb{R}^N;\mathbb{R}) \to \mathbf{L}^1(\mathbb{R}^N;\mathbb{R})$ is $\mathbf{L}^1$ *Gâteaux differentiable in* $u_0 \in \mathbf{L}^1$ *in the direction* $r_0 \in \mathbf{L}^1$ if there exists a continuous linear application $DS(u_0) : \mathbf{L}^1 \to \mathbf{L}^1$ such that

$$\left\| \frac{S(u_0 + hr_0) - S(u_0)}{h} - DS(u_0)(r_0) \right\|_{\mathbf{L}^1} \to_{h \to 0} 0\,.$$

We want to show that the local semi-group giving the solution of the pedestrian traffic is $\mathbf{L}^1$ Gâteaux differentiable. We expect the derivative to be the solution of the linearized problem:

$$\partial_t r + \mathrm{Div}(rV(u) + uDV(u)(r)) = 0\,, \quad r(0,\cdot) = r_0\,.$$

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We introduce the following hypotheses:

(V3) $V : \mathbf{L^1} \to \mathcal{C}^2$ is Fréchet differentiable and there exists $C \in \mathbf{L^\infty_{loc}}$ such that $\forall u, r \in \mathbf{L^1}$,

$$\left\| V(u+r) - V(u) - DV(u)(r) \right\|_{\mathbf{W^{2,\infty}}} \leq C\left(\|u\|_{\mathbf{L^\infty}} + \|u+r\|_{\mathbf{L^\infty}} \right) \|r\|^2_{\mathbf{L^1}} ,$$

$$\left\| DV(u)(r) \right\|_{\mathbf{W^{2,\infty}}} \leq C(\|u\|_{\mathbf{L^\infty}}) \|r\|_{\mathbf{L^1}} .$$

(V4) There exists $C \in \mathbf{L^\infty_{loc}}(\mathbb{R}_+; \mathbb{R}_+)$ such that $\forall u, \tilde{u}, r \in \mathbf{L^1}$

$$\left\| \mathrm{div}\left(V(\tilde{u}) - V(u) - DV(u)(\tilde{u} - u) \right) \right\|_{\mathbf{L^1}} \leq C(\|\tilde{u}\|_{\mathbf{L^\infty}} + \|u\|_{\mathbf{L^\infty}})(\|\tilde{u} - u\|_{\mathbf{L^1}})^2$$

$$\left\| \mathrm{div}\left(DV(u)(r) \right) \right\|_{\mathbf{L^1}} \leq C(\|u\|_{\mathbf{L^\infty}}) \|r\|_{\mathbf{L^1}} .$$

We show that the linearized problem admit a unique weak entropy solution:

Theorem (Linearized — Colombo, Herty, M.)

*Assume that V satisfy **(V1), (V2), (V3)**. Let $u \in \mathcal{C}^0([0, T_{ex}[; \mathbf{W}^{1,\infty} \cap \mathbf{W}^{1,1})$, $r_0 \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^N; \mathbb{R})$. Then, the linearized problem*

$$\partial_t r + \mathrm{Div}(rV(u) + uDV(u)(r)) = 0\,, \qquad with\ r(0,x) = r_0$$

admit a unique weak entropy solution $r \in \mathcal{C}^0([0, T_{ex}[; \mathbf{L}^1(\mathbb{R}^N; \mathbb{R}))$ and we denote $\Sigma_t^u r_0 = r(t, \cdot)$.
If furthermore $r_0 \in \mathbf{W}^{1,1}$, then $\forall t \in [0, T_{ex}[$, $r(t) \in \mathbf{W}^{1,1}$.

Theorem (Gâteaux Derivative — Colombo, Herty, M.)

*Assume that V satisfy **(V1),(V2),(V3),(V4)**. Let $u_0 \in \mathbf{W}^{1,\infty} \cap \mathbf{W}^{2,1}$, $r_0 \in \mathbf{W}^{1,1} \cap \mathbf{L}^\infty$ and let T_{ex} be the time of existence of the initial problem given by Theorem (Traffic). Then for all $t \in [0, T_{ex}[$ the local semi-group of the problem of pedestrian traffic is $\mathbf{L}^1$ Gâteaux differentiable in the direction r_0 and*

$$DS_t(u_0)(r_0) = \Sigma_t^{S_t u_0} r_0 \,.$$

Idea of the proof: Theorem (Flow/Source) allows us to compare the solution associated to the initial condition $u_0 + hr_0$ to the solution $u + hr$.

Let u, u_h be the solutions of the problem $\partial_t u + \mathrm{Div}(uV(u)) = 0$ with initial conditions $u_0, u_0 + hr_0$. Let r be the solution of the linearized problem $\partial_t r + \mathrm{Div}(rV(u) + uDV(u)(r)) = 0$, $r(0) = r_0$ and let $z_h = u + hr$ that satisfies

$$\partial_t z_h + \mathrm{Div}\left(z_h(V(u) + hDV(u)(r))\right) = h^2 \mathrm{Div}(rDV(u)(r)), \quad z_h(0) = u_0 + hr_0 .$$

Then we use Theorem (Flow/Source) to compare u_h and z_h. We obtain

$$\frac{1}{h}\|u_h - z_h\|_{\mathbf{L}^\infty([0,T[,\mathbf{L}^1)} \leq F(T)\left(M\|u_h - u\|^2_{\mathbf{L}^\infty(\mathbf{L}^1)} + \frac{1}{h}\|u_h - z_h\|_{\mathbf{L}^\infty(\mathbf{L}^1)}\right) + hC(\beta)Te^{C(\beta)T}\|r\|_{\mathbf{L}^\infty(\mathbf{W}^{1,1})}\|r\|_{\mathbf{L}^\infty(\mathbf{L}^1)},$$

with F increasing and $F(0) = 0$.

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with F increasing and $F(0) = 0$.

Perspectives

Derivation with respect to geometric term ?
Example of explosion in finite time ?
Change in the model to obtain bounds in $\mathbf{L}^\infty$?

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