L^1 Stability for scalar balance laws.
Control of the continuity equation with a non-local flow.

Magali Mercier

Institut Camille Jordan, Lyon

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Introduction

Scalar balance laws:

$$\begin{cases} \partial_t u + \mathrm{Div} f(t,x,u) = F(t,x,u) & (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^N \\ u(0,x) = u_0(x) \in \mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV} & x \in \mathbb{R}^N, \end{cases}$$

where $f \in \mathcal{C}^2([0,T] \times \mathbb{R}^N \times \mathbb{R}; \mathbb{R}^N)$, $F \in \mathcal{C}^1([0,T] \times \mathbb{R}^N \times \mathbb{R}; \mathbb{R})$.

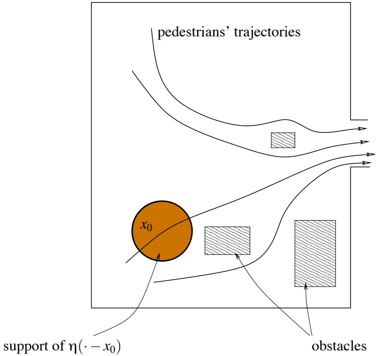
- Existence and uniqueness, dependence w.r.t. initial conditions: Kružkov Theorem;
- Dependence w.r.t. flow and source ?

Continuity equation:

$$\partial_t u + \mathrm{Div}(uV(u(t))) = 0\,, \qquad u(0,\cdot) = u_0 \in \mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV}\,,$$

where $V : \mathbf{L}^1(\mathbb{R}^N;\mathbb{R}) \to \mathcal{C}^2(\mathbb{R}^N;\mathbb{R})$ is a non-local averaging functional, for example, if $v : \mathbb{R} \to \mathbb{R}$ is a regular function:

- $V(u) = v\left(\int_{\mathbb{R}} u \,\mathrm{d}x\right)$ for a supply-chain
- $V(u) = v(\eta *_x u)w(x)$, for pedestrian traffic.



Goal :

- Existence and uniqueness of an entropy solution ?
- Gâteaux differentiability of the semi-group w.r.t. initial conditions ?

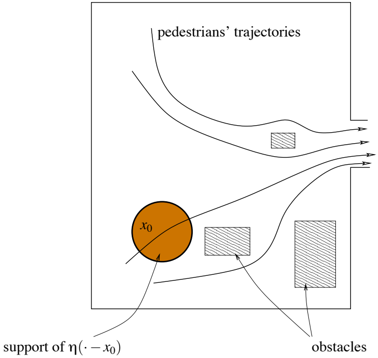


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Table of contents

Plan

1. L^1 Stability with respect to flow and source
   - Previous Results
   - Estimate on the total variation
   - Dependence with respect to flow and source
2. The continuity equation with a non-local flow
   - Existence and uniqueness of a solution
   - Gâteaux derivative of the semi-group
   - Extrema of a Cost Functional

Theorem (Kružkov (1970))

Let us denote $\Omega_A = [0, T] \times \mathbb{R}^N \times [-A, A]$ *for all* $A \geq 0$. *If*

$$(\mathbf{K}) \quad \forall A > 0 \, , \; \partial_u f \in \mathbf{L}^\infty(\Omega_A) \, , \; \partial_u(F - \mathrm{div} f) \in \mathbf{L}^\infty(\Omega_A) \\ \quad and \;\; F - \mathrm{div} f \in \mathbf{L}^\infty(\Omega_A)$$

then for all $u_0 \in (\mathbf{L}^\infty \cap \mathbf{L}^1)(\mathbb{R}^N; \mathbb{R})$ *such that* $\|u_0\|_{\mathbf{L}^\infty} \leq M_0$, *there exists a unique weak entropy solution* $u \in \mathbf{L}^\infty([0, T]; \mathbf{L}^1(\mathbb{R}^N; \mathbb{R}))$ *continuous from the right in time and there exists* $M > 0$ *such that* $\|u\|_{\mathbf{L}^\infty} \leq M$.

Let $v_0 \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^N; \mathbb{R})$ *be such that* $\|v_0\|_{\mathbf{L}^\infty} \leq M_0$, *then*

$$\left\|(u - v)(t)\right\|_{\mathbf{L}^1} \leq e^{\gamma t} \|u_0 - v_0\|_{\mathbf{L}^1} \, ,$$

where $\gamma = \left\|\partial_u(F - \mathrm{div} f)\right\|_{\mathbf{L}^\infty(\Omega_M)}$.

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Theorem (Lucier)

If $f, g : \mathbb{R} \to \mathbb{R}^N$ are globally lipschitz, then $\exists C > 0$ such that $\forall u_0, v_0 \in \mathbf{L}^1 \cap \mathbf{L}^\infty(\mathbb{R}^N; \mathbb{R})$ initial conditions for

$$\partial_t u + \mathrm{Div} f(u) = 0\,, \qquad \partial_t v + \mathrm{Div} g(v) = 0\,.$$

with furthermore $v_0 \in \mathbf{BV}(\mathbb{R}^N; \mathbb{R})$, we have $\forall t \geq 0$,

$$\left\|(u-v)(t)\right\|_{\mathbf{L}^1} \leq \|u_0 - v_0\|_{\mathbf{L}^1} + Ct\,\mathrm{TV}(v_0)\,\mathbf{Lip}\,(f-g)\,.$$

Theorem (Chen & Karlsen)

With $f(t,x,u) = \lambda(x) l(u)$, $g(t,x,v) = \mu(x) m(v)$, no source $F = G = 0$,

$$\begin{aligned}\left\|(u-v)(t)\right\|_{\mathbf{L}^1} \leq \|u_0 - v_0\|_{\mathbf{L}^1} + C_1 t \big(&\|\lambda-\mu\|_{\mathbf{L}^\infty} + \|\lambda-\mu\|_{\mathbf{W}^{1,1}} \\ &+ \|l-m\|_{\mathbf{L}^\infty} + \|l-m\|_{\mathbf{W}^{1,\infty}}\big)\end{aligned}$$

where $C_1 = C\sup_{[0,T]}\left(\mathrm{TV}(u(t)), \mathrm{TV}(v(t))\right)$.

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where $C_1 = C\sup_{[0,T]}\big(\mathrm{TV}(u(t)), \mathrm{TV}(v(t))\big)$.

Total Variation

Definition: For $u \in \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}^N;\mathbb{R})$ we denote

$$\mathrm{TV}(u) = \sup\left\{\int_{\mathbb{R}^N} u \,\mathrm{div}\Psi\,;\quad \Psi \in \mathcal{C}^1_c(\mathbb{R}^N;\mathbb{R}^N)\,,\quad \|\Psi\|_{\mathbf{L}^\infty} \leq 1\right\}\,;$$

and

$$\mathbf{BV}(\mathbb{R}^N;\mathbb{R}) = \left\{u \in \mathbf{L}^1_{\mathbf{loc}};\mathrm{TV}(u) < \infty\right\}\,.$$

Remark : When f and F depend only on u we have

$$u_0 \in \mathbf{L}^\infty \cap \mathbf{BV} \Rightarrow \forall t \geq 0\,,\quad u(t) \in \mathbf{L}^\infty \cap \mathbf{BV}$$

and, denoting $\gamma = \|\partial_u F\|_{\mathbf{L}^\infty(\Omega_M)}$,

$$\mathrm{TV}(u(t)) \leq \mathrm{TV}(u_0)e^{\gamma t}\,.$$

Problem: we do not have in general an estimate on the total variation !

Plan

Estimate on the total variation (Colombo, Mercier, Rosini)

Theorem (TV)

Assume (f, F) *satisfies* ***(K)*** *+* ***(H1)****. Let*
$\kappa_0 = NW_N\left((2N+1)\|\nabla_x \partial_u f\|_{\mathbf{L}^\infty(\Omega_M)} + \|\partial_u F\|_{\mathbf{L}^\infty(\Omega_M)}\right)$*. If* $u_0 \in (\mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^N;\mathbb{R})$*, then* $\forall t \in [0, T]$, $u(t) \in (\mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^N;\mathbb{R})$ *and*

$$\begin{aligned}\mathrm{TV}(u(t)) \leq & \mathrm{TV}(u_0) e^{\kappa_0 t} \\ & + NW_N \int_0^t e^{\kappa_0 (t-\tau)} \int_{\mathbb{R}^N} \left\|\nabla_x (F - \mathrm{div} f)(\tau, x, \cdot)\right\|_{\mathbf{L}^\infty(\mathrm{d}u)} \mathrm{d}x \, \mathrm{d}\tau \,.\end{aligned}$$

(H1) : $\int_0^T \int_{\mathbb{R}^N} \left\|\nabla_x (F - \mathrm{div} f)\right\|_{\mathbf{L}^\infty(\mathrm{d}u)} \mathrm{d}x\mathrm{d}t < \infty$ et $\nabla_x \partial_u f \in \mathbf{L}^\infty(\Omega_M)$

Remark : In some particular cases, we re-obtain known estimates:

- f, F depending only on u,
- f, F not depending on u.

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Theorem (TV)

*Assume (f, F) satisfies **(K) + (H1)**. Let*
$\kappa_0 = NW_N\left((2N+1)\|\nabla_x\partial_u f\|_{\mathbf{L}^\infty(\Omega_M)} + \|\partial_u F\|_{\mathbf{L}^\infty(\Omega_M)}\right)$. *If $u_0 \in (\mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^N;\mathbb{R})$, then $\forall t \in [0,T]$, $u(t) \in (\mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^N;\mathbb{R})$ and*

$$\begin{aligned}\mathrm{TV}(u(t)) \leq& \mathrm{TV}(u_0)e^{\kappa_0 t}\\ &+ NW_N\int_0^t e^{\kappa_0(t-\tau)}\int_{\mathbb{R}^N}\left\|\nabla_x(F - \mathrm{div} f)(\tau,x,\cdot)\right\|_{\mathbf{L}^\infty(\mathrm{d}u)}\mathrm{d}x\,\mathrm{d}\tau\,.\end{aligned}$$

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Idea of the proof

Proposition

Let $\mu \in \mathcal{C}_c^\infty(\mathbb{R}_+;\mathbb{R}_+)$ be such that $\|\mu\|_{\mathbf{L}^1} = 1$ and $\mu' < 0$ on $\mathbb{R}_+^*$. We denote $\mu_\lambda(x) = \frac{1}{\lambda^N}\mu\left(\frac{\|x\|}{\lambda}\right)$. If there exists $C_0 > 0$ such that $\forall \lambda > 0$,

$$\frac{1}{\lambda}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left|u(x+y) - u(x)\right| \mu_\lambda(y) \mathrm{d}x\mathrm{d}y \leq C_0,$$

*then $u \in$ **BV** and*

$$\mathrm{TV}(u)\int_{\mathbb{R}^N} |y_1|\mu(\|y\|)\mathrm{d}y \leq C_0.$$

Let us introduce

$$\mathcal{F}(T,\lambda) = \int_0^T \int_{\mathbb{R}^N}\int_{B(x_0,R+M(T_0-t))} \left|u(x+y) - u(x)\right| \mu_\lambda(y) \mathrm{d}x\mathrm{d}y\mathrm{d}t.$$

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The doubling variables method gives the estimate:

$$\partial_T \mathcal{F}(T,\lambda) \leq \partial_T \mathcal{F}(0,\lambda) + C\lambda\partial_\lambda \mathcal{F}(T,\lambda) + C'\mathcal{F}(T,\lambda) + \lambda \int_0^T A(t)\mathrm{d}t\,,$$

where $A(t) = \int_{\mathbb{R}^N} \left\|\nabla(F - \mathrm{div} f)(t,x\cdot)\right\|_{\mathbf{L}^\infty(\mathrm{d}u)}$.

We integrate in time and divide by $CT\lambda$:

$$0 \leq \frac{1}{C\lambda}\mathcal{F}(0,\lambda) + \partial_\lambda \mathcal{F}(T,\lambda) + \frac{\alpha(T)}{\lambda}\mathcal{F}(T,\lambda) + \frac{1}{C}\int_0^T A(t)\mathrm{d}t\,,$$

where $\alpha(T) = N + C'/C - 1/T \to -\infty$ when $T \to 0$. We choose T small enough and we integrate on $[\lambda, +\infty[$.

We obtain

$$\mathcal{F}(T,\lambda) \leq \frac{\lambda}{-\alpha-1} K\mathrm{TV}(u_0) + \frac{\lambda}{C(-\alpha-1)}\int_0^T A(t)\mathrm{d}t\,.$$

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Dependence with respect to flow and source

Theorem (Flow/Source...)

Assume $(f, F), (g, G)$ *satisfy* ***(K)***, (f, F) *satisfies* ***(H1)*** *and* $(f - g, F - G)$ *satisfies* ***(H2)***. *Let* $u_0, v_0 \in (\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^N; \mathbb{R})$. *We denote*

$$\kappa = 2N \|\nabla_x \partial_u f\|_{\mathbf{L}^\infty(\Omega_M)} + \|\partial_u F\|_{\mathbf{L}^\infty(\Omega_M)} + \left\|\partial_u (F - G)\right\|_{\mathbf{L}^\infty(\Omega_M)}.$$

Let u *and* v *be the solutions associated to* (f, F) *and* (g, G) *respectively and with initial conditions* (u_0, v_0).

(H2) : $\partial_u(F - G) \in \mathbf{L}^\infty(\Omega_M)$, $\partial_u(f - g) \in \mathbf{L}^\infty(\Omega_M)$ and $\int_0^T \int_{\mathbb{R}^N} \left\|F - G - \mathrm{div}(f - g)\right\|_{\mathbf{L}^\infty(\mathrm{d}u)} \mathrm{d}x\mathrm{d}t < \infty$.

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Theorem (...Flow/Source)

then $\forall t \in [0, T]$:

$$\begin{aligned}\left\|(u-v)(t)\right\|_{\mathbf{L}^1} &\leq e^{\kappa t}\left\|u_0 - v_0\right\|_{\mathbf{L}^1} + \frac{e^{\kappa_0 t} - e^{\kappa t}}{\kappa_0 - \kappa}\mathrm{TV}(u_0)\left\|\partial_u(f-g)\right\|_{\mathbf{L}^\infty}\\ &+ \int_0^t \frac{e^{\kappa_0(t-\tau)} - e^{\kappa(t-\tau)}}{\kappa_0 - \kappa}\int_{\mathbb{R}^N}\left\|\nabla_x(F - \mathrm{div} f)(\tau, x, \cdot)\right\|_{\mathbf{L}^\infty(\mathrm{d}u)}\mathrm{d}x\mathrm{d}\tau\\ &\quad \times NW_N\left\|\partial_u(f-g)\right\|_{\mathbf{L}^\infty}\\ &+ \int_0^t e^{\kappa(t-\tau)}\int_{\mathbb{R}^N}\left\|((F-G) - \mathrm{div}(f-g))(\tau, x, \cdot)\right\|_{\mathbf{L}^\infty(\mathrm{d}u)}\mathrm{d}x\mathrm{d}\tau\,.\end{aligned}$$

Remark : As before, we re-obtain known estimates in some particular cases

- f, g depend only on u, $F = G = 0$,
- f, g, F, G not depend on u.

Theorem (...Flow/Source)

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Pedestrian Traffic (Colombo, Herty, Mercier)

We now consider equations of the type

$$\partial_t u + \mathrm{Div}(uV(u)) = 0\,; \quad u_0 \in (\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})(\mathbb{R}^N;\mathbb{R})$$

where $V : \mathbf{L}^1 \to \mathcal{C}^2$ is a non-local functional.

Existence of a solution

Theorem (Traffic)

*If V satisfies **(V1)**, then there exists a time $T_{ex} > 0$ and a unique entropy solution*

$$u \in \mathcal{C}^0([0, T_{ex}[; \mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV})$$

and we denote $S_t u_0 = u(t, \cdot)$.

We can bound by below the time of existence by

$$T_{ex} \geq \sup \left\{ \sum_n \frac{\ln(\alpha_{n+1}/\alpha_n)}{C(\alpha_{n+1})} \,;\, (\alpha_n)_n \text{ strict. increasing}, \, \alpha_0 = \|u_0\|_{\mathbf{L}^\infty} \right\}.$$

*If furthermore, V satisfies **(V2)** then*

$$u_0 \in \mathbf{W}^{2,1} \cap \mathbf{L}^\infty \Rightarrow \forall t \in [0, T_{ex}[, \quad u(t) \in \mathbf{W}^{2,1}.$$

Hypotheses

(V1) There exists $C \in \mathbf{L}^\infty_{\mathbf{loc}}(\mathbb{R}_+;\mathbb{R}_+)$ such that $\forall u \in \mathbf{L}^1(\mathbb{R}^N;\mathbb{R})$

$$V(u) \in \mathbf{L}^\infty, \qquad \left\|\nabla_x V(u)\right\|_{\mathbf{L}^\infty} \leq C(\|u\|_{\mathbf{L}^\infty}),$$

$$\left\|\nabla_x V(u)\right\|_{\mathbf{L}^1} \leq C(\|u\|_{\mathbf{L}^\infty}), \qquad \left\|\nabla_x^2 V(u)\right\|_{\mathbf{L}^1} \leq C(\|u\|_{\mathbf{L}^\infty}),$$

and $\forall u_1, u_2 \in \mathbf{L}^1(\mathbb{R}^N;\mathbb{R})$

$$\left\|V(u_1) - V(u_2)\right\|_{\mathbf{L}^\infty} \leq C(\|u_1\|_{\mathbf{L}^\infty})\|u_1 - u_2\|_{\mathbf{L}^1},$$

$$\left\|\nabla_x (V(u_1) - V(u_2))\right\|_{\mathbf{L}^1} \leq C(\|u_1\|_{\mathbf{L}^\infty})\|u_1 - u_2\|_{\mathbf{L}^1}.$$

(V2) There exists $C \in \mathbf{L}^\infty_{\mathbf{loc}}(\mathbb{R}_+;\mathbb{R}_+)$ such that $\left\|\nabla_x^3 V(u)\right\|_{\mathbf{L}^\infty} \leq C(\|u\|_{\mathbf{L}^\infty})$.

Idea of the proof:

Let us introduce the space $X_\alpha = \mathbf{L}^1 \cap \mathbf{BV}(\mathbb{R}^N; [0, \alpha])$ and the application $\mathcal{Q}$ that associates to $w \in \mathcal{X}_\beta = \mathcal{C}^0([0, T[, X_\beta)$ the solution $u \in \mathcal{X}_\beta$ of the Cauchy problem

$$\partial_t u + \mathrm{Div}(uV(w)) = 0\,, \quad u(0, \cdot) = u_0 \in X_\alpha$$

For w_1, w_2, we obtain thanks to the estimate of Thm (Flow/Source)

$$\left\| \mathcal{Q}(w_1) - \mathcal{Q}(w_2) \right\|_{\mathbf{L}^\infty([0,T[,\mathbf{L}^1)} \leq f(T) \| w_1 - w_2 \|_{\mathbf{L}^\infty([0,T[,\mathbf{L}^1)}\,,$$

where f is increasing , $f(0) = 0$ and $f \to_{T \to \infty} \infty$.
Then we apply the Banach Fixed Point Theorem.

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Plan

Definition : The application $S : \mathbf{L}^1(\mathbb{R}^N;\mathbb{R}) \to \mathbf{L}^1(\mathbb{R}^N;\mathbb{R})$ is said to be $\mathbf{L}^1$ *Gâteaux differentiable in* $u_0 \in \mathbf{L}^1$ *in the direction* $r_0 \in \mathbf{L}^1$ if there exists a linear continuous application $DS(u_0) : \mathbf{L}^1 \to \mathbf{L}^1$ such that

$$\left\| \frac{S(u_0 + hr_0) - S(u_0)}{h} - DS(u_0)(r_0) \right\|_{\mathbf{L}^1} \to_{h\to 0} 0\,.$$

Formally, we expect the Gâteaux derivative of the semi-group to be the solution of the linearized problem:

$$\partial_t r + \mathrm{Div}(rV(u) + uDV(u)(r)) = 0\,, \quad r(0,\cdot) = r_0\,.$$

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We introduce the hypotheses:
(V3) $V : \mathbf{L}^1 \to \mathcal{C}^2$ is differentiable and there exists $C \in \mathbf{L}^\infty_{\mathbf{loc}}$ such that $\forall u, r \in \mathbf{L}^1$,

$$\left\| V(u+r) - V(u) - DV(u)(r) \right\|_{\mathbf{W}^{2,\infty}} \leq C\left(\|u\|_{\mathbf{L}^\infty} + \|u+r\|_{\mathbf{L}^\infty} \right) \|r\|^2_{\mathbf{L}^1},$$

$$\left\| DV(u)(r) \right\|_{\mathbf{W}^{2,\infty}} \leq C(\|u\|_{\mathbf{L}^\infty}) \|r\|_{\mathbf{L}^1}.$$

(V4) There exists $C \in \mathbf{L}^\infty_{\mathbf{loc}}(\mathbb{R}_+;\mathbb{R}_+)$ such that $\forall u, \tilde{u}, r \in \mathbf{L}^1$

$$\left\| \mathrm{div}\left(V(\tilde{u}) - V(u) - DV(u)(\tilde{u}-u) \right) \right\|_{\mathbf{L}^1} \leq C(\|\tilde{u}\|_{\mathbf{L}^\infty} + \|u\|_{\mathbf{L}^\infty})(\|\tilde{u}-u\|_{\mathbf{L}^1})^2$$

$$\left\| \mathrm{div}\left(DV(u)(r) \right) \right\|_{\mathbf{L}^1} \leq C(\|u\|_{\mathbf{L}^\infty}) \|r\|_{\mathbf{L}^1}.$$

We show that the linearised problem has a unique entropy solution:

Theorem (Linearised)

*Assume that V satisfies **(V1), (V2), (V3)**. Let $u \in \mathcal{C}^0([0, T_{ex}[; \mathbf{W}^{1,\infty} \cap \mathbf{W}^{1,1})$, $r_0 \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^N; \mathbb{R})$. Then the linearised problem*

$$\partial_t r + \mathrm{Div}(rV(u) + uDV(u)(r)) = 0\,, \qquad \textit{with } r(0,x) = r_0$$

has a unique entropy solution $r \in \mathcal{C}^0([0, T_{ex}[; \mathbf{L}^1(\mathbb{R}^N; \mathbb{R}))$ and we denote $\Sigma_t^u r_0 = r(t, \cdot)$. If furthermore $r_0 \in \mathbf{W}^{1,1}$, then $\forall t \in [0, T_{ex}[$, $r(t) \in \mathbf{W}^{1,1}$.

Theorem (Gâteaux Derivative)

Assume that V satisfies ***(V1),(V2),(V3),(V4)****. Let $u_0 \in \mathbf{W}^{1,\infty} \cap \mathbf{W}^{2,1}$, $r_0 \in \mathbf{W}^{1,1} \cap \mathbf{L}^{\infty}$ and let T_{ex} be the time of existence for the initial problem given by Thm (Trafic). Then, for all $t \in [0, T_{ex}[$ the local semi-group of the pedestrian traffic problem is $\mathbf{L}^1$ Gâteaux differentiable in the direction r_0 and*

$$DS_t(u_0)(r_0) = \Sigma_t^{S_t u_0} r_0 \,.$$

Idea of the proof: Thm (Flow/Source) allows to compare the solution with initial condition $u_0 + hr_0$ to the solution $u + hr$.

Let u, u_h be the solutions of the Cauchy problem $\partial_t u + \mathrm{Div}(uV(u)) = 0$ with initial conditions $u_0, u_0 + hr_0$. Let r be the solution of the Linerized equation $\partial_t r + \mathrm{Div}(rV(u) + uDV(u)(r)) = 0$, with $r(0) = r_0$. We define then $z_h = u + hr$ that satisfies

$$\partial_t z_h + \mathrm{Div}\left(z_h(V(u) + hDV(u)(r))\right) = h^2 \mathrm{Div}(rDV(u)(r)), \quad z_h(0) = u_0 + hr_0 .$$

Next, we use Thm (Flow/Source) to compare u_h and z_h. We obtain

$$\frac{1}{h}\|u_h - z_h\|_{\mathbf{L}^\infty([0,T[,\mathbf{L}^1)} \leq F(T)\left(\frac{1}{h}\|u_h - u\|^2_{\mathbf{L}^\infty(\mathbf{L}^1)} + \frac{1}{h}\|u_h - z_h\|_{\mathbf{L}^\infty(\mathbf{L}^1)}\right) + hC(\beta)Te^{C(\beta)T}\|r\|_{\mathbf{L}^\infty(\mathbf{W}^{1,1})}\|r\|_{\mathbf{L}^\infty(\mathbf{L}^1)},$$

where F is increasing and $F(0) = 0$.

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Plan

1. L^1 Stability with respect to flow and source
   - Previous Results
   - Estimate on the total variation
   - Dependence with respect to flow and source

2. The continuity equation with a non-local flow
   - Existence and uniqueness of a solution
   - Gâteaux derivative of the semi-group
   - Extrema of a Cost Functional

Let J be a cost functional such that

$$J(\rho_0) = \int_{\mathbb{R}^N} f\left(S_t\rho_0\right)\,\psi(t,x)\mathrm{d}x\,.$$

Proposition

Let $f \in \mathcal{C}^{1,1}(\mathbb{R};\mathbb{R}_+)$ and $\psi \in \mathbf{L}^\infty(I_{\mathrm{ex}} \times \mathbb{R}^N;\mathbb{R})$. Let us assume that $S\colon I \times (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^N;\mathbb{R}) \to (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^N;\mathbb{R})$ is $\mathbf{L}^1$ Gâteaux differentiable. If $\rho_0 \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^N;\mathbb{R})$ is solution of

$$\textit{Find } \min_{\rho_0} J(\rho_0) \textit{ s. t. } \left\{S_t\rho_0 \textit{ is solution of (Traffic)}\right\}.$$

then, for all $r_0 \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)(\mathbb{R}^N;\mathbb{R})$

$$\int_{\mathbb{R}^N} f'(S_t\rho_0)\,\Sigma_t^{\rho_0} r_0\,\psi(t,x)\,\mathrm{d}x = 0\,.$$

References:

- Bouchut, F. and Perthame, B., Kružkov's estimates for scalar conservation laws revisited, Trans. Amer. Math. Soc., 1998
- Chen, G.-Q. and Karlsen, K. H., Quasilinear anisotropic degenerate parabolic equations with time-space dependent diffusion coefficients, Commun. Pure Appl. Anal., 2005
- Colombo, R. M. and M. M. and Rosini, M.D., Stability and total Variation Estimates on General Scalar Balance Laws, Communications in Mathematical Sciences, 2009
- Colombo, R. M. and Herty, M. and M. M., Control of the Continuity Equation with a Non-local Flow, submitted to Esaim-Cocv in 2009
- Kružkov, S. N., First order quasilinear equations with several independent variables. , Mat. Sb. (N.S.), 1970
- Lucier, B. J. , A moving mesh numerical method for hyperbolic conservation laws., Math. Comp. , 1986