A muggle's approach to the Arzelà-Ascoli theorem

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Captatio benevolentiæ. In its most useful part, the Arzelà-Ascoli theorem provides *sufficient* conditions for the pre-compactness of a family of continuous functions. The standard proof of the sufficiency relies on the diagonal process. I present here a quite natural proof *via* coverings. *Since I want to make transparent the approach, I make rather standard assumptions, but I state them in terms of finite coverings, and I do not assume completeness.*

Totally bounded sets. Recall that a subset A of a metric space is *totally bounded* if, for every $r > 0$, A can be covered with a finite number of balls of radius r. A totally bounded set is bounded. (In $\mathbb{R}^n$, totally bounded is the same as bounded.) Recall also that, in a *complete* metric space, A is *pre-compact* (i.e., $\overline{A}$ is compact) if and only if A is totally bounded. Thus in many practical situations (e.g., for subsets of $\mathbb{R}^n$ or of a compact space) total boundedness is the same as pre-compact.

The (metric) Arzelà-Ascoli theorem (sufficiency part). Let (X, d_X) and (Y, d_Y) be metric spaces. Let $\mathscr{F}$ be a family of functions $f : X \to Y$ such that:

(i) X is totally bounded.

(ii) For each $x \in X$, $\{f(x); f \in \mathscr{F}\} \subset Y$ is totally bounded.

(iii) For each $\varepsilon > 0$, there exists some $\delta = \delta(\varepsilon) > 0$ such that

$$[x, y \in X, d_X(x, y) < \delta, f \in \mathscr{F}] \implies d_Y(f(x), f(y)) < \varepsilon. \tag{1}$$

Then:

(a) $\mathscr{F} \subset C_b(X, Y)$. (This first conclusion often occurs as an assumption.)

(b) $\mathscr{F}$ is totally bounded in $C_b(X, Y)$.

Proof. (a) Clearly (by assumption (iii)) $\mathscr{F} \subset C(X, Y)$.

Fix some $z \in Y$. Let $r := \delta(1)$. Consider a finite covering of X with balls $B(x_k, r)$, $x_k \in X$, $1 \le k \le N$. By assumption (ii) and the fact that totally bounded sets are bounded, there exists some $M_k < \infty$ such that

$$d(f(x_k), z) \le M_k, \ \forall f \in \mathscr{F}. \tag{2}$$

Let $x \in X$. Let k be such that $x \in B(x_k, r)$. By assumption (iii) and (2), for every $f \in \mathscr{F}$ we have

$$d_Y(f(x), z) \le d_Y(f(x_k), z) + d_Y(f(x), f(x_k)) \le \sup_j M_j + 1,$$

so that $f \in C_b(X,Y)$.

(b) Let $\varepsilon > 0$ and set $r := \delta(\varepsilon/5)$. Consider a finite covering of X with balls $B(x_k, r)$, $1 \le k \le N$. For each fixed k, consider a finite covering of $\{f(x_k); f \in \mathscr{F}\}$ with balls $B(y_k^j, \varepsilon/5)$, $1 \le j \le M_k$. (This is possible, by assumption (ii).) For each N-tuple

$$J = (j_1, \ldots, j_N) \in L := \{1, \ldots, M_1\} \times \cdots \times \{1, \ldots, M_N\},$$

consider the set

$$\mathscr{F}_J := \left\{ f \in \mathscr{F}; f(x_k) \in B(y_k^{j_k}, \varepsilon/5), \forall\, 1 \le k \le N \right\}.$$

By construction, we have $\mathscr{F} = \cup_{J \in L} \mathscr{F}_J$. Therefore, in order to complete the proof, it suffices to justify the following

Claim. If $g \in \mathscr{F}_J$, then $\mathscr{F}_J \subset B(g, \varepsilon) = \{f \in C_b(X,Y); \|f - g\|_\infty < \varepsilon\}$. (Therefore, either $\mathscr{F}_J$ is empty, or it can be covered, in $C_b(X,Y)$, with a single ball of size ε.)

Proof of the claim. Let $x \in X$. Let $1 \le k \le N$ be such that $d_X(x_k, x) < \delta$. Let $f \in \mathscr{F}_J$. Using the assumption (iii) and the definition of $\mathscr{F}_J$, we find that

$$\begin{aligned} d_Y(f(x), g(x)) \le & d_Y(f(x), f(x_k)) + d_Y(f(x_k), y_k^{j_k}) + d_Y(g(x_k), y_k^{j_k}) + d_Y(g(x), g(x_k)) \\ < & \varepsilon/5 + \varepsilon/5 + \varepsilon/5 + \varepsilon/5 = 4\varepsilon/5, \end{aligned}$$

so that $\|f - g\|_\infty \le 4\varepsilon/5 < \varepsilon$ and thus $f \in B(g, \varepsilon)$. □