The structure of complex unimodular maps. Applications

Variational Methods for Nonlinear Elliptic PDEs, KAIST 2014

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August 5, 2014

Some Sobolev spaces of special interest

$W^{s,p}(\Omega; \mathbb{S}^1) = \{u \in W^{s,p}(\Omega; \mathbb{R}^2);\ |u| = 1\}$
Typically, $\Omega = B(0,1) \subset \mathbb{R}^n$ or $\mathbb{S}^n$

Example #1

$$\min\{E_\varepsilon(U);\ U : G \to \mathbb{C}, \operatorname{tr} U = u : \Omega := \partial G \to \mathbb{S}^1\},$$

with

$$E_\varepsilon(U) = \frac{1}{2}\int_G [|\nabla U|^2 + 1/(2\varepsilon^2)(1 - |U|^2)^2]$$

and $G \subset \mathbb{R}^n$ smooth bounded open set
In this case, the "natural" space for u is $H^{1/2}(\Omega; \mathbb{S}^1)$

Some Sobolev spaces of special interest

Example #2

Let U_ε minimize E_ε

Assume $n \geq 3$

Then $U_\varepsilon \rightharpoonup U$ as $\varepsilon \to 0$, where $U \in W^{1,n/(n-1)}(G; \mathbb{S}^1)$ (Bethuel, Orlandi, Smets 2003)

In this case, the "natural" space for U is $W^{1,n/(n-1)}(G; \mathbb{S}^1)$

Some Sobolev spaces of special interest

Example #3

In micromagnetism, magnetization is described by a function $m : \Omega \to \mathbb{S}^2$

In different asymptotic regimes (in thin ferromagnetic films), $\mathbb{S}^1$-valued maps arise as limits and describe Néel walls, Bloch lines, boundary vortices, mesoscopic Landau states...(see the mémoire HDR Ignat 2011)

In this setting, “natural” function spaces are $BV(\Omega'; \mathbb{S}^1)$, $W^{1,1}(\Omega'; \mathbb{S}^1)$, $H^{1/2}(\Omega'; \mathbb{S}^1)$, $W^{1/p,p}(\Omega'; \mathbb{S}^1)$, $1 < p < \infty$

“Regular” Sobolev spaces

A space is “regular” if the following holds:
Each map $u \in W^{s,p}(\Omega;\mathbb{S}^1)$ can be written as $u = e^{\imath\varphi}$, with $\varphi \in W^{s,p}(\Omega;\mathbb{R})$, and φ is controlled by u

Theorem (Bourgain, Brezis, M. 2000, M., Molnar 2014)

When $n = 1$, all spaces are regular, except $W^{1/p,p}(\Omega;\mathbb{S}^1)$, $1 \leq p < \infty$
When $n = 2$, regular spaces are characterized by $sp < 1$ or $sp \geq 2$
Characterization also known when $n \geq 3$

“Regular” Sobolev spaces

In “regular” spaces, problems are “linearized”

Interesting spaces (*cf* examples) are “irregular”

Two sources of “irregularity”:
Lack of phase $\rightsquigarrow$ functional analytic applications
Lack of control of the phase $\rightsquigarrow$ PDEs applications

Lack of phase

To simplify: $n = 2$, $\Omega = \mathbb{S}^2$ (but answer known in general)
We want to describe the space $W^{s,p}(\mathbb{S}^2; \mathbb{S}^1)$, with $1 \leq sp < 2$
Again to simplify: we further assume that $0 < s < 1$ and
$1 < sp < 2$ (but answer known in general)
Typical example: $H^s(\mathbb{S}^2; \mathbb{S}^1)$, $1/2 < s < 1$

Example of $u \in W^{s,p}(\mathbb{S}^2; \mathbb{S}^1)$ $(0 < s < 1, 1 < sp < 2)$

Example #1

$v = e^{\imath\varphi}$, $\varphi \in W^{s,p}(\mathbb{S}^2; \mathbb{R})$

Example of $u \in W^{s,p}(\mathbb{S}^2; \mathbb{S}^1)$ $(0 < s < 1, 1 < sp < 2)$

Example #2

$w = e^{\imath\psi}$, $\psi \in W^{1,sp}$

Example of $u \in W^{s,p}(\mathbb{S}^2;\mathbb{S}^1)$ $(0 < s < 1, 1 < sp < 2)$

Example #2 relies on the Gagliardo-Nirenberg inequality

$$W^{\sigma,\rho} \cap L^\infty \subset W^{\theta\sigma,\rho/\theta},\ \theta \in (0,1)$$

(true except when $\sigma = \rho = 1$)
Example: a bounded function in $W^{1,2}$ belongs to $W^{1/2,4}$ and to $W^{1/3,6}$

Gagliardo-Nirenberg yields

$$\psi \in W^{1,sp} \Longrightarrow e^{\imath\psi} \in W^{1,sp} \cap L^\infty \Longrightarrow e^{\imath\psi} \in W^{s,p}(\mathbb{S}^2;\mathbb{S}^1)$$

Example of $u \in W^{s,p}(\mathbb{S}^2; \mathbb{S}^1)$ $(0 < s < 1, 1 < sp < 2)$

Example #3

$\xi(z) = \dfrac{z}{|z|}$ (with $\mathbb{S}^2 \approx \mathbb{C} \cup \{\infty\}$)

Heuristics

ξ homogeneous of degree 0 $\implies D^s\xi$ homogeneous of degree $-s \implies D^s\xi \in L^p$ (since $sp < 2$=space dimension)

Examples of $u \in W^{s,p}(\mathbb{S}^2;\mathbb{S}^1)$ $(0 < s < 1, 1 < sp < 2)$

Summary

- $v = e^{\imath\varphi}$, $\varphi \in W^{s,p}(\mathbb{S}^2;\mathbb{R})$
- $w = e^{\imath\psi}$, $\psi \in W^{1,sp}(\mathbb{S}^2;\mathbb{R})$
- $\xi(z) = \dfrac{z}{|z|}$

Examples of $u \in W^{s,p}(\mathbb{S}^2; \mathbb{S}^1)$ $(0 < s < 1, 1 < sp < 2)$

Summary

- $v = e^{\imath\varphi}$, $\varphi \in W^{s,p}(\mathbb{S}^2; \mathbb{R})$
- $w = e^{\imath\psi}$, $\psi \in W^{1,sp}(\mathbb{S}^2; \mathbb{R})$
- $\xi(z) = \dfrac{z}{|z|}$

"Theorem"

Assume

- space dimension $n = 2$
- $0 < s < 1$
- $1 < sp < 2$

"Then the three above examples are the only possible ones"

Singularities of u

If u is “sufficiently nice” (*i.e.* if u has a finite number of singularities, and u is not too singular near its singularities), then one can “hear” the singularities of u:
the distribution defined by

$$\langle T(u), \zeta \rangle := \frac{1}{2\pi} \int (u \wedge du) \wedge d\zeta$$

satisfies

$$T(u) = \sum_{a \text{ singularity of } u} \deg\,(u, a)\, \delta_a$$

(Brezis, Coron, Lieb 1986; special case Ball 1977)

Singularities of u

Theorem (All maps have a singular set; Bourgain, Brezis, M. 2004, 2005; Bousquet 2007)

- *"Sufficiently nice" maps are dense in* $W^{s,p}(\mathbb{S}^2; \mathbb{S}^1)$
- *The map* $u \mapsto T(u)$ *extends by density+continuity to* $W^{s,p}(\mathbb{S}^2; \mathbb{S}^1)$
- *The result is of the form*

$$T(u) = \sum_{j=1}^{\infty} (\delta_{P_j} - \delta_{N_j})$$

Range of the mapping $u \mapsto T(u)$?

Assume $u \in W^{1,q}$ (with $1 < q < 2$)

$$\langle T(u), \zeta \rangle := \frac{1}{2\pi} \int (u \wedge du) \wedge d\zeta$$

$\Longrightarrow T(u)$ acts on $W^{1,q'}$

Assume $u \in W^{q,1}$
Then $T(u)$ acts on $W^{1,q'} \cap C^{2-q}$

Range of the mapping $u \mapsto T(u)$?

Theorem (Bousquet 2007)

The conditions on the previous slides are nsc:
A sum of the form

$$T = \sum_{j=1}^{\infty} (\delta_{P_j} - \delta_{N_j})$$

can be realized as

- $T = T(u)$ *for some* $u \in W^{1,q}$ *iff* T *acts on* $W^{1,q'}$
- $T = T(u)$ *for some* $u \in W^{q,1}$ *iff* T *acts on* $W^{1,q'} \cap C^{2-q}$

Range of the mapping $u \mapsto T(u)$?

Theorem (M.)

For a sum of the form $T = \sum_{j=1}^{\infty}(\delta_{P_j} - \delta_{N_j})$*, acting on* $W^{1,q'}$ *is the same as acting on* $W^{1,q'} \cap C^{2-q}$
More generally,

$$\{T(u)\ ;\ u \in W^{s,p}(\mathbb{S}^2;\mathbb{S}^1)\}$$

depends only on the value of the product sp, not on s or p

The proof of the above theorem implies a very strange property of sums of Dirac masses

Theorem

Let $m \in \mathbb{N}^*$, $P_1, \ldots, P_m, N_1, \ldots, N_m \in \mathbb{S}^2$ *and*

$$T := \sum_{j=1}^{m} (\delta_{P_j} - \delta_{N_j})$$

Then (with C *independent of* m *and of the points)*

$$\|T\|_{(\dot{C}^{1/2})^*} \leq C \|T\|^{3/2}_{(\dot{W}^{1,3})^*}$$

Range of the mapping $u \mapsto T(u)$?

The proof of "acts on… $\iff$ acts on…" relies on

Theorem (M.)

Assume that $n \geq 1$ *and* $sp \geq 1$
Then every $u \in W^{s,p}(\mathbb{S}^2; \mathbb{S}^1)$ *splits as*

$$u = ve^{\imath\varphi}, \ v \in W^{sp,1}, \ \varphi \in W^{s,p}$$

Special cases due to Bourgain-Brezis 2003, Nguyen 2008

Analysis of $W^{s,p}(\mathbb{S}^2;\mathbb{S}^1)$

Theorem (Splitting according to Examples $\#1$ to $\#3$)

Assume

- $n = 2$
- $1 < sp < 2$
- $0 < s < 1$

Then we may explicitly decompose every $u \in W^{s,p}(\mathbb{S}^2;\mathbb{S}^1)$ as follows

- *Compute $T(u)$*
- *Associate to $T(u)$ a "canonical mapping" ξ s.t. $T(\xi) = T(u)$*
- *Find explicitly $\varphi \in W^{s,p}$, $\psi \in W^{1,sp}$ s.t.*

$$u = v\,w\,\xi, \quad \text{with } v = e^{\imath\varphi},\ w = e^{\imath\psi}$$

This also works the other way around...

Overview of the general case (any n, s, p)

Structure of $W^{s,p}(\mathbb{S}^2;\mathbb{S}^1)$ for the other values of s and p $\rightsquigarrow$ OK

Structure of $W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$ $\rightsquigarrow$ OK
Points become $(n-2)$-manifolds

An application: trace theory

Problem

Describe

$$X := \text{trace } W^{s+1/p,p}(\mathbb{S}^n \times (0,1); \mathbb{S}^1)$$

In general, this trace is not the full space $W^{s,p}(\mathbb{S}^n; \mathbb{S}^1)$
Reasons: obvious topological obstructions, and less obvious analytical obstructions (Bethuel, Demengel 1995)

The structure theory of unimodular maps allows the characterization of the space X

Lack of control

Occurs only in the spaces $W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$, $1 \leq p < \infty$
The presence of $\mathbb{S}^1$ as a source space creates topological obstructions, but lack of control appears even in absence of topological obstructions

Lack of control: “the” example

Let $\mathbb{D}$ be the unit disc in $\mathbb{R}^2$ and set

$$M_a(z) = \frac{z-a}{1-\overline{a}z},\ z \in \overline{\mathbb{D}},\ a \in \mathbb{D}$$

(the Moebius transform)

Let $u_a(z) = \overline{z}\, M_a(z)$, $z \in \mathbb{S}^1$, $a \in \mathbb{D}$
Then u_a is bounded in $W^{1/p,p}$, but “its” phase blows up in $W^{1/p,p}$ as $|a| \to 1$

No other blow up mechanism known.
Probably no other possible mechanism.

Recovering phase control

We work in $H^{1/2}(\mathbb{S}^1;\mathbb{S}^1)$, with the energy

$$|u|^2_{H^{1/2}} = \frac{1}{2}\int_{\mathbb{D}} |\nabla U|^2,$$

with U the harmonic extension of u

No loss control below the energy of a Moebius map:
If $u : \mathbb{S}^1 \to \mathbb{S}^1$ and $|u|^2_{H^{1/2}} \leq \pi - \delta$, then $u = e^{\imath\varphi}$ and

$$|\varphi|_{H^{1/2}} \leq F(\delta).$$

Reason

The harmonic extension U of u stays far away from 0

Recovering phase control

Theorem (Berlyand, M., Rybalko, Sandier 2014)

Assume that $|u|^2_{H^{1/2}} \leq 2\pi - \delta$ (i.e., no room for 2 Moebius transforms M_a) and $\deg u = 1$

Then we may find $a = a(u)$ s.t. $u = M_a e^{\imath\varphi}$, with $|\varphi|_{H^{1/2}} \leq F(\delta)$

Idea of proof

There is no place for two “remoted” zeros of the harmonic extension U of u

From this, control the region where $|U| \not\approx 1$, then the phase of U, then (by taking traces) the one of u

From control to weak compactness

Equivalently

The set

$$X = \{u \in H^{1/2}(\mathbb{S}^1; \mathbb{S}^1); \deg u = 1\}$$

is not weakly closed

However:

$$\{u \in X; |u|^2_{H^{1/2}} \leq 2\pi - \delta\}/\text{Moebius group}$$

is weakly closed

Higher degrees analog

Theorem (Berlyand, M., Rybalko, Sandier 2014)

Let $d = 2, 3, \ldots$
Then there is some $\varepsilon > 0$ such that

$$u : \mathbb{S}^1 \to \mathbb{S}^1, |u|^2_{H^{1/2}} \leq \pi d + \varepsilon, \ \deg u = d \Longrightarrow$$

$u = M_{a_1} \ldots M_{a_d} e^{\imath\varphi}$, with $|\varphi|_{H^{1/2}} \leq C$.

Remark

Probably $\forall\, \varepsilon < 2\pi$ works...

Idea of proof

Induction, relying on the case $d = 0$ + Wente estimates in order to obtain almost orthogonal decomposition of the energy + classification of maps at energy level $\pi\, d$

An application: Ginzburg-Landau with prescribed degrees

General problem

Given:
$\Omega \subset \mathbb{R}^2$, with boundary $\Gamma_1 \cup \ldots \cup \Gamma_k$
Integers $d_1, \ldots, d_k \in \mathbb{Z}$
Minimize (or find critical points of) E_ε subject to

$$|u| = 1 \text{ and } \deg u = d_j \text{ on } \Gamma_j$$

An application: GL with prescribed degrees

Big picture (partly conjectural)

In any domain, existence of critical points (Berlyand-Rybalko 2010, Dos Santos 2009, Berlyand-M.-Rybalko-Sandier 2014, Lamy-M. 2014)
In thin domains, existence of minimizers (Golovaty-Berlyand 2002, Berlyand-M. 2006)
In thick domains, non existence of minimizers (Berlyand-Golovaty-Rybalko 2008, M. 2010)

Simply connected domain requires different techniques
Similar to no holes (Brezis-Nirenberg 1983)-holes (Bahri-Coron 1988) situation for critical exponent

An application: GL with prescribed degrees

Big picture in simply connected domains (partly conjectural)

No minimizer
But critical points for large ε (via variational methods)
And for small ε (via IFT methods)

An application: GL with prescribed degrees

Theorem (Berlyand, M., Rybalko, Sandier 2014)

Assume Ω simply connected, prescribed degree = 1
Then E_ε has critical points for large ε

Probably also for $d \geq 2$, but no proof

Sketch of proof

Min-max method: consider

$$\min_F \max\{E_\varepsilon(F(u));\ F \in C(\mathbb{D}; H^1), F(a) = M_a \text{ for } |a| \approx 1\}.$$

Establish mountain pass geometry (via the description of the low energy maps of degree 1)
Prove that the energy functional is C^1
Next establish behavior of PS sequences. Requires killing the Moebius group (rescaling) and the use of the Hopf differential
Prove convergence of the energy to the energy of the rescaled map
Last step leads to compactness of PS sequences

All steps but the last can be performed for arbitrary domains and degrees (via additional Wente type estimates).
This leads to bubbling analysis like in Brezis-Coron or Struwe, but not to compactness

Application: How much it takes to wind once

Examine the existence of

$$m_p = \min\{|u|^p_{W^{1/p,p}};\ u : \mathbb{S}^1 \to \mathbb{S}^1, \deg u = 1\}$$

$$|u|^p_{W^{1/p,p}} = \iint_{\mathbb{S}^1 \times \mathbb{S}^1} \frac{|u(x) - u(y)|^p}{|x - y|^2}\, dxdy,\ 1 < p < \infty$$

Application: How much it takes to wind once

Main difficulty: non compact (energy invariant by the Moebius group)

Easy cases

$p = 1$: $m_1 = 2\pi$, minimizers are $W^{1,1}$-maps with non decreasing phases
$p = 2$: $m_2 = 4\pi^2$, minimizers are Moebius transforms

Application: How much it takes to wind once

Theorem (M.)

There exists some $\varepsilon > 0$ such that m_p is attained when $p \in (2-\varepsilon, 2)$

Sketch of proof

For such p,

$$\{u \in W^{1/p,p};\, |u|^p_{W^{1/p,p}} \approx m_p\}/\text{Moebius group}$$

is weakly closed

Do not know what happens when $d \geq 2$

GL with prescribed degree for small ε

Theorem (Lamy, M. 2014)

In a "generic" simply connected domain, E_ε has critical points with prescribed degree 1

GL with prescribed degree for small ε

Sketch of the proof

Naive analysis of the behavior of critical points as $\varepsilon \to 0$
Identification of the limit u_0 and of its trace g on $\partial\Omega$
Construction of the candidates (à la Pacard-Rivière 2000; see also del Pino, Kowalczyk, Musso 2006) with $u_\varepsilon \approx u_0$ and $g \approx g_0$
Existence of an exact solution (Schauder fixed point theorem)
The two last items require nondegeneracy assumptions
We finally prove that the nondegeneracy assumptions are stable and generically satisfied