Final exam topics
SOLVING THE LAPLACE EQUATION WITH DIVERGENCE FORM DATUM
A UNIQUENESS RESULT

SOLVING THE LAPLACE EQUATION WITH DIVERGENCE FORM DATUM. The purpose is to establish the following

Theorem. Let $\Omega \in C^2$ be a bounded domain, and $1 < p < \infty$. For $F \in L^p(\Omega; \mathbb{R}^N)$, the equation

$$-\Delta u = \operatorname{div} F \text{ in } \mathscr{D}'(\Omega) \tag{1}$$

has a unique solution $u \in W_0^{1,p}(\Omega)$. In addition, with some finite constant C independent of F, we have the estimate $\|\nabla u\|_p \le C\|F\|_p$.

In the proof, one can freely use the L^p-regularity theorem.

Here are some suggested steps.

1. If p, q are conjugated exponents, prove that

$$[u \in W_0^{1,p}(\Omega), -\Delta u = 0, v \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)] \implies \int_\Omega u(-\Delta v) = 0,$$

and derive the uniqueness, in $W_0^{1,p}(\Omega)$, of a solution of (1).

2. Start with $F \in C_c^\infty(\Omega; \mathbb{R}^N)$, and for such F derive, using a smallness assumption condition on the covering of $\overline{\Omega}$, the *a priori* estimate $\|u\|_{W^{1,p}} \le C\|F\|_p + C\|u\|_p$.

3. With F as above, derive the *a priori* estimate $\|u\|_{W^{1,p}} \le C\|F\|_p$, and conclude.

A UNIQUENESS RESULT. Let B denote the unit ball in $\mathbb{R}^N$ and set, $\forall\, r > 0$, $B_r := \{x \in \mathbb{R}^N;\ |x| < r\}$, $S_r := \{x \in \mathbb{R}^N;\ |x| = r\}$. We want to prove that

$$[u \in W_0^{1,1}(B), -\Delta u = 0] \implies u = 0.$$

1. Let $g \in C_c^\infty(B)$ and let $v \in H_0^1(B)$ solve $-\Delta v = g$. For $0 < r < 1$, prove that

$$\begin{aligned}\left|\int_{B_r} ug\right| &\le \|\nabla v\|_{L^\infty(B)} \int_{S_r} |u| + \|v\|_{L^\infty(S_r)} \int_{S_r} |\nabla u| \\ &\le \|\nabla v\|_{L^\infty(B)} \int_{B\setminus B_r} |\nabla u| + (1-r)\|\nabla v\|_{L^\infty(B)} \int_{S_r} |\nabla u|.\end{aligned}$$

2. Conclude, using an appropriate sequence $r_j \to 1$.