Lecture # 1
THE DIRECT METHOD: A FEW EXAMPLES

In what follows, $\Omega \subset \mathbb{R}^N$ is a bounded open set. Additional smoothness, if needed, is explicitly assumed.

(a) Functional analytical preliminaries

A [4, Corollary 3.9] Let E be a Banach space. Let $\varphi : E \to (-\infty, \infty]$ be convex and lower semicontinuous. Then φ is weakly lower semicontinuous.

B [4, Corollary 3.23] Let E be a reflexive Banach space. Let $A \subset E$ be a closed convex set. Let $\varphi : A \to (-\infty, \infty]$ be convex and lower semicontinuous. Assume that:

(a) Either A is bounded.

(b) Or $\lim_{x\in A, \|x\|\to\infty} \varphi(x) = \infty$.

Then φ achieves its minimum on A.

C [4, Theorem 3.18] Let E be a reflexive Banach space. Let $(x_n) \subset E$ be a bounded sequence. Then (x_n) contains a weakly convergent subsequence.

D **Fundamental exercise.** Let $1 < p < \infty$. Let $\varphi : W^{1,p}(\Omega) \to \mathbb{R}$ be convex, lower semicontinuous, and *coercive*, i.e., $\lim_{\|u\|\to\infty} \varphi(u) = \infty$. Then φ achieves its (global) minimum.

E **Fundamental exercise.** Let $1 < p < \infty$. Prove that a bounded sequence $(u_n) \subset W_0^{1,p}(\Omega)$ contains a subsequence (u_{n_j}) such that:

(a) $u_{n_j} \to u$ a.e., for some $u \in W_0^{1,p}(\Omega)$.

(b) $\nabla u_{n_j} \rightharpoonup \nabla u$ in $L^p(\Omega)$.

Same for $W^{1,p}(\Omega)$ if Ω is assumed Lipschitz.

Can one replace, in item (b), weak convergence with strong convergence?

(b) Basic examples

In items A, B, C, $a \in C(\overline{\Omega})$, $a \geq 0$, and $f \in C(\overline{\Omega})$.

A The problem

$$\begin{cases} -\Delta u + a(x)u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique weak solution $u \in H_0^1(\Omega)$.

B Same for the problem

$$\begin{cases} -\Delta u + a(x)|u|^{q-1}\,\operatorname{sgn} u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}, \text{ with } 1 < q < \infty.$$

In this case, give a meaning to the notion of solution, and specify a space in which this solution is unique.

Useful results:

Exercise. Let $1 < q < \infty$. Then

$$L^q(X, \mathscr{T}, \mu) \ni u \mapsto G(u) := |u|^{q-1}\,\operatorname{sgn} u \in L^{q/(q-1)}(X, \mathscr{T}, \mu)$$

is continuous.

Lemma. Let $1 < q < \infty$. Then

$$L^q(X, \mathscr{T}, \mu) \ni u \mapsto F(u) := \int_X |u|^q\, d\mu \text{ is } C^1, \text{ and}$$

$$F'(u)(\varphi) = q \int_\Omega |u|^{q-1}\,(\operatorname{sgn} u)\,\varphi,\ \forall\, u, \varphi \in L^q(X, \mathscr{T}, \mu).$$

C The problem

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + a(x)|u|^{q-1}\,\operatorname{sgn} u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases},$$

with $1 < p, q < \infty$, has a unique distributional solution in the space $u \in W_0^{1,p}(\Omega) \cap L^q(\Omega)$.

Useful result:

Exercise. Let $1 < p < \infty$. Then

$$L^p(X, \mathscr{T}, \mu; \mathbb{R}^d) \ni f \mapsto F(f) := \int_X |f|^p\, d\mu \text{ is } C^1, \text{ and}$$

$$F'(f)(g) = p \int_\Omega |f|^{p-2}\, f \cdot g,\ \forall\, f, g \in L^p(X, \mathscr{T}, \mu; \mathbb{R}^d).$$

D **Definition.** A *Carathéodory function* is a function $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^d$ such that

(i) $x \mapsto f(x, u, \xi)$ is (Lebesgue) measurable, $\forall\, (u, \xi) \in \mathbb{R}^m \times \mathbb{R}^d$.

(ii) $(u, \xi) \mapsto f(x, u, \xi)$ is continuous, for a.e. $x \in \Omega$.

Theorem. (Tonelli, Mac Shane, Morrey, ...) Let $1 \le p, q \le \infty$. Let f be a Carathéodory function such that:

a) $f(x,u,\xi) \geq a(x)\cdot u + b(x)\cdot\xi$, $\forall\, u,\xi$, for a.e. x, for some $a \in L^{q'}(\Omega;\mathbb{R}^m)$, $b \in L^{p'}(\Omega;\mathbb{R}^d)$.

b) $\xi \mapsto f(x,u,\xi)$ is convex for a.e. $x \in \Omega$ and every $u \in \Omega$.

Set

$$L^q(\Omega;\mathbb{R}^m) \times L^p(\Omega;\mathbb{R}^d) \ni (u,\xi) \mapsto L(u,\xi) := \int_\Omega f(x,u(x),\xi(x))\,dx \in \mathbb{R}\cup\{\infty\}.$$

Then

$$[u_j \to u \text{ in } L^q(\Omega;\mathbb{R}^m),\ \xi_j \rightharpoonup \xi \text{ in } L^p(\Omega;\mathbb{R}^d)] \implies \liminf L(u_j,\xi_j) \geq L(u,\xi).$$

(When $p=\infty$, we may replace $\rightharpoonup$ by $\overset{*}{\rightharpoonup}$.)

Useful results:

Exercise. If f is a Carathéodory function and $(u,\xi) : \Omega \to \mathbb{R}^m \times \mathbb{R}^d$ is measurable, prove that $\Omega \ni x \mapsto f(x,u(x),\xi(x))$ is measurable. (Hint: start with the case where u and ξ are step functions.)

Exercise. If f is a non-negative Carathéodory function, $u : \Omega \to \mathbb{R}^d$ is measurable, $1 \leq p < \infty$, and $\xi_j \rightharpoonup \xi$ in $L^p(\Omega)$, then

$$\int_\Omega f(x,u(x),\xi(x))\,dx \leq \liminf \int_\Omega f(x,u(x),\xi_j(x))\,dx.$$

Exercise.

1. Let f be a Carathéodory function. Prove that, for each $\varepsilon, M > 0$, there exist: some $\delta = \delta(\varepsilon,M) > 0$ and some compact set $K = K(\varepsilon,M) \subset \Omega$ such that:
   i. $|\Omega \setminus K| < \varepsilon$.
   ii. $[x \in K, u,v \in \mathbb{R}^m, \xi,\eta \in \mathbb{R}^d, |u| \leq M, |\xi| \leq M, |u-v| \leq \delta, |\xi-\eta| \leq \delta] \Rightarrow |f(x,u,\xi) - f(x,v,\eta)| \leq \varepsilon$.

   (Hint: prove first the statement for some Lebesgue measurable (instead of compact) set.)
2. Prove the *Scorza-Dragoni theorem*: f is a Carathéodory function if and only if for each $\varepsilon > 0$ there exists some compact set $L_\varepsilon \subset \Omega$ such that:
   i. $|\Omega \setminus L_\varepsilon| < \varepsilon$.
   ii. f is continuous on $L_\varepsilon \times \mathbb{R}^m \times \mathbb{R}^d$.

   (Hint: Consider u, ξ with rational coordinates and use Vitali's theorem to find a large set $L \subset \Omega$ such that $L \ni x \mapsto f(x,u,\xi)$ is continuous.)

Useful references: [6, Theorem 3.4, Section 3.3.1], [4, Corollary 3.9], [3, Theorem 2.2.10].

(c) Notions of convexity

A **Definition.** A continuous function $f:\mathbb{R}^{Nm}\to\mathbb{R}$ is *quasi-convex* if

$$|U|\,f(\xi)\le\int_U f(\xi+D\varphi(x))\,dx,\;\forall\,U\subset\mathbb{R}^N \text{ bounded open set,}\qquad \forall\,\xi\in\mathbb{R}^{Nm},\;\forall\,\varphi\in C_c^\infty(U;\mathbb{R}^m). \tag{1}$$

Exercise. Prove that a convex function is quasi-convex.

Exercise. Prove that the f is quasi-convex iff (1) is satisfied for *one* non empty U.

Exercise. Assume that U is bounded and convex.

1. Prove that $W^{1,\infty}(U)=\mathrm{Lip}\,(U)$.
2. Prove that (1) still holds when $\varphi\in W_c^{1,\infty}(U,\mathbb{R}^m)$.
3. Prove that, with $(\varphi_j)\subset W^{1,\infty}(U;\mathbb{R}^m)$, we have $\varphi_j\overset{*}{\rightharpoonup}0$ iff (φ_j) has uniformly bounded Lipschitz constants and $\varphi_j\to 0$ uniformly on U.

Lemma. (Morrey) If f is quasi-convex and $Q\subset\mathbb{R}^N$ is a cube, then

$$[(\varphi_j)\subset W^{1,\infty}(Q;\mathbb{R}^m),\;\varphi_j\overset{*}{\rightharpoonup}0]\implies \liminf\int_Q f(\xi+D\varphi_j(x))\,dx\ge|Q|\,f(\xi),\qquad\forall\,\xi\in\mathbb{R}^{Nm}.$$

Useful reference: [11, Lemma 2.2].

Exercise. Prove a version of Morrey's lemma with Q replaced with a finite volume open set.

B **Theorem.** (Morrey, ..., Acerbi-Fusco) Let f be a Carathéodory function on $\Omega\times\mathbb{R}^m\times\mathbb{R}^{Nm}$ such that:

a) for a.e. $x\in\Omega$ and each $u\in\mathbb{R}^m$, $\mathbb{R}^{Nm}\ni\xi\mapsto f(x,u,\xi)$ is quasi-convex.
b) $0\le f(x,u,\xi)\le a(x)+b(u,\xi)$, with $a\in L^1(\Omega)$, $b\in L^\infty_{loc}(\mathbb{R}^m\times\mathbb{R}^{Nm})$.

If $(u_j)\subset W^{1,\infty}(\Omega;\mathbb{R}^m)$ and $u_j\overset{*}{\rightharpoonup}u$, then

$$\liminf\int_\Omega f(x,u_j(x),Du_j(x))\,dx\ge\int_\Omega f(x,u(x),Du(x))\,dx.$$

Useful result:

Exercise. (Easy version of Lebesgue's differentiation theorem) Let $Q:=(0,1)^N$ and let $g\in L^1(Q)$. Let $\ell\ge 1$ be an integer and

$$g_\ell(x):=\fint_C g(y)\,dy \text{ if } x \text{ belongs to the dyadic cube } C \text{ of size } 2^{-\ell}.$$

Then, up to a subsequence $\ell_n \to \infty$, $g_\ell \to g$ a.e.

Useful references: [1, Theorem II.1], [13, Corollary, p. 13].

Theorem. (Acerbi-Fusco, [1, Theorem II.4]) Let $1 \le p < \infty$. Let f be a Carathéodory function on $\Omega \times \mathbb{R}^m \times \mathbb{R}^{Nm}$ such that:

a) for a.e. $x \in \Omega$ and each $u \in \mathbb{R}^m$, $\mathbb{R}^{Nm} \ni \xi \mapsto f(x, u, \xi)$ is quasi-convex.

b) $0 \le f(x, u, \xi) \le a(x) + C(|u|^p + |\xi|^p)$, with $a \in L^1(\Omega)$ and C finite.

If $(u_j) \subset W^{1,p}(\Omega; \mathbb{R}^m)$ and $u_j \rightharpoonup u$, then

$$\liminf \int_\Omega f(x, u_j, Du_j(x))\, dx \ge \int_\Omega f(x, u, Du(x))\, dx.$$

C **Theorem.** (Morrey) Let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{Nm}$ be continuous. If, for every open set $U \subset \Omega$,

$$[u_j \overset{*}{\rightharpoonup} u \text{ in } W^{1,\infty}(U)] \implies$$
$$\liminf \int_U f(x, u_j(x), Du_j(x))\, dx \ge \int_U f(x, u(x), Du(x))\, dx,$$

then, for each $x \in \Omega$ and $u \in \mathbb{R}^m$, $\mathbb{R}^{Nm} \ni \xi \mapsto f(x, u, \xi)$ is quasi-convex.

Useful result:

Lemma. Let $Q := (0, 1)^N$ and let $\zeta \in C_c^\infty(Q; \mathbb{R}^m)$, extended as a smooth 1-periodic function to $\mathbb{R}^m$. Let $U \subset \Omega$ be relatively compact. Let $u_0 \in C(\Omega; \mathbb{R}^m)$, $\xi_0 \in C(\Omega; \mathbb{R}^{Nm})$. Set $\zeta_j(x) := 2^{-j}\zeta(2^j x)$, $\forall\, j \ge 1$, $\forall\, x \in \mathbb{R}^N$. Then

$$\lim \int_U f(x, u_0(x), \xi_0(x) + D\zeta_j(x))\, dx = \int_U \int_Q f(x, u_0(x), \xi_0(x) + D\zeta(y))\, dy dx$$

and

$$\lim \int_U f(x, u_0(x) + \zeta_j(x), \xi_0(x) + D\zeta_j(x))\, dx$$
$$= \int_U \int_Q f(x, u_0(x), \xi_0(x) + D\zeta(y))\, dy dx.$$

Theorem. (Acerbi-Fusco, [1, Theorem II.2]) Let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{Nm}$ be a Carathéodory function such that $0 \le f(x, u, \xi) \le a(x) + b(u, \xi)$, $\forall\, x \in \Omega$, $\forall\, (u, \xi) \in \mathbb{R}^m \times \mathbb{R}^{Nm}$, where $a \in L^1(\Omega)$ and $b \in L^\infty_{loc}(\mathbb{R}^m \times \mathbb{R}^{Nm})$. If, for every open set $U \subset \Omega$,

$$[u_j \overset{*}{\rightharpoonup} u \text{ in } W^{1,\infty}(U)] \implies$$
$$\liminf \int_U f(x, u_j(x), Du_j(x))\, dx \ge \int_U f(x, u(x), Du(x))\, dx,$$

then, for a.e. $x \in \Omega$ and each $u \in \mathbb{R}^m$, $\mathbb{R}^{Nm} \ni \xi \mapsto f(x, u, \xi)$ is quasi-convex.

D **Proposition.** Assume that $N = 1$ and let $f : \mathbb{R}^m \to \mathbb{R}$ be continuous. Then f is quasi-convex if and only if f is convex.

Theorem. [6, Theorem 3.1, Section 3.3.1] Assume that $m = 1$ (i.e., we work with scalar functions u) and let $f : \mathbb{R}^N \to \mathbb{R}$. Then f is quasi-convex if and only if f is convex.

E We identify $\mathbb{R}^{Nm}$ with $M_{m,N}(\mathbb{R})$. Let $A \in M_{m,N}(\mathbb{R})$. Given $1 \le \ell \le K := \min(m, N)$, and $I = \{i_1 < i_2 < \ldots < i_\ell\} \subset \{1, \ldots, m\}$, $J = \{j_1 < j_2 < \ldots < j_\ell\} \subset \{1, \ldots, N\}$, let $A_{I,J}$ denote the minor of order ℓ of A formed with the rows $i_1, \ldots, i_\ell$, respectively the columns $j_1, \ldots, j_\ell$. Let M be the number of all possible minors. We order the minors as $A^1, \ldots, A^M$.

Definition. (Morrey, Ball) A function $f : \mathbb{R}^{Nm} \to \mathbb{R}$ is *polyconvex* if there exists some convex function $g : \mathbb{R}^M \to \mathbb{R}$ such that $f(A) = g(A^1, \ldots, A^M)$.

Proposition. (Morrey, Ball) A polyconvex function is quasi-convex.

A useful result:

Lemma.

1. If $\Omega \subset \mathbb{R}^k$ is open bounded and $u, v \in C^\infty(\overline{\Omega}; \mathbb{R}^k)$ are such that $u = v$ near $\partial\Omega$, then
$$\int_\Omega \det(\nabla u)(x)\, dx = \int_\Omega \det(\nabla v)(x)\, dx.$$
2. Let $U \subset \mathbb{R}^N$ be open bounded. If I, J are as above, $A \in M_{m,N}(\mathbb{R})$ and $\zeta \in C_c^\infty(U; \mathbb{R}^m)$, then
$$\int_U (A + D\varphi(x))_{I,J}\, dx = |U|\, A_{I,J}.$$

Useful references: [2, Section 4], [6, Section 4.1].

(d) Passing to the weak limits in nonlinear quantities

A **Theorem** (Reshetnyak) If $u^j, u \in W^{1,N}(\Omega, \mathbb{R}^N)$ and $u^j \rightharpoonup u$ in $W^{1,N}$, then
$$\det(\nabla u^j) \to \det(\nabla u) \text{ in } \mathscr{D}'(\Omega).$$

Useful reference: [12].

B **Definition.** (Ball) Let $u = (u_1, \ldots, u_N) \in W^{1,N^2/(N+1)}(\Omega, \mathbb{R}^N)$. Then
$$\operatorname{Det}(\nabla u) := *d(u_1 du_2 \wedge \cdots \wedge du_N) \in \mathscr{D}'(\Omega).$$

Exercise. Using the Sobolev embeddings, prove that the above definition makes sense.

Exercise. If $u \in W^{1,N}(\Omega, \mathbb{R}^N)$, prove that $\operatorname{Det}(\nabla u) = \det(\nabla u)$.

Equivalently, prove that, if $u \in W^{1,N}(\Omega, \mathbb{R}^N)$, then
$$\int_\Omega \det(\nabla u)\, \varphi = -\int_\Omega \det(\varphi, u_2, \ldots, u_N)\, u_1, \forall\, \varphi \in C_c^\infty(\Omega, \mathbb{R}).$$

C **Theorem.** (Reshetnyak, Ball, Brezis-Nguyen) Let $N^2/(N+1) < p \le N$. Let $u^j, u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be such that $u^j \rightharpoonup u$ in $W^{1,p}$. Then

$$\mathrm{Det}\,(\nabla u^j) \to \mathrm{Det}\,(\nabla u) \text{ in } \mathscr{D}'(\Omega).$$

Useful result:

Lemma. Let $p \ge N-1$ and let $q \ge 1$ satisfy $(N-1)/p + 1/q = 1$. If $u, v \in C^\infty(\overline{\Omega}, \mathbb{R}^N)$, then

$$\left| \int_\Omega [\det\,(\nabla v) - \det\,(\nabla u)]\, \varphi \right| \le C_{N,\Omega} \|v-u\|_q \, (\|\nabla u\|_p + \|\nabla v\|_p)^{N-1} \|\nabla \varphi\|_\infty,$$
$$\forall\, \varphi \in C_c^\infty(\Omega, \mathbb{R}).$$

Useful reference: [5, Theorem 1].

Exercise. When $N = 2$, establish the above theorem by proving the following stronger statement: if $p > 4/3$ and $u^j = (u_1^j, u_2^j) \rightharpoonup u = (u_1, u_2)$ in $W^{1,p}(\Omega, \mathbb{R}^2)$, then $u_1^j\, \nabla u_2^j \to u_1\, \nabla u_2$ in $\mathscr{D}'(\Omega)$.

D **Theorem.** (Edelsen, Ericksen, Ball) Let $f : \mathbb{R}^{Nm} \to \mathbb{R}$ be a continuous function such that, for some $1 \le p < \infty$,

$$[u^j \rightharpoonup u \text{ in } W^{1,p}(\Omega, \mathbb{R}^m)] \implies [f(Du^j) \to f(Du) \text{ in } \mathscr{D}'(\Omega)].$$

Then f is an affine function of the minors of Du. Similarly when $p = \infty$, for the $\overset{*}{\rightharpoonup}$ convergence.

Useful reference: [6, Theorem 1.5 in Section 4.1.2, and Section 4.2.2].

E **Gap (or Lavrentiev) phenomen**

Theorem. (Maniá) Let

$$F(x) := \int_0^1 (x^3(t) - t)^2\, x'^6(t)\, dt, \ \forall\, x \in W^{1,1}((0,1)) \text{ with } x(0) = 0 \text{ and } x(1) = 1.$$

Then we have the following *Lavrentiev phenomenon*

$$\inf\{F(x);\ x \in C^1([0,1])\} > \inf\{F(x);\ x \in W^{1,1}((0,1))\}.$$

Useful reference: [8].

(e) Concentration-compactness

Useful general reference: [14, Section I.4].

A **Exercise.** Let $F_m : [0,\infty) \to [0,1]$, $m \ge 0$, be *non decreasing* functions. Prove that, up to a subsequence, F_m converges simply.

First concentration-compactness lemma (Lions) Let (μ_m) be a sequence of Borel probability measures on $\mathbb{R}^N$. Then, up to a subsequence, one of the following holds:

(a) (Compactness) There exists a sequence $(x_m) \subset \mathbb{R}^N$ such that, for every $\varepsilon > 0$, there exists some $R = R(\varepsilon)$ satisfying $\mu_m(B_R(x_m)) > 1 - \varepsilon, \forall\, m$.

(b) (Vanishing) For every $R > 0$, $\sup_{x \in \mathbb{R}^N} \mu_m(B_R(x)) \to 0$ as $m \to \infty$.

(c) (Dichotomy) There exists some $0 < \lambda < 1$ and sequences $(x_m) \subset \mathbb{R}^N$, $R_m \to \infty$ such that

$$\mu_m(B_{R_m}(x_m)) \to \lambda,\ \mu_m(\mathbb{R}^N \setminus \overline{B}_{2R_m}(x_m)) \to 1 - \lambda,$$
$$\mu_m(\overline{B}_{2R_m}(x_m) \setminus B_{R_m}(x_m)) \to 0.$$

Moreover, in the above we may replace $2R_m$ with any $\rho_m > R_m$.

B **Brezis-Lieb lemma** Let $(X, \mathscr{T}, \mu)$ be a measured space and $0 < p < \infty$. Let $f_j, f : X \to \mathbb{C}$ be measurable functions such that:

(i) $f_j \to f$ a.e.

(ii) For some finite C, $\displaystyle\int_X |f_j|^p \le C, \forall\, j$.

Then

$$\int_X ||f_j|^p - |f|^p - |f_j - f|^p| \to 0,$$

In particular, if $p \ge 1$, $X = \mathbb{R}^N$ with the Lebesgue measure, and we set

$$\mu_j := (|f_j|^p - |f|^p - |f_j - f|^p)\, dx,$$

then $\mu_j \overset{*}{\rightharpoonup} 0$ in the sense of measures.

Useful references: [7, Theorem 1.9], [10, Exercice de synthèse $\#10$].

C **Theorem** (Lions) Let $a = a(x) \in C(\mathbb{R}^N, (0, \infty))$ be such that

$$\lim_{|x| \to \infty} a(x) = a_\infty > 0.$$

Let $1 < p < \dfrac{N+2}{N-2}$ and set

$$I := \inf \left\{ \int_{\mathbb{R}^N} (|\nabla u|^2 + a u^2);\ u \in H^1(\mathbb{R}^N),\ \int_{\mathbb{R}^N} |u|^{p+1} = 1 \right\},$$
$$I_\infty := \inf \left\{ \int_{\mathbb{R}^N} (|\nabla u|^2 + a_\infty u^2);\ u \in H^1(\mathbb{R}^N),\ \int_{\mathbb{R}^N} |u|^{p+1} = 1 \right\}.$$

If $I < I_\infty$, then the $\inf$ in I is attained. Up to a multiplicative constant, a minimizer is a non trivial solution $u \in H^1(\mathbb{R}^N)$ of

$$-\Delta u + a u = |u|^{p-1} u \text{ in } \mathbb{R}^N.$$

D **Exercise.** Let μ be a finite *diffuse* Borel measure in $\mathbb{R}^N$. Prove that

$$\lim_{r\to 0}\sup_{x\in\mathbb{R}^N}\mu(B_r(x)) = 0.$$

Exercise. Let ω, λ be finite Borel measures in $\mathbb{R}^N$ and $1 \le p < q < \infty$. Assume that, for some $0 < S < \infty$, we have

$$S\left(\int_{\mathbb{R}^N}|f|^q\,d\omega\right)^{p/q} \le \int_{\mathbb{R}^N}|f|^p\,d\lambda,\ \forall\text{ Borel function } f:\mathbb{R}^N\to\mathbb{R}. \tag{2}$$

Prove that:

(a) ω is a purely atomic measure, i.e., there exist $\alpha_j > 0$, $x_j \in \mathbb{R}^N$ such that $\omega = \sum_j \alpha_j\delta_{x_j}$.

(b) $\sum_j(\alpha_j)^{p/q} < \infty$.

(c) $\lambda \ge S\sum_j(\alpha_j)^{p/q}\delta_{x_j}$.

(d) (2) holds if and only if it holds for $f \in C_c^\infty(\mathbb{R}^N)$.

Hint. *Step 1.* Assume first that λ is diffuse. Using the previous exercise, prove that, for every cube $C \subset \mathbb{R}^N$, $\omega(C) = 0$, and thus $\omega = 0$.

Step 2. Apply Step 1 to ω_0 and λ_0, where ω_0, respectively λ_0, is the diffuse part of ω, respectively λ.

Exercise. Let $1 \le p < \infty$ and $k \ge 1$ be such that $kp < N$. Let $\dfrac{1}{q} := \dfrac{1}{p} - \dfrac{k}{N}$.

Set

$$\dot{W}^{k,p} := \{u \in \mathscr{D}'(\mathbb{R}^N);\ D^k u \in L^p, u \in L^q\}.$$

Prove that, if we endow $\dot{W}^{k,p}$ with the norm $u \mapsto \|D^k u\|_p$, then $C_c^\infty(\mathbb{R}^N)$ is dense in $\dot{W}^{k,p}$. In particular, prove that we have the Sobolev inequality

$$S\|u\|_q^p \le \|D^k u\|_p^p,\ \forall\, u \in \dot{W}^{k,p}, \tag{3}$$

for some (optimal Sobolev constant) $0 < S < \infty$.

Useful reference for $k = 1$: [9, Lemma 14]. Hint for $k \ge 2$: prove the following result:

Exercise. Let k, p, and q be as above. For $R > 0$, set $A_R := \{x \in \mathbb{R}^N;\ R \le |x| \le 2R\}$. If $v \in C^\infty(A_R)$, then for every $\varepsilon > 0$ there exists some finite $C(\varepsilon)$ (independent of R and v) such that

$$\sum_{\ell=0}^{k-1} R^{-(k-\ell)}\|D^\ell v\|_{L^p(A_R)} \le \varepsilon\|D^k v\|_{L^p(A_R)} + C(\varepsilon)\|v\|_{L^q(A_R)}.$$

Exercise. Let μ be a finite measure on X and $1 \le p < q \le \infty$. If $(f_m) \subset L^q(X)$ is bounded and $f_m \to 0$ a.e., then $f_m \to 0$ in $L^p(X)$.

Second concentration-compactness lemma (Lions) Let $1 < p < \infty$, k, q, and S be as above. Let $(u_m) \subset \dot{W}^{k,p}$ and $u \in \dot{W}^{k,p}$ be such that:

(i) $u_m \rightharpoonup u$ in $\dot{W}^{k,p}$ and $u_m \to u$ a.e.

(ii) $|u_m|^q\,dx \overset{*}{\rightharpoonup} |u|^q dx + \omega$ in the sense of measures, for some (non-negative) Borel measure ω.

(iii) $|D^k u_m|^p dx \overset{*}{\rightharpoonup} |D^k u|^p dx + \mu$ in the sense of measures, for some (non-negative) Borel measure μ.

Then:

(a) ω is a purely atomic measure: $\omega = \sum_j \alpha_j \delta_{x_j}$, with $\alpha_j > 0$, $x_j \in \mathbb{R}^N$.

(b) We have $\sum_j (\alpha_j)^{p/q} < \infty$.

(c) We have $\mu \geq S \sum_j (\alpha_j)^{p/q} \delta_{x_j}$.

E **Theorem** (Aubin, Talenti, Lions) Let $k \geq 1$ and $1 < p < \infty$ be such that $kp < N$. Then there exists some $u \in \dot{W}^{k,p} \setminus \{0\}$ such that equality holds in (3).

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