Lecture # 2
MAXIMUM PRINCIPLES AND APPLICATIONS

(a) Basic maximum principles for positive elliptic operators

Here, we work in a bounded domain $\Omega \subset \mathbb{R}^N$ (domain=connected open set), and consider functions $c, u : \Omega \to \mathbb{R}$ such that $u \in C^2(\Omega)$ and

$$Lu := -\Delta u + c\,u \leq 0 \text{ in } \Omega, \tag{1}$$

$$c \geq 0 \text{ in } \Omega \text{ and } c \text{ is bounded.} \tag{2}$$

The non-negativity condition in (2) will be removed later, under additional assumptions on u and/or Ω.

General references for this section: [5, Sections 3.1–3.3], [6, Sections 2.2–2.3]

A **Exercise.** Let $A = (a_{ij})_{1\leq i,j\leq N}$ satisfy the *(weak) ellipticity condition*

$$\sum_{1\leq i,j\leq N} a_{ij}\xi_i\xi_j \geq 0,\ \forall\,\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N.$$

Let $B = (b_{ij})_{1\leq i,j\leq N}$ be a positive (symmetric) matrix. Then $\displaystyle\sum_{1\leq i,j\leq N} a_{ij}\, b_{ij} \geq 0.$

Use the above exercise to undertake the following

General exercise. Extend the results in this section to more general operators of the form

$$Lu = -\sum_{1\leq i,j\leq N} a_{ij}\partial_{ij}u + \sum_{1\leq j\leq N} b_j\partial_j u + cu,$$

with $b_j = b_j(x)$, $\forall\, j$, $c = c(x) \geq 0$ (possibly b_j bounded), $(a_{ij})_{1\leq i,j\leq N}$ weakly elliptic, or possibly satisfying the *uniform ellipticity condition*

$$\lambda\,|\xi|^2 \leq \sum_{1\leq i,j\leq N} a_{ij}(x)\,\xi_i\xi_j \leq \Lambda\,|\xi|^2,\ \forall\,\xi \in \mathbb{R}^N,\ \forall\, x \in \Omega,$$

where $0 < \lambda \leq \Lambda < \infty$.

B (Basic lemma; no sign or boundedness condition on c) If $Lu < 0$ and if u has a local maximum point $x_0 \in \Omega$, then $c(x_0)\,u(x_0) < 0$.

C (Weak maximum principle; no boundedness condition on c) If $u \in C(\overline{\Omega})$, then $u(x) \leq \sup_{\partial\Omega} u^+$, $\forall\, x \in \Omega$.

D (Hopf lemma) If:

(a) Ω is a ball

(b) $x_0 \in \partial\Omega$

(c) $u \in C^1(\overline{\Omega})$

(d) $u(x) < u(x_0), \forall\, x \in \Omega$

(e) $u(x_0) \geq 0$,

then $\dfrac{\partial u}{\partial \nu}(x_0) > 0$, where ν is the outward unit normal at Ω at x_0.

In the above and in all the versions below of the Hopf lemma, we may replace the assumption $u \in C^1(\overline{\Omega})$ by the continuity of u at x_0 and the existence of $\dfrac{\partial u}{\partial \nu}(x_0)$. We may also require boundedness of c only near x_0.

E (Strong maximum principle) If $u \in C(\overline{\Omega})$ and u has an interior maximum point x_0 such that $u(x_0) \geq 0$, then u is constant.

F (Comparison principle) If $u \in C(\overline{\Omega})$ and $u \leq 0$ on $\partial\Omega$, then either $u = 0$ in Ω, or $u < 0$ in Ω.

G (Hopf boundary lemma)

a) Assume that Ω is of class C^2. If $u \in C^1(\overline{\Omega})$, u is non constant, and $x_0 \in \partial\Omega$ is a boundary maximum point of u such that $u(x_0) \geq 0$, then $\dfrac{\partial u}{\partial \nu}(x_0) > 0$, where ν is the outward unit normal to Ω at x_0.

b) In particular, if $u \in C^1(\overline{\Omega})$, u is non constant, and $u = 0$ on $\partial\Omega$, then $\dfrac{\partial u}{\partial \nu}(x_0) > 0$ at each point $x_0 \in \partial\Omega$.

c) If we don't assume Ω of class C^2, then the above conclusions hold at each point $x_0 \in \partial\Omega$ such that there exists an open ball $B \subset \Omega$ with $\overline{B} \cap \partial\Omega = \{x_0\}$.

Useful result:

Exercise. Let Ω be a C^2 domain. (Or even a $C^{1,1}$ domain.) Then Ω satisfies the *interior ball condition* and the *exterior ball condition*: for every $x_0 \in \partial\Omega$, there exist open balls B and C such that: $B \subset \Omega$, $C \subset \mathbb{R}^N \setminus \overline{\Omega}$, $\overline{B} \cap \partial\Omega = \{x_0\}$, $\overline{C} \cap \partial\Omega = \{x_0\}$.

In addition, we may choose balls of same radius.

Useful reference: [7, Lemma 2.31]

(b) Maximum principles for elliptic operators

Here, we don't have any sign assumption on c, but we still assume that $-\Delta u + c\,u \leq 0$ in Ω.

General references for this section: [5, Section 9.1], [6, Section 2.5]

A (Serrin's maximum principle) If $u \leq 0$ in Ω, then either $u = 0$ in Ω, or $u < 0$ in Ω.

B (Maximum principle in narrow domains) Assume that $c \geq -M$ for some $M \geq 0$. Then there exists some $\delta > 0$ and $C > 0$ depending only on M such that, if Ω is contained in a strip of width $\leq \delta$ and $u \in C(\overline{\Omega})$, then $u(x) \leq C \sup_{\partial\Omega} u^+, \forall\, x \in \Omega$.

C (Hopf boundary lemma) Let $u \in C^1(\overline{\Omega})$ satisfy $u \leq 0$ on $\partial\Omega$ and $u < 0$ in Ω. Then, at each point $x_0 \in \partial\Omega$ such that $u(x_0) = 0$ and there exists an open ball $B \subset \Omega$ with $\overline{B} \cap \partial\Omega = \{x_0\}$, we have $\dfrac{\partial u}{\partial \nu}(x_0) > 0$.

D (Supercritical maximum principle) Let $N < p \leq \infty$. Assume that:

(a) $c \geq 0$

(b) $u \in C(\overline{\Omega})$

(c) $-\Delta u + c\, u \leq f$, with $f \in L^p(\Omega, \mathbb{R}_+)$.

If $d := \operatorname{diam}(\Omega)$, then

$$u(x) \leq \sup_{\partial\Omega} u^+ + C_{p,N} d^{2-p/N} \|f\|_p. \tag{3}$$

E (Varadhan's maximum principle; maximum principle in small domains) Fix two numbers $M \geq 0$, $D > 0$. Then there exists some $\delta = \delta(M, D) > 0$ such that: if $c \geq -M$, $\operatorname{diam}(\Omega) \leq D$, $|\Omega| < \delta$, then: for every non constant $u \in C(\overline{\Omega})$ such that $-\Delta u + c\, u \leq 0$ in Ω and $u \leq 0$ on $\partial\Omega$, we have $u < 0$ in Ω.

F **Exercise** (Alexandroff's maximum principle) With the notation in item D, we assume this time that $f \in L^N(\Omega, \mathbb{R}_+)$. Prove the estimate

$$u(x) \leq \sup_{\partial\Omega} u^+ + C_N d \|f\|_N, \tag{4}$$

which formally corresponds to $p = N$ in (3), following these lines:

(a) Let $M \subset \Omega$ be a Borel set. Prove that

$$\int_M |\det(D^2 u)(x)|\, dx \geq |Du(M)|.$$

Hint: use the area formula/Banach indicatrix formula. [4, Section 3.4.3, Theorem 2], [8, Exercices de synthèse et avancés, Exercice 28].

(b) Let

$$M := \{x \in \Omega;\ u(x) \geq \sup_{\partial\Omega} u_+ \text{ and } u(y) \leq u(x) + Du(x) \cdot (y - x),\ \forall\, y \in \Omega\}.$$

Prove that, at each $x \in M$:

i. $D^2 u(x)$ is non-positive

ii. $|\det(D^2 u)(x)| \leq \left(\dfrac{-\Delta u(x)}{N}\right)^N \leq \left(\dfrac{f(x)}{N}\right)^N$.

(c) Prove that, if $a \in \mathbb{R}^N$ is such that

$$|a| < \frac{\sup_{\Omega} u - \sup_{\partial\Omega} u_+}{d},$$

then $a \in Du(M)$.
Hint: prove that $\overline{\Omega} \ni x \mapsto u(x) - a \cdot x$ achieves its maximum in Ω.

(d) Derive (4).

G (Stampacchia's maximum principle) Here, Ω is of class C^1 (Lipschitz would suffice). If:

(a) $c = c(x) > -\lambda_1(\Omega)$ for a.e. $x \in \Omega$, with $\lambda_1(\Omega)$ the least eigenvalue of $-\Delta$ in $H_0^1(\Omega)$

(b) c is bounded from above

(c) $u \in H^1(\Omega)$ satisfies $-\Delta u + c\,u \geq 0$ in Ω and $u \geq 0$ on $\partial\Omega$,

then $u \geq 0$ in Ω.

Here, the meaning of $-\Delta u + c\,u \geq 0$ is

$$\int_\Omega \nabla u \cdot \nabla \varphi + \int_\Omega c\,u\,\varphi \geq 0,\ \forall\,\varphi \in C_c^\infty(\Omega, \mathbb{R}_+).$$

Useful results: the *de la Vallée Poussin chain rule* and its consequences [7, Theorem 3.16, Corollary 3.17, Lemma 3.20], and basic trace theory [7, Section 1.5], [2, Theorem 9.17]

(c) Moving planes

General references: [6, Section 2.6], [1]

A **Theorem.** (Gidas-Ni-Nirenberg) Let Ω be a convex bounded domain, symmetric with respect to the hyperplane $\{x_1 = 0\}$. If $f \in \mathrm{Lip}_{loc}(\mathbb{R}, \mathbb{R})$ and if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies

$$\begin{cases} -\Delta u + f(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\ , \\ u > 0 & \text{in } \Omega \end{cases} \tag{5}$$

then:

(a) $u(-x_1, x_2, \ldots, x_N) = u(x_1, \ldots, x_N), \forall\, x = (x_1, \ldots, x_N) \in \Omega$

(b) $\partial_1 u < 0$ in the set $\{x \in \Omega;\ x_1 > 0\}$.

B **Theorem.** (Gidas-Ni-Nirenberg) Let $\Omega = B_R(0)$, and f and u as above. Then there exists some $g \in C^2([0, R)) \cap C([0, R])$ such that:

(a) $u(x) = g(|x|), \forall\, x \in \overline{\Omega}$

(b) $g'(r) < 0, \forall\, 0 < r < R$.

(d) Sub- and supersolutions

General reference: [3, Section 9.3]
We consider the problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}. \tag{6}$$

We assume that:

1. Ω is connected and of class C^3.
2. $f \in C^1(\overline{\Omega} \times \mathbb{R})$.

A **Theorem.** Assume that there exist $\underline{u}, \overline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$ such that:

(a) $-\Delta \underline{u} \leq f(x, \underline{u})$ in Ω and $\underline{u} \leq 0$ on $\partial\Omega$ ($\underline{u}$ is a *subsolution*)
(b) $-\Delta \overline{u} \geq f(x, \overline{u})$ in Ω and $\overline{u} \geq 0$ on $\partial\Omega$ ($\overline{u}$ is a *supersolution*)
(c) $\underline{u} \leq \overline{u}$ in Ω.

Then the set

$$M := \{u \in C^2(\overline{\Omega});\ u \text{ solves (6) and } \underline{u} \leq u \leq \overline{u}\}$$

is non-empty, and contains a minimal element and a maximal element.

B Same conclusion if $\underline{u}, \overline{u} \in H^1(\Omega) \cap C(\overline{\Omega})$.

C **Exercise.** [3, Section 9.7, exercise 6] Assume $f = f(u)$, with $f(0) = 0$, $f'(0) > \lambda_1(\Omega)$, and f bounded from above on $\mathbb{R}_+$. Prove that the problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega \end{cases} \tag{7}$$

has a solution $u \in C^2(\overline{\Omega})$.

(e) Uniqueness

In this section, Ω is of class C^2, and we work with functions $u \in C^2(\overline{\Omega})$.
General reference: [3, Section 9.4.2]

A **Theorem.** (Krasnoselskii) Let $f : [0, \infty) \to [0, \infty)$ be a continuous function such that

$$(0, \infty) \ni t \mapsto \frac{f(t)}{t} \text{ is (strictly) decreasing.}$$

Then the problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega \end{cases} \tag{8}$$

has at most one solution.

B **Theorem** (Pohozaev) Assume $N \geq 3$ and $\dfrac{N+2}{N-2} < p < \infty$. Assume also that Ω is convex. Then the problem

$$\begin{cases} -\Delta u = |u|^{p-1} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{9}$$

has only the trivial solution $u = 0$.

References

[1] Henri Berestycki and Louis Nirenberg. On the method of moving planes and the sliding method. *Bol. Soc. Brasil. Mat. (N.S.)*, 22(1):1–37, 1991.

[2] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.

[3] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[4] Lawrence C. Evans and Ronald F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[5] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*, volume 224 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1983.

[6] Qing Han and Fanghua Lin. *Elliptic partial differential equations*, volume 1 of *Courant Lecture Notes in Mathematics*. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1997.

[7] Petru Mironescu. *Sobolev spaces. Elliptic equations* (Partial lecture notes). `http://math.univ-lyon1.fr/~mironescu/resources/sobolev_spaces_elliptic_equations_2010.pdf`, 2010.

[8] Petru Mironescu. *Mesure et intégration*. `http://math.univ-lyon1.fr/~mironescu/resources/complet_mesure_integration.pdf`, 2020.