Lecture # 2
TOPOLOGICAL METHODS

In what follows, $\Omega \subset \mathbb{R}^N$ is a bounded domain. (Domain = open, connected.)

**(a) Mountain pass solutions**

Useful references: [1, Chapter 4], [3, Chapter 1]

A **Ekeland's Variational Principle** Let $(M, d)$ be a complete metric space, $\Phi : M \to \mathbb{R}$, and $\varepsilon > 0$. Assume that:

(i) $\Phi$ is l. s. c. and $m := \inf \Phi > -\infty$.
(ii) $u \in M$ is such that $\Phi(u) < m + \varepsilon^2$.

Then there exists some $v \in M$ such that:

(a) $\Phi(v) \le \Phi(u)$.
(b) $\Phi(w) > \Phi(v) - \varepsilon\, d(w, v), \forall\, w \in M \setminus \{v\}$.
(c) $d(v, u) < \varepsilon$.

In the special case where $(M, \| \|)$ is a Banach space and $\Phi$ is Gâteaux differentiable at the above $v$, we have

$$\left| \frac{\partial \Phi}{\partial y}(v) \right| \le \varepsilon \|y\|,\ \forall\, y \in M.$$

B **Minimax principle** (Shi) Consider:

(i) A Banach space $X$ and $J \in C^1(X, \mathbb{R})$.
(ii) A compact metric space $(K, d)$, a compact subspace $K_0 \subset K$, and a continuous map $\zeta : K_0 \to X$.

Set

$$\begin{aligned} &M := \{\gamma \in C(K, X);\ \gamma_{|K_0} = \zeta\}, \\ &\Phi(\gamma) := \max_K J \circ \gamma,\ \forall\, \gamma \in M, \\ &c := \inf_M \Phi, \\ &c_0 := \max_{K_0} J \circ \zeta. \end{aligned}$$

If $c > c_0$, then there exists a sequence $(x_j) \subset X$ such that:

(a) $J(x_j) \to c$.
(b) $J'(x_j) \to 0$ in $X'$.

**Corollary.** Assume that, with $c$ as above, $J$ satisfies the *Palais-Smale condition* $(\mathrm{PS})_c$ *at level* $c$: any sequence $(x_j) \subset X$ satisfying (a) and (b) contains a convergent subsequence. Then $J$ has a critical point $x$ such that $J(x) = c$.

**Exercise.** Prove that the above application $\Phi$ is continuous.

**Exercise.** Let $K$, $K_0$, $X$, $J$ be as above. Let $S \subset K \setminus K_0$ be a compact set. Let $F \in C(K, X)$. Assume that $\|J' \circ F(t)\| > 1, \forall\, t \in S$. Prove that there exist an integer $N$, functions $\zeta_j \in C(K, [0,1])$, vectors $x_j \in X$, $j = 1, \ldots, N$, and a number $\delta > 0$ such that:

(a) $\operatorname{supp} \zeta_j \subset K \setminus K_0, \forall\, j$;
(b) $0 \le \sum_j \zeta_j \le 1$;
(c) $\sum_j \zeta_j(t) = 1, \forall\, t \in S$;
(d) $\|x_j\| = 1, \forall\, j$;
(e) $\forall\, j, \forall\, t \in K, \forall\, u \in X, [\zeta_j(t) \neq 0,\ \|F(t) - u\| < \delta] \implies J'(u)x_j > 1$.

C **Exercise.** Prove the *Ambrosetti-Rabinowitz Mountain pass theorem*. Let $X$ be a Banach space and $J \in C^1(X, \mathbb{R})$. Assume that there exist $R > 0$ and $x_0 \in X$ such that:

(i) $\max\{J(0),\ J(x_0)\} < \inf\{J(x);\ \|x\| = R\}$.
(ii) $\|x_0\| > R$.

Set

$$c := \inf\left\{ \max_{t \in [0,1]} J(\gamma(t));\ \gamma \in C([0,1], X),\ \gamma(0) = 0,\ \gamma(1) = x_0, \right\}.$$

If $J$ satisfies the $(\mathrm{PS})_c$ condition, then $J$ has a critical point $x$ such that $J(x) = c$.

**Theorem.** Let $\Omega \subset \mathbb{R}^N$ be a bounded $C^{1,1}$-domain and $1 < p < \dfrac{N+2}{N-2}$.

Prove that the problem

$$\begin{cases} -\Delta u = \lambda u + u^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega \end{cases}$$

has a classical solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ if and only if $\lambda < \lambda_1(\Omega)$.
Useful reference: [3, Theorem 1.19]

**Exercise.** Let $p$ be as above. If $u_j \rightharpoonup u$ and $\varphi_j \rightharpoonup \varphi$ in $H^1_0(\Omega)$, prove that

$$\int_\Omega [(u_j)_+]^p \varphi_j \to \int_\Omega [u_+]^p \varphi,\ \forall\, \varphi \in H^1_0(\Omega).$$

What if $p = \dfrac{N+2}{N-2}$?

**(b) A glimpse of other topological methods**

A **Exercise.** Prove the *Rabinowitz saddle point theorem*. Let $X$ be a Banach space and $J \in C^1(X, \mathbb{R})$. Let $X = X^- \oplus X^+$, with $X^-$ finite dimensional and $X^+$ closed.

For fixed $R > 0$, let

$$K := \{x \in X^-;\ \|x\| \le R\} \text{ and } K_0 := \{x \in X^-;\ \|x\| = R\}.$$

Assume that:

(i) $\max_{K_0} J < \inf_{X^+} J$.

(ii) $J$ satisfies the (PS)$_c$ condition, where

$$c := \inf\{\max J \circ g;\ g \in C(K, X), g(x) = x \text{ if } x \in K_0\}.$$

Then $J$ has a critical point $x$ such that $J(x) = c$.

Useful references: [1, Theorem 4.7], [2]

**Exercise.** Let $K$ be a compact subset of a finite dimensional normed space $X$. Let $g \in C(K, X)$ be such that $g(x) = x, \forall\, x \in \partial K$. Prove that $g(K) \supset K$.

**Theorem.** Let $N \ge 3$ and $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Let $a = a(x) \in L^{(2N)/(N+2)}(\Omega)$, $a \ge 0$, $a \not\equiv 0$. Prove that the problem

$$\begin{cases} -\Delta u = a \dfrac{u + \cos u}{\sqrt{1 + 2\sin u + u^2}} & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

has a weak solution $u \in H^1(\Omega)$.

**Exercise.** Let $N \ge 3$ and $a \in L^{(2N)/(N+2)}(\Omega)$. Let $f \in C^1(\mathbb{R}, \mathbb{R})$ be a Lipschitz function. Set

$$F(u) := \int_\Omega a(x) f(u(x))\, dx,\ \forall\, u \in H^1(\Omega).$$

Prove that $F \in C^1(H^1(\Omega), \mathbb{R})$ and that

$$F'(u)\varphi = \int_\Omega a(x) f'(u(x))\varphi(x)\, dx,\ \forall\, u \in H^1(\Omega),\ \forall\, \varphi \in H^1(\Omega).$$

**Exercise.** Assume that $\Omega$ is Lipschitz. Prove the following *Poincaré inequality*: there exists some finite $C = C_\Omega$ such that

$$\int_\Omega u^2 \le C \int_\Omega |\nabla u|^2,\ \forall\, u \in H^1(\Omega) \text{ s.t. } \int_\Omega u = 0.$$

B **Rabinowitz linking theorem** Let $X$, $X^-$, $X^+$, and $J$ be as above.

For fixed $R > \rho > 0$ and $z \in X^+ \setminus \{0\}$, let

$$K := \{u = x + tz;\ x \in X^-,\ t \geq 0,\ \|u\| \leq R\},$$
$$K_0 := \partial K \text{ (where } K \text{ is considered as a subset of } X^- \oplus \mathbb{R}z),$$
$$L := \{x \in X^+;\ \|x\| = \rho\}.$$

Assume that:

(i) $\max_{K_0} J < \min_L J$.

(ii) $J$ satisfies the (PS)$_c$ condition, where

$$c := \inf\{\max J \circ g;\ g \in C(K, X), g(x) = x \text{ if } x \in K_0\}.$$

Then $J$ has a critical point $x$ such that $J(x) = c$.

Useful reference: [3, Theorem 2.12]

**Theorem.** Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $\lambda \in \mathbb{R}$, and $1 < p < \dfrac{N+2}{N-2}$. The equation

$$-\Delta u = \lambda u + |u|^{p-1} u \text{ in } \Omega$$

has a non trivial solution $u \in H_0^1(\Omega)$.

Useful reference: [3, Theorem 2.18, Corollary 2.19]

**Exercise.** Let $p$ be as above. Let $Y \subset H_0^1(\Omega)$ be a finite dimensional subspace. Let $(u_j) \subset Y$ be a bounded sequence such that $u_j \to 0$ in $L^{p+1}(\Omega)$. Prove that $u_j \to 0$ in $H_0^1(\Omega)$.

C For the record:

**Theorem** (**Lusternik/Ljusternik**, **Schnirelman**, Rabinowitz) Let $X$ be a Banach space and $G$ a discrete subgroup of $X$ spanning an $N$-dimensional subspace of $X$.

Let $J \in C^1(X, \mathbb{R})$ be such that:

(i) $J(x + g) = J(x), \forall\, x \in X, \forall\, g \in G$.

(ii) $J : X/G \to \mathbb{R}$ satisfies the (PS)$_c$ condition at any level $c \in \mathbb{R}$.

(iii) $J$ is bounded from below.

Then $J$ has at least $N + 1$ critical orbits, i.e., there exist $x_1, \dots, x_{N+1} \in X$ such that:

(a) $J'(x_j) = 0, \forall\, j$.

(b) $x_j - x_k \notin G$ if $j \neq k$.

Useful reference: [1, Section 4.6]

# References

[1] Jean Mawhin and Michel Willem. *Critical point theory and Hamiltonian systems*, volume 74 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1989.

[2] John Milnor. Analytic proofs of the "hairy ball theorem" and the Brouwer fixed-point theorem. *Amer. Math. Monthly*, 85(7):521–524, 1978.

[3] Michel Willem. *Minimax theorems*, volume 24 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA, 1996.